ON THE GLUING PROBLEM FOR DIRAC OPERATORS ON MANIFOLDS WITH CYLINDRICAL ENDS

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ABSTRACT. Combining elements of the b-calculus and the theory of elliptic boundary value problems, we solve the gluing problem for b-determinants of Dirac type operators on manifolds with cylindrical ends. As a corollary of our proof, we derive a gluing formula for the b-eta invariant and also a relative invariant formula relating the b-spectral invariants on a manifold with cylindrical end to the spectral invariants with the augmented APS boundary condition on the corresponding compact manifold with boundary.

1. INTRODUCTION

Two central objects of study in the spectral geometry of Dirac operators are the eta invariant and ζ-determinant. In particular, the behavior of eta invariants under "gluing" or "surgery" of the underlying manifolds has enjoyed great research activity within the past several years, cf. Brüning and Lesch [5], Bunke [7], Dai and Freed [11], Hassell, Mazzeo, and Melrose [16, 17], Kirk and Lesch [19], Loya and Park [25], Mazzeo and Melrose [28], Müller [31], Park and Wojciechowski [34], Wojciechowski [43, 44]; see the survey articles by Bleecker and Booß-Bavnbek [3] and Mazzeo and Piazza [29]. The gluing problem for the ζ-determinant of Laplace type operators was pioneered by Burghelea, Friedlander, and Kappeler [8] and has been further developed by Carron [10], Hassell [15], Hassell and Zelditch [18], Lee [20], Loya and Park [24], Vishik [42], and others. However, only recently was the gluing problem for the ζ-determinant of Dirac type operators solved for closed manifolds by the authors in [25]. The purpose of this paper is to extend this result to *manifolds with cylindrical end*. To accomplish this we enhance our technique initiated in [24, 25] with Melrose's b-calculus [30] in order to circumvent many new features and difficulties not found in the compact case, especially in connection to the presence of continuous spectrum. The technical advantage of this approach is that we can derive the corresponding gluing formula for the eta invariant as a simple byproduct of our ζ-determinant proof; this is because the gluing formulas of these two invariants represent just two facets (the phase and modulus) of one spectral data.

The set up of our problem is as follows. Let X be a n-dimensional Riemannian manifold with a cylindrical end, that is, we have a decomposition

$$X = M \cup Z,$$

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where M is a compact manifold with boundary Y and $Z = [0, \infty) \times Y$ is a half infinite cylinder. We also assume that M has a tubular neighbourhood $N = [-1, 0] \times Y$ of Y. Let $\mathcal{D}$ be a Dirac type operator acting on $C^\infty(X, S)$ where S is a Clifford bundle over X. We assume that all geometric structures are of product type over $\hat{Z} := N \cup Z = [-1, \infty)_u \times Y$, where u is the cylindrical variable; in particular, over $\hat{Z}$, the Dirac operator takes the product form

$$\mathcal{D} = G(\partial_u + D_Y),$$

where G is a unitary map on $S_0 := S|_Y$ and D_Y is a Dirac type operator acting on $C^\infty(Y, S_0)$ such that $G^2 = -\mathrm{Id}$ and $D_Y G = -G D_Y$. Then

$$\mathcal{D} : H^1(X, S) \longrightarrow L^2(X, S)$$

is self-adjoint and has a finite-dimensional kernel, but in general is not Fredholm, and even worse, has continuous spectrum equal to all of $\mathbb{R}$! At this point, it is worth mentioning that the study of Dirac operators on manifolds with cylindrical ends has intensified since the publication of Atiyah, Patodi, and Singer's seminal paper [1] and after Melrose's [30] recasting of their index theorem in terms of his b-calculus.

We now discuss the spectral invariants — the eta invariant and ζ-determinant. Consider first the case of the compact manifold with boundary M. In order to define the spectral invariants for this case, we need to impose pseudodifferential boundary conditions at Y. A natural choice is the Calderón projector $\mathcal{P}_M$ [9], which is the orthogonal projector onto the closure in $L^2(Y, S_0)$ of the *Cauchy data space* of the restriction of $\mathcal{D}$ to M:

$$\{\, \phi|_Y \mid \phi \in C^\infty(M, S)\ ,\ \mathcal{D}\phi = 0 \,\} \subset C^\infty(Y, S_0).$$

Then, imposing boundary condition given by $\mathcal{P}_M$ for $\mathcal{D}|_M$, we obtain a self adjoint Fredholm operator,

$$\mathcal{D}_{\mathcal{P}_M} : \mathrm{dom}(\mathcal{D}_{\mathcal{P}_M}) \longrightarrow L^2(M, S), \tag{1.1}$$

where $\mathrm{dom}(\mathcal{D}_{\mathcal{P}_M}) := \{\, \phi \in H^1(M, S) \mid \mathcal{P}_M(\phi|_Y) = 0 \,\}$. The η-function of $\mathcal{D}_{\mathcal{P}_M}$ and the ζ-function of $\mathcal{D}^2_{\mathcal{P}_M}$ are defined through the heat operator $e^{-t\mathcal{D}^2_{\mathcal{P}_M}}$ via

$$\eta_{\mathcal{D}_{\mathcal{P}_M}}(s) = \frac{1}{\Gamma(\frac{s+1}{2})}\Big(\int_0^1 + \int_1^\infty\Big)\, t^{\frac{s-1}{2}}\, \mathrm{Tr}(\mathcal{D}_{\mathcal{P}_M} e^{-t\mathcal{D}^2_{\mathcal{P}_M}})\, dt, \tag{1.2}$$

$$\zeta_{\mathcal{D}^2_{\mathcal{P}_M}}(s) = \frac{1}{\Gamma(s)}\Big(\int_0^1 + \int_1^\infty\Big)\, t^{s-1}\, \mathrm{Tr}(e^{-t\mathcal{D}^2_{\mathcal{P}_M}})\, dt, \tag{1.3}$$

where the integrals $\int_0^1$ are defined a priori for $\Re s \gg 0$ and the integrals $\int_1^\infty$ a priori for $\Re s \ll 0$ respectively and both of which extend to be meromorphic functions on $\mathbb{C}$ that are regular at $s = 0$, see Grubb [13], [14] and Wojciechowski [44]; in this case, the second integrals are actually entire, but we present these general definition because these work later for b-eta invariants and b-zeta determinant. Then the eta invariant of $\mathcal{D}_{\mathcal{P}_M}$ is defined by

$$\eta(\mathcal{D}_{\mathcal{P}_M}) := \eta_{\mathcal{D}_{\mathcal{P}_M}}(0),$$

and the ζ-determinant of $\mathcal{D}^2_{\mathcal{P}_M}$ is defined by

$$\det{}_\zeta \mathcal{D}^2_{\mathcal{P}_M} := \exp\Big(-\frac{d}{ds}\Big|_{s=0} \zeta_{\mathcal{D}^2_{\mathcal{P}_M}}(s)\Big). \tag{1.4}$$

The eta invariant was introduced in Atiyah, Patodi, and Singer's paper [1] as the boundary correction term in their index formula for manifolds with boundary and the ζ-determinant was introduced by Ray and Singer in [35] in their study of analytic torsion.

For the noncompact manifold with cylindrical end X, the heat operators $\mathcal{D}e^{-t\mathcal{D}^2}$ and $e^{-t\mathcal{D}^2}$ are not of trace class. In particular, the definitions (1.2) and (1.3) cannot be used directly to define the corresponding η-function for $\mathcal{D}$ and ζ-function for $\mathcal{D}^2$. There are two main routes that one can follow to make sense of these invariants. One way is to define so-called *relative invariants* as in Bruneau [6], Carron [10], and Müller [32], and others, whereby we subtract off certain operators that make the difference of the heat operators trace class. However, we shall follow Melrose's path using the b-trace [30], ${}^b\mathrm{Tr}$, which is a natural substitute for the trace. In particular, $\mathcal{D}e^{-t\mathcal{D}^2}$ and $e^{-t\mathcal{D}^2}$ are b-trace class. Moreover, ${}^b\mathrm{Tr}\,\mathcal{D}e^{-t\mathcal{D}^2}$ and ${}^b\mathrm{Tr}\,e^{-t\mathcal{D}^2}$ have asymptotic expansions in half-integer powers of t as $t \to 0$ and $t \to \infty$. It follows that the ${}^b\eta$-function ${}^b\eta_{\mathcal{D}}(s)$ and ${}^b\zeta$-function ${}^b\zeta_{\mathcal{D}^2}(s)$ can be defined exactly as in the formulas (1.2) and (1.3), respectively, where we replace $\mathcal{D}_{\mathcal{P}_M}$ with $\mathcal{D}$ and Tr with ${}^b\mathrm{Tr}$. Furthermore, ${}^b\eta_{\mathcal{D}}(s)$ and ${}^b\zeta_{\mathcal{D}^2}(s)$ extend to define meromorphic functions on $\mathbb{C}$ that are regular at $s = 0$, so we can define the b-eta invariant of $\mathcal{D}$ by ${}^b\eta(\mathcal{D}) := {}^b\eta_{\mathcal{D}}(0)$ and the b-determinant of $\mathcal{D}^2$, $\det_{{}^b\zeta}\mathcal{D}^2$, by the formula (1.4) using ${}^b\zeta_{\mathcal{D}^2}(s)$. An introduction to the b-trace is presented in Section 2.

We now discuss the b-spectral invariants on Z, having discussed the spectral invariants on the whole manifold X and its compact part M. Because $Z = [0, \infty) \times Y$ is a manifold with boundary, we need to impose boundary conditions. With this in mind, we ask: What is the natural boundary condition? The answer is to look at the "Cauchy data space" of $\mathcal{D}$ over the whole manifold X:

$$\{\, \phi|_Y \mid \phi \in C^\infty(X, S)\ ,\ \mathcal{D}\phi = 0 \,\} \subset C^\infty(Y, S_0).$$

It turns out (see [30, 31]) that an element ψ of this space is the restriction $\psi = \phi|_Y$ of an L^2 section ϕ over X if and only if $\big(\Pi_< + \frac{\mathrm{Id}+\sigma}{2}\Pi_0\big)\psi = 0$, where $\Pi_<$ is the orthogonal projection onto the eigenspaces of D_Y with negative eigenvalues, Π_0 is the orthogonal projection onto $\ker(D_Y)$, and where σ is *the* unitary map on $\ker(D_Y)$ such that $\sigma^2 = \mathrm{Id}$ and $\sigma G = -G\sigma$ determined by the *scattering matrix.* (Let us note that we have $\mathcal{P}_M\Pi_0 = \frac{\mathrm{Id}+\sigma}{2}\Pi_0$ by definition of $\mathcal{P}_M$.) For more details about the scattering matrix, we refer to [30, Ch. 6], [31, Sec. 4]. For this reason, the natural projection on Z is

$$\mathcal{P}_Z := \Pi_> + \frac{\mathrm{Id} - \sigma}{2}\Pi_0$$

where $\Pi_>$ is the orthogonal projection onto the eigenspaces of D_Y with positive eigenvalues. We can now define

$$\mathcal{D}_{\mathcal{P}_Z} : \mathrm{dom}(\mathcal{D}_{\mathcal{P}_Z}) \longrightarrow L^2(Z, S),$$

where $\mathrm{dom}(\mathcal{D}_{\mathcal{P}_Z}) := \{\, \phi \in H^1(Z, S) \mid \mathcal{P}_Z(\phi|_Y) = 0 \,\}$. Using the b-trace, we can define the b-spectral invariants ${}^b\eta(\mathcal{D}_{\mathcal{P}_Z})$ and $\det_{{}^b\zeta}\mathcal{D}^2_{\mathcal{P}_Z}$.

We now have all the ingredients necessary to state the gluing problem for our manifold. The *gluing problem* is to describe the "defects"

$$\frac{\det_{{}^b\zeta}\mathcal{D}^2}{\det_\zeta \mathcal{D}^2_{\mathcal{P}_M} \cdot \det_{{}^b\zeta}\mathcal{D}^2_{\mathcal{P}_Z}} = \boxed{?} \quad , \quad {}^b\eta(\mathcal{D}) - \eta(\mathcal{D}_{\mathcal{P}_M}) - {}^b\eta(\mathcal{D}_{\mathcal{P}_Z}) = \boxed{?}$$

in terms of recognizable data. Before describing our solution, we first introduce some natural operators. The Calderón projectors $\mathcal{P}_M$ and $\mathcal{P}_Z$ have the following matrix forms [37]:

$$\mathcal{P}_M = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_M^{-1} \\ \kappa_M & \mathrm{Id} \end{pmatrix} \quad , \quad \mathcal{P}_Z = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_Z^{-1} \\ \kappa_Z & \mathrm{Id} \end{pmatrix} \tag{1.5}$$

with respect to the decomposition $L^2(Y, S_0) = L^2(Y, S^+) \oplus L^2(Y, S^-)$ where $S^\pm$ are the subbundles of S_0 equal to the $\pm i$ eigenspaces of G, and where κ_M and κ_Z are isometries from $L^2(Y, S^+)$ onto $L^2(Y, S^-)$. In particular, the operator $U := -\kappa_M \kappa_Z^{-1}$ is a unitary operator over $L^2(Y, S^-)$. We denote by $\widehat{U}$ the restriction of U to the orthogonal complement of its (-1)-eigenspace. We also put

$$\mathcal{L} := \sum_{k=1}^{h_X} \gamma_0 U_k \otimes (\gamma_0 U_k)^* \tag{1.6}$$

where $h_X = \dim \ker_{L^2}(\mathcal{D})$ (the L^2-kernel of $\mathcal{D}$), γ_0 is the restriction map from X to Y, $\{U_k\}$ is an orthonormal basis of $\ker_{L^2}(\mathcal{D})$, and $(\gamma_0 U_k)^* := \langle \cdot\,, \gamma_0 U_k \rangle_Y$ where $\langle \cdot\,, \cdot \rangle_Y$ is the inner product on $L^2(Y, S_0)$. Then $\mathcal{L}$ is a positive operator on the finite-dimensional vector space $\gamma_0(\ker_{L^2}(\mathcal{D}))$. The following theorem is our first main result.

Theorem 1.1. *The following ζ-determinant gluing formula holds:*

$$\frac{\det_{{}^b\zeta} \mathcal{D}^2}{\det_\zeta \mathcal{D}^2_{\mathcal{P}_M} \cdot \det_{{}^b\zeta} \mathcal{D}^2_{\mathcal{P}_Z}} = 2^{-\zeta_{D_Y^2}(0) - h_Y} (\det \mathcal{L})^{-2} \det{}_F \Big(\frac{2\mathrm{Id} + \widehat{U} + \widehat{U}^{-1}}{4} \Big)$$

where $\zeta_{D_Y^2}(s)$ is the ζ-function of D_Y^2, $h_Y = \dim \ker(D_Y)$, and $\det_F$ denotes the Fredholm determinant.

This formula appears the same as the gluing formula in the compact case [25], which may seem quite remarkable due to the decidedly nontrivial issue of the continuous spectrum of $\mathcal{D}$. One of the main accomplishments of this paper is the analysis of this issue relying in part on certain "miraculous cancellations", see Section 6. The proof of Theorem 1.1 is achieved through two crucial ingredients. The first is the introduction of an operator $K(\lambda)$ that links the Cauchy data spaces of $\mathcal{D} - \lambda$ on the two parts M and Z of X with the resolvents $(\mathcal{D} - \lambda)^{-1}$, $(\mathcal{D}_{\mathcal{P}_M} - \lambda)^{-1}$, $(\mathcal{D}_{\mathcal{P}_Z} - \lambda)^{-1}$. The next component is to compare these objects on X with corresponding objects, $\hat{K}(\lambda)$ and resolvents, for an *auxiliary model problem* on $\hat{Z}$. The model gluing problem for the partition of $\hat{Z}$ into N and Z is explicitly solvable (cf. [26]), and this facilitates the derivation of our gluing formula with no undetermined constants. This model problem also enables us to establish the trace class nature of the differences of the resolvents on X and $\hat{Z}$.

We can recast the formula in Theorem 1.1 in various ways. For example, some computations (see Section 2) show that $\det_{{}^b\zeta} \mathcal{D}^2_{\mathcal{P}_Z} = 2^{\frac{1}{2}\zeta_{D_Y^2}(0)}$, so after substituting this equality into our main theorem, we can obtain another version of our formula without the $\det_{{}^b\zeta} \mathcal{D}^2_{\mathcal{P}_Z}$ term. We can get another recasting of our theorem in terms of the relative ζ-determinant. One can show that $\det_{{}^b\zeta} \mathcal{D}^2 / \det_{{}^b\zeta} \mathcal{D}^2_{\mathcal{P}_Z} = \det_\zeta(\mathcal{D}^2, \mathcal{D}^2_{\mathcal{P}_Z})$, the relative ζ-determinant of the pair $(\mathcal{D}^2, \mathcal{D}^2_{\mathcal{P}_Z})$, so that our main formula can be written with the relative ζ-determinant on the left-hand side. As byproduct of our proof of Theorem 1.1, we also obtain the gluing formula for the eta invariant.

Corollary 1.2 (of proof). *The following eta invariant gluing formula holds:*

$$ {}^b\tilde{\eta}(\mathcal{D}) - \tilde{\eta}(\mathcal{D}_{\mathcal{P}_M}) - {}^b\tilde{\eta}(\mathcal{D}_{\mathcal{P}_Z}) \;=\; {}^b\tilde{\eta}(\mathcal{D}) - \tilde{\eta}(\mathcal{D}_{\mathcal{P}_M}) \;=\; \frac{1}{2\pi i}\log\det{}_F U \qquad \mathrm{mod}\;\; \mathbb{Z} $$

where the tildes denote reduced invariants, e.g. ${}^b\tilde{\eta}(\mathcal{D}) = ({}^b\eta(\mathcal{D}) + \dim\ker_{L^2}(\mathcal{D}))/2$.

The second aim of this paper is to study the relative invariant problem (cf. [27], [37], [36]). Recall that $\mathcal{P}_Z$ is the natural choice of the projection on the cylinder. This suggests that the natural boundary projector on M taking into consideration the infinite cylinder should be

$$ \mathcal{P}_\sigma := \Pi_< + \frac{\mathrm{Id}+\sigma}{2}\Pi_0, $$

instead of the Calderón projector $\mathcal{P}_M$. The projector $\mathcal{P}_\sigma$ is called the *augmented APS spectral projector* (cf. [17]) and it plays the central role in the gluing problem of the eta invariant in several of the aforementioned works. We can define $\mathcal{D}_{\mathcal{P}_\sigma}$ on M just as we defined $\mathcal{D}_{\mathcal{P}_M}$ in (1.1). The *relative invariant problem* is to describe the "defects"

$$ \frac{\det_{{}^b\zeta}\mathcal{D}^2}{\det_\zeta \mathcal{D}^2_{\mathcal{P}_\sigma}} = \boxed{?} \quad , \quad {}^b\eta(\mathcal{D}) - \eta(\mathcal{D}_{\mathcal{P}_\sigma}) = \boxed{?} $$

in terms of recognizable data. The following theorem solves this problem for the ζ-determinant.

Theorem 1.3. *The following relative ζ-determinant formula holds:*

$$ \frac{\det_{{}^b\zeta}\mathcal{D}^2}{\det_\zeta \mathcal{D}^2_{\mathcal{P}_\sigma}} \;=\; 2^{-\frac{1}{2}\zeta_{D_Y^2}(0) - h_Y} \left(\frac{\det\mathcal{L}}{\det\mathcal{L}_{\mathcal{P}_\sigma}}\right)^{-2}, $$

where $\mathcal{L}_{\mathcal{P}_\sigma}$ is defined by (1.6) *but with $\{U_k\}$ an orthonormal basis for* $\ker(\mathcal{D}_{\mathcal{P}_\sigma})$.

As a byproduct of our proof of Theorems 1.1 and 1.3, we also obtain the corresponding formula for the eta invariant.

Corollary 1.4 (of proof). *The following formula holds:*

$$ {}^b\eta(\mathcal{D}) = \eta(\mathcal{D}_{\mathcal{P}_\sigma}) \qquad \mathrm{mod}\;\; 2\mathbb{Z}. $$

Thus, the b-eta invariant of $\mathcal{D}$ and the eta invariant of $\mathcal{D}_{\mathcal{P}_\sigma}$ are the same modulo $2\mathbb{Z}$ while the ζ-determinants differ by terms on Y and the global data given by the kernels of $\mathcal{D}$ and $\mathcal{D}_{\mathcal{P}_\sigma}$. Because the ζ-determinant is highly nonlocal, one would not expect the ratio to be unity, and one might even conjecture that the ratio involves some globally defined objects. This is indeed the case as shown in Theorem 1.3. In the case of *compatible* Dirac operators, a similar formula for the eta invariant with $\mathcal{D}_{\mathcal{P}_{-\sigma}}$ and without the integer ambiguity was first proved by Müller [31] using a completely different method.

We now outline our paper. In Section 2, we review some basic material of the b-trace that is needed to define the b-spectral invariants. With some technical computations, we also obtain the explicit values of the b-spectral invariants over Z. In Section 3, we explain the basic theory of elliptic boundary problems for our Dirac type operator and we introduce the *auxiliary model problem* over $\hat{Z}$. In Section 4 we analyze and examine the structure of the resolvents of our Dirac type operators and in Section 5, we relate $K(\lambda)$ and $\hat{K}(\lambda)$ with the relative trace of the resolvents. This relation is one of the main ingredients in our proofs of the main results. In Section 6, we study the asymptotics of $\det_F(K(\lambda)\hat{K}(\lambda)^{-1})$ for small and large λ.

In Section 7, we express the relative b-spectral invariants in terms of resolvents. Finally, in Section 8, we prove the main theorems of this paper combining the results proved in the previous sections.

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2. The b-trace and the spectral invariants for the cylinder

In this section, we give an elementary introduction to Melrose's b-trace [30]. We continue to use the same notation set up in the introduction.

To understand the need for the b-trace, we first describe the restriction of the heat operator $e^{-t\mathcal{D}^2}$ to the cylindrical part $Z = [0,\infty)_u \times Y$. Restricting the heat kernel, defined in the spatial variables over all of X^2, to the product cylinder Z^2, one can show that (see [30, Ch. 7])

$$e^{-t\mathcal{D}^2}(u,u',y,y') = \frac{1}{\sqrt{4\pi t}}\, e^{-(u-u')^2/4t} e^{-tD_Y^2} + H(t,u,u',y,y'),$$

where, for fixed $t>0$, $H(t,u,u',y,y') = \mathcal{O}(e^{-u/2}\, e^{-u'/2})$ and where the (un)primed coordinates are the coordinates on the (left)right factor of Z^2. Restricting this Schwartz kernel to the diagonal in Z^2 and taking the fiber-wise trace, we obtain

$$\operatorname{tr} e^{-t\mathcal{D}^2}|_{\mathrm{Diag}} = \frac{1}{\sqrt{4\pi t}} \operatorname{tr} e^{-tD_Y^2}(y,y) + \operatorname{tr} H(t,u,u,y,y). \tag{2.1}$$

Observe that $H(t,u,u,y,y) = \mathcal{O}(e^{-u})$, which is integrable on the infinite cylinder Z, but the first term is constant with respect to u, so is not integrable on the infinite cylinder. In particular, the heat trace defined via the Lidskiĭ [22] trace formula is not defined. This shows that in order to develop *heat kernel methods* on manifolds with cylindrical ends, another notion of "trace" is needed. One such notion is the b-trace introduced by Melrose [30] and is described as follows. Let ϕ be a locally integrable function on X and suppose that on the infinite cylinder Z, we can write $\phi(u,y) = \varphi(y) + \psi(u,y)$ where $\varphi(y)$ is constant in u and $\psi(u,y)$ is integrable. Then the function $\varphi(y)$ is exactly the obstruction to ϕ being integrable on X. We define the *b-integral* of ϕ by simply omitting this obstruction:

$${}^b\!\!\int_X \phi := \int_M \phi + \int_Z \psi(u,y)\, du\, dy,$$

where dy is the measure on Y, and we say that ϕ is *b-integrable*. From the decomposition (2.1), we see that $\operatorname{tr} e^{-t\mathcal{D}^2}|_{\mathrm{Diag}}$ is b-integrable. We define the *b-trace* of the heat operator $e^{-t\mathcal{D}^2}$ via the following "b-Lidskiĭ formula":

$${}^b\mathrm{Tr}(e^{-t\mathcal{D}^2}) := {}^b\!\!\int_X \operatorname{tr} e^{-t\mathcal{D}^2}|_{\mathrm{Diag}}.$$

The b-trace of the heat operator has the long time asymptotic expansion (see [30, Sec. 7.8] or [15, Appendix])

$${}^b\mathrm{Tr}(e^{-t\mathcal{D}^2}) \sim \sum_{k=0}^{\infty} b_k\, t^{-\frac{k}{2}} \qquad \text{as } t\to\infty, \tag{2.2}$$

where $b_0 = h_X$ with $h_X = \dim \ker_{L^2}(\mathcal{D})$. Also, there is the usual short time asymptotic expansion (see [30]):

$$
{}^b\mathrm{Tr}(e^{-t\mathcal{D}^2}) \sim \sum_{k=0}^{\infty} a_k \, t^{k-\frac{n}{2}} \qquad \text{as } t \to 0, \tag{2.3}
$$

where $n = \dim X$ and the pointwise trace of the heat kernel on the diagonal also has such an expansion. Using (2.2) and (2.3), a straightforward computation shows that

$$
{}^b\zeta_{\mathcal{D}^2}(s) = \frac{1}{\Gamma(s)} \Big(\int_0^1 + \int_1^\infty \Big) t^{s-1} \, {}^b\mathrm{Tr}(e^{-t\mathcal{D}^2}) \, dt,
$$

where the first integral is defined a priori for $\Re s \gg 0$ and the second one for $\Re s \ll 0$, extends to be a meromorphic function on $\mathbb{C}$ that is regular at $s = 0$. This shows that $\log \det_{{}^b\zeta}(\mathcal{D}^2) := -\frac{d}{ds} {}^b\zeta_{\mathcal{D}^2}(s)\big|_{s=0}$ is well-defined. The b-trace of $\mathcal{D}e^{-t\mathcal{D}^2}$ has a related long time expansion (see [30, Ch. 9.7]):

$$
{}^b\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2}) \sim \sum_{k=0}^{\infty} \tilde{b}_k \, t^{-1-\frac{k}{2}} \qquad \text{as } t \to \infty. \tag{2.4}
$$

There is the short time asymptotic expansion of the same form as (2.3) (this follows from [23, Cor. 6.20]; cf. also Müller [31, Lem. 1.17]):

$$
{}^b\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2}) \sim \sum_{k=0}^{\infty} \tilde{a}_k \, t^{k-\frac{n}{2}} \qquad \text{as } t \to 0; \tag{2.5}
$$

as before, $n = \dim X$ and the pointwise trace of the heat kernel on the diagonal also has such an expansion. Using (2.4) and (2.5), one can show that

$$
{}^b\eta_{\mathcal{D}}(s) = \frac{1}{\Gamma(\frac{s+1}{2})} \Big(\int_0^1 + \int_1^\infty \Big) t^{\frac{s-1}{2}} \, {}^b\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2}) \, dt,
$$

where the first integral is defined a priori for $\Re s \gg 0$ and the second one for $\Re s \ll 0$, extends to be a meromorphic function on $\mathbb{C}$ that is regular at $s = 0$, as long as n is *even.* If n is *odd* and $\mathcal{D}$ is *compatible*, then the expansion (2.5) starts at $t^{1/2}$ [30, Sec. 8.13], so for the case of compatible Dirac operators, ${}^b\eta_{\mathcal{D}}(s)$ is also regular at $s = 0$. Here, the Dirac operator $\mathcal{D}$ is compatible if $\mathcal{D} = c \cdot \nabla$ where $c(\cdot)$ is Clifford multiplication on T^*X and ∇ is a unitary connection on S (of product type on the cylinder $\hat{Z} = [-1, \infty)_u \times Y$) such that for $\phi \in C^\infty(X, T^*X)$ and $\psi \in C^\infty(X, S)$,

$$
\nabla(c(\phi)\psi) = c(\nabla^{LC}\phi)\psi + c(\phi)\nabla\psi
$$

where $\nabla^{LC}\phi$ is the Levi-Civita connection. In Theorem 7.4, we show that ${}^b\eta_{\mathcal{D}}(s)$ is regular at $s = 0$ for any n and general Dirac type operators. In all cases, ${}^b\eta(\mathcal{D}) := {}^b\eta_{\mathcal{D}}(0)$ is well-defined.

We can also apply the b-trace to the heat operator $e^{-t\mathcal{D}_{\mathcal{P}_Z}^2}$ over Z. In this case, if $\{(\mu_k, \varphi_k)\}$ is the spectral resolution of D_Y, then we know that [1], [4]

$$
\begin{aligned}
e^{-t\mathcal{D}_{\mathcal{P}_Z}^2} &= \sum_{\mu_k>0} \frac{e^{-t\mu_k^2}}{\sqrt{4\pi t}} \Big[e^{-(u-u')^2/4t} - e^{-(u+u')^2/4t} \Big] \varphi_k(y) \otimes \varphi_k(y') \\
&+ \sum_{\mu_k>0} \Big\{ \frac{e^{-t\mu_k^2}}{\sqrt{4\pi t}} \Big[e^{-(u-u')^2/4t} + e^{-(u+u')^2/4t} \Big] \\
&- \mu_k \, e^{\mu_k(u+u')} \mathrm{erfc}\Big(\frac{u+u'}{2\sqrt{t}} + \mu_k \sqrt{t} \Big) \Big\} G\varphi_k(y) \otimes G\varphi_k(y') \\
&+ \Pi_0 \frac{1}{\sqrt{4\pi t}} \Big\{ e^{-(u-u')^2/4t} + \sigma \, e^{-(u+u')^2/4t} \Big\} \Pi_0.
\end{aligned} \tag{2.6}
$$

Using this formula, we can prove the following

Lemma 2.1. *${}^b\eta_{\mathcal{D}_{\mathcal{P}_Z}}(s) = 0$ for all $s \in \mathbb{C}$; in particular, ${}^b\eta(\mathcal{D}_{\mathcal{P}_Z}) = 0$.*

Proof. Since $\big[\partial_u e^{-t(u-u')^2/4t}\big]|_{u=u'} = 0$, using the explicit formula for $e^{-t\mathcal{D}_{\mathcal{P}_Z}^2}$ in (2.6) and the fact that $\mathcal{D} = G(\partial_u + D_Y)$, we obtain

$$
\begin{aligned}
(\mathcal{D}e^{-t\mathcal{D}_{\mathcal{P}_Z}^2})\Big|_{u=u', y=y'} &= \sum_{\mu_k>0} f_k \, G\varphi_k(y) \otimes \varphi_k(y) + g_k \, \varphi_k(y) \otimes G\varphi_k(y) \\
&\qquad - \sum_{\mu_k=0} \frac{1}{\sqrt{4\pi t}} \frac{1}{2t} \, G\sigma e^{-u^2/t} \varphi_k(y) \otimes \varphi_k(y)
\end{aligned}
$$

for some scalar functions f_k, g_k whose exact forms are not important. Since $G\varphi_k$ and φ_k are orthogonal and since $G\sigma = -\sigma G$, it follows that

$$
\mathrm{Tr}_Y \, G\varphi_k \otimes \varphi_k = \mathrm{Tr}_Y \, \varphi_k \otimes G\varphi_Y = \mathrm{Tr}_Y \, G\sigma = 0,
$$

where Tr_Y is the trace over Y. This implies that $\mathrm{Tr}_Y(\mathcal{D}_{\mathcal{P}_Z} e^{-t\mathcal{D}_{\mathcal{P}_Z}^2}) = 0$. Hence, ${}^b\mathrm{Tr}(\mathcal{D}_{\mathcal{P}_Z} e^{-t\mathcal{D}_{\mathcal{P}_Z}^2}) = 0$, and so ${}^b\eta_{\mathcal{D}_{\mathcal{P}_Z}}(s) = 0$. □

Now restricting $e^{-t\mathcal{D}_{\mathcal{P}_Z}^2}$ in (2.6) to the diagonal, and then integrating over Y (taking the trace over Y), and using that $\mathrm{Tr}_Y \, \varphi_k \otimes \varphi_k = \mathrm{Tr}_Y \, G\varphi_k \otimes G\varphi_k = 1$ and $\mathrm{Tr}_Y \, \sigma = 0$, we see that

$$
\begin{aligned}
\mathrm{Tr}_Y(e^{-t\mathcal{D}_{\mathcal{P}_Z}^2})(u,u) &= \sum_{\mu_k>0} \frac{e^{-t\mu_k^2}}{\sqrt{4\pi t}} \Big[1 - e^{-u^2/t} \Big] \\
&+ \sum_{\mu_k>0} \Big\{ \frac{e^{-t\mu_k^2}}{\sqrt{4\pi t}} \Big[1 + e^{-u^2/t} \Big] - \mu_k e^{2\mu_k u} \mathrm{erfc}\Big(\frac{u}{\sqrt{t}} + \mu_k \sqrt{t} \Big) \Big\} + \frac{h_Y}{\sqrt{4\pi t}}.
\end{aligned} \tag{2.7}
$$

Since the terms involving $e^{-u^2/t}$ cancel each other and the b-trace, by definition, kills the constant term in u, we get

$$
{}^b\mathrm{Tr}(e^{-t\mathcal{D}_{\mathcal{P}_Z}^2}) = - \sum_{\mu_k>0} \mu_k \int_0^\infty e^{2\mu_k u} \mathrm{erfc}\Big(\frac{u}{\sqrt{t}} + \mu_k \sqrt{t} \Big) du.
$$

We can simplify this integral using that $\frac{d}{dx}\mathrm{erfc}(x) = -\frac{2}{\sqrt{\pi}}e^{-x^2}$ and integration by parts to obtain

$$
{}^b\mathrm{Tr}(e^{-t\mathcal{D}^2_{\mathcal{P}_Z}}) = \frac{1}{2}\sum_{\mu_k>0}\mathrm{erfc}\Big(\mu_k\sqrt{t}\Big) - \frac{1}{2}\sum_{\mu_k>0}e^{-\mu_k^2 t}. \tag{2.8}
$$

Using this equation, we can find $\det_{{}^b\zeta}(\mathcal{D}^2_{\mathcal{P}_Z})$.

Lemma 2.2. *We have*

$$
\det_{{}^b\zeta}(\mathcal{D}^2_{\mathcal{P}_Z}) = 2^{\frac{1}{2}\zeta_{D^2_Y}(0)}.
$$

Proof. We compute

$$
\begin{aligned}
\int_0^\infty t^{s-1}\,\mathrm{erfc}(\mu_k\sqrt{t})\,dt &= \frac{1}{s}\int_0^\infty \frac{d}{dt}(t^s)\,\mathrm{erfc}(\mu_k\sqrt{t})\,dt \\
&= \frac{t^s}{s}\,\mathrm{erfc}(\mu_k\sqrt{t})\big|_0^\infty - \frac{1}{s}\int_0^\infty t^s\,\mathrm{erfc}'(\mu_k\sqrt{t})\frac{\mu_k}{2\sqrt{t}}\,dt \\
&= -\frac{\mu_k}{2s}\int_0^\infty t^{s-\frac{1}{2}}\Big(-\frac{2}{\sqrt{\pi}}e^{-\mu_k^2 t}\Big)\,dt \\
&= \frac{\mu_k}{\sqrt{\pi}s}\int_0^\infty t^{s-\frac{1}{2}}e^{-\mu_k^2 t}\,dt = \frac{\Gamma(s+1/2)}{\sqrt{\pi}s}\,\mu_k^{-2s}.
\end{aligned}
$$

Hence,

$$
\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{erfc}\Big(\mu_k\sqrt{t}\Big)dt = \frac{\Gamma(s+1/2)}{\sqrt{\pi}\Gamma(s+1)}\,\mu_k^{-2s}.
$$

Now taking the Mellin transform of (2.8), and dividing the result by $\Gamma(s)$, we obtain

$$
{}^b\zeta_{\mathcal{D}^2_{\mathcal{P}_Z}}(s) = \left(\frac{\Gamma(s+1/2)}{2\sqrt{\pi}\,\Gamma(s+1)} - \frac{1}{2}\right)\sum_{\mu_k>0}\frac{1}{\mu_k^{2s}} = \frac{1}{2}\left(\frac{\Gamma(s+1/2)}{2\sqrt{\pi}\,\Gamma(s+1)} - \frac{1}{2}\right)\zeta_{D^2_Y}(s).
$$

Therefore,

$$
{}^b\zeta'_{\mathcal{D}^2_{\mathcal{P}_Z}}(0) = \frac{1}{4}\left(\frac{\Gamma'(1/2)}{\sqrt{\pi}} - \Gamma'(1)\right)\zeta_{D^2_Y}(0).
$$

Differentiating both sides of the identity $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\cdot\Gamma(z+\frac{1}{2})\cdot\Gamma(z)$ and setting $z = \frac{1}{2}$ (see p. 1423 in [33] for the details) shows that the number in parenthesis equals $-2\log 2$. Thus,

$$
\log\det_{{}^b\zeta}(\mathcal{D}^2_{\mathcal{P}_Z}) = \frac{1}{2}\zeta_{D^2_Y}(0)\log 2,
$$

from which our lemma follows after exponentiation. □

3. Calderón projectors and the model cylinder

In this section, we introduce some basic theory of elliptic boundary problems for our Dirac type operator over X. We also introduce the *auxiliary model problem* over $\hat{Z} = [-1,\infty)_u \times Y$. References for this section include Seeley [38, 39], Grubb [12], or Booß–Bavnbek and Wojciechowski [4].

Near the cutting hypersurface $\{0\}\times Y$, the trace map γ_ε is defined by

$$
\phi \mapsto \gamma_\varepsilon(\phi) := \phi\,|_{Y_\varepsilon} \;:\; C^\infty(X,S) \longrightarrow C^\infty(Y_\varepsilon,S_0)
$$

where $Y_\varepsilon := \{\varepsilon\}\times Y \subset \hat{Z} \subset X$. The trace map γ_ε extends to a well-defined map,

$$
\gamma_\varepsilon \;:\; H^k(X,S) \longrightarrow H^{k-\frac{1}{2}}(Y_\varepsilon,S_0)
$$

for $k > \frac{1}{2}$. Throughout this paper, we fix a union of sectors $\Lambda \subset \mathbb{C}$ of the form

$$\Lambda = \{\lambda \in \mathbb{C} \setminus \{0\} \mid \varepsilon_0 \leq \arg\lambda \leq \pi - \varepsilon_0 \quad \text{or} \quad \pi + \varepsilon_0 \leq \arg\lambda \leq 2\pi - \varepsilon_0\},$$

where $0 < \varepsilon_0 < \pi/8$. For $\lambda \in \Lambda$, we define

$$\mathcal{D}(\lambda) = \mathcal{D} - \lambda \ : \ H^1(X,S) \longrightarrow L^2(X,S)$$

and for $\lambda \in \Lambda$, its inverse operator $\mathcal{D}(\lambda)^{-1}$ from $L^2(X,S)$ to $H^1(X,S)$ is defined. We now define the *Calderón projectors* over $L^2(Y,S_0)$ as follows:

$$P_M(\lambda) = \ -\gamma_{0-}\mathcal{D}(\lambda)^{-1}\gamma_0^* G \quad , \quad P_Z(\lambda) = \ \gamma_{0+}\mathcal{D}(\lambda)^{-1}\gamma_0^* G$$

where $\gamma_{0\pm} \ = \lim_{\varepsilon\downarrow 0}\gamma_{\pm\varepsilon}$ and γ_0^* is the adjoint map of γ_0 at $\{0\} \times Y$. A slight modification of the work of Seeley [38, Th. 5] (cf. Grubb [12] or copy the proof of Lemma 3.1) shows that these operators are pseudodifferential projections satisfying

$$P_M(\lambda) + P_Z(\lambda) = \mathrm{Id} \tag{3.1}$$

and the images of $P_M(\lambda)$ and $P_Z(\lambda)$ coincide with the closures in $L^2(Y,S_0)$ of the *Cauchy data spaces*

$$\mathcal{H}_M(\lambda) = \big\{\gamma_{0-}\phi_M \mid \phi_M \in C^\infty(M,S)\ ,\ \mathcal{D}_M(\lambda)\phi_M = 0\big\},$$
$$\mathcal{H}_Z(\lambda) = \big\{\gamma_{0+}\phi_Z \mid \phi_Z \in C^\infty(Z,S)\cap L^2(Z,S)\ ,\ \mathcal{D}_Z(\lambda)\phi_Z = 0\big\},$$

respectively, where $\mathcal{D}_M(\lambda) := \mathcal{D}(\lambda)|_M$, $\mathcal{D}_Z(\lambda) := \mathcal{D}(\lambda)|_Z$. It is evident that $\mathcal{H}_M(\lambda)$, $\mathcal{H}_Z(\lambda)$ depend only on the restrictions of $\mathcal{D}(\lambda)$ to M, Z. On the other hand, the Calderón projectors $P_M(\lambda)$, $P_Z(\lambda)$ onto these spaces depend on the extrinsic data out of M, Z through $\mathcal{D}(\lambda)^{-1}$. We remark that $P_M(\lambda)$ and $P_Z(\lambda)$ are not *orthogonal* projections, but in some situations we need to make these projections orthogonal. To do so, we recall that for an arbitrary projection P, the operator (cf. [2], [4])

$$P^o := PP^*(PP^* + (\mathrm{Id} - P^*)(\mathrm{Id} - P))^{-1}$$

is an orthogonal projection onto the range of P. Using this formula, we can define the orthogonalized projections of $P_M(\lambda)$, $P_Z(\lambda)$ and we denote them by $P^o_M(\lambda)$, $P^o_Z(\lambda)$, respectively.

We recall that $\mathcal{P}_M$ denotes the orthogonal projector onto the closure of the Cauchy data space $\mathcal{H}_M(0)$. Let us now define

$$S_M(\lambda) := \mathcal{P}_M P_M(\lambda) \ : \ L^2(Y,S_0) \longrightarrow L^2(Y,S_0),$$

which induces a bijective map between $\mathrm{Im}(P_M(\lambda))$ and $\mathrm{Im}(\mathcal{P}_M)$ for $\lambda \in \Lambda$. We define $S_M(\lambda)^{-1}$ over $L^2(Y,S_0)$ by

$$S_M(\lambda)^{-1} := P^o_M(\lambda)[\mathcal{P}_M P^o_M(\lambda) + (\mathrm{Id} - \mathcal{P}_M)(\mathrm{Id} - P^o_M(\lambda))]^{-1}\mathcal{P}_M,$$

then this operator has the following properties:

$$S_M(\lambda)S_M(\lambda)^{-1} = \mathcal{P}_M \quad , \quad S_M(\lambda)^{-1}S_M(\lambda) = P_M(\lambda). \tag{3.2}$$

Let us remark $S_M(\lambda)^{-1}$ does not mean the inverse of $S_M(\lambda)$, but the operator satisfying the relations with $S_M(\lambda)$ in (3.2). This inverse notation, however, seems to be standard and has been used in [36], [37].

The corresponding operators on Z are defined as follows. Recall that

$$\mathcal{P}_Z := \Pi_> + \frac{\mathrm{Id} - \sigma}{2}\Pi_0$$

where σ is the involution over $\ker(D_Y)$ determined by the scattering matrix or by $\mathcal{P}_M\Pi_0 = \frac{\mathrm{Id}+\sigma}{2}\Pi_0$. Then the corresponding operators $S_Z(\lambda)$ and $S_Z(\lambda)^{-1}$ over $L^2(Y,S_0)$ are defined by

$$S_Z(\lambda) := \mathcal{P}_Z P_Z(\lambda),$$
$$S_Z(\lambda)^{-1} := P_Z^o(\lambda)[\mathcal{P}_Z P_Z^o(\lambda) + (\mathrm{Id}-\mathcal{P}_Z)(\mathrm{Id}-P_Z^o(\lambda))]^{-1}\mathcal{P}_Z.$$

These operators satisfy similar identities that are in (3.2).

For the Dirac type operators $\mathcal{D}_M$, $\mathcal{D}_Z$, we impose the boundary conditions $\mathcal{P}_M$, $\mathcal{P}_Z$, respectively, and denote the resulting operators by

$$\mathcal{D}_{\mathcal{P}_M} : \operatorname{dom}(\mathcal{D}_{\mathcal{P}_M}) \longrightarrow L^2(M,S) \quad \text{and} \quad \mathcal{D}_{\mathcal{P}_Z} : \operatorname{dom}(\mathcal{D}_{\mathcal{P}_Z}) \longrightarrow L^2(Z,S),$$

where

$$\operatorname{dom}(\mathcal{D}_{\mathcal{P}_M}) := \{\ \phi \in H^1(M,S) \mid \mathcal{P}_M\gamma_0\phi = 0\ \},$$
$$\operatorname{dom}(\mathcal{D}_{\mathcal{P}_Z}) := \{\ \phi \in H^1(Z,S) \mid \mathcal{P}_Z\gamma_0\phi = 0\ \}.$$

The boundary conditions $\mathcal{P}_M$, $\mathcal{P}_Z$ imply that $\mathcal{D}_{\mathcal{P}_M}$, $\mathcal{D}_{\mathcal{P}_Z}$ are self-adjoint operators such that for all $\lambda \in \Lambda$, the resolvents $\mathcal{D}_{\mathcal{P}_M}(\lambda)^{-1}$ and $\mathcal{D}_{\mathcal{P}_Z}(\lambda)^{-1}$ exist, where $\mathcal{D}_{\mathcal{P}_M}(\lambda) = \mathcal{D}_{\mathcal{P}_M} - \lambda$ and $\mathcal{D}_{\mathcal{P}_Z}(\lambda) = \mathcal{D}_{\mathcal{P}_Z} - \lambda$. We can give formulas for these resolvents as follows. We denote the restriction map from $L^2(X,S)$ to $L^2(M,S)$, $L^2(Z,S)$ by r_M, r_Z, respectively. We also define e_M, e_Z from $L^2(M,S)$, $L^2(Z,S)$ to $L^2(X,S)$ to be the extension maps by zero out of the concerned submanifolds. Using these maps, we define

$$\begin{aligned} \mathcal{D}_M(\lambda)^{-1} &= r_M\mathcal{D}(\lambda)^{-1}e_M \quad , \quad \mathcal{D}_Z(\lambda)^{-1} = r_Z\mathcal{D}(\lambda)^{-1}e_Z, \\ \mathcal{K}_M(\lambda) &= -r_M\mathcal{D}(\lambda)^{-1}\gamma_0^*G \quad , \quad \mathcal{K}_Z(\lambda) = r_Z\mathcal{D}(\lambda)^{-1}\gamma_0^*G. \end{aligned} \tag{3.3}$$

Then the inverses of $\mathcal{D}_{\mathcal{P}_M}(\lambda)$, $\mathcal{D}_{\mathcal{P}_Z}(\lambda)$ are given by (see [12])

$$\begin{aligned} \mathcal{D}_{\mathcal{P}_M}(\lambda)^{-1} &= \mathcal{D}_M(\lambda)^{-1} - \mathcal{K}_{\mathcal{P}_M}(\lambda)\mathcal{P}_M\gamma_0\mathcal{D}_M(\lambda)^{-1}, \\ \mathcal{D}_{\mathcal{P}_Z}(\lambda)^{-1} &= \mathcal{D}_Z(\lambda)^{-1} - \mathcal{K}_{\mathcal{P}_Z}(\lambda)\mathcal{P}_Z\gamma_0\mathcal{D}_Z(\lambda)^{-1}. \end{aligned} \tag{3.4}$$

Here, $\mathcal{K}_{\mathcal{P}_M}(\lambda) := \mathcal{K}_M(\lambda)S_M(\lambda)^{-1}$, $\mathcal{K}_{\mathcal{P}_Z}(\lambda) := \mathcal{K}_Z(\lambda)S_Z(\lambda)^{-1}$ are the *Poisson operators* of $\mathcal{D}_{\mathcal{P}_M}(\lambda)$, $\mathcal{D}_{\mathcal{P}_Z}(\lambda)$, respectively, which satisfy the equalities

$$\mathcal{P}_M\gamma_{0-}\mathcal{K}_{\mathcal{P}_M}(\lambda) = \mathcal{P}_M P_M(\lambda)\ S_M(\lambda)^{-1} = \mathcal{P}_M,$$
$$\mathcal{P}_Z\gamma_{0+}\mathcal{K}_{\mathcal{P}_Z}(\lambda) = \mathcal{P}_Z P_Z(\lambda)\ S_Z(\lambda)^{-1} = \mathcal{P}_Z.$$

Using these equalities, one can check that the images of the right-hand operators in (3.4) lie in $\operatorname{dom}(\mathcal{D}_{\mathcal{P}_M}(\lambda)) := \operatorname{dom}(\mathcal{D}_{\mathcal{P}_M})$ and $\operatorname{dom}(\mathcal{D}_{\mathcal{P}_Z}(\lambda)) := \operatorname{dom}(\mathcal{D}_{\mathcal{P}_Z})$, respectively.

We now consider the *model problem* on the submanifold $\hat{Z} = [-1,\infty) \times Y$ of X and the decomposition of $\hat{Z}$ into $N = [-1,0] \times Y$ and Z. The Dirac type operator $\mathcal{D}$ restricts to $G(\partial_u + D_Y)$ over $\hat{Z}$. For this operator, we impose the boundary condition defined by $\mathcal{P}_Z$ at $\{-1\} \times Y$. We denote by $\hat{\mathcal{D}}$ the resulting operator with this boundary condition at $\{-1\} \times Y$:

$$\hat{\mathcal{D}} : \operatorname{dom}(\hat{\mathcal{D}}) \longrightarrow L^2(\hat{Z},S)$$

where

$$\operatorname{dom}(\hat{\mathcal{D}}) := \{\ \phi \in H^1(\hat{Z},S) \mid \mathcal{P}_Z\gamma_{-1}\phi = 0\ \}.$$

As with the operator $\mathcal{D}_{\mathcal{P}_Z}$, the operator $\hat{\mathcal{D}}$ is a self-adjoint operator and for all $\lambda \in \Lambda$, the resolvent $\hat{\mathcal{D}}(\lambda)^{-1}$ exists, where $\hat{\mathcal{D}}(\lambda) = \hat{\mathcal{D}} - \lambda$. We denote by $\hat{\mathcal{D}}_N(\lambda)$ and $\hat{\mathcal{D}}_Z(\lambda)$ the restrictions of $\hat{\mathcal{D}}(\lambda)$ to N and Z, respectively. If $\hat{\mathcal{K}}(\lambda) = \hat{\mathcal{D}}(\lambda)^{-1}\gamma_0^*G$, then the usual arguments using the rational symbolic structure of $\hat{\mathcal{D}}(\lambda)^{-1}$ (see [38, pp. 795–796] or [4, pp. 84–85]) show that if $\varphi \in C^\infty(Y, S_0)$, then $\hat{\mathcal{K}}(\lambda)\varphi\big|_{N,Z} \in \mathrm{dom}(\hat{\mathcal{D}}_{N,Z}(\lambda))$ and is smooth up to each side of Y with at most a jump discontinuity at Y. In particular, we can define the *Calderón projectors* $\hat{P}_N(\lambda)$ for $\hat{\mathcal{D}}_N(\lambda)$ and $\hat{P}_Z(\lambda)$ for $\hat{\mathcal{D}}_Z(\lambda)$ over $L^2(Y, S_0)$ by

$$\hat{P}_N(\lambda) = -\gamma_{0^-}\hat{\mathcal{D}}(\lambda)^{-1}\gamma_0^* G \quad , \quad \hat{P}_Z(\lambda) = \gamma_{0^+}\hat{\mathcal{D}}(\lambda)^{-1}\gamma_0^* G.$$

The rational symbolic structure of $\hat{\mathcal{D}}(\lambda)^{-1}$ also implies that $\hat{P}_N(\lambda)$ and $\hat{P}_Z(\lambda)$ are pseudodifferential operators of order zero. The following lemma shows that $\hat{P}_N(\lambda)$ and $\hat{P}_Z(\lambda)$ deserve to be called Calderón projectors.

Lemma 3.1. *For all $\lambda \in \Lambda$, the operators $\hat{P}_N(\lambda)$ and $\hat{P}_Z(\lambda)$ over $L^2(Y, S_0)$ are projections satisfying*

$$\hat{P}_N(\lambda) + \hat{P}_Z(\lambda) = \mathrm{Id}$$

and their images coincide with the closures in $L^2(Y, S_0)$ of the Cauchy data spaces

$$\hat{\mathcal{H}}_N(\lambda) = \big\{\gamma_{0^-}\phi_N \mid \phi_N \in C^\infty(N, S)\ ,\ \mathcal{P}_Z\gamma_{-1}\phi_N = 0\ ,\ \hat{\mathcal{D}}_N(\lambda)\phi_N = 0\big\},$$
$$\hat{\mathcal{H}}_Z(\lambda) = \big\{\gamma_{0^+}\phi_Z \mid \phi_Z \in C^\infty(Z, S) \cap L^2(Z, S)\ ,\ \ \hat{\mathcal{D}}_Z(\lambda)\phi_Z = 0\big\}.$$

Proof. The proof of this lemma is similar to Seeley [38, Th. 5] (see also Grubb [12]) for the compact case. We shall prove that $\hat{P}_N(\lambda)$ over $C^\infty(Y, S_0)$ is a projection with image $\hat{\mathcal{H}}_N(\lambda)$; a similar proof works for $\hat{P}_Z(\lambda)$. We first show that $\hat{P}_N(\lambda) = \mathrm{Id}$ on $\hat{\mathcal{H}}_N(\lambda)$. Let $\varphi = \gamma_{0^-}\phi_N$, where $\phi_N \in C^\infty(N, S)$, $\mathcal{P}_Z\gamma_{-1}\phi_N = 0$ and $\hat{\mathcal{D}}_N(\lambda)\phi_N = 0$, and define

$$\phi := \begin{cases} \phi_N & \text{on } N \\ 0 & \text{on } \hat{Z} \setminus N. \end{cases}$$

Since $\hat{\mathcal{D}}_N(\lambda)\phi_N = 0$ and $\hat{\mathcal{D}}(\lambda) = G(\partial_u + D_Y) - \lambda$, and the derivative of the Heaviside function is the delta distribution, it follows that

$$\hat{\mathcal{D}}(\lambda)\phi = -\delta_Y \otimes G\varphi = -\gamma_0^* G\varphi,$$

since $\gamma_0^* = \delta_Y \otimes \cdot$ with δ_Y the delta distribution concentrated at $\{0\} \times Y$. Thus, $\phi = -\hat{\mathcal{D}}(\lambda)^{-1}\gamma_0^* G\varphi$, and so

$$\hat{P}_N(\lambda)\varphi := -\gamma_{0^-}\big(\hat{\mathcal{D}}(\lambda)^{-1}\gamma_0^* G\varphi\big) = \gamma_{0^-}\big(\phi\big) = \varphi.$$

Hence, $\hat{P}_N(\lambda) = \mathrm{Id}$ on $\hat{\mathcal{H}}_N(\lambda)$. We now show that $\hat{P}_N(\lambda)^2 = \hat{P}_N(\lambda)$. Let $\varphi \in C^\infty(Y, S_0)$. Then by definition of $\hat{P}_N(\lambda)$, we have

$$\hat{P}_N(\lambda)\varphi = \gamma_{0^-}\phi_N \quad , \quad \phi_N = -\big(\hat{\mathcal{D}}(\lambda)^{-1}\gamma_0^* G\varphi\big)\big|_N.$$

Note that $\phi_N \in C^\infty(N, S)$, $\mathcal{P}_Z\gamma_{-1}\phi_N = 0$ and $\hat{\mathcal{D}}_N(\lambda)\phi_N = 0$. Thus, $\gamma_{0^-}\phi_N \in \hat{\mathcal{H}}_N(\lambda)$, so as we know that $\hat{P}_N(\lambda) = \mathrm{Id}$ on $\hat{\mathcal{H}}_N(\lambda)$, it follows that

$$\hat{P}_N(\lambda)^2\varphi = \hat{P}_N(\lambda)\big(\hat{P}_N(\lambda)\varphi\big) = \hat{P}_N(\lambda)\big(\gamma_{0^-}\phi_N\big) = \gamma_{0^-}\phi_N = \hat{P}_N(\lambda)\varphi.$$

We now prove that $\hat{P}_N(\lambda) + \hat{P}_Z(\lambda) = \mathrm{Id}$. Let $\phi \in C_c^\infty((-1, \infty) \times Y, S)$ and let $\psi \in C^\infty(Y, S_0)$. Denote the L^2-pairing on Y by $\langle\ ,\ \rangle$ and denote the distributional pairing on $\hat{Z}$ by parentheses. If $\hat{\mathcal{K}}(\lambda) = \hat{\mathcal{D}}(\lambda)^{-1}\gamma_0^* G$, then

$$\begin{aligned}(3.5)\quad \langle \gamma_0\phi, G\psi\rangle &= (\gamma_0^* G\psi)(\phi) = \big(\hat{\mathcal{D}}(\lambda)\hat{\mathcal{K}}(\lambda)\psi\big)(\phi) \\ &= \big(\hat{\mathcal{K}}(\lambda)\psi\big)\big(\hat{\mathcal{D}}(\lambda)^*\phi\big) = \int_{-1}^{\infty} \langle \hat{\mathcal{D}}(\lambda)^*\phi, \hat{\mathcal{K}}(\lambda)\psi\rangle\, du.\end{aligned}$$

Since $\hat{\mathcal{K}}(\lambda)\psi$ is smooth off of Y with at most a jump discontinuity at Y, we can write

$$\int_{-1}^{\infty} \langle \hat{\mathcal{D}}(\lambda)^*\phi, \hat{\mathcal{K}}(\lambda)\psi\rangle\, du = \lim_{\varepsilon\downarrow 0} \int_{|u|>\varepsilon} \langle \hat{\mathcal{D}}(\lambda)^*\phi, \hat{\mathcal{K}}(\lambda)\psi\rangle\, du.$$

Now observe that

$$\begin{aligned}\int_{|u|>\varepsilon} \langle \hat{\mathcal{D}}(\lambda)^*\phi, \hat{\mathcal{K}}(\lambda)\psi\rangle\, du &= \int_{|u|>\varepsilon} \langle \big(G(\partial_u + D_Y) - \overline{\lambda}\big)\phi, \hat{\mathcal{K}}(\lambda)\psi\rangle\, du \\ &= -\int_{|u|>\varepsilon} \langle \partial_u\phi, G\hat{\mathcal{K}}(\lambda)\psi\rangle\, du + \int_{|u|>\varepsilon} \langle \phi, \big(GD_Y - \lambda\big)\hat{\mathcal{K}}(\lambda)\psi\rangle\, du \\ &= -\int_{|u|>\varepsilon} \partial_u\langle \phi, G\hat{\mathcal{K}}(\lambda)\psi\rangle\, du + \int_{|u|>\varepsilon} \langle \phi, \hat{\mathcal{D}}(\lambda)\hat{\mathcal{K}}(\lambda)\psi\rangle\, du \\ (3.6)\qquad &= -\langle \gamma_{-\varepsilon}\phi, G\gamma_{-\varepsilon}\hat{\mathcal{K}}(\lambda)\psi\rangle + \langle \gamma_{\varepsilon}\phi, G\gamma_{\varepsilon}\hat{\mathcal{K}}(\lambda)\psi\rangle,\end{aligned}$$

where at the last step we used that $\hat{\mathcal{D}}(\lambda)\hat{\mathcal{K}}(\lambda) = 0$ off of Y and the fundamental theorem of calculus, recalling that ϕ is supported on the interior of $\hat{Z}$. Taking $\varepsilon \downarrow 0$ in (3.6) and equating this with (3.5), and using the definition of $\hat{P}_N(\lambda)$ and $\hat{P}_Z(\lambda)$, we conclude that

$$\langle \gamma_0\phi, G\psi\rangle = \langle \gamma_0\phi, G\hat{P}_N(\lambda)\psi\rangle + \langle \gamma_0\phi, G\hat{P}_Z(\lambda)\psi\rangle.$$

Since $\phi \in C_c^\infty((-1,\infty)\times Y, S)$ and $\psi \in C^\infty(Y, S_0)$ were arbitrary, it follows that $\mathrm{Id} = \hat{P}_N(\lambda) + \hat{P}_Z(\lambda)$, and our proof is now complete. □

Let us note that $\hat{\mathcal{H}}_Z(\lambda) = \mathcal{H}_Z(\lambda)$, because $\mathcal{D}(\lambda)|_Z = \hat{\mathcal{D}}(\lambda)|_Z$. Thus, the projections $\hat{P}_Z(\lambda)$ and $P_Z(\lambda)$ have the same image, but they are not the same projections because they are defined through different resolvents.

We put

$$\mathcal{P}_N := \mathrm{Id} - \mathcal{P}_Z = \Pi_< + \frac{\mathrm{Id}+\sigma}{2}\Pi_0;$$

this is just the projection $\mathcal{P}_\sigma$ in the introduction. This projection defines a well-posed boundary condition for $\hat{\mathcal{D}}_N(\lambda)$ at $\{0\}\times Y$ and we denote the resulting operator by $\hat{\mathcal{D}}_{\mathcal{P}_N}(\lambda)$. From $\hat{\mathcal{D}}(\lambda)$ we can also define $\hat{\mathcal{D}}_N(\lambda)^{-1}$, $\hat{\mathcal{D}}_Z(\lambda)^{-1}$, $\hat{\mathcal{K}}_N(\lambda)$, and $\hat{\mathcal{K}}_Z(\lambda)$ just like in (3.3) and we can define operators $\hat{S}_N(\lambda)$, $\hat{S}_Z(\lambda)$, $\hat{S}_N(\lambda)^{-1}$, $\hat{S}_Z(\lambda)^{-1}$, $\hat{\mathcal{K}}_{\mathcal{P}_N}(\lambda)$, and $\hat{\mathcal{K}}_{\mathcal{P}_Z}(\lambda)$ with the obvious meanings. As in (3.4), we have

$$(3.7)\qquad \begin{aligned}\hat{\mathcal{D}}_{\mathcal{P}_N}(\lambda)^{-1} &= \hat{\mathcal{D}}_N(\lambda)^{-1} - \hat{\mathcal{K}}_{\mathcal{P}_N}(\lambda)\mathcal{P}_N\gamma_0\hat{\mathcal{D}}_N(\lambda)^{-1}, \\ \hat{\mathcal{D}}_{\mathcal{P}_Z}(\lambda)^{-1} &= \hat{\mathcal{D}}_Z(\lambda)^{-1} - \hat{\mathcal{K}}_{\mathcal{P}_Z}(\lambda)\mathcal{P}_Z\gamma_0\hat{\mathcal{D}}_Z(\lambda)^{-1}.\end{aligned}$$

Lemma 3.2. *We have*

$$\partial_\lambda \mathcal{K}_{\mathcal{P}_M}(\lambda) = \mathcal{D}_{\mathcal{P}_M}(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_M}(\lambda) \quad , \quad \partial_\lambda \mathcal{K}_{\mathcal{P}_Z}(\lambda) = \mathcal{D}_{\mathcal{P}_Z}(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_Z}(\lambda).$$

The same formulas hold for $\hat{\mathcal{K}}_{\mathcal{P}_N}(\lambda)$, $\hat{\mathcal{K}}_{\mathcal{P}_Z}(\lambda)$.

Proof. Taking the derivatives of the equalities

$$\mathcal{D}_M(\lambda)\mathcal{K}_{\mathcal{P}_M}(\lambda) = 0 \quad , \quad \mathcal{P}_M\gamma_0\mathcal{K}_{\mathcal{P}_M}(\lambda) = \mathrm{Id},$$

we obtain

$$\mathcal{D}_M(\lambda)\partial_\lambda\mathcal{K}_{\mathcal{P}_M}(\lambda) = \mathcal{K}_{\mathcal{P}_M}(\lambda) \quad , \quad \mathcal{P}_M\gamma_0\partial_\lambda\mathcal{K}_{\mathcal{P}_M}(\lambda) = 0.$$

The second equality means that $\partial_\lambda\mathcal{K}_{\mathcal{P}_M}(\lambda)$ is in the domain of $\mathcal{D}_{\mathcal{P}_M}(\lambda)$. Hence,

$$\partial_\lambda\mathcal{K}_{\mathcal{P}_M}(\lambda) = \mathcal{D}_{\mathcal{P}_M}(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_M}(\lambda).$$

In the same way, we can derive the formula

$$\partial_\lambda\mathcal{K}_{\mathcal{P}_Z}(\lambda) = \mathcal{D}_{\mathcal{P}_Z}(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_Z}(\lambda).$$

The same proof works for the " ˆ " case. □

4. Relative traces of resolvents

In this section, we study relative traces of resolvents. We begin with the following proposition.

Proposition 4.1. *The following operators are smoothing operators:*

$$P_M(\lambda) - \hat{P}_N(\lambda) \ , \ P_Z(\lambda) - \hat{P}_Z(\lambda) \ : \ L^2(Y, S_0) \ \longrightarrow \ L^2(Y, S_0).$$

Proof. We introduce a smooth even function $\rho(a,b) : [-1,1] \to [0,1]$ equal to 0 for $-a \le u \le a$ and equal to 1 for $b \le |u|$. We use $\rho(a,b)(u)$ to define

$$\begin{aligned} \phi_1 &= 1 - \rho(5/7, 6/7) \quad , \quad \psi_1 = 1 - \psi_2 \quad , \\ \phi_2 &= \rho(1/7, 2/7) \quad , \quad \psi_2 = \rho(3/7, 4/7) \end{aligned}$$

and then we extend these functions to the whole manifold X in the obvious way. Now we define a parametrix $Q(\lambda)$ for the operator $\mathcal{D}(\lambda)^{-1}$ by

$$Q(\lambda)(x,z) = \phi_1(x)\hat{\mathcal{D}}(\lambda)^{-1}(x,z)\psi_1(z) + \phi_2(x)\mathcal{D}(\lambda)^{-1}(x,z)\psi_2(z). \tag{4.1}$$

Then we have

$$\begin{aligned} \mathcal{D}(\lambda)Q(\lambda)(x,z) \ &= \ \mathrm{Id} + G\partial_u\phi_1(x)\hat{\mathcal{D}}(\lambda)^{-1}(x,z)\psi_1(z) \\ &\qquad + G\partial_u\phi_2(x)\mathcal{D}(\lambda)^{-1}(x,z)\psi_2(z). \end{aligned}$$

Since the supports of $\partial_u\phi_i$ and ψ_i are disjoint, it follows that

$$\mathcal{D}(\lambda)Q(\lambda) \ = \ \mathrm{Id} + \mathcal{S}(\lambda),$$

where $\mathcal{S}(\lambda)$ is a smoothing operator. Thus,

$$\mathcal{D}(\lambda)^{-1} - Q(\lambda) \ = \ \widetilde{\mathcal{S}}(\lambda) \tag{4.2}$$

where $\widetilde{\mathcal{S}}(\lambda) = -\mathcal{D}(\lambda)^{-1}\mathcal{S}(\lambda)$ is a smoothing operator. Then by definition of $P_M(\lambda)$ and $\hat{P}_N(\lambda)$, the equalities (4.1) and (4.2) imply that

$$P_M(\lambda) - \hat{P}_N(\lambda) \ = \ -\gamma_{0^-}(\mathcal{D}(\lambda)^{-1} - \hat{\mathcal{D}}(\lambda)^{-1})\gamma_0^*G \ = \ -\gamma_0\widetilde{\mathcal{S}}(\lambda)\gamma_0^*G.$$

Hence, $P_M(\lambda) - \hat{P}_N(\lambda)$ is a smoothing operator. Similarly,

$$P_Z(\lambda) - \hat{P}_Z(\lambda) \ = \ \gamma_0\widetilde{\mathcal{S}}(\lambda)\gamma_0^*G$$

is a smoothing operator. □

We decompose the Hilbert spaces $L^2(X,S)$, $L^2(\hat{Z},S)$ into the following orthogonal direct sums:

$$L^2(X,S) = L^2(M,S) \oplus L^2(Z,S) \quad , \quad L^2(\hat{Z},S) = L^2(N,S) \oplus L^2(Z,S).$$

Then we can consider $\mathcal{D}_{\mathcal{P}_M}(\lambda)^{-1}$, $\mathcal{D}_{\mathcal{P}_Z}(\lambda)^{-1}$, $\hat{\mathcal{D}}_{\mathcal{P}_N}(\lambda)^{-1}$, $\hat{\mathcal{D}}_{\mathcal{P}_Z}(\lambda)^{-1}$, and our other operators defined on M, Z, N, and $\hat{Z}$, as linear operators on $L^2(X,S)$ by simply extending them to be zero off their domains and codomains. Thus, if we put

$$\mathcal{D}_{\mathcal{P}}(\lambda)^{-1} = \mathcal{D}_{\mathcal{P}_M}(\lambda)^{-1} \oplus \mathcal{D}_{\mathcal{P}_Z}(\lambda)^{-1}, \qquad \hat{\mathcal{D}}_{\mathcal{P}}(\lambda)^{-1} = \hat{\mathcal{D}}_{\mathcal{P}_N}(\lambda)^{-1} \oplus \hat{\mathcal{D}}_{\mathcal{P}_Z}(\lambda)^{-1},$$

then we can consider $\mathcal{D}_{\mathcal{P}}(\lambda)^{-1}$ and $\hat{\mathcal{D}}_{\mathcal{P}}(\lambda)^{-1}$ as operators on $L^2(X,S)$. In this view point, a similar argument as we used to prove Proposition 4.1 gives

Lemma 4.2. *The operators $\mathcal{P}_M - \mathcal{P}_N$, $S_M(\lambda) - \hat{S}_N(\lambda)$, $S_M(\lambda)^{-1} - \hat{S}_N(\lambda)^{-1}$ are smoothing on $L^2(Y,S_0)$. Also, each difference $\mathcal{K}_M(\lambda) - \hat{\mathcal{K}}_N(\lambda)$, $\mathcal{K}_{\mathcal{P}_M}(\lambda) - \hat{\mathcal{K}}_{\mathcal{P}_N}(\lambda) : L^2(Y,S_0) \to L^2(M,S)$ has a smoothing Schwartz kernel apart from a jump discontinuity at $\{-1\} \times Y$ in M, and $\gamma_0(\mathcal{D}_M(\lambda)^{-1} - \hat{\mathcal{D}}_N(\lambda)^{-1}) : L^2(M,S) \to L^2(M,S)$ has the same property. These statements hold when M and N are replaced by Z.*

Theorem 4.3. *For all $\lambda \in \Lambda$, the operator*

$$\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{P}}(\lambda)^{-1} - (\hat{\mathcal{D}}(\lambda)^{-1} - \hat{\mathcal{D}}_{\mathcal{P}}(\lambda)^{-1}) : L^2(X,S) \longrightarrow L^2(X,S)$$

is of trace class and the following equality holds:

$$\begin{aligned} &\operatorname{Tr}\left(\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{P}}(\lambda)^{-1} - (\hat{\mathcal{D}}(\lambda)^{-1} - \hat{\mathcal{D}}_{\mathcal{P}}(\lambda)^{-1})\right) \\ &\quad = \operatorname{Tr}\left(\mathcal{K}_{\mathcal{P}_M}(\lambda)\gamma_0\mathcal{D}_M(\lambda)^{-1} - \hat{\mathcal{K}}_{\mathcal{P}_N}(\lambda)\gamma_0\hat{\mathcal{D}}_N(\lambda)^{-1}\right) \\ &\qquad + \operatorname{Tr}\left(\mathcal{K}_{\mathcal{P}_Z}(\lambda)\gamma_0\mathcal{D}_Z(\lambda)^{-1} - \hat{\mathcal{K}}_{\mathcal{P}_Z}(\lambda)\gamma_0\hat{\mathcal{D}}_Z(\lambda)^{-1}\right). \end{aligned} \tag{4.3}$$

Proof. By the formulas (3.4) and (3.7), we have

$$\begin{aligned} &\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{P}}(\lambda)^{-1} - (\hat{\mathcal{D}}(\lambda)^{-1} - \hat{\mathcal{D}}_{\mathcal{P}}(\lambda)^{-1}) \\ &= \left(\mathcal{D}(\lambda)^{-1} - \mathcal{D}_M(\lambda)^{-1} - \mathcal{D}_Z(\lambda)^{-1}\right) - \left(\hat{\mathcal{D}}(\lambda)^{-1} - \hat{\mathcal{D}}_N(\lambda)^{-1} - \hat{\mathcal{D}}_Z(\lambda)^{-1}\right) \\ &\quad + \mathcal{K}_{\mathcal{P}_M}(\lambda)\gamma_0\mathcal{D}_M(\lambda)^{-1} - \hat{\mathcal{K}}_{\mathcal{P}_N}(\lambda)\gamma_0\hat{\mathcal{D}}_N(\lambda)^{-1} \\ &\quad + \mathcal{K}_{\mathcal{P}_Z}(\lambda)\gamma_0\mathcal{D}_Z(\lambda)^{-1} - \hat{\mathcal{K}}_{\mathcal{P}_Z}(\lambda)\gamma_0\hat{\mathcal{D}}_Z(\lambda)^{-1}. \end{aligned} \tag{4.4}$$

By Lemma 4.2 and the fact that the Schwartz kernels of $\mathcal{D}_Z(\lambda)^{-1}$ and $\hat{\mathcal{D}}_Z(\lambda)^{-1}$ are identical up to a smoothing term decaying exponentially along the cylinder cf. [30], it follows that the last four operators here define an operator of trace class. Notice that $\mathcal{D}(\lambda)^{-1} - \mathcal{D}_M(\lambda)^{-1} - \mathcal{D}_Z(\lambda)^{-1}$ is off diagonal with respect to the orthogonal decomposition $L^2(X,S) = L^2(M,S) \oplus L^2(Z,S)$ and $\hat{\mathcal{D}}(\lambda)^{-1} - \hat{\mathcal{D}}_N(\lambda)^{-1} - \hat{\mathcal{D}}_Z(\lambda)^{-1}$ is off diagonal with respect to the orthogonal decomposition $L^2(\hat{Z},S) = L^2(N,S) \oplus L^2(Z,S)$. Hence, the operator on the first line on the right-hand side of (4.4) is of trace class with trace zero. It follows that $\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{P}}(\lambda)^{-1} - (\hat{\mathcal{D}}(\lambda)^{-1} - \hat{\mathcal{D}}_{\mathcal{P}}(\lambda)^{-1})$ is of trace class and taking the trace of (4.4) yields the formula (4.3). □

5. Linking Calderón projectors and the resolvents

In this section, we define the key operators of this paper, $K(\lambda)$ and $\hat{K}(\lambda)$, over Y, which are defined through our various Calderón projectors. In Theorem 5.3, we relate these operators with the relative trace of the resolvents $\mathcal{D}(\lambda)^{-1}$, $\mathcal{D}_{\mathcal{P}_M}(\lambda)^{-1}$, $\mathcal{D}_{\mathcal{P}_Z}(\lambda)^{-1}$, and the resolvents over $\hat{Z}$.

Recalling the maps $\kappa_M, \kappa_Z : L^2(Y, S^+) \to L^2(Y, S^-)$ from (1.5) in the introduction, we define a unitary operator on $L^2(Y, S_0)$ by

$$V = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\kappa_M \kappa_Z^{-1} \end{pmatrix} \tag{5.1}$$

where the matrix is written with respect to $L^2(Y, S^+) \oplus L^2(Y, S^-)$. Then, by definition, V satisfies

$$V^{-1}(\mathrm{Id} - \mathcal{P}_M)V = \mathcal{P}_Z.$$

For $\lambda \in \Lambda$, the following operators, which are pseudodifferential operators of order zero defined over $L^2(Y, S_0)$, play the central roles in this paper:

$$K(\lambda) = S_M(\lambda)^{-1} V \mathcal{P}_N + S_Z(\lambda)^{-1}, \qquad \hat{K}(\lambda) = \hat{S}_N(\lambda)^{-1} + \hat{S}_Z(\lambda)^{-1}.$$

These operators link the Cauchy data spaces between M and Z, and N and Z, respectively. We first note that these operators are invertible.

Proposition 5.1. *For $\lambda \in \Lambda$, the operators $K(\lambda)$ and $\hat{K}(\lambda)$ are invertible with inverses given by*

$$K(\lambda)^{-1} = \mathcal{P}_N V^{-1} S_M(\lambda) + S_Z(\lambda), \qquad \hat{K}(\lambda)^{-1} = \hat{S}_N(\lambda) + \hat{S}_Z(\lambda).$$

Proof. First, using that $\mathcal{P}_N \mathcal{P}_Z = 0 = \mathcal{P}_Z \mathcal{P}_N$ and the identities (3.1) and (3.2), we obtain

$$\begin{aligned} K(\lambda)\big(\mathcal{P}_N V^{-1} S_M(\lambda) + S_Z(\lambda)\big) &= \big(S_M(\lambda)^{-1} V \mathcal{P}_N + S_Z(\lambda)^{-1}\big) \\ &\qquad \circ \big(\mathcal{P}_N V^{-1} S_M(\lambda) + S_Z(\lambda)\big) \\ &= S_M(\lambda)^{-1} S_M(\lambda) + S_Z(\lambda)^{-1} S_Z(\lambda) = P_M(\lambda) + P_Z(\lambda) = \mathrm{Id}. \end{aligned}$$

Second, using that $P_M(\lambda) P_Z(\lambda) = 0 = P_Z(\lambda) P_M(\lambda)$ which follow from (3.1), we also have

$$\begin{aligned} \big(\mathcal{P}_N V^{-1} S_M(\lambda) + S_Z(\lambda)\big) K(\lambda) &= \big(\mathcal{P}_N V^{-1} S_M(\lambda) + S_Z(\lambda)\big) \\ &\qquad \circ \big(S_M(\lambda)^{-1} V \mathcal{P}_N + S_Z(\lambda)^{-1}\big) \\ &= \mathcal{P}_N V^{-1} \mathcal{P}_M V \mathcal{P}_N + \mathcal{P}_Z = \mathcal{P}_N + \mathcal{P}_Z = \mathrm{Id}. \end{aligned}$$

A similar (but easier) computation shows that $\hat{S}_N(\lambda) + \hat{S}_Z(\lambda)$ is the inverse of $\hat{K}(\lambda)$. This completes the proof. □

We next note that $K(\lambda)\,\hat{K}(\lambda)^{-1}$ is of *Fredholm determinant class*.

Proposition 5.2. *For $\lambda \in \Lambda$, $K(\lambda)\,\hat{K}(\lambda)^{-1} - \mathrm{Id}$ is a smoothing operator over $L^2(Y, S_0)$.*

Proof. Let us observe that

$$\begin{aligned} K(\lambda) - \hat{K}(\lambda) =& S_M(\lambda)^{-1} V \mathcal{P}_N + S_Z(\lambda)^{-1} - \hat{S}_N(\lambda)^{-1} - \hat{S}_Z(\lambda)^{-1} \\ =& S_M(\lambda)^{-1} V \mathcal{P}_N - \hat{S}_N(\lambda)^{-1}, \end{aligned}$$

because $S_Z(\lambda)^{-1} = \hat{S}_Z(\lambda)^{-1}$. By Lemma 4.2, $\mathcal{P}_M - \mathcal{P}_N = \mathcal{P}_M + \mathcal{P}_Z - \mathrm{Id}$ is smoothing, which implies that V has a form $\mathrm{Id} + \mathcal{S}$ for a smoothing operator $\mathcal{S}$, so that $V\mathcal{P}_N - \mathcal{P}_N$ is a smoothing operator. Also by Lemma 4.2, $S_M(\lambda)^{-1} - \hat{S}_N(\lambda)^{-1}$ is a smoothing operator. These facts imply that the difference $K(\lambda) - \hat{K}(\lambda)$ is a smoothing operator. Multiplying this difference by $\hat{K}(\lambda)^{-1}$, the claim follows. □

By Proposition 5.2, we can define $\det_F(K(\lambda)\hat{K}(\lambda)^{-1})$, which is a holomorphic function over Λ. Since each connected component of Λ is simply connected, we can henceforth fix a logarithm $\log\det_F K(\lambda)$ depending holomorphically for λ over each connected component of Λ. Now we can state the main result of this section.

Theorem 5.3. *For $\lambda \in \Lambda$, the following variation formula holds:*

$$\partial_\lambda \log\det_F(K(\lambda)\hat{K}(\lambda)^{-1}) = -\operatorname{Tr}\big(\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{P}}(\lambda)^{-1} - (\hat{\mathcal{D}}(\lambda)^{-1} - \hat{\mathcal{D}}_{\mathcal{P}}(\lambda)^{-1})\big).$$

Proof. We have

$$\partial_\lambda \log\det_F(K(\lambda)\hat{K}(\lambda)^{-1}) = \operatorname{Tr}\big(K(\lambda)^{-1}\partial_\lambda K(\lambda) - \hat{K}(\lambda)^{-1}\partial_\lambda \hat{K}(\lambda)\big).$$

Recalling that $K(\lambda)^{-1} = \mathcal{P}_N V^{-1} S_M(\lambda) + S_Z(\lambda)$, we can write

$$\begin{aligned} & K(\lambda)^{-1}\partial_\lambda K(\lambda) \\ =& \big(\mathcal{P}_N V^{-1} S_M(\lambda) + S_Z(\lambda)\big)\ \partial_\lambda\big(S_M(\lambda)^{-1} V\mathcal{P}_N + S_Z(\lambda)^{-1}\big) \\ =& \mathcal{P}_N V^{-1} S_M(\lambda)\ \partial_\lambda S_M(\lambda)^{-1} V\mathcal{P}_N + S_Z(\lambda)\ \partial_\lambda S_Z(\lambda)^{-1} \\ & +\ \text{off diagonal terms,} \end{aligned}$$

where "off diagonal" means with respect to $L^2(Y, S_0) = \operatorname{Im}(\mathcal{P}_N) \oplus \operatorname{Im}(\mathcal{P}_Z)$. By definition, we have

$$\gamma_0\mathcal{K}_{\mathcal{P}_M}(\lambda) = \gamma_0\mathcal{K}_M(\lambda)S_M(\lambda)^{-1} = P_M(\lambda)S_M(\lambda)^{-1} = S_M(\lambda)^{-1}$$

and by Lemma 3.2, we have $\partial_\lambda\mathcal{K}_{\mathcal{P}_M}(\lambda) = \mathcal{D}_{\mathcal{P}_M}(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_M}(\lambda)$. From these equalities, we obtain

$$\begin{aligned} & S_M(\lambda)\ \partial_\lambda S_M(\lambda)^{-1} \\ =& S_M(\lambda)\gamma_0\big(\mathcal{D}_M(\lambda)^{-1} - \mathcal{K}_{\mathcal{P}_M}(\lambda)\gamma_0\mathcal{D}_M(\lambda)^{-1}\big)\mathcal{K}_{\mathcal{P}_M}(\lambda) \\ =& S_M(\lambda)\gamma_0\mathcal{D}_M(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_M}(\lambda) - \mathcal{P}_M\gamma_0\mathcal{D}_M(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_M}(\lambda), \end{aligned}$$

where we used the first equality in the identities (3.4). A similar formula holds for $S_Z(\lambda)\ \partial_\lambda S_Z(\lambda)^{-1}$, hence

$$\begin{aligned} & K(\lambda)^{-1}\partial_\lambda K(\lambda) \\ =\ & \mathcal{P}_N V^{-1}\big(S_M(\lambda)\gamma_0\mathcal{D}_M(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_M}(\lambda) - \mathcal{P}_M\gamma_0\mathcal{D}_M(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_M}(\lambda)\big)V\mathcal{P}_N \\ & + S_Z(\lambda)\gamma_0\mathcal{D}_Z(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_Z}(\lambda) - \mathcal{P}_Z\gamma_0\mathcal{D}_Z(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_Z}(\lambda) \\ & + \text{off diagonal terms.} \end{aligned}$$

Using a similar formula for $\hat{K}(\lambda)$, we get

$$\begin{aligned} & \partial_\lambda \log\det_F(K(\lambda)\hat{K}(\lambda)^{-1}) \\ =& \operatorname{Tr}\big(\mathcal{P}_N V^{-1}\big(S_M(\lambda)\gamma_0\mathcal{D}_M(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_M}(\lambda) - \mathcal{P}_M\gamma_0\mathcal{D}_M(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_M}(\lambda)\big)V\mathcal{P}_N \\ & \quad - \hat{S}_N(\lambda)\gamma_0\hat{\mathcal{D}}_N(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_N}(\lambda) - \mathcal{P}_N\gamma_0\hat{\mathcal{D}}_N(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_N}(\lambda)\big) \\ & + \operatorname{Tr}\big(S_Z(\lambda)\gamma_0\mathcal{D}_Z(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_Z}(\lambda) - \mathcal{P}_Z\gamma_0\mathcal{D}_Z(\lambda)^{-1}\mathcal{K}_{\mathcal{P}_Z}(\lambda) \\ & \qquad - \hat{S}_Z(\lambda)\gamma_0\hat{\mathcal{D}}_Z(\lambda)^{-1}\hat{\mathcal{K}}_{\mathcal{P}_Z}(\lambda) - \mathcal{P}_Z\gamma_0\hat{\mathcal{D}}_Z(\lambda)^{-1}\hat{\mathcal{K}}_{\mathcal{P}_Z}(\lambda)\big). \end{aligned}$$

Thus,

$$\begin{aligned}
&\partial_\lambda \log \det_F (K(\lambda)\hat{K}(\lambda)^{-1}) \\
= \ & \operatorname{Tr}\left(\mathcal{K}_{\mathcal{P}_M}(\lambda)S_M(\lambda)\gamma_0\mathcal{D}_M(\lambda)^{-1} - \mathcal{K}_{\mathcal{P}_N}(\lambda)\hat{S}_N(\lambda)\gamma_0\hat{\mathcal{D}}_N(\lambda)^{-1}\right) \\
&- \operatorname{Tr}\left(\mathcal{K}_{\mathcal{P}_M}(\lambda)\gamma_0\mathcal{D}_M(\lambda)^{-1} - \mathcal{K}_{\mathcal{P}_N}(\lambda)\gamma_0\hat{\mathcal{D}}_N(\lambda)^{-1}\right) \\
&+ \operatorname{Tr}\left(\mathcal{K}_{\mathcal{P}_Z}(\lambda)S_Z(\lambda)\gamma_0\mathcal{D}_Z(\lambda)^{-1} - \hat{\mathcal{K}}_{\mathcal{P}_Z}(\lambda)\hat{S}_Z(\lambda)\gamma_0\hat{\mathcal{D}}_Z(\lambda)^{-1}\right) \\
&- \operatorname{Tr}\left(\mathcal{K}_{\mathcal{P}_Z}(\lambda)\gamma_0\mathcal{D}_Z(\lambda)^{-1} - \hat{\mathcal{K}}_{\mathcal{P}_Z}(\lambda)\gamma_0\hat{\mathcal{D}}_Z(\lambda)^{-1}\right).
\end{aligned}$$

By definition, $\mathcal{K}_{\mathcal{P}_M}(\lambda) = \mathcal{K}_M(\lambda)S_M(\lambda)^{-1}$ along with similar formulas for the other Poisson operators, so we can rewrite the above equation as

$$\begin{aligned}
&\partial_\lambda \log \det_F (K(\lambda)\hat{K}(\lambda)^{-1}) \\
= \ & \operatorname{Tr}\left(\mathcal{K}_M(\lambda)P_M(\lambda)\gamma_0\mathcal{D}_M(\lambda)^{-1} - \mathcal{K}_N(\lambda)\hat{P}_N(\lambda)\gamma_0\hat{\mathcal{D}}_N(\lambda)^{-1}\right) \\
&- \operatorname{Tr}\left(\mathcal{K}_{\mathcal{P}_M}(\lambda)\gamma_0\mathcal{D}_M(\lambda)^{-1} - \mathcal{K}_{\mathcal{P}_N}(\lambda)\gamma_0\hat{\mathcal{D}}_N(\lambda)^{-1}\right) \\
&+ \operatorname{Tr}\left(\mathcal{K}_Z(\lambda)P_Z(\lambda)\gamma_0\mathcal{D}_Z(\lambda)^{-1} - \hat{\mathcal{K}}_Z(\lambda)\hat{P}_Z(\lambda)\gamma_0\hat{\mathcal{D}}_Z(\lambda)^{-1}\right) \\
&- \operatorname{Tr}\left(\mathcal{K}_{\mathcal{P}_Z}(\lambda)\gamma_0\mathcal{D}_Z(\lambda)^{-1} - \hat{\mathcal{K}}_{\mathcal{P}_Z}(\lambda)\gamma_0\hat{\mathcal{D}}_Z(\lambda)^{-1}\right).
\end{aligned}$$

Finally, a proof similar to Lemma 3.2 shows that

$$\partial_\lambda P_M(\lambda) = \partial_\lambda \gamma_0 \mathcal{K}_M(\lambda) = \gamma_0\mathcal{D}_M(\lambda)^{-1}\mathcal{K}_M(\lambda)$$

with similar formulas hold for the other Calderón projections, hence recalling that $P_M(\lambda) + P_Z(\lambda) = \mathrm{Id}$ and $\hat{P}_N(\lambda) + \hat{P}_Z(\lambda) = \mathrm{Id}$, we obtain

$$\begin{aligned}
&\partial_\lambda \log \det_F (K(\lambda)\hat{K}(\lambda)^{-1}) \\
= \ & \frac{1}{2}\partial_\lambda \operatorname{Tr}\left(P_M(\lambda) + P_Z(\lambda) - (\hat{P}_N(\lambda) + \hat{P}_Z(\lambda))\right) \\
&- \operatorname{Tr}\left(\mathcal{K}_{\mathcal{P}_M}(\lambda)\gamma_0\mathcal{D}_M(\lambda)^{-1} - \mathcal{K}_{\mathcal{P}_N}(\lambda)\gamma_0\hat{\mathcal{D}}_N(\lambda)^{-1}\right) \\
&- \operatorname{Tr}\left(\mathcal{K}_{\mathcal{P}_Z}(\lambda)\gamma_0\mathcal{D}_Z(\lambda)^{-1} - \hat{\mathcal{K}}_{\mathcal{P}_Z}(\lambda)\gamma_0\hat{\mathcal{D}}_Z(\lambda)^{-1}\right) \\
= \ & - \operatorname{Tr}\left(\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{P}}(\lambda)^{-1} - (\hat{\mathcal{D}}(\lambda)^{-1} - \hat{\mathcal{D}}_{\mathcal{P}}(\lambda)^{-1})\right),
\end{aligned}$$

where we used (4.3) at the last line. This completes the proof. □

6. Asymptotics of $\det_F (K(\lambda)\hat{K}(\lambda)^{-1})$

In this section, we investigate the asymptotics of $\det_F (K(\lambda)\hat{K}(\lambda)^{-1})$ for small and large λ, which enable us to extract the $\mathcal{L}$ and $\widehat{U}$ contributions to our main Theorem 1.1. We begin by studying the asymptotics of $P_M(\lambda)$ and $P_Z(\lambda)$ as $\lambda \to 0$.

Lemma 6.1. *For $\lambda \in \Lambda$, we have*

$$P_M(\lambda) = \lambda^{-1}\mathcal{L}\, G + Q_M(\lambda) \quad , \quad P_Z(\lambda) = -\lambda^{-1}\mathcal{L}\, G + Q_Z(\lambda),$$

where $\mathcal{L} = \sum_{k=1}^{h_X} \gamma_0 U_k \otimes (\gamma_0 U_k)^*$ is defined in (1.6) and $Q_M(\lambda), Q_Z(\lambda)$ are pseudodifferential operators over Y that are regular at $\lambda = 0$.

Proof. From Melrose [30, Ch. 6] or Vaillant [41], we know that

$$(\mathcal{D} - \lambda)^{-1} = -\lambda^{-1}\sum_{k=1}^{h_X} U_k \otimes U_k^* + R(\lambda) \tag{6.1}$$

where $U_k^* := \langle\,\cdot\,, U_k\rangle_X$ where $\langle\,\cdot\,,\,\cdot\,\rangle_X$ is the inner product on $L^2(X,S)$ and $R(\lambda)$ is a b-pseudodifferential operator over X that is regular at $\lambda = 0$. Applying this equality to the definition of $P_M(\lambda)$, we obtain

$$P_M(\lambda) = -\gamma_{0^-}(\mathcal{D}-\lambda)^{-1}\gamma_0^* G = \lambda^{-1}\sum_{k=1}^{h_X}\gamma_0 U_k \otimes U_k^*\gamma_0^* G + Q_M(\lambda),$$

where $Q_M(\lambda)$ is regular at $\lambda = 0$. This completes our proof for $P_M(\lambda)$. A similar proof works for $P_Z(\lambda)$. □

Remark 6.2. We remark that the operator $R(\lambda)$ in (6.1) has two analytic continuations to a neighborhood of $\lambda = 0$, one from $\Im\lambda > 0$ and the other from $\Im\lambda < 0$, but when $\ker(D_Y) \neq 0$, they have different values at $\lambda = 0$! Thus, technically speaking, we should write $Q_M^{\pm}(\lambda)$ to emphasize the fact that $Q_M(\lambda)$ may have different values at $\lambda = 0$ depending on its continuation from $\Im\lambda > 0$ or $\Im\lambda < 0$. We emphasize, however, that these different values *play no role* in the analysis that follows; below the important fact that we will need is that $P_M(\lambda) = \lambda^{-1}\mathcal{L}\,G$ modulo bounded at $\lambda = 0$. (A similar remark holds for $Q_Z(\lambda)$.)

Let $P^{\pm} = \frac{\mathrm{Id}\mp iG}{2}$, which are the orthogonal projections onto $S^{\pm}$. Next, we need the following observation.

Lemma 6.3. *Let* $W = \mathrm{Im}(\mathcal{P}_M)\cap\mathrm{Im}(\mathcal{P}_Z)$. *Then*

$$W = \mathrm{Im}(\mathcal{P}_M)\cap\mathrm{Im}(\Pi_>) = \gamma_0(\ker_{L^2}(\mathcal{D})).$$

Moreover, $\dim W = h_X = \dim\ker_{L^2}(\mathcal{D})$ *and* $V = -iG$ *on* $W \oplus GW$.

Proof. By definition of $\mathcal{P}_M$ and $\mathcal{P}_Z$, it follows that $\mathrm{Im}(\mathcal{P}_M)\cap\mathrm{Im}(\mathcal{P}_Z) = \mathrm{Im}(\mathcal{P}_M)\cap\mathrm{Im}(\Pi_>)$ and elements of the intersection $\mathrm{Im}(\mathcal{P}_M)\cap\mathrm{Im}(\Pi_>)$ are exactly the restrictions of elements in $\ker_{L^2}(\mathcal{D})$. This proves that $W = \mathrm{Im}(\mathcal{P}_M)\cap\mathrm{Im}(\Pi_>) = \gamma_0(\ker_{L^2}(\mathcal{D}))$. Thus, $\dim W = \dim\ker_{L^2}(\mathcal{D})$ by the unique continuation theorem for $\mathcal{D}$. From the expressions (1.5) for $\mathcal{P}_M$ and $\mathcal{P}_Z$, it follows that $\kappa_M = \kappa_Z$ over $P^+W = P^+GW$. Thus, from the definition of V in (5.1), we can see that acting on W or GW, we have

$$V = \begin{pmatrix}\mathrm{Id} & 0\\ 0 & -\mathrm{Id}\end{pmatrix}.$$

Recalling that $G = \begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}$ completes our proof. □

Since X is a noncompact manifold with cylindrical end, we obtain some unusual phenomena when we consider the behavior of $K(\lambda)$ near $\lambda = 0$, features which are nonexistent in the compact case [25]. For instance, consider

Proposition 6.4. *Putting* $\Pi_\sigma := \frac{\mathrm{Id}-\sigma}{2}\Pi_0$, *for small* λ *near* 0, *we have*

$$K(\lambda) \sim_{\Im(\lambda)\to\pm 0} \mathcal{P}_M V\mathcal{P}_N + \Pi_> + 2P^{\mp}\Pi_\sigma + o(1),$$
$$\hat{K}(\lambda) \sim_{\Im(\lambda)\to\pm 0} \mathcal{P}_N + \Pi_> + 2P^{\mp}\Pi_\sigma + o(1).$$

Proof. Let us recall the definition $K(\lambda) = S_M(\lambda)^{-1}V\mathcal{P}_N + S_Z(\lambda)^{-1}$. Since M is compact it follows that $S_M(\lambda)^{-1}V\mathcal{P}_N$ has the form $\mathcal{P}_M V\mathcal{P}_N + o(1)$ for small λ near 0. Now recall that

$$S_Z(\lambda)^{-1} := P_Z^o(\lambda)[\mathcal{P}_Z P_Z^o(\lambda) + (\mathrm{Id}-\mathcal{P}_Z)(\mathrm{Id}-P_Z^o(\lambda))]^{-1}\mathcal{P}_Z,$$

which depends only on the intrinsic data over Z. It is not difficult to see that $S_Z(\lambda)^{-1}\Pi_> = \Pi_> + o(1)$ for small λ near 0. Now it remains to show

$$S_Z(\lambda)^{-1}\Pi_0 \sim_{\Im(\lambda)\to\pm 0} 2P^{\mp}\Pi_\sigma + o(1).$$

To see this, we need to determine the intersection $\mathcal{H}_Z(\lambda) \cap \mathrm{Im}(\Pi_0)$. To this end, let $\phi \in C^\infty(Z, \ker(D_Y))$, and note that $(\mathcal{D} - \lambda)\phi = (G\partial_u - \lambda)\phi = 0$ if and only if $\phi = (ae^{-i\lambda u}, be^{i\lambda u})$ for some constants a, b, where ϕ is decomposed into the $\pm i$ eigenspaces of G. For ϕ to be in L^2 we need $a = 0$ if $\Im\lambda > 0$ and $b = 0$ if $\Im\lambda < 0$. It follows that

$$\mathcal{H}_Z(\lambda) \cap \mathrm{Im}(\Pi_0) = \begin{cases} P^-\Pi_0 & \text{for } \Im\lambda > 0, \\ P^+\Pi_0 & \text{for } \Im\lambda < 0. \end{cases}$$

Since $P^o_Z(\lambda)$ is the unique orthogonal projector onto $\mathcal{H}_Z(\lambda)$ it follows that $P^o_Z(\lambda)\Pi_0 = P^{\mp}\Pi_0$ for $\Im\lambda > 0$ and $\Im\lambda < 0$, respectively. Thus,

$$S_Z(\lambda)^{-1}\Pi_0 = P^{\mp}[\Pi_\sigma P^{\mp} + \Pi_{-\sigma}P^{\pm}]^{-1}\Pi_\sigma,$$

for $\Im\lambda > 0$ and $\Im\lambda < 0$, respectively. An easy computation shows that the right-hand side is exactly $2P^{\mp}\Pi_\sigma$. The same arguments can be applied to $\hat{K}(\lambda)$. This completes our proof. □

For convenience, from now on we shall use the notation

$$K(\pm i0) := \lim_{\Im\lambda\to 0^\pm} K(\lambda) = \mathcal{P}_M V \mathcal{P}_N + \mathcal{P}^{\mp}_Z \qquad \text{where} \quad \mathcal{P}^{\pm}_Z := \Pi_> + 2P^{\pm}\Pi_\sigma.$$

Let us remark that $K(0)$ is *not* well-defined if $\ker(D_Y)$ is nontrivial; this stands in noteworthy contrast to the compact case considered in [25]. Now we consider a decomposition of $K(\pm i0)$, which will be used later.

Proposition 6.5. *Let $\mathcal{W} = W^+ \oplus W^-$ where $W^\pm = P^\pm W = P^\pm GW$ with $W = \gamma_0(\ker_{L^2}(\mathcal{D})) = \mathrm{Im}(\mathcal{P}_M) \cap \mathrm{Im}(\mathcal{P}_Z)$. Then $K(\pm i0)$ takes the matrix form*

$$K(\pm i0) = \begin{pmatrix} A & 0 \\ 0 & P_{\mathcal{W}^\perp} K(\pm i0) P_{\mathcal{W}^\perp} \end{pmatrix}$$

with respect to the decomposition $L^2(Y, S_0) = \mathcal{W} \oplus \mathcal{W}^\perp$ and the operator $A : \mathcal{W} \to \mathcal{W}$ is of the form $A = \begin{pmatrix} \mathrm{Id} & 0 \\ \kappa_M|_{W^+} & 0 \end{pmatrix}$ with respect to the decomposition $\mathcal{W} = W^+ \oplus W^-$.

Proof. For $\varphi \in W = \mathrm{Im}(\mathcal{P}_M) \cap \mathrm{Im}(\mathcal{P}_Z) = \mathrm{Im}(\mathcal{P}_M) \cap \mathrm{Im}(\Pi_>)$, using the fact that $\mathcal{P}_M G = G(\mathrm{Id} - \mathcal{P}_M)$ and $\mathcal{P}_Z G = G(\mathrm{Id} - \mathcal{P}_Z)$, we have

$$\mathcal{P}_M\varphi = \mathcal{P}_Z\varphi = \varphi \quad , \quad \mathcal{P}_M G\varphi = \mathcal{P}_Z G\varphi = 0.$$

Using these formulas and the fact that $V = -iG$ over W and GW by Lemma 6.3, we find that

$$\begin{aligned} K(\pm i0)P^+\varphi &= \big(\mathcal{P}_M V(\mathrm{Id} - \mathcal{P}_Z) + \mathcal{P}^{\mp}_Z\big)\frac{\mathrm{Id} - iG}{2}\varphi = \varphi, \\ K(\pm i0)P^-\varphi &= \big(\mathcal{P}_M V(\mathrm{Id} - \mathcal{P}_Z) + \mathcal{P}^{\mp}_Z\big)\frac{\mathrm{Id} + iG}{2}\varphi = 0. \end{aligned} \tag{6.2}$$

These equations show that $K(\pm i0) : \mathcal{W} \to \mathcal{W}$, and if $A := K(\pm i0)|_{\mathcal{W}}$, then with respect to the decomposition $\mathcal{W} = W^+ \oplus W^-$, we have $A = \begin{pmatrix} \mathrm{Id} & 0 \\ \kappa_M|_{W^+} & 0 \end{pmatrix}$ since $\varphi = (P^+\varphi, \kappa_M P^+\varphi)$. Thus, our proof is finished once we show that $P_{\mathcal{W}^\perp} K(\pm i0) P_{\mathcal{W}} = 0$ and $P_{\mathcal{W}} K(\pm i0) P_{\mathcal{W}^\perp} = 0$. That $P_{\mathcal{W}^\perp} K(\pm i0) P_{\mathcal{W}} = 0$ follows from the fact that

$K(\pm i0) : \mathcal{W} \to \mathcal{W}$. To prove that $P_{\mathcal{W}} K(\pm i0) P_{\mathcal{W}^\perp} = 0$ it suffices to consider adjoints and prove that $P_{\mathcal{W}^\perp} K(\pm i0)^* P_{\mathcal{W}} = 0$. However, the exact same argument shown in (6.2) can be used to show that

$$K(\pm i0)^* = (\mathrm{Id} - \mathcal{P}_Z) V^{-1} \mathcal{P}_M + (\mathcal{P}_Z^{\mp})^* : \mathcal{W} \longrightarrow \mathcal{W},$$

which in turn proves that $P_{\mathcal{W}^\perp} K(\pm i0)^* P_{\mathcal{W}} = 0$. □

We now consider the asymptotics of $K(\lambda)$ as $\lambda \to 0$. To this end, we use Lemma 6.1 to write

$$\begin{aligned} K(\lambda)^{-1} &= \mathcal{P}_N V^{-1} S_M(\lambda) + S_Z(\lambda) \\ &= \mathcal{P}_N V^{-1} \mathcal{P}_M(\lambda^{-1} \mathcal{L}\, G + Q_M(\lambda)) + \mathcal{P}_Z(-\lambda^{-1} \mathcal{L}\, G + Q_Z(\lambda)) \\ &= \lambda^{-1}(V^{-1} \mathcal{L}\, G - \mathcal{L}\, G) + Q(\lambda) \end{aligned} \tag{6.3}$$

where $Q(\lambda)$ is a pseudodifferential operator over Y that is regular at $\lambda = 0$. By Lemma 6.3, $V = -iG$ over the space spanned by $\{\gamma_0 U_k\}$, that is, the space W, so if we look at the first term in (6.3) in more detail, we see that

$$\begin{aligned} V^{-1} \mathcal{L}\, G - \mathcal{L}\, G &= (-iG - \mathrm{Id}) \mathcal{L}\, G \\ &= -i(\mathrm{Id} + iG) G \mathcal{L}\, G = -2i P^- G \mathcal{L}\, G. \end{aligned} \tag{6.4}$$

Now it is straightforward to check that

$$P^{\pm} G : \gamma_0(\ker_{L^2}(\mathcal{D})) \longrightarrow P^{\pm} G \gamma_0(\ker_{L^2}(\mathcal{D}))$$

is an isomorphism, and directly from the definition of $\mathcal{L}$, we have

$$\mathcal{L}_{\pm} := -2P^- G \mathcal{L}\, G P^{\pm} = (P^- G)\, \mathcal{L}\, (P^{\pm} G)^{-1} \, : \, W^{\pm} \longrightarrow W^-. \tag{6.5}$$

Then in view of (6.3), (6.4), and the definition of $\mathcal{L}_{\pm}$, we can write

$$K(\lambda)^{-1} = i\lambda^{-1} \mathcal{L}_+ + i\lambda^{-1} \mathcal{L}_- + Q(\lambda) \tag{6.6}$$

where $Q(\lambda)$ is a pseudodifferential operator on Y that is regular at $\lambda = 0$. Then we have

Corollary 6.6. *Let $\mathcal{W} = W^+ \oplus W^-$. Then with respect to the decomposition $L^2(Y, S_0) = \mathcal{W} \oplus \mathcal{W}^\perp$, for for $\Im\lambda > 0$ and $\Im\lambda < 0$ small, we have*

$$K(\lambda) = \begin{pmatrix} A(\lambda) & \mathcal{O}(\lambda) \\ \mathcal{O}(\lambda) & P_{\mathcal{W}^\perp} K(\pm i0) P_{\mathcal{W}^\perp} + \mathcal{O}(\lambda) \end{pmatrix},$$

where $A(\lambda)$ satisfies

$$A(\lambda)^{-1} = \begin{pmatrix} \mathrm{Id} + b_{\pm}\, \mathcal{L}_-^{-1} \mathcal{L}_+ + \mathcal{O}(\lambda) & b_{\pm} + \mathcal{O}(\lambda) \\ i\lambda^{-1} \mathcal{L}_+ + p_{\pm}(\lambda) & i\lambda^{-1} \mathcal{L}_- + q_{\pm}(\lambda) \end{pmatrix}$$

with respect to the decomposition $\mathcal{W} = W^+ \oplus W^-$ where $p_{\pm}(\lambda), q_{\pm}(\lambda)$ are regular at $\lambda = 0$ and $b_{\pm} : W^- \to W^+$.

Proof. The statement for $K(\lambda)$ just follows from Proposition 6.5. To derive the matrix form of $A(\lambda)^{-1}$ recall that a matrix of operators $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with a invertible, is invertible if and only if $D := d - c\, a^{-1} b$ is invertible, in which case [41, p. 53]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} + a^{-1} b D^{-1} c\, a^{-1} & -a^{-1} b D^{-1} \\ -D^{-1} c\, a^{-1} & D^{-1} \end{pmatrix}. \tag{6.7}$$

From Proposition 6.5, we know that $A(\lambda) = \begin{pmatrix} a(\lambda) & \lambda\, b(\lambda) \\ c(\lambda) & \lambda\, d(\lambda) \end{pmatrix}$ with respect to the decomposition $\mathcal{W} = W^+ \oplus W^-$, where $a(0) = \mathrm{Id}$. In particular, $a(0) = \mathrm{Id}$ is invertible, so we can apply the formula (6.7) to find $A(\lambda)^{-1}$ for λ near 0. Doing this and using (6.6), we get, after a long and tedious but very elementary computation, our desired formula for $A(\lambda)^{-1}$. □

We are now ready to compute the asymptotics of $\det_F\,(K(\lambda)\hat{K}(\lambda)^{-1})$ as $\lambda \to 0$. First, from Corollary 6.6, it follows that as $\lambda \to 0$ from $\Im\lambda > 0$ and $\Im\lambda < 0$,

$$\det{}_F\,(K(\lambda)\hat{K}(\lambda)^{-1}) = \det A(\lambda) \cdot \det{}_F\,\big(P_{\mathcal{W}^\perp} K(\pm i0) P_{\mathcal{W}^\perp}\big)\big(1 + o(1)\big). \tag{6.8}$$

To find $\det A(\lambda)$, observe that by Corollary 6.6, we can write

$$A(\lambda)^{-1} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & i\lambda^{-1}\mathcal{L}_- \end{pmatrix} \left[\begin{pmatrix} \mathrm{Id} + b_\pm\, \mathcal{L}_-^{-1}\mathcal{L}_+ & b_\pm \\ \mathcal{L}_-^{-1}\mathcal{L}_+ & \mathrm{Id} \end{pmatrix} + \mathcal{O}(\lambda) \right].$$

From this expression it may seem like $\lim_{\lambda\to 0} \det A(\lambda)$ depends on the sign of $\Im\lambda$, but miraculously it does not because of the identity

$$\begin{pmatrix} \mathrm{Id} + b_\pm\, \mathcal{L}_-^{-1}\mathcal{L}_+ & b_\pm \\ \mathcal{L}_-^{-1}\mathcal{L}_+ & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & b_\pm\, \mathcal{L}_-^{-1} \\ 0 & \mathcal{L}_-^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ \mathcal{L}_+ & \mathcal{L}_- \end{pmatrix},$$

which implies that for small λ near 0,

$$\begin{aligned} \det A(\lambda)^{-1} &= (i\lambda^{-1})^{h_X} (\det \mathcal{L}_-)(\det \mathcal{L}_-^{-1})(\det \mathcal{L}_-)\big(1 + o(1)\big) \\ &= (i\lambda^{-1})^{h_X} (\det \mathcal{L}_-)\big(1 + o(1)\big). \end{aligned}$$

Since $\mathcal{L}_- = (P^-G)\,\mathcal{L}\,(P^-G)^{-1}$ by the definition (6.5), we have $\det \mathcal{L}_- = \det \mathcal{L}$. Thus, for small λ near 0 we have

$$\det A(\lambda) = (-i\lambda)^{h_X} (\det\, \mathcal{L})^{-1}\,\big(1 + o(1)\big).$$

Second, by Proposition 6.4, $K(\pm i0)$ and $\hat{K}(\pm i0)$ preserve $\ker(D_Y)$ and with respect to the decomposition $\ker(D_Y) = \mathrm{Im}(\frac{\mathrm{Id}+\sigma}{2}) \oplus \mathrm{Im}(\frac{\mathrm{Id}-\sigma}{2})$, using the formulas in Proposition 6.4 one can show that

$$K(\pm i0)|_{\ker(D_Y)} = \begin{pmatrix} 1 & \mp 1 \\ 0 & 1 \end{pmatrix} \quad , \quad \hat{K}(\pm i0)|_{\ker(D_Y)} = \begin{pmatrix} 1 & \mp 1 \\ 0 & 1 \end{pmatrix}.$$

These matrices contribute unity to the asymptotics of $\det_F\,(K(\lambda)\hat{K}(\lambda)^{-1})$ as $\Im\lambda \to 0$. Therefore, we can assume that $\ker(D_Y) = 0$ for the remaining computations in this section. Under this condition $K(0)$ is well defined and we shall use this notation for the following part of the proof. In particular, in view of (6.8) and our analysis of $\det A(\lambda)$, for small λ near 0 we have

$$\det{}_F\,(K(\lambda)\hat{K}(\lambda)^{-1}) = (-i\lambda)^{h_X} (\det\, \mathcal{L})^{-1} \det{}_F\,\big(P_{\mathcal{W}^\perp} K(0) P_{\mathcal{W}^\perp}\big)\big(1 + o(1)\big). \tag{6.9}$$

Third, let us consider the equality

$$K(0) = V(\mathrm{Id} - \mathcal{P}_Z) + \mathcal{P}_Z = \begin{pmatrix} \mathcal{P}_Z & \mathcal{P}_Z V(\mathrm{Id} - \mathcal{P}_Z) \\ 0 & (\mathrm{Id} - \mathcal{P}_Z)V(\mathrm{Id} - \mathcal{P}_Z) \end{pmatrix},$$

which is written with respect to $L^2(Y, S_0) = \mathrm{Im}(\mathcal{P}_Z) \oplus \mathrm{Im}(\mathrm{Id} - \mathcal{P}_Z)$. Recall that $\mathcal{W} = W^+ \oplus W^-$ where $W^\pm = P^\pm W = P^\pm G W$, and we have the inclusions $W \subset \mathrm{Im}(\mathcal{P}_Z)$ and $GW \subset \mathrm{Im}(\mathrm{Id} - \mathcal{P}_Z)$. Hence,

$$\det{}_F\,\big(P_{\mathcal{W}^\perp} K(0) P_{\mathcal{W}^\perp}\big) = \det{}_F\,\big(P_{GW^\perp}(\mathrm{Id} - \mathcal{P}_Z)V(\mathrm{Id} - \mathcal{P}_Z)P_{GW^\perp}\big) \tag{6.10}$$

where $P_{GW^\perp}$ is the orthogonal projection on $GW^\perp \cap \operatorname{Im}(\mathrm{Id} - \mathcal{P}_Z)$. For $\varphi = (x, -\kappa_Z x) \in \operatorname{Im}(\mathrm{Id} - \mathcal{P}_Z)$ written as a column vector, using the formulas (1.5) and (5.1) for $\mathcal{P}_M$, $\mathcal{P}_Z$, and V, we have

$$\begin{aligned}(\mathrm{Id} - \mathcal{P}_Z)V(\mathrm{Id} - \mathcal{P}_Z)\varphi &= \frac{1}{2}\begin{pmatrix}\mathrm{Id} & -\kappa_Z^{-1}\\ -\kappa_Z & \mathrm{Id}\end{pmatrix}\begin{pmatrix}\mathrm{Id} & 0\\ 0 & -\kappa_M\kappa_Z^{-1}\end{pmatrix}\begin{pmatrix}x\\ -\kappa_Z x\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}\mathrm{Id} & -\kappa_Z^{-1}\\ -\kappa_Z & \mathrm{Id}\end{pmatrix}\begin{pmatrix}x\\ \kappa_M x\end{pmatrix} = \frac{1}{2}\begin{pmatrix}(\mathrm{Id} - \kappa_Z^{-1}\kappa_M)x\\ (\mathrm{Id} - \kappa_M\kappa_Z^{-1})(-\kappa_Z x)\end{pmatrix}.\end{aligned}$$

In other words, over $\operatorname{Im}(\mathrm{Id} - \mathcal{P}_Z)$,

$$(\mathrm{Id} - \mathcal{P}_Z)V(\mathrm{Id} - \mathcal{P}_Z) = \begin{pmatrix}\kappa_Z^{-1}\frac{1}{2}\big(\mathrm{Id} - \kappa_M\kappa_Z^{-1}\big)\kappa_Z & 0\\ 0 & \frac{1}{2}\big(\mathrm{Id} - \kappa_M\kappa_Z^{-1}\big)\end{pmatrix}. \tag{6.11}$$

We recall that $U := -\kappa_M\kappa_Z^{-1}$ and observe that for $\psi \in L^2(Y, S^-)$,

$$U\psi = -\psi \qquad \text{if and only if} \qquad \psi \in W^- = P^- W = P^- GW.$$

In other words, the (-1)-eigenspace of U is exactly W^-. Thus, if we define $\widehat{U}$ as the restriction of U to the orthogonal complement of its (-1)-eigenspace, then $\mathrm{Id} + \widehat{U}$ is invertible on its domain. Now by (6.10) and (6.11), we obtain

$$\det{}_F\left(P_{\mathcal{W}^\perp} K(0) P_{\mathcal{W}^\perp}\right) = \det{}_F\Big(\frac{\mathrm{Id} + \widehat{U}}{2}\Big).$$

Combining this formula with the expansion (6.9) we conclude

Theorem 6.7. *For $\lambda \in \Lambda$ near 0, we have*

$$\det{}_F\left(K(\lambda)\hat{K}(\lambda)^{-1}\right) = (-i\lambda)^{h_X}(\det \mathcal{L})^{-1}\det{}_F\Big(\frac{\mathrm{Id} + \widehat{U}}{2}\Big)\big(1 + o(1)\big).$$

We now consider the limits of $K(\lambda)$ and $\hat{K}(\lambda)$ as $\Im\lambda \to \pm\infty$.

Proposition 6.8. *For $\lambda \in \Lambda$, we have*

$$\lim_{\Im\lambda\to\infty} K(\lambda) = \begin{pmatrix}\mathrm{Id} & -\kappa_Z^{-1}\\ \kappa_Z & \mathrm{Id}\end{pmatrix}, \quad \lim_{\Im\lambda\to-\infty} K(\lambda) = \begin{pmatrix}\mathrm{Id} & \kappa_Z^{-1}\\ \kappa_M & -\kappa_M\kappa_Z^{-1}\end{pmatrix},$$
$$\lim_{\Im\lambda\to\infty} \hat{K}(\lambda) = \begin{pmatrix}\mathrm{Id} & -\kappa_Z^{-1}\\ \kappa_Z & \mathrm{Id}\end{pmatrix}, \quad \lim_{\Im\lambda\to-\infty} \hat{K}(\lambda) = \begin{pmatrix}\mathrm{Id} & \kappa_Z^{-1}\\ -\kappa_Z & \mathrm{Id}\end{pmatrix}$$

where the matrices are written with respect to $L^2(Y, S_0) = L^2(Y, S^+) \oplus L^2(Y, S^-)$.

Proof. By Corollary 6.2 in [25], we have

$$\begin{aligned}&\lim_{\Im\lambda\to\infty} S_M(\lambda)^{-1} = \begin{pmatrix}\mathrm{Id} & \kappa_M^{-1}\\ 0 & 0\end{pmatrix}, \quad \lim_{\Im\lambda\to\infty} S_Z(\lambda)^{-1} = \begin{pmatrix}0 & 0\\ \kappa_Z & \mathrm{Id}\end{pmatrix},\\ &\lim_{\Im\lambda\to-\infty} S_M(\lambda)^{-1} = \begin{pmatrix}0 & 0\\ \kappa_M & \mathrm{Id}\end{pmatrix}, \quad \lim_{\Im\lambda\to-\infty} S_Z(\lambda)^{-1} = \begin{pmatrix}\mathrm{Id} & \kappa_Z^{-1}\\ 0 & 0\end{pmatrix}.\end{aligned} \tag{6.12}$$

and

$$\begin{aligned}&\lim_{\Im\lambda\to\infty} \hat{S}_N(\lambda)^{-1} = \begin{pmatrix}\mathrm{Id} & -\kappa_Z^{-1}\\ 0 & 0\end{pmatrix}, \quad \lim_{\Im\lambda\to\infty} \hat{S}_Z(\lambda)^{-1} = \begin{pmatrix}0 & 0\\ \kappa_Z & \mathrm{Id}\end{pmatrix},\\ &\lim_{\Im\lambda\to-\infty} \hat{S}_N(\lambda)^{-1} = \begin{pmatrix}0 & 0\\ -\kappa_Z & \mathrm{Id}\end{pmatrix}, \quad \lim_{\Im\lambda\to-\infty} \hat{S}_Z(\lambda)^{-1} = \begin{pmatrix}\mathrm{Id} & \kappa_Z^{-1}\\ 0 & 0\end{pmatrix}.\end{aligned} \tag{6.13}$$

Hence, as $\Im\lambda \to \infty$,

$$K(\lambda) = S_M(\lambda)^{-1} V \mathcal{P}_N + S_Z(\lambda)^{-1}$$
$$\to \begin{pmatrix} \mathrm{Id} & \kappa_M^{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\kappa_M \kappa_Z^{-1} \end{pmatrix} \frac{1}{2} \begin{pmatrix} \mathrm{Id} & -\kappa_Z^{-1} \\ -\kappa_Z & \mathrm{Id} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \kappa_Z & \mathrm{Id} \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{Id} & -\kappa_Z^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \kappa_Z & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & -\kappa_Z^{-1} \\ \kappa_Z & \mathrm{Id} \end{pmatrix}.$$

The limit as $\Im\lambda \to -\infty$ can be computed using a similar argument. The limits of $\hat{K}(\lambda)$ are obtained in the exact same way, but we use (6.13). ☐

Corollary 6.9. *For $\lambda \in \Lambda$, we have*

$$\lim_{\Im\lambda\to\infty} \det{}_F\,(K(\lambda)\hat{K}(\lambda)^{-1}) = 1 \quad , \quad \lim_{\Im\to-\infty} \det{}_F\,(K(\lambda)\hat{K}(\lambda)^{-1}) = \det{}_F U.$$

Proof. By Proposition 6.8, we have

$$\lim_{\Im\lambda\to\infty} \det{}_F\,(K(\lambda)\hat{K}(\lambda)^{-1}) = \det{}_F \begin{pmatrix} \mathrm{Id} & -\kappa_Z^{-1} \\ \kappa_Z & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & -\kappa_Z^{-1} \\ \kappa_Z & \mathrm{Id} \end{pmatrix}^{-1}$$
$$= \det{}_F \mathrm{Id} = 1$$

and

$$\lim_{\Im\lambda\to-\infty} \det{}_F\,(K(\lambda)\hat{K}(\lambda)^{-1}) = \det{}_F \begin{pmatrix} \mathrm{Id} & \kappa_Z^{-1} \\ \kappa_M & -\kappa_M\kappa_Z^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & \kappa_Z^{-1} \\ -\kappa_Z & \mathrm{Id} \end{pmatrix}^{-1}$$
$$= \det{}_F \begin{pmatrix} \mathrm{Id} & \kappa_Z^{-1} \\ \kappa_M & -\kappa_M\kappa_Z^{-1} \end{pmatrix} \frac{1}{2} \begin{pmatrix} \mathrm{Id} & -\kappa_Z^{-1} \\ \kappa_Z & \mathrm{Id} \end{pmatrix}$$
$$= \det{}_F \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\kappa_M\kappa_Z^{-1} \end{pmatrix} = \det{}_F\,(-\kappa_M\kappa_Z^{-1}) = \det{}_F U.$$

We remark that we can interchange the limits above with $\det_F$ because $K(\lambda)\hat{K}(\lambda)^{-1}$ converges as $\Im\lambda \to \pm\infty$ within the determinant class operators due to the pseudo-differential nature of these operators studied in Section 6 of [25]. ☐

This corollary along with the theorem 7.3 in [25] imply the following theorem,

Theorem 6.10. *For $\lambda \in \Lambda$, we have*

$$\det{}_F\,(K(\lambda)\hat{K}(\lambda)^{-1}) \ \sim_{\Im\lambda\to\pm\infty} \ \sum_{k=0}^{\infty} a_k^{\pm}\,\lambda^{-k},$$

where $a_0^+ = 1$ and $a_0^- = \det_F U$.

7. Resolvents and the spectral invariants

In this section, we relate $\log\det_F\,(K(\lambda)\hat{K}(\lambda)^{-1})$ with the spectral invariants through the resolvents via Theorem 5.3. As before, let us put

$$\mathcal{D}_{\mathcal{P}} = \mathcal{D}_{\mathcal{P}_M} \oplus \mathcal{D}_{\mathcal{P}_Z}, \qquad \hat{\mathcal{D}}_{\mathcal{P}} = \hat{\mathcal{D}}_{\mathcal{P}_N} \oplus \hat{\mathcal{D}}_{\mathcal{P}_Z},$$

which are considered as (unbounded) operators on $L^2(X,S)$. From Theorem 5.3, we can derive the following result.

Proposition 7.1. *For $\lambda \in \Lambda$, we have*

$$\begin{aligned}
&\partial_\lambda\big(\, \log \det{}_F\, (K(\lambda)\hat{K}(\lambda)^{-1}) - \log \det{}_F\, (K(\lambda)\hat{K}(-\lambda)^{-1})\,\big)\\
&= \; -2\,\mathrm{Tr}\bigg(\, \frac{\mathcal{D}}{\mathcal{D}^2-\lambda^2} - \frac{\mathcal{D}_{\mathcal{P}}}{\mathcal{D}_{\mathcal{P}}^2-\lambda^2} - \frac{\hat{\mathcal{D}}}{\hat{\mathcal{D}}^2-\lambda^2} + \frac{\hat{\mathcal{D}}_{\mathcal{P}}}{\hat{\mathcal{D}}_{\mathcal{P}}^2-\lambda^2}\,\bigg),
\end{aligned}$$

$$\begin{aligned}
&\partial_\lambda\big(\, \log \det{}_F\, (K(\lambda)\hat{K}(\lambda)^{-1}) + \log \det{}_F\, (K(-\lambda)\hat{K}(-\lambda)^{-1})\,\big)\\
&= \; -2\,\mathrm{Tr}\bigg(\, \frac{\lambda}{\mathcal{D}^2-\lambda^2} - \frac{\lambda}{\mathcal{D}_{\mathcal{P}}^2-\lambda^2} - \frac{\lambda}{\hat{\mathcal{D}}^2-\lambda^2} + \frac{\lambda}{\hat{\mathcal{D}}_{\mathcal{P}}^2-\lambda^2}\,\bigg).
\end{aligned}$$

Proof. The proofs of these formulas are similar, so we shall only prove the second formula. If $F(\lambda) = \log \det_F\, (K(\lambda)\hat{K}(\lambda)^{-1})$, then from Theorem 5.3, we have

$$\begin{aligned}
&\partial_\lambda\big(\, F(\lambda) + F(-\lambda)\,\big) = \partial_\lambda F(\lambda) - (\partial_\lambda F)(-\lambda)\\
&= -\,\mathrm{Tr}\,\big(\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{P}}(\lambda)^{-1} - (\hat{\mathcal{D}}(\lambda)^{-1} - \hat{\mathcal{D}}_{\mathcal{P}}(\lambda)^{-1})\big)\\
&\quad + \mathrm{Tr}\,\big(\mathcal{D}(-\lambda)^{-1} - \mathcal{D}_{\mathcal{P}}(-\lambda)^{-1} - (\hat{\mathcal{D}}(-\lambda)^{-1} - \hat{\mathcal{D}}_{\mathcal{P}}(-\lambda)^{-1})\big).
\end{aligned}$$

Since

$$\mathcal{D}(\lambda)^{-1} - \mathcal{D}(-\lambda)^{-1} = (\mathcal{D}-\lambda)^{-1} - (\mathcal{D}+\lambda)^{-1} = \frac{2\lambda}{\mathcal{D}^2-\lambda^2},$$

with similar formulas holding for the resolvents of the other Dirac operators, we get our second equality. □

Since for any holomorphic branch of log around a point c, we have $\log(c+z) \sim \sum_{k=0}^\infty c_k z^k$ as $z \to 0$, by Theorem 6.10 it follows that $\log \det_F\, (K(\lambda)\hat{K}(\lambda)^{-1})$ has expansions as $\Im\lambda \to \pm\infty$ that resemble the expansions in Theorem 6.10. Replacing λ by μ with $\mu = \lambda^2$ in Proposition 7.1, then taking k-derivatives of the asymptotic expansions of $\log \det_F\, (K(\lambda)\hat{K}(\lambda)^{-1})$ with $\mu = \lambda^2$, we immediately obtain

Corollary 7.2. *As $|\mu| \to \infty$ for $\sqrt{\mu} \in \Lambda$, we have*

$$\partial_\mu^k\,\mathrm{Tr}\bigg(\, \frac{\mathcal{D}}{\mathcal{D}^2-\mu} - \frac{\mathcal{D}_{\mathcal{P}}}{\mathcal{D}_{\mathcal{P}}^2-\mu} - \frac{\hat{\mathcal{D}}}{\hat{\mathcal{D}}^2-\mu} + \frac{\hat{\mathcal{D}}_{\mathcal{P}}}{\hat{\mathcal{D}}_{\mathcal{P}}^2-\mu}\,\bigg) = \mathcal{O}(\mu^{-1-k}).$$

$$\partial_\mu^k\,\mathrm{Tr}\bigg(\, \frac{1}{\mathcal{D}^2-\mu} - \frac{1}{\mathcal{D}_{\mathcal{P}}^2-\mu} - \frac{1}{\hat{\mathcal{D}}^2-\mu} + \frac{1}{\hat{\mathcal{D}}_{\mathcal{P}}^2-\mu}\,\bigg) = \mathcal{O}(\mu^{-3/2-k}),$$

This corollary will be used to derive the following estimates.

Lemma 7.3. *For $0 < t < 1$, we have*

$$\begin{aligned}
&|\;\, {}^b\mathrm{Tr}\,\big(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_{\mathcal{P}}e^{-t\mathcal{D}_{\mathcal{P}}^2} - \hat{\mathcal{D}}e^{-t\hat{\mathcal{D}}^2} + \hat{\mathcal{D}}_{\mathcal{P}}e^{-t\hat{\mathcal{D}}_{\mathcal{P}}^2}\big)\;|\le c_1,\\
&|\;\, {}^b\mathrm{Tr}\,\big(e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_{\mathcal{P}}^2} - e^{-t\hat{\mathcal{D}}^2} + e^{-t\hat{\mathcal{D}}_{\mathcal{P}}^2}\big)\;| \le c_2\sqrt{t},
\end{aligned}$$

for positive constants c_1, c_2.

Proof. Let Γ be the contour $\Gamma = -1 + \{\mu \in \mathbb{C} \mid \arg\mu = \pi/4,\, 7\pi/4\}$. Consider the first inequality and define

$$\begin{aligned}
f(t) &= \mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_{\mathcal{P}}e^{-t\mathcal{D}_{\mathcal{P}}^2} - \hat{\mathcal{D}}e^{-t\hat{\mathcal{D}}^2} + \hat{\mathcal{D}}_{\mathcal{P}}e^{-t\hat{\mathcal{D}}_{\mathcal{P}}^2}\\
&= \frac{i}{2\pi}\int_\Gamma e^{-t\mu}\bigg(\frac{\mathcal{D}}{\mathcal{D}^2-\mu} - \frac{\mathcal{D}_{\mathcal{P}}}{\mathcal{D}_{\mathcal{P}}^2-\mu} - \frac{\hat{\mathcal{D}}}{\hat{\mathcal{D}}^2-\mu} + \frac{\hat{\mathcal{D}}_{\mathcal{P}}}{\hat{\mathcal{D}}_{\mathcal{P}}^2-\mu}\bigg)d\mu,
\end{aligned}$$

where we used Cauchy's formula to write the heat operators as contour integrals over Γ. To prove the first inequality, we want to formally pass ${}^b\mathrm{Tr}$ through the integral sign. We can justify this by the following integration-by-parts trick. First, integrating by parts k-times, as an operator we can write

$$f(t) = \frac{i}{2\pi} \cdot t^{-k} k! \int_\Gamma e^{-t\mu} \Big(\frac{\mathcal{D}}{(\mathcal{D}^2 - \mu)^{k+1}} - \frac{\mathcal{D}_{\mathcal{P}}}{(\mathcal{D}_{\mathcal{P}}^2 - \mu)^{k+1}} - \frac{\hat{\mathcal{D}}}{(\hat{\mathcal{D}}^2 - \mu)^{k+1}} + \frac{\hat{\mathcal{D}}_{\mathcal{P}}}{(\hat{\mathcal{D}}_{\mathcal{P}}^2 - \mu)^{k+1}} \Big) d\mu.$$

For k sufficiently large, the inner operators are individually b-trace class and vanish as $|\mu| \to \infty$ sufficiently rapidly so that we can interchange the b-trace with the integral:

$${}^b\mathrm{Tr}(f(t)) = \frac{i}{2\pi} \cdot t^{-k} k! \int_\Gamma e^{-t\mu}\, \mathrm{Tr} \Big(\frac{\mathcal{D}}{(\mathcal{D}^2 - \mu)^{k+1}} - \frac{\mathcal{D}_{\mathcal{P}}}{(\mathcal{D}_{\mathcal{P}}^2 - \mu)^{k+1}} - \frac{\hat{\mathcal{D}}}{(\hat{\mathcal{D}}^2 - \mu)^{k+1}} + \frac{\hat{\mathcal{D}}_{\mathcal{P}}}{(\hat{\mathcal{D}}_{\mathcal{P}}^2 - \mu)^{k+1}} \Big) d\mu.$$

where we used that the operator inside the integral is of sufficient regularity (cf. the proof of Theorem 4.3) so the b-trace of this operator reduces to the usual trace. Now we can integrate by parts k-times reversing what was done before, using Corollary 7.2 to justify this integration, obtaining

$$(7.1) \qquad {}^b\mathrm{Tr}(f(t)) = \frac{i}{2\pi} \int_\Gamma e^{-t\mu}\, \mathrm{Tr} \Big(\frac{\mathcal{D}}{\mathcal{D}^2 - \mu} - \frac{\mathcal{D}_{\mathcal{P}}}{\mathcal{D}_{\mathcal{P}}^2 - \mu} - \frac{\hat{\mathcal{D}}}{\hat{\mathcal{D}}^2 - \mu} + \frac{\hat{\mathcal{D}}_{\mathcal{P}}}{\hat{\mathcal{D}}_{\mathcal{P}}^2 - \mu} \Big) d\mu.$$

The asymptotics in Corollary 7.2 now imply the first inequality. To prove the second inequality, we proceed in the same way, defining

$$\begin{aligned} F(t) &= e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_{\mathcal{P}}^2} - e^{-t\hat{\mathcal{D}}^2} + e^{-t\hat{\mathcal{D}}_{\mathcal{P}}^2} \\ &= \frac{i}{2\pi} \int_\Gamma e^{-t\mu} \Big(\frac{1}{\mathcal{D}^2 - \mu} - \frac{1}{\mathcal{D}_{\mathcal{P}}^2 - \mu} - \frac{1}{\hat{\mathcal{D}}^2 - \mu} + \frac{1}{\hat{\mathcal{D}}_{\mathcal{P}}^2 - \mu} \Big) d\mu, \end{aligned}$$

and then using the same integration-by-parts trick to write

$$(7.2) \qquad {}^b\mathrm{Tr}(F(t)) = \frac{i}{2\pi} \int_\Gamma e^{-t\mu}\, \mathrm{Tr} \Big(\frac{1}{\mathcal{D}^2 - \mu} - \frac{1}{\mathcal{D}_{\mathcal{P}}^2 - \mu} - \frac{1}{\hat{\mathcal{D}}^2 - \mu} + \frac{1}{\hat{\mathcal{D}}_{\mathcal{P}}^2 - \mu} \Big) d\mu.$$

The second inequality now follows from this equation and the second estimate in Corollary 7.2. □

In the following two theorems, we express the spectral invariants in terms of the relative traces of our resolvents.

Theorem 7.4. *${}^b\eta_{\mathcal{D}}(s)$ is regular at $s = 0$; in particular, ${}^b\eta(\mathcal{D}) := {}^b\eta_{\mathcal{D}}(0)$ is well-defined. Moreover,*

$${}^b\eta(\mathcal{D}) - {}^b\eta(\mathcal{D}_{\mathcal{P}}) = \frac{2}{\pi} \int_0^\infty \mathrm{Tr}\Big(\frac{\mathcal{D}}{\mathcal{D}^2 + \nu^2} - \frac{\mathcal{D}_{\mathcal{P}}}{\mathcal{D}_{\mathcal{P}}^2 + \nu^2} - \frac{\hat{\mathcal{D}}}{\hat{\mathcal{D}}^2 + \nu^2} + \frac{\hat{\mathcal{D}}_{\mathcal{P}}}{\hat{\mathcal{D}}_{\mathcal{P}}^2 + \nu^2} \Big)\, d\nu.$$

Proof. By Lemma 2.1 it follows that ${}^b\eta_{\hat{\mathcal{D}}}(s)$ and ${}^b\eta_{\mathcal{D}_{\mathcal{P}_Z}}(s)$ are identically zero and by Lemmas 2.2 and 2.3 in [21] we also have $\eta_{\hat{\mathcal{D}}_{\mathcal{P}_N}}(s)=0$. Thus,

$$\begin{aligned}
&{}^b\eta_{\mathcal{D}}(s) - {}^b\eta_{\mathcal{D}_{\mathcal{P}}}(s) = {}^b\eta_{\mathcal{D}}(s) - {}^b\eta_{\mathcal{D}_{\mathcal{P}}}(s) - {}^b\eta_{\hat{\mathcal{D}}}(s) + {}^b\eta_{\hat{\mathcal{D}}_{\mathcal{P}}}(s)\\
&\quad= \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^1 t^{\frac{s-1}{2}}\, {}^b\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_{\mathcal{P}}e^{-t\mathcal{D}_{\mathcal{P}}^2} - \hat{\mathcal{D}}e^{-t\hat{\mathcal{D}}^2} + \hat{\mathcal{D}}_{\mathcal{P}}e^{-t\hat{\mathcal{D}}_{\mathcal{P}}^2})\, dt\\
&\qquad + \frac{1}{\Gamma(\frac{s+1}{2})}\int_1^\infty t^{\frac{s-1}{2}}\, {}^b\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_{\mathcal{P}}e^{-t\mathcal{D}_{\mathcal{P}}^2} - \hat{\mathcal{D}}e^{-t\hat{\mathcal{D}}^2} + \hat{\mathcal{D}}_{\mathcal{P}}e^{-t\hat{\mathcal{D}}_{\mathcal{P}}^2})\, dt.
\end{aligned}$$

Here the small (large) time contribution is defined a priori from the half plane with $\Re s \gg 0$ ($\Re s \ll 0$). By Lemma 7.3, we know that the integral $\int_0^1$ is absolutely convergent for s near 0 and by the expansion (2.4), so is the integral $\int_1^\infty$. In conclusion, ${}^b\eta_{\mathcal{D}}(s) - {}^b\eta_{\mathcal{D}_{\mathcal{P}}}(s) = {}^b\eta_{\mathcal{D}}(s) - \eta_{\mathcal{D}_{\mathcal{P}_M}}(s)$ is regular at $s=0$. Since $\eta_{\mathcal{D}_{\mathcal{P}_M}}(s)$ is also regular at $s=0$ by Theorem 5.5 in [27] (see also [13], [14], [44]), it follows that ${}^b\eta_{\mathcal{D}}(s)$ is regular at $s=0$. Moreover, we have

$$\begin{aligned}
&{}^b\eta(\mathcal{D}) - {}^b\eta(\mathcal{D}_{\mathcal{P}})\\
&\qquad = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-\frac{1}{2}}\, {}^b\mathrm{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_{\mathcal{P}}e^{-t\mathcal{D}_{\mathcal{P}}^2} - \hat{\mathcal{D}}e^{-t\hat{\mathcal{D}}^2} + \hat{\mathcal{D}}_{\mathcal{P}}e^{-t\hat{\mathcal{D}}_{\mathcal{P}}^2}\right) dt.
\end{aligned}$$

Using the equality $\frac{1}{\sqrt{\pi}}t^{-\frac{1}{2}} = \frac{2}{\pi}\int_0^\infty e^{-t\nu^2}\, d\nu$, we can write

$$ {}^b\eta(\mathcal{D}) - {}^b\eta(\mathcal{D}_{\mathcal{P}}) = \frac{2}{\pi}\int_0^\infty \int_0^\infty {}^b\mathrm{Tr}\left(f(t,\nu)\right)\, d\nu\; dt,$$

where

$$f(t,\nu) = \mathcal{D}e^{-t(\mathcal{D}^2+\nu^2)} - \mathcal{D}_{\mathcal{P}}e^{-t(\mathcal{D}_{\mathcal{P}}^2+\nu^2)} - \hat{\mathcal{D}}e^{-t(\hat{\mathcal{D}}^2+\nu^2)} + \hat{\mathcal{D}}_{\mathcal{P}}e^{-t(\hat{\mathcal{D}}_{\mathcal{P}}^2+\nu^2)}.$$

As we already mentioned, ${}^b\mathrm{Tr}\left(f(t,\nu)\right)$ is absolutely integrable, so we can switch the order of integration, obtaining

$$ {}^b\eta(\mathcal{D}) - {}^b\eta(\mathcal{D}_{\mathcal{P}}) \;=\; \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty {}^b\mathrm{Tr}(f(t,\nu))\, dt\right) d\nu. \tag{7.3}$$

We can relate ${}^b\mathrm{Tr}(f(t,\nu))$ to the resolvents as follows. Let $\Upsilon\subset\mathbb{C}$ be the contour $\Upsilon = \nu^2/2 + \{\mu\in\mathbb{C} \mid \arg\mu = \pi/4,\, 7\pi/4\}$. Then just as we proved (7.1), we find that

$$\begin{aligned}
{}^b\mathrm{Tr}(f(t,\nu)) = \frac{i}{2\pi}\int_\Upsilon e^{-t\mu}\,\mathrm{Tr}\Bigg(&\frac{\mathcal{D}}{\mathcal{D}^2+\nu^2-\mu} - \frac{\mathcal{D}_{\mathcal{P}}}{\mathcal{D}_{\mathcal{P}}^2+\nu^2-\mu}\\
&- \frac{\hat{\mathcal{D}}}{\hat{\mathcal{D}}^2+\nu^2-\mu} + \frac{\hat{\mathcal{D}}_{\mathcal{P}}}{\hat{\mathcal{D}}_{\mathcal{P}}^2+\nu^2-\mu}\Bigg) d\mu.
\end{aligned}$$

Hence, ${}^b\mathrm{Tr}(f(t,\nu)) = \partial_t g(t,\nu)$, where

$$\begin{aligned}
g(t,\nu) = -\frac{i}{2\pi}\int_\Upsilon \frac{e^{-t\mu}}{\mu}\,\mathrm{Tr}\Bigg(&\frac{\mathcal{D}}{\mathcal{D}^2+\nu^2-\mu} - \frac{\mathcal{D}_{\mathcal{P}}}{\mathcal{D}_{\mathcal{P}}^2+\nu^2-\mu}\\
&- \frac{\hat{\mathcal{D}}}{\hat{\mathcal{D}}^2+\nu^2-\mu} + \frac{\hat{\mathcal{D}}_{\mathcal{P}}}{\hat{\mathcal{D}}_{\mathcal{P}}^2+\nu^2-\mu}\Bigg) d\mu.
\end{aligned}$$

By the first estimate in Corollary 7.2 it follows that $\lim_{t\to\infty} g(t,\nu) = 0$ and $g(t,\nu)$ is continuous at $t = 0$ with value given by setting $t = 0$ in the previous integral and then applying Cauchy's formula:

$$g(0,\nu) = -\operatorname{Tr}\left(\frac{\mathcal{D}}{\mathcal{D}^2+\nu^2} - \frac{\mathcal{D}_{\mathcal{P}}}{\mathcal{D}_{\mathcal{P}}^2+\nu^2} - \frac{\hat{\mathcal{D}}}{\hat{\mathcal{D}}^2+\nu^2} + \frac{\hat{\mathcal{D}}_{\mathcal{P}}}{\hat{\mathcal{D}}_{\mathcal{P}}^2+\nu^2}\right).$$

Finally, using that ${}^b\mathrm{Tr}(f(t,\nu)) = \partial_t g(t,\nu)$ and applying the fundamental theorem of calculus to (7.3) proves our theorem. ☐

Theorem 7.5. *We have*

(1)
$$\lim_{\nu\to\infty}\Big[\log\big(\det{}_{{}^b\zeta}(\mathcal{D}^2+\nu^2)\det{}_{{}^b\zeta}(\mathcal{D}_{\mathcal{P}}^2+\nu^2)^{-1}\big) - \log\big(\det{}_{{}^b\zeta}(\hat{\mathcal{D}}_{\mathcal{P}}^2+\nu^2)\det{}_{{}^b\zeta}(\hat{\mathcal{D}}^2+\nu^2)^{-1}\big)\Big] = 0,$$

(2)
$$\partial_\nu\Big[\log\big(\det{}_{{}^b\zeta}(\mathcal{D}^2+\nu^2)\det{}_{{}^b\zeta}(\mathcal{D}_{\mathcal{P}}^2+\nu^2)^{-1}\big) - \log\big(\det{}_{{}^b\zeta}(\hat{\mathcal{D}}_{\mathcal{P}}^2+\nu^2)\det{}_{{}^b\zeta}(\hat{\mathcal{D}}^2+\nu^2)^{-1}\big)\Big] = 2\operatorname{Tr}\Big(\frac{\nu}{\mathcal{D}^2+\nu^2} - \frac{\nu}{\mathcal{D}_{\mathcal{P}}^2+\nu^2} - \frac{\nu}{\hat{\mathcal{D}}^2+\nu^2} + \frac{\nu}{\hat{\mathcal{D}}_{\mathcal{P}}^2+\nu^2}\Big).$$

Proof. First let us put

$$F(t,\nu) = e^{-t(\mathcal{D}^2+\nu^2)} - e^{-t(\mathcal{D}_{\mathcal{P}}^2+\nu^2)} - e^{-t(\hat{\mathcal{D}}^2+\nu^2)} + e^{-t(\hat{\mathcal{D}}_{\mathcal{P}}^2+\nu^2)}.$$

Then, by Proposition A.1 in [33], the meromorphic extension of

$$\int_0^\infty t^{s-1}\,{}^b\mathrm{Tr}(F(t,\nu))\,dt,$$

which is a priori defined for $\Re(s) \gg 0$, is regular at $s = 0$. Hence, denoting the difference of the logarithms in (1) by $F(\nu)$, according to Singer's formula [40], we can write

$$F(\nu) = -\int_0^\infty t^{-1}\,{}^b\mathrm{Tr}(F(t,\nu))\,dt.$$

By the second estimate in Lemma 7.3 and the asymptotics (2.2) and (2.8), it follows that the integrand in $F(\nu)$ is absolutely integrable and vanishes exponentially as $\nu\to\infty$. This implies (1). To prove (2), we take the derivative of $F(\nu)$:

$$\partial_\nu F(\nu) = 2\nu\int_0^\infty {}^b\mathrm{Tr}(F(t,\nu))\,dt. \tag{7.4}$$

Let $\Upsilon \subset \mathbb{C}$ be the contour $\Upsilon = \nu^2/2 + \{\mu\in\mathbb{C} \mid \arg\mu = \pi/4,\ 7\pi/4\}$. Then exactly as we proved (7.2), we can show that

$${}^b\mathrm{Tr}(F(t,\nu)) = \frac{i}{2\pi}\int_\Upsilon e^{-t\mu}\operatorname{Tr}\Big(\frac{1}{\mathcal{D}^2+\nu^2-\mu} - \frac{1}{\mathcal{D}_{\mathcal{P}}^2+\nu^2-\mu} - \frac{1}{\hat{\mathcal{D}}^2+\nu^2-\mu} + \frac{1}{\hat{\mathcal{D}}_{\mathcal{P}}^2+\nu^2-\mu}\Big)d\mu.$$

Hence, ${}^b\mathrm{Tr}(F(t,\nu)) = \partial_t G(t,\nu)$, where

$$G(t,\nu) = -\frac{i}{2\pi}\int_\Upsilon \frac{e^{-t\mu}}{\mu}\,\mathrm{Tr}\left(\frac{1}{\mathcal{D}^2+\nu^2-\mu} - \frac{1}{\mathcal{D}_{\mathcal{P}}^2+\nu^2-\mu} - \frac{1}{\hat{\mathcal{D}}^2+\nu^2-\mu} + \frac{1}{\hat{\mathcal{D}}_{\mathcal{P}}^2+\nu^2-\mu}\right)d\mu.$$

By the first estimate in Corollary 7.2 it follows that $\lim_{t\to\infty} G(t,\nu) = 0$ and $G(t,\nu)$ is continuous at $t = 0$ with value given by setting $t = 0$ in the previous integral and then applying Cauchy's formula:

$$G(0,\nu) = -\,\mathrm{Tr}\left(\frac{1}{\mathcal{D}^2+\nu^2} - \frac{1}{\mathcal{D}_{\mathcal{P}}^2+\nu^2} - \frac{1}{\hat{\mathcal{D}}^2+\nu^2} + \frac{1}{\hat{\mathcal{D}}_{\mathcal{P}}^2+\nu^2}\right).$$

Finally, using that ${}^b\mathrm{Tr}(F(t,\nu)) = \partial_t G(t,\nu)$ and applying the fundamental theorem of calculus to (7.4) proves our theorem. □

The following corollary, which follows from Propositions 7.1, 7.4, and 7.5, relates the spectral invariants to $\log\det_F\,(K(\lambda)\hat{K}(\lambda)^{-1})$.

Corollary 7.6. *We have*

(1) $${}^b\eta(\mathcal{D}) - {}^b\eta(\mathcal{D}_{\mathcal{P}}) = -\frac{1}{\pi i}\int_0^\infty \partial_\nu\Big(\log\det_F\,(K(i\nu)\hat{K}(i\nu)^{-1}) - \log\det_F\,(K(-i\nu)\hat{K}(-i\nu)^{-1})\Big)\,d\nu,$$

(2) $$\partial_\nu\Big[\log\big(\det_{{}^b\zeta}(\mathcal{D}^2+\nu^2)\det_{{}^b\zeta}(\mathcal{D}_{\mathcal{P}}^2+\nu^2)^{-1}\big) - \log\big(\det_{{}^b\zeta}(\hat{\mathcal{D}}_{\mathcal{P}}^2+\nu^2)\det_{{}^b\zeta}(\hat{\mathcal{D}}^2+\nu^2)^{-1}\big)\Big]$$
$$= \partial_\nu\Big(\log\det_F\,(K(i\nu)\hat{K}(i\nu)^{-1}) + \log\det_F\,(K(-i\nu)\hat{K}(-i\nu)^{-1})\Big).$$

8. Proof of the main Theorems

In this section we prove Theorems 1.1 and 1.3. To do so, we first need to prove two lemmas.

Lemma 8.1. *As $\nu \to 0^+$, we have*

$$\log\det_{{}^b\zeta}(\mathcal{D}^2+\nu^2) = 2h_X\,\log\nu + \log\det_{{}^b\zeta}\mathcal{D}^2 + o(1),$$
$$\log\det_{{}^b\zeta}(\mathcal{D}_{\mathcal{P}}^2+\nu^2) = \log\det_{{}^b\zeta}\mathcal{D}_{\mathcal{P}}^2 + o(1).$$

Proof. By definition, ${}^b\zeta(s,\mathcal{D}^2+\nu^2) = {}^b\zeta_1(s,\nu) + {}^b\zeta_2(s,\nu) + h_X\nu^{-2s}$, where

$${}^b\zeta_1(s,\nu) = \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\,e^{-t\nu^2}\big({}^b\mathrm{Tr}(e^{-t\mathcal{D}^2}) - h_X\big)\,dt,$$
$${}^b\zeta_2(s,\nu) = \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\,e^{-t\nu^2}\big({}^b\mathrm{Tr}(e^{-t\mathcal{D}^2}) - h_X\big)\,dt$$

where the first (second) one has the meromorphic extensions over $\mathbb{C}$ from $\Re s \gg 0$ ($\Re s \ll 0$). Since ${}^b\mathrm{Tr}(e^{-t\mathcal{D}^2}) - h_X = \mathcal{O}(t^{-1/2})$ as $t\to\infty$ by (2.2), ${}^b\zeta_1(s,\nu) + {}^b\zeta_2(s,\nu)$ converges uniformly to ${}^b\zeta_{\mathcal{D}^2}(s)$ as $\nu\to 0^+$ for s in a compact neighborhood of 0 in $\mathbb{C}$. Then, taking the derivative with respect to s for the equality ${}^b\zeta(s,\mathcal{D}^2+\nu^2) = {}^b\zeta_1(s,\nu) + {}^b\zeta_2(s,\nu) + h_X\nu^{-2s}$, we obtain the first equality.

We now prove the second equality. To do so, observe that

$$\log \det_{{}^b\zeta}(\mathcal{D}_{\mathcal{P}}^2 + \nu^2) = \log \det_{{}^b\zeta}(\mathcal{D}_{\mathcal{P}_M}^2 + \nu^2) + \log \det_{{}^b\zeta}(\mathcal{D}_{\mathcal{P}_Z}^2 + \nu^2).$$

By definition, $\mathcal{D}_{\mathcal{P}_M}$ is invertible so that $\log \det_\zeta(\mathcal{D}_{\mathcal{P}_M}^2 + \nu^2) = \log \det_\zeta \mathcal{D}_{\mathcal{P}_M}^2 + o(1)$. By (2.8) we can write

$${}^b\mathrm{Tr}(e^{-t\mathcal{D}_{\mathcal{P}_Z}^2}) = \frac{1}{\sqrt{\pi}} \sum_{\mu_k>0} \int_{\mu_k\sqrt{t}}^\infty e^{-x^2}\, dx - \frac{1}{2}\sum_{\mu_k>0} e^{-\mu_k^2 t}$$

where $\{\mu_k\}$ are the positive eigenvalues of D_Y. Everything here is exponentially decreasing as $t \to \infty$. This implies that

$${}^b\zeta_{(\mathcal{D}_{\mathcal{P}_Z}^2+\nu^2)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, e^{-t\nu^2}\, {}^b\mathrm{Tr}(e^{-t\mathcal{D}_{\mathcal{P}_Z}^2})\, dt$$

is smooth in ν down to $\nu = 0$. Taking the derivative with respect to s and using that the μ_k's are positive, prove our theorem. □

Lemma 8.2. *As $\nu \to 0^+$, we have*

$$\log \det_{{}^b\zeta}(\hat{\mathcal{D}}^2 + \nu^2) = \log \det_{{}^b\zeta} \hat{\mathcal{D}}^2 + o(1),$$
$$\log \det_{{}^b\zeta}(\hat{\mathcal{D}}_{\mathcal{P}}^2 + \nu^2) = \log \det_{{}^b\zeta} \hat{\mathcal{D}}_{\mathcal{P}}^2 + o(1).$$

Proof. Using the explicit expression (2.7) for the trace of $e^{-t\hat{\mathcal{D}}_{\mathcal{P}_Z}^2}$ and the corresponding formula involving $e^{-t\hat{\mathcal{D}}^2}$ obtained by replacing u with $u+1$, we find that

$$\text{(8.1)} \qquad {}^b\mathrm{Tr}(e^{-t\hat{\mathcal{D}}^2}) - {}^b\mathrm{Tr}(e^{-t\hat{\mathcal{D}}_{\mathcal{P}_Z}^2}) = \frac{1}{\sqrt{4\pi t}}\, \mathrm{Tr}_Y\left(e^{-tD_Y^2}\right).$$

At the end of the proof of Lemma 8.1, we found that $\log \det_{{}^b\zeta}(\hat{\mathcal{D}}_{\mathcal{P}_Z}^2 + \nu^2) = \log \det_{{}^b\zeta} \hat{\mathcal{D}}_{\mathcal{P}_Z}^2 + o(1)$; then from Equation (8.1) it follows that the same must hold for $\hat{\mathcal{D}}^2$. This proves our first equality. Since

$$\log \det_{{}^b\zeta}(\hat{\mathcal{D}}_{\mathcal{P}}^2 + \nu^2) = \log \det_\zeta(\hat{\mathcal{D}}_{\mathcal{P}_N}^2 + \nu^2) + \log \det_{{}^b\zeta}(\hat{\mathcal{D}}_{\mathcal{P}_Z}^2 + \nu^2),$$

where $\hat{\mathcal{D}}_{\mathcal{P}_N}$ is invertible and $\log \det_{{}^b\zeta}(\hat{\mathcal{D}}_{\mathcal{P}_Z}^2 + \nu^2) = \log \det_{{}^b\zeta} \hat{\mathcal{D}}_{\mathcal{P}_Z}^2 + o(1)$, our second equality follows. □

Proof of Theorem 1.1: From the second formula in Corollary 7.6, we have

$$\text{(8.2)} \qquad \left[\frac{\det_{{}^b\zeta}(\mathcal{D}^2 + \nu^2)}{\det_{{}^b\zeta}(\mathcal{D}_{\mathcal{P}}^2 + \nu^2)} \cdot \frac{\det_{{}^b\zeta}(\hat{\mathcal{D}}_{\mathcal{P}}^2 + \nu^2)}{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2 + \nu^2)}\right] = C \cdot \det_F\left(K(i\nu)\hat{K}(i\nu)^{-1}\right) \cdot \det_F\left(K(-i\nu)\hat{K}(-i\nu)^{-1}\right)$$

where C is a constant. By Theorem 7.5, we know that the left-hand side approaches to 1 as $\nu \to \infty$, and by Corollary 6.9, we know that

$$\text{(8.3)} \quad \lim_{\nu\to\infty} \det_F\left(K(i\nu)\hat{K}(i\nu)^{-1}\right) = 1 \quad , \quad \lim_{\nu\to\infty} \det_F\left(K(-i\nu)\hat{K}(-i\nu)^{-1}\right) = \det_F U.$$

Therefore, $C = \det_F U^{-1} = (-1)^{h_X} \det_F \widehat{U}^{-1}$. Substituting this value of C into the identity (8.2), and then using the asymptotics as $\nu \to 0^+$:

$$\text{(8.4)} \quad \begin{aligned} \det_F\left(K(i\nu)\hat{K}(i\nu)^{-1}\right) &= \nu^{h_X} (\det \mathcal{L})^{-1} \det_F\left(\frac{\mathrm{Id} + \widehat{U}}{2}\right)\big(1 + o(1)\big), \\ \det_F\left(K(-i\nu)\hat{K}(-i\nu)^{-1}\right) &= (-\nu)^{h_X} (\det \mathcal{L})^{-1} \det_F\left(\frac{\mathrm{Id} + \widehat{U}}{2}\right)\big(1 + o(1)\big), \end{aligned}$$

which come from Theorem 6.7, we see that for $\nu > 0$ small,

$$\Big[\frac{\det_{{}^b\zeta}(\mathcal{D}^2+\nu^2)}{\det_{{}^b\zeta}(\mathcal{D}^2_{\mathcal{P}}+\nu^2)}\cdot\frac{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2_{\mathcal{P}}+\nu^2)}{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2+\nu^2)}\Big] \tag{8.5}$$
$$= \nu^{2h_X}(\det\mathcal{L})^{-2}\det_F\widehat{U}^{-1}\det_F\Big(\frac{\mathrm{Id}+\widehat{U}}{2}\Big)^2\big(1+o(1)\big)$$
$$= \nu^{2h_X}(\det\mathcal{L})^{-2}\det_F\Big(\frac{2\mathrm{Id}+\widehat{U}+\widehat{U}^{-1}}{4}\Big)\big(1+o(1)\big).$$

According to Lemmas 8.1 and 8.2, for $\nu > 0$ small, we have

$$\Big[\frac{\det_{{}^b\zeta}(\mathcal{D}^2+\nu^2)}{\det_{{}^b\zeta}(\mathcal{D}^2_{\mathcal{P}}+\nu^2)}\cdot\frac{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2_{\mathcal{P}}+\nu^2)}{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2+\nu^2)}\Big] = \nu^{2h_X}\Big[\frac{\det_{{}^b\zeta}(\mathcal{D}^2)}{\det_{{}^b\zeta}(\mathcal{D}^2_{\mathcal{P}})}\cdot\frac{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2_{\mathcal{P}})}{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2)}\Big]\big(1+o(1)\big).$$

Substituting these expressions into the left-hand side of the equation (8.5), and then taking $\nu \to 0^+$, we obtain

$$\Big[\frac{\det_{{}^b\zeta}(\mathcal{D}^2)}{\det_{{}^b\zeta}(\mathcal{D}^2_{\mathcal{P}})}\cdot\frac{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2_{\mathcal{P}})}{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2)}\Big] = (\det\mathcal{L})^{-2}\det_F\Big(\frac{2\mathrm{Id}+\widehat{U}+\widehat{U}^{-1}}{4}\Big). \tag{8.6}$$

In the following lemma, we compute the ratio of ζ-determinants over the cylinder; this also completes the proof for the ζ-determinant gluing formula in Theorem 1.1.

Lemma 8.3. *We have*

$$\frac{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2)}{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2_{\mathcal{P}})} = 2^{-\zeta_{D^2_Y}(0)-h_Y}.$$

Proof. Let us consider a Dirac operator defined by $G(\partial_u + D_Y)$ over the half infinite cylinder $[-R,\infty)\times Y$ (with $R > 1$). We impose the boundary condition given by $\mathcal{P}_Z$ at $\{-R\}\times Y$ and denote the resulting operator by $\mathcal{D}_R$. We also denote by $\mathcal{D}_{R,\mathcal{P}}$ the Dirac operator obtained by the restrictions of $\mathcal{D}_R$ to $[-R,0]\times Y \sqcup [0,\infty)\times Y$ with the boundary conditions given by $\mathcal{P}_N \sqcup \mathcal{P}_Z$ at the two copies of $\{0\}\times Y$. Now we apply the equality (8.6) to the pairs $(\mathcal{D}^2_R, \mathcal{D}^2_{R,\mathcal{P}})$, $(\hat{\mathcal{D}}^2, \hat{\mathcal{D}}^2_{\mathcal{P}})$. Then the equality (8.6) in this case is

$$\frac{\det_{{}^b\zeta}(\mathcal{D}^2_R)}{\det_{{}^b\zeta}(\mathcal{D}^2_{R,\mathcal{P}})} = \frac{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2)}{\det_{{}^b\zeta}(\hat{\mathcal{D}}^2_{\mathcal{P}})}.$$

This equality means that the ratio of the b-determinants $\frac{\det_{{}^b\zeta}(\mathcal{D}^2_R)}{\det_{{}^b\zeta}(\mathcal{D}^2_{R,\mathcal{P}})}$ is independent of R! To compute this ratio we take $R \to \infty$ and use [33], [34] to get $2^{-\zeta_{D^2_Y}(0)-h_Y}$. Since these computations are explained thoroughly in [33], we omit the computations, but we refer the reader to [26] for a tutorial on the adiabatic method. □

Proof of Theorem 1.3: In Lemma 2.2 we showed that $\det_{{}^b\zeta}\mathcal{D}^2_{\mathcal{P}_Z} = 2^{\frac{1}{2}\zeta_{D^2_Y}(0)}$. Thus, combining this with Theorem 1.1, we have

$$\frac{\det_{{}^b\zeta}\mathcal{D}^2}{\det_{\zeta}\mathcal{D}^2_{\mathcal{P}_M}} = 2^{-\frac{1}{2}\zeta_{D^2_Y}(0)-h_Y}(\det\mathcal{L})^{-2}\det_F\Big(\frac{2\mathrm{Id}+\widehat{U}+\widehat{U}^{-1}}{4}\Big). \tag{8.7}$$

We now determine $\det_\zeta\mathcal{D}^2_{\mathcal{P}_\sigma}$ where $\mathcal{P}_\sigma = \Pi_< + \frac{\mathrm{Id}+\sigma}{2}\Pi_0$. To do so, for $R > 0$ we define $M_R := M \cup Z_R$, where $Z_R := [0,R]\times Y$, with the Dirac operator $\mathcal{D}_R$ which is $\mathcal{D}$ restricted to M_R with the boundary condition $\mathcal{P}_\sigma$ at $\{u = R\}$. We shall apply the gluing formula of [25] to the *compact* manifold M_R decomposed into M and

Z_R. Here, the gluing formula of [25] is identical to that of Theorem 1.1 but for the compact situation. (Although the details in [25] were proved for the compact boundaryless case, the proof easily goes through for the situation of a compact manifold with boundary where a well-posed boundary condition is imposed, which is what we have here.) Since $\mathcal{D}_R$ has the boundary condition $\mathcal{P}_\sigma = \Pi_< + \frac{\mathrm{Id}+\sigma}{2}\Pi_0$ at $\{u = R\}$, one can check that the Calderón projector at $\{u = 0\}$ of the restriction of $\mathcal{D}_R$ over Z_R is exactly $\mathcal{P}_Z = \Pi_> + \frac{\mathrm{Id}-\sigma}{2}\Pi_0$. Thus, by the main result of [25], we have

$$\frac{\det_\zeta \mathcal{D}_R^2}{\det_\zeta \mathcal{D}_{\mathcal{P}_M}^2 \cdot \det_\zeta \mathcal{D}_{R,\mathcal{P}_Z}^2} = 2^{-\zeta_{D_Y^2}(0)-h_Y} (\det \mathcal{L}_R)^{-2} \det{}_F\Big(\frac{2\mathrm{Id} + \widehat{U} + \widehat{U}^{-1}}{4}\Big), \tag{8.8}$$

where

$$\mathcal{L}_R := \sum_{k=1}^{h_X} \gamma_0 U_k(R) \otimes (\gamma_0 U_k(R))^*$$

with $\{U_k(R)\}$ an orthonormal basis for $\ker(\mathcal{D}_R)$ and $(\gamma_0 U_k(R))^* := \langle\,\cdot\,, \gamma_0 U_k(R)\rangle_{M_R}$ where $\langle\,\cdot\,,\,\cdot\,\rangle_{M_R}$ is the inner product on $L^2(M_R, S)$. Since $\mathcal{D}_{R,\mathcal{P}_Z}$ is the Dirac operator $G(\partial_u + D_Y)$ over $[0, R]\times Y$ with the boundary condition $\mathcal{P}_\sigma = \Pi_< + \frac{\mathrm{Id}+\sigma}{2}\Pi_0$ at $\{u = R\}$ and $\mathcal{P}_Z = \Pi_> + \frac{\mathrm{Id}-\sigma}{2}\Pi_0$ at $\{u = 0\}$, by the main result of [26], we have $\det_\zeta \mathcal{D}_{R,\mathcal{P}_Z}^2 = e^{CR}\, 2^{\zeta_{D_Y^2}(0)+h_Y}$, $C = -(2\sqrt{\pi})^{-1}(\Gamma(s)^{-1}\Gamma(s - 1/2)\zeta_{D_Y^2}(s - 1/2))'(0)$. Substituting this equation into (8.8), we find that

$$\frac{\det_\zeta \mathcal{D}_R^2}{\det_\zeta \mathcal{D}_{\mathcal{P}_M}^2} = e^{CR} (\det \mathcal{L}_R)^{-2} \det{}_F\Big(\frac{2\mathrm{Id} + \widehat{U} + \widehat{U}^{-1}}{4}\Big).$$

As $R \to 0$, $\det_\zeta \mathcal{D}_R^2 \to \det_\zeta \mathcal{D}_{\mathcal{P}_\sigma}^2$ and $\det \mathcal{L}_R \to \det \mathcal{L}_0 =: \det \mathcal{L}_{\mathcal{P}_\sigma}$, so we conclude that

$$\frac{\det_\zeta \mathcal{D}_{\mathcal{P}_\sigma}^2}{\det_\zeta \mathcal{D}_{\mathcal{P}_M}^2} = (\det \mathcal{L}_{\mathcal{P}_\sigma})^{-2} \det{}_F\Big(\frac{2\mathrm{Id} + \widehat{U} + \widehat{U}^{-1}}{4}\Big). \tag{8.9}$$

Finally, taking the quotient of (8.7) and (8.9), we obtain Theorem 1.3.

Proof of Corollaries 1.2 and 1.4: We now prove Corollaries 1.2 and 1.4 by following almost *verbatim* the proofs for the ζ-determinant. First, from formula (1) in Corollary 7.6, we have

$$\begin{aligned}
&{}^b\eta(\mathcal{D}) - {}^b\eta(\mathcal{D}_{\mathcal{P}}) \\
&\quad = -\frac{1}{\pi i}\Big(\lim_{\nu\to\infty} \big(\log\det{}_F\,(K(i\nu)\hat{K}(i\nu)^{-1}) - \log\det{}_F\,(K(-i\nu)\hat{K}(-i\nu)^{-1})\big) \\
&\qquad\quad - \lim_{\nu\to 0^+} \big(\log\det{}_F\,(K(i\nu)\hat{K}(i\nu)^{-1}) - \log\det{}_F\,(K(-i\nu)\hat{K}(-i\nu)^{-1})\big)\Big).
\end{aligned}$$

In view of the limits (8.3) as $\nu \to \infty$, we have, modulo $2\pi i\mathbb{Z}$,

$$\lim_{\nu\to\infty} \big(\log\det{}_F\,(K(i\nu)\hat{K}(i\nu)^{-1}) - \log\det{}_F\,(K(-i\nu)\hat{K}(-i\nu)^{-1})\big) = -\log\det{}_F U$$

and in view of the asymptotics (8.4) for positive ν near 0, we have, modulo $2\pi i\mathbb{Z}$,

$$\lim_{\nu\to 0^+} \big(\log\det{}_F\,(K(i\nu)\hat{K}(i\nu)^{-1}) - \log\det{}_F\,(K(-i\nu)\hat{K}(-i\nu)^{-1})\big) = -h_X\pi i.$$

These equalities imply that

$${}^b\eta(\mathcal{D}) - \eta(\mathcal{D}_{\mathcal{P}_M}) - {}^b\eta(\mathcal{D}_{\mathcal{P}_Z}) = \frac{1}{\pi i}\log\det{}_F U - h_X \mod 2\mathbb{Z}. \tag{8.10}$$

This completes the proof of Corollary 1.2 combining ${}^b\eta(\mathcal{D}_{\mathcal{P}_Z}) = 0$ proved in Lemma 2.1. By the main result in [27] we have

$$\eta(\mathcal{D}_{\mathcal{P}_\sigma}) - \eta(\mathcal{D}_{\mathcal{P}_M}) = \frac{1}{\pi i} \log \det_F U - h_X \mod 2\mathbb{Z}.$$

From this equality and the equality (8.10), we conclude that ${}^b\eta(\mathcal{D}) = \eta(\mathcal{D}_{\mathcal{P}_\sigma})$ mod $2\mathbb{Z}$. This completes the proof of Corollary 1.4.

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