ADIABATIC DECOMPOSITION OF THE ζ-DETERMINANT AND SCATTERING THEORY.

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1. Introduction and Statement of the Results

Let $\mathcal{D} : C^\infty(M,S) \to C^\infty(M,S)$ be a compatible Dirac operator acting on sections of a Clifford bundle S over a closed manifold M of dimension n. The operator $\mathcal{D}$ is a self-adjoint operator, with discrete spectrum $\{\lambda_k\}_{k\in\mathbb{Z}}$. The ζ-determinant of the Dirac Laplacian $\mathcal{D}^2$ is given by the formula

$$\det{}_\zeta \mathcal{D}^2 = e^{-\zeta'_{\mathcal{D}^2}(0)} \quad , \tag{1.1}$$

where $\zeta_{\mathcal{D}^2}(s)$ is defined as follows:

$$\zeta_{\mathcal{D}^2}(s) = \sum_{\lambda_k \neq 0} (\lambda_k^2)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} [\ \mathrm{Tr}\,(e^{-t\mathcal{D}^2}) - \dim\ker\mathcal{D}\ ]\ dt \quad . \tag{1.2}$$

This is a holomorphic function of s for $\Re(s) \gg 0$ and has the meromorphic extension to the complex plane with $s = 0$ as a regular point.

Let us consider a decomposition of M as $M_1 \cup M_2$, where M_1 and M_2 are compact manifolds with boundaries such that

$$M = M_1 \cup M_2\,, \quad Y = M_1 \cap M_2 = \partial M_1 = \partial M_2\,. \tag{1.3}$$

In this paper, we study the adiabatic decomposition of the ζ-determinant of $\mathcal{D}^2$, which describes the contributions in $\det_\zeta \mathcal{D}^2$ coming from the submanifolds M_1 and M_2. Throughout the paper, we assume that the manifold M and the operator $\mathcal{D}$ have product structures in a neighborhood of the cutting hypersurface Y. Hence, there is a bicollar neighborhood $N \cong [-1,1]_u \times Y$ of $Y \cong \{0\} \times Y$ in M such that the Riemannian structure on M and the Hermitian structure on S are products of the corresponding structures over $[-1,1]_u$ and Y when restricted to N, so that $\mathcal{D}$ has the following form,

$$\mathcal{D} = G(\partial_u + B) \quad \text{over} \quad N\ . \tag{1.4}$$

Here u denotes the normal variable, $G : S|_Y \to S|_Y$ is a bundle automorphism, and B is a corresponding Dirac operator on Y. Moreover, G and B do not depend on u and they satisfy

$$G^* = -G\ \ , \quad G^2 = -\mathrm{Id}\ , \quad B = B^* \quad \text{and} \quad GB = -BG\ . \tag{1.5}$$

To prove the adiabatic decomposition formula of $\det_\zeta \mathcal{D}^2$, we follow the original Douglas-Wojciechowski proof of the decomposition formula for the η-invariant in

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[10]. However, we face two new problems, not present in the case of the η-invariant. First, $\det_\zeta \mathcal{D}^2$ is a much more *non-local* invariant than the η-invariant. This results, for instance, in the fact that the value of $\det_\zeta \mathcal{D}^2$ varies with the length of the cylinder. Second, the contribution of $\det_\zeta \mathcal{D}^2$ over the cylindrical part is now non-trivial. We still follow the idea of the paper [10] and we stretch our manifold M to separate M_1 and M_2. For this, let us introduce a manifold M_R equal to the manifold M with N replaced by $N_R \cong [-R, R]_u \times Y$. By assumption of product structures over N, we can extend the bundle S to M_R. Furthermore, using (1.4), we can extend $\mathcal{D}$ to the Dirac operator $\mathcal{D}_R$ over M_R. Now we decompose M_R by the hypersurface $\{0\} \times Y$ into two submanifolds $M_{1,R}$, $M_{2,R}$ and we obtain $\mathcal{D}_{1,R}$, $\mathcal{D}_{2,R}$ by restricting $\mathcal{D}_R$ to $M_{1,R}$, $M_{2,R}$ respectively.

To formulate the decomposition formula for the ζ-determinant, we have to describe the invariant on a manifold with boundary which enters the picture at this point. The tangential operator B has discrete spectrum with infinitely many positive and infinitely many negative eigenvalues. Let $\Pi_>$, $\Pi_<$ denote the Atiyah-Patodi-Singer (APS) spectral projections onto the subspaces spanned by the eigensections of B corresponding to the positive, negative eigenvalues respectively. We select two involutions σ_1, σ_2 on kernel of B, which satisfy $G\sigma_i = -\sigma_i G$ and define $\pi_i = \frac{1-\sigma_i}{2}$ the orthogonal projections onto -1 eigenspaces of σ_i . We define

$$P_1 = \Pi_< + \pi_1 \quad , \quad P_2 = \Pi_> + \pi_2 \ , \tag{1.6}$$

which provides us with the *ideal boundary condition* introduced by Cheeger in [6], [7]. The projection P_i imposes an elliptic boundary condition for $\mathcal{D}_{i,R}$ (see [1]; see [2] for an exposition of the theory of elliptic boundary problems for Dirac operators). This means that the associated operator

$$(\mathcal{D}_{i,R})_{P_i} \ = \ \mathcal{D}_{i,R} : \mathrm{dom}\,(\mathcal{D}_{i,R})_{P_i} \to L^2(M_{i,R}, S)$$

where

$$\mathrm{dom}\,(\mathcal{D}_{i,R})_{P_i} = \{s \in H^1(M_{i,R}, S) \mid P_i(s|_Y) = 0\}$$

is a self-adjoint Fredholm operator with $\ker(\mathcal{D}_{i,R})_{P_i} \subset C^\infty(M_{i,R}, S)$ and discrete spectrum.

The main concern of this paper is to consider the limit of the following ratio of the ζ-determinants,

$$\frac{\det_\zeta \mathcal{D}_R^2}{\det_\zeta (\mathcal{D}_{1,R})^2_{P_1} \cdot \det_\zeta (\mathcal{D}_{2,R})^2_{P_2}} \quad \text{as} \quad R \to \infty \ , \tag{1.7}$$

which we call as the *adiabatic decomposition* of the ζ-determinant of $\mathcal{D}^2$.

The eigenvalues of $\mathcal{D}_R$ fall into three different categories as $R \to \infty$. There are infinitely many large eigenvalues (*l-values*) bounded away from 0 and infinitely many small eigenvalues (*s-values*) of the size $O(R^{-1})$. Besides these, there are finitely many eigenvalues which decay exponentially with R (*e-values*). The number h_M of *e-values* is given by

$$h_M = \dim \ker_{L^2} \mathcal{D}_{1,\infty} + \dim \ker_{L^2} \mathcal{D}_{2,\infty} + \dim L_1 \cap L_2 \ , \tag{1.8}$$

where $\mathcal{D}_{i,\infty}$ is the operator defined from $\mathcal{D}$ in a natural way over the manifold $M_{i,\infty}$, which is equal to M_i with the half infinite cylinder $[0,\infty)\times Y$ or $(-\infty,0]\times Y$ attached. More precisely, the operator $\mathcal{D}_i = \mathcal{D}|_{M_i}$ extends in a natural way to the manifold $M_{i,\infty}$. It has a unique closed self-adjoint extension in $L^2(M_{i,\infty}, S)$, which we denote by $\mathcal{D}_{i,\infty}$. The subspaces $L_i \subset \ker B$ are the spaces of limiting values of extended L^2-solutions of $\mathcal{D}_{i,\infty}$. The decomposition of the eigenvalues of the operator $\mathcal{D}_R$ into different classes was discussed by Cappell, Lee and Miller (see [5]). The corresponding analysis for the operator $(\mathcal{D}_{i,R})_{P_i}$ was provided by Müller (see [17]). The spectrum of the operators $(\mathcal{D}_{i,R})_{P_i}$ splits in the same way as the spectrum of $\mathcal{D}_R$. The only difference is that the operators $(\mathcal{D}_{i,R})_{P_i}$ do not have nonzero *e-values* and the dimension of the space of the solutions of $(\mathcal{D}_{i,R})_{P_i}$ is equal to

$$h_i = \dim\ker(\mathcal{D}_{i,R})_{P_i} = \dim\ker_{L^2}\mathcal{D}_{i,\infty} + \dim L_i \cap \ker(\sigma_i - 1) \ . \tag{1.9}$$

In the adiabatic limit process, the different types of eigenvalues make their contribution at different time intervals of the integral representation of $\zeta_{\mathcal{D}^2}(s)$ in (1.2). The contribution made by *l-values* comes from the time interval $[0, R^{2-\varepsilon}]$, where ε is a sufficiently small positive number, and we fix ε from now on. More precisely, it is not difficult to show that the *l-values* contribution to the adiabatic limit of (1.7) from the time interval $[R^{2-\varepsilon}, \infty]$ disappears as $R \to \infty$ (see Section 2). The contribution made by *l-values* was discussed in [19]. To be more precise, in [19] we discussed the case of the operator $\mathcal{D}$, such that $\mathcal{D}_{i,\infty}$ and B have trivial kernels. These conditions imply that there are no *e-values* and *s-values.* This allows us to reduce the computation of the quotient in (1.7) to the corresponding quotient on the cylinder, hence one can show that the limit of (1.7) as $R \to \infty$ is equal to $2^{-\zeta_{B^2}(0)}$. Actually, even in the presence of *e-values* and *s-values*, we are able to show that in the adiabatic limit the contribution of *l-values* comes only from the time interval $[0, R^{2-\varepsilon}]$ so that we can reduce to the cylinder as in [19]. The method we use to prove this combines Duhamel's principle and Finite propagation speed property of the wave operators. Details are presented in Section 2.

The *s-values* contribution comes from the time interval $[R^{2-\varepsilon}, R^{2+\varepsilon}]$. The computation of the *s-values* contribution is the main achievement of this paper. We follow Müller (see [17]) and use the *Scattering theory* to get a description of the *s-values.* The operators $\mathcal{D}_{i,\infty}$ on $M_{i,\infty}$ determine scattering matrices $C_i(\lambda)$. It turned out that the matrix $C_{12} = C_1(0)\circ C_2(0)$ on $\ker B \cap \ker(G+i)$ determines the contribution of *s-values* of the operator $\mathcal{D}_R$ in the adiabatic limit. Similarly the finite-dimensional unitary matrix S_{σ_i} on $\ker(\sigma_i + 1)$, which is defined by the scattering matrix $C_i(0)$ and the involution σ_i, determines the contribution of *s-values* of the operators $(\mathcal{D}_{i,R})_{P_i}$. The exact correspondence is stated in Section 3.

Finally we have to discuss *e-values* of $\mathcal{D}_R$. The number of *e-values* is equal to h_M which, as remarked above, is constant. On the other hand, the set of zero eigenvalues of $\mathcal{D}_R$, which is a subset of *e-values* by definition, is very unstable with respect to R . Hence, without making additional assumptions we are not able to control the adiabatic limit of the determinant of $\mathcal{D}_R^2$ due to the finite number of nonzero *e-values*. Hence, we assume that all the *e-values* are zero eigenvalues in order to avoid the technical difficulty of the nonzero *e-values*. One of the important examples of such situation is the case of the operator

$$d_\rho + d_\rho^* \;:\; \oplus_{i=0}^n \Omega^i(M, V_\rho) \;\to\; \oplus_{i=0}^n \Omega^i(M, V_\rho)$$

where V_ρ denotes the flat vector bundle defined by the unitary representation ρ of $\pi_1(M)$ (see Proposition 3.9). For the operator $L : W \to W$ acting on a finite dimensional vector space W, we denote by $\det^* L$ the determinant of the operator L restricted to the subspace $(\ker L)^\perp$. Now we are ready to formulate the main result of the paper.

Theorem 1.1. *When all the* e-values *of $\mathcal{D}_R$ are zero eigenvalues, the following formula holds:*

$$(1.10) \quad \lim_{R\to\infty} R^{-2h} \cdot \frac{\det_\zeta \mathcal{D}_R^2}{\det_\zeta (\mathcal{D}_{1,R})_{P_1}^2 \cdot \det_\zeta (\mathcal{D}_{2,R})_{P_2}^2}$$

$$= 2^{-\zeta_{B^2}(0) - h_Y + 2h_M} \cdot \det{}^*\left(\frac{2\,\mathrm{Id} - C_{12} - C_{12}^{-1}}{4}\right) \cdot \prod_{i=1}^{2} \det{}^*\left(\frac{2\,\mathrm{Id} - S_{\sigma_i} - S_{\sigma_i}^{-1}}{4}\right)^{-1}$$

where $h = h_M - h_1 - h_2$ and $h_Y = \dim\ker B$.

Remark 1.2. In [12] and [11], the reduced normal operators corresponding to our operators C_{12}, S_{σ_i} were introduced in the framework of *b-calculus* and used in the analysis of *s-values* for the analytic surgery of the η-invariant and analytic torsion.

To prove Theorem 1.1, we consider the following relative ζ-function and its derivative at $s = 0$,

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} [\,\mathrm{Tr}\,(e^{-t\mathcal{D}_R^2} - e^{-t(\mathcal{D}_{1,R})_{P_1}^2} - e^{-t(\mathcal{D}_{2,R})_{P_2}^2}) - h\,]\; dt \;\;,$$

which we decompose into two parts,

$$\zeta_s^R(s) = \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} (\cdot)\; dt \;\;,\;\; \zeta_l^R(s) = \frac{1}{\Gamma(s)} \int_{R^{2-\varepsilon}}^\infty (\cdot)\; dt \;\;,$$

where ε is the fixed sufficiently small positive number. The derivatives of $\zeta_s^R(s)$ and $\zeta_l^R(s)$ at $s = 0$ give the small and large time contribution in (1.10).

In Section 2 we deal with the small time contribution and prove that this equal $2^{-\zeta_{B^2}(0)}$, which gives the first factor on the right side of (1.10). In Section 3 we explain some basic description of the small eigenvalues. We follow [17] and use scattering theory in order to get a description of the *s-values* of $\mathcal{D}_R$ and $(\mathcal{D}_{i,R})_{P_i}$, which allows us to make a comparison of *s-values* of those operators with the eigenvalues of certain model operators over $\mathbb{S}^1$. This is the central part of this paper. In Section 4 we use the results of Section 3 to show that, in the adiabatic limit, the large time contribution to the quotient (1.7) is equal to

$$2^{-h_Y + 2h_M} \cdot \det{}^*\left(\frac{2\mathrm{Id} - C_{12} - C_{12}^{-1}}{4}\right) \cdot \prod_{i=1}^{2} \det{}^*\left(\frac{2\mathrm{Id} - S_{\sigma_i} - S_{\sigma_i}^{-1}}{4}\right)^{-1} \quad .$$

This is the second factor on the right side of (1.10). The zero eigenvalues make their presence via the factor R^{-2h} on the left side of (1.10).

In Section 5 we review the decomposition formula for the η-invariant and offer a new proof based on the method developed in order to prove Theorem 1.1. This proof is more complicated than other proofs presented in [4], [9], [12], [18], [23], [3], [13], [16]. However, it is a nice illustration of the differences we encounter when we deal with the ζ-determinant instead of the η-invariant.

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2. Small Time Contribution

In this section we determine the small time contribution, which is done in two steps. First, we use Duhamel's principle and Finite propagation speed property of the wave operator to show that we can reduce the problem to computations on the cylinder. Then, we perform the explicit calculations on the cylinder. Both parts are fairly standard. The cylinder contribution has been recently computed in [19]. Therefore, we only discuss the reduction scheme and refer to [19] for the explicit computation on the cylinder.

Let $\mathcal{E}_R(t;x,y)$ denote the kernel of the operator $e^{-t\mathcal{D}_R^2}$. We introduce the specific parametrix for $\mathcal{E}_R(t;x,y)$, which fits our main purpose to *localize* the contribution coming from the cylinder $[-R,R]_u \times Y$ and the interior of M_R . In fact, the interesting point here is that we use $\mathcal{E}_R(t;x,y)$ to construct this parametrix. Let $\mathcal{E}_c(t;x,y)$ denote the kernel of the operator $e^{-t(-\partial_u^2+B^2)}$ on the infinite cylinder $\mathbb{R} \times Y$. We introduce a smooth, increasing function $\rho(a,b)$: $[0,\infty) \to [0,1]$ equal to 0 for $0 \le u \le a$ and equal to 1 for $b \le u$. We use $\rho(a,b)(u)$ to define

$$\phi_{1,R} = 1 - \rho(\frac{5}{7}R, \frac{6}{7}R) \quad , \quad \psi_{1,R} = 1 - \psi_{2,R} \quad ,$$
$$\phi_{2,R} = \rho(\frac{1}{7}R, \frac{2}{7}R) \quad , \quad \psi_{2,R} = \rho(\frac{3}{7}R, \frac{4}{7}R) \quad .$$

We extend these functions to symmetric functions on the whole real line. These functions are constant outside the interval $[-R,R]_u$ and we use them to define the corresponding functions on a manifold M_R, which are denoted by the same notations. Now, we define $Q_R(t;x,y)$ a *parametrix* for the kernel $\mathcal{E}_R(t;x,y)$ by

$$Q_R(t;x,y) = \phi_{1,R}(x)\mathcal{E}_c(t;x,y)\psi_{1,R}(y) + \phi_{2,R}(x)\mathcal{E}_R(t;x,y)\psi_{2,R}(y) \quad . \tag{2.1}$$

It follows from the Duhamel's principle that

$$\mathcal{E}_R(t;x,y) = Q_R(t;x,y) + (\mathcal{E}_R * \mathcal{C}_R)(t;x,y) \quad , \tag{2.2}$$

where $\mathcal{E}_R * \mathcal{C}_R$ is the convolution given by

$$(\mathcal{E}_R * \mathcal{C}_R)(t;x,y) = \int_0^t ds \int_{M_R} dz\, \mathcal{E}_R(s;x,z)\, \mathcal{C}_R(t-s;z,y) \quad ,$$

and the error term $\mathcal{C}_R(t;x,y)$ is given by the formula

$$\begin{aligned}\mathcal{C}_R(t;x,y) = &- \partial_u^2\phi_{1,R}(x)\,\mathcal{E}_c(t;x,y)\,\psi_{1,R}(y) - \partial_u\,\phi_{1,R}(x)\,\partial_u\mathcal{E}_c(t;x,y)\,\psi_{1,R}(y)\\ &- \partial_u^2\phi_{2,R}(x)\,\mathcal{E}_R(t;x,y)\,\psi_{2,R}(y) - \partial_u\phi_{2,R}(x)\,\partial_u\mathcal{E}_R(t;x,y)\,\psi_{2,R}(y)\ .\end{aligned}$$

The following elementary lemma follows from the construction of $Q_R(t;x,y)$,

Lemma 2.1. *For a fixed y, the support of the error term $\mathcal{C}_R(t;x,y)$ as a function of x is a subset of $\left([-\frac{6}{7}R,-\frac{1}{7}R]_u \cup [\frac{1}{7}R,\frac{6}{7}R]_u\right)\times Y$. Moreover it is equal to 0 if the distance between x and y is smaller than $\frac{R}{7}$.*

Now, following Cheeger, Gromov and Taylor (see [8]; see also Section 3 of [4]), we use the Finite propagation speed property for the wave operator. The technique introduced in [8] allows us to compare the heat kernel of the operator $\mathcal{D}_R^2$ over M_R with the heat kernel of the operator $-\partial_u^2+B^2$ on the cylinder $\mathbb{R}\times Y$. We describe the case we need in our work. Let X_1 and X_2 be Riemannian manifolds of dimension n and S_i be spinors bundle with Dirac operators $\mathcal{D}_i$ over X_i. Assume that there exists a decomposition $X_i = K_i \cup U_i$, where U_i is an open subset of X_i . Moreover, we assume that there exists an isometry $h: U_1 \to U_2$ covered by the unitary bundle isomorphism $\Phi_h : S_1|_{U_1} \to S_2|_{U_2}$, which intertwines Dirac operators $\mathcal{D}_1|_{U_1}$ and $\mathcal{D}_2|_{U_2}$. We identify

$$U \cong U_1 \cong U_2\ \ ,$$

so that X_1 and X_2 have a common open subset U . Let $\mathcal{E}_i(t;x,y)$ denote the kernel of the operator $e^{-t\mathcal{D}_i^2}$. Then we have the following estimate on the difference of the heat kernels on U as in Lemma 3.6 in [4],

Proposition 2.2. *For $x,y\in U$ and $t>0$, there exist positive constants c_1,c_2 such that*

$$\|\partial_u^j\mathcal{E}_1(t;x,y) - \partial_u^j\mathcal{E}_2(t;x,y)\| \le c_1 e^{-c_2\frac{r^2}{t}} \tag{2.3}$$

where $j=0,1$ and $r=\min(d(x,K_1),d(y,K_1))$.

In our situation, $X_1=M_R$, $X_2=\mathbb{R}\times Y$ and $U=[-R,R]_u\times Y$. Note that the heat kernel $\mathcal{E}_c(t;x,y)$ over $\mathbb{R}\times Y$ satisfies the standard estimate. More precisely, for $t>0$ we have

$$\|\partial_u^j\mathcal{E}_c(t;(u,w),(v,z))\| \le c_1|u-v|^j t^{-\frac{n}{2}-j}e^{-c_3\frac{(u-v)^2}{t}}\ \ , \tag{2.4}$$

where $j=0,1,\ u,v\in\mathbb{R}$ and $w,z\in Y$. This follows from the corresponding estimate for the heat kernel of B^2 over the closed manifold Y (see Proposition 4.1 in [21]) and explicit form of the heat kernel of $-\partial_u^2$ over $\mathbb{R}$. We are going to use (2.3) and (2.4) in the following proposition.

Proposition 2.3. *There exist constants $c_1,c_2>0$ such that for any t with $0<t<R^{2-\varepsilon}$ and $((u,w),(v,z))\in\operatorname{supp}\mathcal{C}_R(t;\cdot,\cdot)$,*

$$\|\mathcal{E}_R(t;(u,w),(v,z))\| \le c_1 e^{-c_2\frac{R^2}{t}}\ , \qquad \|\mathcal{C}_R(t;(u,w),(v,z))\| \le c_1 e^{-c_2\frac{R^2}{t}}\ . \tag{2.5}$$

Proof. For $j = 0, 1$, we have

$$\begin{aligned}\|\partial_u^j \mathcal{E}_R(t;(u,w),(v,z))\| \;\le\; & \|\partial_u^j \mathcal{E}_c(t;(u,w),(v,z))\| \\ & + \|\partial_u^j \mathcal{E}_R(t;(u,w),(v,z)) - \partial_u^j \mathcal{E}_c(t;(u,w),(v,z))\| \ .\end{aligned}$$

By Lemma 2.1 and (2.3), (2.4), there exist some constants $c_1, c_2 > 0$ such that for $(u,w),(v,z) \in \operatorname{supp} \mathcal{C}_R(t;\cdot,\cdot)$, both summands on the right side satisfy the desired estimate. This estimate for $j = 0$ ($j = 1$) implies the first (second) estimate in (2.5). □

Now we are ready to prove the following technical result,

Proposition 2.4.

$$\lim_{R\to\infty} \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} t^{s-1} dt \int_{M_R} \operatorname{tr}(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)\ dx = 0 \ . \tag{2.6}$$

Proof. By Lemma 2.1 and Proposition 2.3,

$$\begin{aligned}|\operatorname{tr}(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)| \;&\le\; \|(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)\| \\ &\le \int_0^t ds \int_{[-\frac{6}{7}R,\frac{6}{7}R]_u \times Y} \|\mathcal{E}_R(s;x,z)\,\mathcal{C}_R(t-s;z,x)\|\ dz \\ &\le c_1^2 \cdot \int_0^t ds \int_{[-\frac{6}{7}R,\frac{6}{7}R]_u \times Y} e^{-c_2\frac{R^2}{s}} e^{-c_2\frac{R^2}{t-s}}\ dz \\ &\le c_3 R \cdot \int_0^t e^{-c_2\frac{tR^2}{s(t-s)}} ds \;\le\; c_3 R \cdot \int_0^{\frac{t}{2}} e^{-2c_2\frac{R^2}{s}} ds \;\le\; c_3 R\, \frac{t}{2}\, e^{-4c_2\frac{R^2}{t}} \ ,\end{aligned}$$

where the last estimate is a consequence of the elementary inequality

$$\int_0^t e^{-\frac{c}{s}} ds \le t\, e^{-\frac{c}{t}} \ .$$

Hence we have proved

$$|\operatorname{tr}\ (\mathcal{E}_R * \mathcal{C}_R)(t;x,x)| \le c_4 R\, t\, e^{-c_5\frac{R^2}{t}} \ . \tag{2.7}$$

This allows us to estimate as follows

$$\begin{aligned}&\Big|\frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} t^{s-1} dt \int_{M_R} \operatorname{tr}(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)\ dx\Big| \\ &\qquad\qquad \le\; c_6 R^2 \cdot \Big|\frac{1}{\Gamma(s)}\Big| \int_0^{R^{2-\varepsilon}} |t^s| e^{-c_5\frac{R^2}{t}}\ dt \ .\end{aligned}$$

As $R \to \infty$, the function of s on the right side uniformly converges to zero for s in any compact set in $\mathbb{C}$. Hence, the derivative at $s = 0$ of the meromorphic function on the left side converges to zero as $R \to \infty$. This completes the proof. □

The corresponding result for the operator $(\mathcal{D}_{i,R})^2_{P_i}$ to Proposition 2.4 can be carried out in exactly the same manner. First, as for $\mathcal{D}_R$ over M_R, we can construct the parametrices for the heat kernels of $e^{-t(\mathcal{D}_{i,R})^2_{P_i}}$ using $\mathcal{E}_R(t;x,y)$ and the heat kernels of $\big(G(\partial_u + B)\big)^2_{P_i}$ over $[0,\infty)_u \times Y$ or $(-\infty, 0]_u \times Y$. Second, one can obtain the corresponding estimate to (2.3) using the explicit form of the heat kernel of $(G(\partial_u + B))^2_{P_i}$. Third, one can also have the corresponding estimate to Proposition 2.3 for $(\mathcal{D}_{i,R})^2_{P_i}$ since the similar estimate as in Proposition 2.2 holds over the support of the error terms. (see Lemma 3.6 in [4]). All these imply that the similar estimate to Proposition 2.4 holds for $(\mathcal{D}_{i,R})^2_{P_i}$. Now we are ready to prove the following main result of this section,

Proposition 2.5.

$$\lim_{R\to\infty}\Big(\frac{d}{ds}\Big|_{s=0}\zeta_s^R(s) + h(\gamma + (2-\varepsilon)\cdot\log R)\Big) = \zeta_{B^2}(0)\cdot\log 2 \quad . \tag{2.8}$$

Proof. First we observe

$$\frac{d}{ds}\Big|_{s=0}\Big(\frac{h}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}\,dt\Big) = h(\gamma + (2-\varepsilon)\log R) \quad . \tag{2.9}$$

Hence we need to compute the limit as $R \to \infty$ of the following remaining part of $\zeta_s^R(s)$,

$$\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}\mathrm{Tr}\,(e^{-t\mathcal{D}_R^2} - e^{-t(\mathcal{D}_{1,R})^2_{P_1}} - e^{-t(\mathcal{D}_{2,R})^2_{P_2}})\;dt.$$

By Proposition 2.4 and corresponding results for $(\mathcal{D}_{i,R})^2_{P_i}$, it is sufficient to consider the limit as $R \to \infty$ of

$$\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}dt\int_{M_R}\mathrm{tr}\,\big(Q_R(t;x,x) - Q_{1,R}(t;x,x) - Q_{2,R}(t;x,x)\big)dx$$

where $Q_{i,R}(t;x,y)$ denotes the parametrix for $e^{-t(\mathcal{D}_{i,R})^2_{P_i}}$. Now, the interior contributions to the different parametrices, all determined by the kernel $\mathcal{E}_R(t;x,y)$, cancel out and we are left only with the cylinder contribution. Hence we have to deal with the limit as $R \to \infty$ of

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}dt\int_{M_R}\mathrm{tr}\,\big(&\psi_{1,R}\mathcal{E}_c(t;x,x)\\ &- \psi_{1,R}\mathcal{E}_{c,1}(t;x,x) - \psi_{1,R}\mathcal{E}_{c,2}(t;x,x)\big)dx\end{aligned} \tag{2.10}$$

where $\mathcal{E}_{c,i}(t;x,y)$ denotes the heat kernel of $(G(\partial_u + B))^2_{P_i}$ over the half cylinder. We repeat computations in Section 2 of [19] where we assumed the conditions that B is invertible and Y is even dimensional. But we can easily derive the same formula following Section 2 of [19] without these assumptions. So we can show that for s in a compact subset of $\mathbb{C}$ the integral part in (2.10) uniformly converges to the following function as $R \to \infty$,

$$2\Big(\frac{\Gamma(s)}{4} - \frac{\Gamma(s+\frac{1}{2})}{4s\sqrt{\pi}}\Big)\cdot\zeta_{B^2}(s) \quad .$$

Hence, we obtain

$$\lim_{R\to\infty} \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} t^{s-1}\mathrm{Tr}\,(e^{-t\mathcal{D}_R^2} - e^{-t(\mathcal{D}_{1,R})^2_{P_1}} - e^{-t(\mathcal{D}_{2,R})^2_{P_2}})\, dt \tag{2.11}$$
$$= \frac{d}{ds}\Big|_{s=0} \frac{2}{\Gamma(s)} \Big(\frac{\Gamma(s)}{4} - \frac{\Gamma(s+\frac{1}{2})}{4s\sqrt{\pi}}\Big)\cdot\zeta_{B^2}(s) = \zeta_{B^2}(0)\cdot\log 2 \ .$$

Combining (2.9) and (2.11) completes the proof.

□

3. SMALL EIGENVALUES AND SCATTERING MATRICES

In this section we investigate the relation between the *s-values* of the operators $\mathcal{D}_R$, $(\mathcal{D}_{i,R})_{P_i}$ and the scattering matrices $C_i(\lambda)$ determined by the operators $\mathcal{D}_{i,\infty}$ on $M_{i,\infty}$ for $i = 1, 2$. We refer to Section 4 and Section 8 in [17] for a more detailed exposition of the elements of *Scattering theory* that we use in this paper.

Let us recall that M_R has the cylindrical part $N_R = [-R, R]_u \times Y$. Hence $M_{1,R}, M_{2,R}$ have the cylindrical part $[-R, 0]_u \times Y$, $[0, R]_u \times Y$ respectively. But, in order to consider $M_{i,R}$ as a submanifold of $M_{i,\infty}$ which is obtained by attaching $[0,\infty)_v\times Y$ or $(-\infty, 0]_v\times Y$ to M_i, we change the variable by $v = u+R$ or $v = u-R$ so that the cylindrical part of $M_{i,R}$ is given by $[0, R]_v \times Y$ or $[-R, 0]_v \times Y$. Throughout this section, we will use this convention when it is needed.

For any $\psi \in \ker B$ and $\lambda \in \mathbb{C} - (-\infty, -\mu_1] \cup [\mu_1, +\infty)$ where μ_1 denotes the lowest positive eigenvalue of the tangential operator B, there exists a generalized eigensection $E(\psi, \lambda)$ of $\mathcal{D}_{1,\infty}$ over $M_{1,\infty}$ determined by the couple (ψ, λ) (see Section 4 in [17]) in the following sense,

$$\mathcal{D}_{1,\infty}E(\psi, \lambda) = \lambda\, E(\psi, \lambda) \ .$$

The section $E(\psi, \lambda)$ has the following form over $[0,\infty)_v \times Y$,

$$E(\psi, \lambda) = e^{-i\lambda v}(\psi - iG\psi) + e^{i\lambda v}C_1(\lambda)(\psi - iG\psi) + \theta(\psi, \lambda) \tag{3.1}$$

where θ is a square integrable section such that, for each v , $\theta(\psi, \lambda, (v, \cdot))$ is orthogonal to $\ker B$. The operator $C_1(\lambda) : \ker B \to \ker B$ is regular and unitary for $|\lambda| < \mu_1$ and equals the *Scattering matrix* such that

$$C_1(\lambda)C_1(-\lambda) = \mathrm{Id} \ , \quad C_1(\lambda)G = -GC_1(\lambda) \ ,$$

which imply

$$C_1(0)^2 = \mathrm{Id} \ , \quad C_1(0)G = -GC_1(0) \ .$$

Therefore $C_1(0)$ gives a distinguished unitary involution of $\ker B$. In fact, the space of the limiting values of the extended L^2-solutions of $\mathcal{D}_{1,\infty}$, $L_1 \subset \ker B$ is equal to the $(+1)$-eigenspace of $C_1(0)$, that is, $L_1 = \ker(C_1(0)-1)$. The following proposition is a basic tool to deal with $E(\psi, \lambda)$,

Proposition 3.1. (Maass-Selberg) *The following equality holds,*

$$\langle E(\phi, \lambda), E(\psi, \lambda)\rangle_{M_{1,R}}$$
$$= 4R\langle\phi, \psi\rangle_Y - i\langle C_1(-\lambda)C_1'(\lambda)(\phi - iG\phi), \psi - iG\psi\rangle_Y + O(e^{-cR})$$

where $\phi, \psi \in \ker B$.

Proof. By Green's formula, we have

$$
\begin{aligned}
(3.2)\qquad & h\langle E(\phi,\lambda+h),E(\psi,\lambda)\rangle_{M_{1,R}} \\
&= \langle \mathcal{D}_{1,R}E(\phi,\lambda+h),E(\psi,\lambda)\rangle_{M_{1,R}} - \langle E(\phi,\lambda+h),\mathcal{D}_{1,R}E(\psi,\lambda)\rangle_{M_{1,R}} \\
&= \langle GE(\phi,\lambda+h)|_{\partial(M_{1,R})},E(\psi,\lambda)|_{\partial(M_{1,R})}\rangle_{\partial(M_{1,R})} \ .
\end{aligned}
$$

Using (3.1), the last line has the following form,

$$
\begin{aligned}
& i\, e^{-ihR}\langle\phi-iG\phi,\psi-iG\psi\rangle_Y \\
& - i\, e^{ihR}\langle C_1(\lambda+h)(\phi-iG\phi),C_1(\lambda)(\psi-iG\psi)\rangle_Y + O(e^{-cR}) \\
= \ & i\, e^{-ihR}\langle\phi-iG\phi,\psi-iG\psi\rangle_Y - i\, e^{ihR}\langle\phi-iG\phi,\psi-iG\psi\rangle_Y \\
& + i\, e^{ihR}\langle C_1(\lambda)(\phi-iG\phi),C_1(\lambda)(\psi-iG\psi)\rangle_Y \\
& - i\, e^{ihR}\langle C_1(\lambda+h)(\phi-iG\phi),C_1(\lambda)(\psi-iG\psi)\rangle_Y + O(e^{-cR}) \ .
\end{aligned}
$$

Now, dividing the right side by h and taking the limit $h\to 0$, we obtain

$$
\begin{aligned}
& 2R\langle\phi-iG\phi,\psi-iG\psi\rangle_Y - i\langle C_1'(\lambda)(\phi-iG\phi),C_1(\lambda)(\psi-iG\psi)\rangle_Y + O(e^{-cR}) \\
= \ & 4R\langle\phi,\psi\rangle_Y - i\langle C_1(-\lambda)C_1'(\lambda)(\phi-iG\phi),(\psi-iG\psi)\rangle_Y + O(e^{-cR}) \ .
\end{aligned}
$$

Comparing this with (3.2) (divided by h) completes the proof. □

Now we shall analyze the *s-values* of $\mathcal{D}_R$ over M_R. Let us consider a *s-value* $\lambda=\lambda(R)$ of $\mathcal{D}_R$ such that

$$|\lambda(R)|\le R^{-\kappa} \quad \text{for sufficiently large } R$$

where κ is a fixed constant with $0<\kappa<1$. Let Ψ_R denote a *normalized* eigensection of $\mathcal{D}_R$ corresponding to *s-value* λ, that is,

$$\mathcal{D}_R\Psi_R=\lambda\Psi_R \ , \quad \|\Psi_R\|=1 \ .$$

Over the cylindrical part $[-R,R]_u\times Y$ in M_R, the eigensection Ψ_R corresponding to *s-value* λ of $\mathcal{D}_R$ has the following form,

$$\Psi_R=e^{-i\lambda u}\psi_1+e^{i\lambda u}\psi_2+\hat{\Psi}_R \tag{3.3}$$

where $\psi_1\in\ker B\cap\ker(G-i)$, $\psi_2\in\ker B\cap\ker(G+i)$ and $\hat{\Psi}_R$ is orthogonal to $\ker B$.

Lemma 3.2. *We have the following estimates*

$$||\hat{\Psi}_R|_{\{u\}\times Y}||_Y\le c_1e^{-c_2R} \qquad \textit{for} \quad -\frac{3}{4}R\le u\le\frac{3}{4}R$$

where c_1,c_2 are positive constants independent of R .

The proof of this lemma is same as the one of Lemma 2.1 in [22]. Now we can prove

Proposition 3.3. *The zero eigenmode $e^{-i\lambda u}\psi_1+e^{i\lambda u}\psi_2$ of the eigensection Ψ_R of* s-value *$\lambda(R)$ of $\mathcal{D}_R$ is non-trivial.*

Proof. We follow the proof of Theorem 2.2 in [22], so we assume that the zero eigenmode of Ψ_R is trivial, which will contradict to the fact $\lambda(R)$ is a *s-value*. Throughout the proof, we regard $M_{1,R}$ as a submanifold of $M_{1,\infty}$ using the change of variable $v = u + R$. We define a section Φ_R on $M_{1,\infty}$ by

$$\Phi_R = \begin{cases} h_R(x)\Psi_R(x) & \text{for} \quad x \in M_{1,R} \\ 0 & \text{for} \quad x \in M_{1,\infty} \setminus M_{1,R} \end{cases}$$

where h_R is a smooth function on $M_{1,\infty}$, equal to 1 for $x \in M_1 \cup [0, \frac{R}{2}]_v \times Y$ and equal to 0 for $x \in [\frac{3}{4}R, \infty)_v \times Y$ with $|\frac{\partial^j h}{\partial v^j}| \le CR^{-j}$ for a constant $C > 0$. Let $H^1(M_{1,\infty}, S)$ denote the first Sobolev space. For any $a \ge 0$, we introduce a closed subspace of $H^1(M_{1,\infty}, S)$ by

$$\begin{aligned} &H^1_a(M_{1,\infty}, S) \\ &\qquad = \{\Phi \in H^1(M_{1,\infty}, S) \mid \langle \Phi(v, \cdot), \phi_k \rangle = 0 \quad \text{for} \quad v \ge a,\ k = 1, \dots, h_Y\} \end{aligned}$$

where $\phi_1, \dots, \phi_{h_Y}$ denotes an orthonormal basis of $\ker B$. Consider the quadratic form,

$$Q(\Phi) = \|D\Phi\|^2 \quad \text{for} \quad \Phi \in H^1_a(M_{1,\infty}, S)$$

where D denotes the differential operator over $M_{1,\infty}$ whose self adjoint extension is $\mathcal{D}_{1,\infty}$. Then this quadratic form is represented by a positive self adjoint operator H_a in the closure of $H^1_a(M_{1,\infty}, S)$ in $L^2(M_{1,\infty}, S)$. Then H_a has pure point spectrum near 0 and $\ker H_a = \ker_{L^2} \mathcal{D}_{1,\infty}$ for any $a \ge 0$ by Proposition 8.7 in [17]. Following the proof of Proposition 2.4 in [22], we can prove that there exist positive constants c_1, c_2 such that

$$|\langle \Phi_R, s \rangle| \le c_1 e^{-c_2 R} \|s\| \tag{3.4}$$

for $s \in \ker H_a$. Now let $\tilde{\Phi}_R := \Phi_R - \sum_{k=1}^{h_{1,\infty}} \langle \Phi_R, s_k \rangle s_k$ where $\{s_k\}_{k=1}^{h_{1,\infty}}$ denotes an orthonormal basis of $\ker H_a$ with $h_{1,\infty} := \dim \ker H_a$. Hence, $\tilde{\Phi}_R$ is orthogonal to $\ker H_a$, and by (3.4) there is a positive constant C independent of R such that $\|\tilde{\Phi}_R\| \ge \frac{1}{2}\|\Phi_R\| \ge C > 0$ for sufficiently large R. Noting that $\tilde{\Phi}_R \in \operatorname{dom} H_a$, and by the mini-max principle, we have

$$\langle H_a \tilde{\Phi}_R, \tilde{\Phi}_R \rangle \ge \nu^2 C^2 \tag{3.5}$$

where ν^2 is the smallest nonzero eigenvalue of H_a. Now we have

$$\begin{aligned} &\lambda(R)^2 = \langle \mathcal{D}_R^2 \Psi_R, \Psi_R \rangle = \int_{M_R} \|\mathcal{D}_R \Psi_R(x)\|^2 \, dx \\ &\ge \int_{M_{1,R}} \|\mathcal{D}_R \Psi_R(x)\|^2 \, dx = \int_{M_{1,R}} \|\mathcal{D}_R\big(h_R \Psi_R + (1 - h_R)\Psi_R\big)(x)\|^2 \, dx \\ &\ge \int_{M_{1,\infty}} \|H_a \Phi_R(x)\| \, dx - \int_{M_{1,R}} \|\mathcal{D}_R (1 - h_R)\Psi_R(x)\|^2 \, dx \ . \end{aligned}$$

By (3.5), the first term has the lower bound $\nu^2 C^2$ since $H_a\Phi_R = H_a\tilde{\Phi}_R$. For the second term, we have

$$\begin{aligned}
&\int_{M_{1,R}} \|\mathcal{D}_R(1-h_R)\Psi_R(x)\|^2\,dx \\
=&\int_{M_{1,R}} \|(1-h_R)(x)\mathcal{D}_R\Psi_R(x) - G(\partial_u h_R)(x)\Psi_R(x)\|^2\,dx \\
\le& 2\int_{M_{1,R}} \|\lambda(R)(1-h_R)(x)\Psi_R(x)\|^2 + \|G(\partial_u h_R)(x)\Psi_R(x)\|^2\,dx \ .
\end{aligned}$$

By applying Lemma 3.2 with $v = u + R$ to each term of the last line, we have

$$\int_{M_{1,R}} \|\mathcal{D}_R(1-h_R)\Psi_R(x)\|^2\,dx \le c_3 e^{-c_4 R}$$

for positive constants c_3, c_4. Hence these inequalities imply that $\lambda(R)^2 \ge \frac{1}{2}\nu^2C^2$ for sufficiently large R. This completes the proof.

□

Changing to the variable $v = u + R$, we regard that the cylindrical part N_R of M_R is given by $[0,2R]_v \times Y$. In particular, we have the new expression for Ψ_R from (3.3),

$$\Psi_R = e^{-i\lambda v}\phi_1^1 + e^{i\lambda v}\phi_2^1 + \hat{\Psi}_R \tag{3.6}$$

where $\phi_1^1 = e^{i\lambda R}\psi_1$, $\phi_2^1 = e^{-i\lambda R}\psi_2$. Let $(\ker B)_\pm$ denote the $\pm i$ eigenspace of $G : \ker B \to \ker B$. We need the following lemma,

Lemma 3.4. *Let σ be an involution over $\ker B$ such that $G\sigma = -\sigma G$. Then for any element $\phi \in (\ker B)_\pm$, there exists a unique $\psi \in \mathrm{Im}(\sigma+1)$ such that*

$$\phi = \psi \mp iG\psi.$$

Proof. For a given $\phi \in (\ker B)_+$, let $\psi := \frac{1}{2}(1+\sigma)\phi$, which lies in $\mathrm{Im}(\sigma+1)$ by definition. Then we have

$$\begin{aligned}
\psi - iG\psi =& \frac{1}{2}\big((1-iG)\phi + (\sigma - iG\sigma)\phi\big) \\
=& \frac{1}{2}\big((1-iG)\phi + (\sigma + i\sigma G)\phi\big) = \frac{1}{2}\cdot 2\phi = \phi \ .
\end{aligned}$$

This completes the proof for the case of $(+)$ and the other case of $(-)$ can be proved in the same way. □

By Proposition 3.3, one of ϕ_1^1 and ϕ_2^1 in (3.6) is nontrivial. First we assume that ϕ_1^1 is nontrivial. Now, since $L_1 = \mathrm{Im}(C_1(0)+1)$ and $C_1(0)$ is an involution over $\ker B$, by Lemma 3.4 we can choose $\psi \in L_1$ such that $\phi_1^1 = \psi - iG\psi$. Then the generalized eigensection $E(\psi,\lambda)$ over $M_{1,\infty}$ associated to ψ has the following expression

$$E(\psi,\lambda) = e^{-i\lambda v}(\psi - iG\psi) + e^{i\lambda v}C_1(\lambda)(\psi - iG\psi) + \theta(\psi,\lambda)$$

over $[0,\infty)_v \times Y$. Following [17], we introduce

$$F = \Psi_R|_{M_{1,R}} - E(\psi,\lambda)|_{M_{1,R}} \ .$$

Green's formula gives

$$0 = \langle \mathcal{D}_{1,R} F, F \rangle_{M_{1,R}} - \langle F, \mathcal{D}_{1,R} F \rangle_{M_{1,R}} = \int_{\partial(M_{1,R})} \langle GF|_{\partial(M_{1,R})}, F|_{\partial(M_{1,R})} \rangle \, dy.$$

On the other hand, Lemma 3.2 shows that

$$\int_{\partial(M_{1,R})} \langle GF|_{\partial(M_{1,R})}, F|_{\partial(M_{1,R})} \rangle \, dy = -i \| \, C_1(\lambda)\phi_1^1 - \phi_2^1 \|^2 + O(e^{-c_3 R})$$

for some positive constant c_3. This produces the estimate

$$\| \, C_1(\lambda)\phi_1^1 - \phi_2^1 \, \| \le e^{-cR} \tag{3.7}$$

for a positive constant c. Therefore, for $R \gg 0$, if ϕ_1^1 is nontrivial, then ϕ_2^1 is also nontrivial. In the same way, one can show its inverse. Hence we can conclude that both ϕ_1^1, ϕ_2^1 in (3.6) are nontrivial for $R \gg 0$.

Now we want to get the corresponding estimate involving the scattering matrix $C_2(\lambda)$. For this, we change the variable by $v = u - R$ and regard the cylindrical part as $[-2R, 0]_v \times Y$. Then we have the corresponding expression for Ψ_R,

$$\Psi_R = e^{-i\lambda v}\phi_1^2 + e^{i\lambda v}\phi_2^2 + \hat{\Psi}_R$$

where $\phi_1^2 = e^{-i\lambda R}\psi_1$, $\phi_2^2 = e^{i\lambda R}\psi_2$. For the given $\phi_2^2 \in (\ker B)_-$, using Lemma 3.4, we choose $\psi \in L_2 = \mathrm{Im}(C_2(0) + 1)$ such that $\phi_2^2 = \psi + iG\psi$. The generalized eigensection $E(\psi, \lambda)$ over $M_{2,\infty}$ attached to the couple (ψ, λ) has the following expression

$$E(\psi, \lambda) = e^{i\lambda v}(\psi + iG\psi) + e^{-i\lambda v} C_2(\lambda)(\psi + iG\psi) + \theta(\psi, \lambda)$$

over $(-\infty, 0]_v \times Y$. As above, comparing Ψ_R and $E(\psi, \lambda)$, we obtain

$$\| \, C_2(\lambda)\phi_2^2 - \phi_1^2 \, \| \le e^{-cR} \tag{3.8}$$

for a positive constant c. By definition, we have

$$\phi_1^1 = e^{2i\lambda R}\phi_1^2 \quad , \quad \phi_2^1 = e^{-2i\lambda R}\phi_2^2 \quad . \tag{3.9}$$

Now, combining (3.7), (3.8) and (3.9), we get

$$\| \, e^{4i\lambda R} C_1(\lambda) \circ C_2(\lambda)\phi_2^1 - \phi_2^1 \, \| \le e^{-cR} \quad . \tag{3.10}$$

We define the operator $C_{12}(\lambda)$ by

$$C_{12}(\lambda) := C_1(\lambda) \circ C_2(\lambda)|_{(\ker B)_-} \ : \ (\ker B)_- \to (\ker B)_- \quad .$$

The operator $C_{12}(\lambda)$ is a unitary operator and is an analytic function of λ for $\lambda \in (-\delta, \delta)$ for a small $\delta > 0$ since the unitary operators $C_1(\lambda)$, $C_2(\lambda)$ are analytic functions of λ for $\lambda \in (-\delta, \delta)$. Furthermore, there exist real analytic functions $\alpha_j(\lambda)$ for $1 \le j \le \frac{h_Y}{2}$ of $\lambda \in (-\delta, \delta)$ such that $\exp(i\alpha_j(\lambda))$ are the corresponding eigenvalues of $C_{12}(\lambda)$ and $\alpha_j(\lambda)$ has the following expansion at $\lambda = 0$,

$$\alpha_j(\lambda) = \alpha_{j0} + \alpha_{j1}\lambda + \alpha_{j2}\lambda^2 + \alpha_{j3}\lambda^3 + \dots \quad . \tag{3.11}$$

We now introduce

$$\Omega(R) := \{ \, \rho \in \mathbb{R} - \{0\} \mid \det(e^{4i\rho R} C_{12}(\rho) - \mathrm{Id}) = 0 \ , \ |\rho| \le R^{-\kappa} \, \} \quad . \tag{3.12}$$

The following theorem is a main result of this section,

Theorem 3.5. *Assume that all the* e-values *of $\mathcal{D}_R$ are zero eigenvalues. Let $\lambda_1(R) \le \lambda_2(R) \le \ldots \le \lambda_{p(R)}(R)$ be the nonzero eigenvalues, counted to multiplicity, of $\mathcal{D}_R$ satisfying $|\lambda_k(R)| \le R^{-\kappa}$, and let $\rho_1(R) \le \rho_2(R) \le \ldots \le \rho_{m(R)}(R)$ be the nonzero element , counted to multiplicity, of $\Omega(R)$. Then there exist R_0 and $c > 0$, independent of R, such that for $R \ge R_0$, $p(R) = m(R)$ and*

$$|\lambda_k(R) - \rho_k(R)| \le e^{-cR} \quad \textit{for} \quad k = 1, \ldots, p(R) \ .$$

Proof. The proof of this theorem consists of two steps.

Step I: Let $\lambda = \lambda(R)$ be a given *s-value* with the multiplicity $m(\lambda)$. By Proposition 3.3, we get $m(\lambda)$ linearly independent vectors $\phi_1, \ldots, \phi_{m(\lambda)}$ in $(\ker B)_-$, which satisfies (3.10). Since $C_{12}(\lambda)$ is unitary, the eigenvalues of $e^{4i\lambda R}C_{12}(\lambda) - \mathrm{Id}$ have the form $e^{i\theta} - 1$ for $\theta \in \mathbb{R}$. Let $0 \le \zeta$ be the smallest eigenvalue of $(e^{4i\lambda R}C_{12}(\lambda) - \mathrm{Id})(e^{4i\lambda R}C_{12}(\lambda) - \mathrm{Id})^*$; then

$$\zeta = \min_{\phi \in (\ker B)_-} \frac{\|(e^{4i\lambda R}C_{12}(\lambda) - \mathrm{Id})\phi\|^2}{\|\phi\|^2} \ .$$

Combined with (3.10), this implies that $\zeta \le e^{-cR}$. Hence $e^{4i\lambda R}C_{12}(\lambda)$ has an eigenvalue $e^{i\theta}$ satisfying $|1 - \cos\theta| \le e^{-cR}$, and there exists $k \in \mathbb{Z}$ such that $|2\pi k - \theta| \le e^{-cR}$. Therefore, by definition of $\alpha_j(\lambda)$, the following holds

$$| \, 4\lambda R + \alpha_j(\lambda) - 2\pi k \, | \le e^{-cR} \tag{3.13}$$

for pairwise distinct branches $\alpha_1, \ldots, \alpha_{m(\lambda)}$. Now, let us fix δ_1 with $0 < \delta_1 < \delta$ and let

$$m_j = \max_{\lambda \in (-\delta_1, \delta_1)} |\alpha_j'(\lambda)| \ .$$

Then the function $f(\lambda) = 4\lambda R + \alpha_j(\lambda)$ is strictly increasing for $|\lambda| < \delta_1$ and $R \ge m_j$. Choose R_1 such that $R_1 \ge \max(m_j, \delta_1^{-\frac{1}{\kappa}})$ for any $j = 1, \ldots, \frac{h_Y}{2}$. For $R \ge R_1$ and $k \in \mathbb{Z}$, there exists at most one solution $\rho_{j,k}$ of

$$4\lambda R + \alpha_j(\lambda) = 2\pi k \ , \qquad |\lambda| \le R^{-\kappa} \ . \tag{3.14}$$

Let $k_{j,\max}$ be the maximal k for which (3.14) has a solution; then by (3.14),

$$|k_{j,\max}| \le \frac{2R^{1-\kappa}}{\pi} + C \le R^{1-\kappa} \ . \tag{3.15}$$

Then, for $R \ge R_1$, any element ρ in $\Omega(R)$ is given by $\rho = \rho_{j,k}$ for some $1 \le j \le \frac{h_Y}{2}$, and $|k| \le k_{j,\max}$. Therefore, if $R \ge R_1$, for a given λ satisfying (3.13) with $|\lambda| \le R^{-\kappa}$, there is a unique solution $\rho_{j,k}$ of (3.14) such that

$$|\lambda - \rho_{j,k}| \le e^{-cR} \ . \tag{3.16}$$

In conclusion, if $R \ge R_1$, for a given *s-value* $\lambda = \lambda(R)$ of $\mathcal{D}_R$ with the multiplicity $m(\lambda)$ satisfying $|\lambda| \le R^{-\kappa}$, there exist $m(\lambda)$-number of elements $\rho_{j,k}$'s in $\Omega(R)$ with the relation (3.16), in particular, $p(R) \le m(R)$.

Step II: To complete the proof, we need to prove that $m(R) \le p(R)$. For k with $1 \le k \le m(R)$, we choose $\psi_k \in (\ker B)_-$ with the following properties,

(1) $e^{4i\rho_k R}C_{12}(\rho_k)\psi_k = \psi_k \ , \quad |\rho_k| \le R^{-\kappa} \ ,$

(2) When $\rho_k = \rho_{k+1} = \ldots = \rho_{k+\ell}$, $\psi_k, \psi_{k+1} \ldots, \psi_{k+\ell}$ form an orthonormal system of vectors of $(\ker B)_-$.

For a given pair (ψ_k, ρ_k) for some k, we put

$$\phi_k^1 = e^{-i\rho_k R} C_1(-\rho_k)\psi_k , \quad \phi_k^2 = e^{i\rho_k R}\psi_k \ . \tag{3.17}$$

Now we consider the generalized eigensection $E(\phi_k^1, \rho_k)$ over $M_{1,\infty}$ and $E(\phi_k^2, \rho_k)$ over $M_{2,\infty}$, which have the following forms,

$$\begin{aligned} E(\phi_k^1, \rho_k) &= e^{-i\rho_k v}\phi_k^1 + e^{i\rho_k v} C_1(\rho_k)\phi_k^1 + O(e^{-cv}) \quad \text{over} \quad [0,\infty)_v \times Y \subset M_{1,\infty} \ , \\ E(\phi_k^2, \rho_k) &= e^{i\rho_k v}\phi_k^2 + e^{-i\rho_k v} C_2(\rho_k)\phi_k^2 + O(e^{-cv}) \quad \text{over} \quad (-\infty, 0]_v \times Y \subset M_{2,\infty} \ . \end{aligned}$$

(Here we use abuse notations for simplicity since the correct notation for $E(\phi_k^i, \rho_k)$ is $E(\tilde{\phi}_k^i, \rho_k)$ with $\phi_k^i = \tilde{\phi}_k^i + (-1)^i \sqrt{-1} G \tilde{\phi}_k^i$ by Lemma 3.4.) Restricting $E(\phi_k^i, \rho_k)$ to $M_{i,R}$, we obtain sections over $M_{i,R}$. Let $f_{1,R}$ be the restriction to $M_{1,R}$ of the smooth function h_R over $M_{1,\infty}$ defined in the proof of Proposition 3.3 and $f_{2,R}$ be a smooth function over $M_{2,R}$ defined in a similar way. These functions have the obvious extension over M_R. Denoting by $E_0(\phi_k^i, \rho_k)$ the zero eigenmode of $E(\phi_k^i, \rho_k)$ and using (3.17) and $e^{4i\rho_k R} C_{12}(\rho_k)\psi_k = \psi_k$, we have

$$\begin{aligned} E_0(\phi_k^1, \rho_k) &= e^{-i\rho_k u} e^{-2i\rho_k R} C_1(-\rho_k)\psi_k + e^{i\rho_k u}\psi_k \\ &= e^{i\rho_k u}\psi_k + e^{-i\rho_k u} e^{2i\rho_k R} C_2(\rho_k)\psi_k = E_0(\phi_k^2, \rho_k) \ . \end{aligned} \tag{3.18}$$

Hence we can see that $E_0(\phi_k^1, \rho_k)$ and $E_0(\phi_k^2, \rho_k)$ define a smooth section over N_R, which we denote by $E_0(\psi_k, \rho_k)$. Let us define

$$\begin{aligned} \tilde{\Psi}_k := {}& f_{1,R}(E(\phi_k^1, \rho_k) - \chi_{[-R,0]_u} E_0(\phi_k^1, \rho_k)) \\ &+ f_{2,R}(E(\phi_k^2, \rho_k) - \chi_{[0,R]_u} E_0(\phi_k^2, \rho_k)) + \chi_{[-R,R]_u} E(\psi_k, \rho_k) \end{aligned} \tag{3.19}$$

where $\chi_{[a,b]_u}$ is the characteristic function of the u-variable over $[a,b]_u \times Y \subset N_R$. By (3.18), $\tilde{\Psi}_k$ is a smooth section over M_R. Put $\Psi_k := \tilde{\Psi}_k / \|\tilde{\Psi}_k\|$ and

$$\hat{\Psi}_k = \Psi_k - \pi_R \Psi_k, \quad \text{for} \quad k = 1, \ldots, m(R) \ ,$$

where π_R denote the orthogonal projection of $L^2(M_R, S)$ onto $\ker \mathcal{D}_R$. Let us recall that $\ker \mathcal{D}_R$ equals the space spanned by eigensections of *e-values* by our assumption, so that the dimension of this space is constant with respect to R. Combining this fact and Lemma 3.6, we have

$$|\langle \hat{\Psi}_k, \hat{\Psi}_\ell \rangle - \delta_{k\ell}| \le e^{-cR} \qquad \text{for} \quad k, \ell = 1, \ldots, m(R) \ .$$

From this and (3.15), it follows that $\{\hat{\Psi}_k\}_{k=1}^{m(R)}$ are linearly independent for $R \gg 0$. Now let $0 < \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \ldots \le \tilde{\lambda}_{p(R)}$ denote the nonzero eigenvalues, counted with multiplicity, of $\mathcal{D}_R^2$, which are $\le R^{-2\kappa}$. Let $k_1, \ldots, k_{m(R)}$ be a permutation of $\{1, \ldots, m(R)\}$ such that $0 < \rho_{k_1}^2 \le \ldots \le \rho_{k_{m(R)}}^2$. By the mini-max principle, we have

$$\tilde{\lambda}_\ell = \min_W \max_{\phi \in W} \frac{\|\mathcal{D}_R \phi\|^2}{\|\phi\|^2}$$

where W runs over all ℓ-dimensional subspaces of $L^2(M_R,S)$ which are orthogonal to $\ker(\mathcal{D}_R)$. Let W_ℓ be the subspace of $L^2(M_R,S)$ spanned by $\hat\Psi_{k_1},\dots,\hat\Psi_{k_\ell}$. Then, by Lemma 3.6, we have

$$\tilde\lambda_\ell \le \max_{\phi\in W_\ell}\frac{\|\mathcal{D}_R\phi\|^2}{\|\phi\|^2} \le \rho_{k_\ell}^2(1+Ce^{-cR})$$

for some constants $C,c>0$. Hence, there exists R_2 such that $m(R)\le p(R)$ for $R\ge R_2$. Putting $R_0=\max(R_1,R_2)$, this completes the proof of Theorem 3.5. □

Lemma 3.6. *Assume that all the* e-values *of $\mathcal{D}_R$ are zero eigenvalues. Then there exist $c_1,c_2>0$ such that*

$$|\langle\Psi_k,\Psi_\ell\rangle|\le c_1e^{-c_2R} \quad \textit{for} \quad k\ne\ell,\ k,\ell=1,\dots,m(R)\ ,$$
$$|\langle\Psi_k,\Psi\rangle|\le c_1e^{-c_2R} \quad \textit{for} \quad k=1,\dots,m(R),\ \textit{and}\ \Psi\in\ker\mathcal{D}_R\ \textit{with}\ ||\Psi||=1\ .$$

Proof. For a couple (ψ_k,ρ_k) and ϕ_k^i satisfying (3.17), we put

$$E_k^\perp = E(\phi_k^1,\rho_k)|_{M_{1,R}} - \chi_{[-R,0]_u}E_0(\phi_k^1,\rho_k) + E(\phi_k^2,\rho)|_{M_{2,R}} - \chi_{[0,R]_u}E_0(\phi_k^2,\rho_k)\ ,$$
$$E_{k,0} = E(\psi_k,\rho_k) = \chi_{[-R,0]_u}E_0(\phi_k^1,\rho_k) + \chi_{[0,R]_u}E_0(\phi_k^2,\rho_k)\ .$$

Putting $f_R=f_{1,R}+f_{2,R}$, it is easy to see that $\tilde\Psi_k$ defined in (3.19) has the form $f_RE_k^\perp+\chi_{[-R,R]_u}E_{k,0}$. Now we have

$$\begin{aligned}
&\langle\tilde\Psi_k,\tilde\Psi_\ell\rangle = \langle fE_k^\perp+\chi E_{k,0}, fE_\ell^\perp+\chi E_{\ell,0}\rangle\\
&= \langle fE_k^\perp, fE_\ell^\perp\rangle + \langle\chi E_{k,0},\chi E_{\ell,0}\rangle\\
&= \langle E_k^\perp-(1-f)E_k^\perp, E_\ell^\perp-(1-f)E_\ell^\perp\rangle + \langle\chi E_{k,0},\chi E_{\ell,0}\rangle\\
&= \langle E_k^\perp,E_\ell^\perp\rangle - \langle E_k^\perp,(1-f)E_\ell^\perp\rangle - \langle(1-f)E_k^\perp,E_\ell^\perp\rangle\\
&\quad + \langle(1-f)E_k^\perp,(1-f)E_\ell^\perp\rangle + \langle\chi E_{k,0},\chi E_{\ell,0}\rangle\\
&= \langle E_k,E_\ell\rangle - \langle E_k^\perp,(1-f)E_\ell^\perp\rangle - \langle(1-f)E_k^\perp,E_\ell^\perp\rangle\\
&\quad + \langle(1-f)E_k^\perp,(1-f)E_\ell^\perp\rangle
\end{aligned}\tag{3.20}$$

where $f=f_R,\chi=\chi_{[-R,R]_u}$. Since $\operatorname{supp}(1-f_R)\subset[-\frac{R}{2},\frac{R}{2}]_u\times Y$, where $E_k^\perp,E_\ell^\perp$ are $O(e^{-cR})$, the last three terms in (3.20) are $O(e^{-cR})$. Now we consider the first term in (3.20), which can be written as

$$\langle E_k,E_\ell\rangle = \langle E(\phi_k^1,\rho_k),E(\phi_\ell^1,\rho_\ell)\rangle_{M_{1,R}} + \langle E(\phi_k^2,\rho_k),E(\phi_\ell^2,\rho_\ell)\rangle_{M_{2,R}}\ .\tag{3.21}$$

When $\rho_k\ne\rho_\ell$, as in the proof of Proposition 3.1, we apply Green formula to each term on the right side of (3.21), then these equal

$$(\rho_k-\rho_\ell)^{-1}\langle GE(\phi_k^1,\rho_k)|_{\partial(M_{1,R})},E(\phi_\ell^1,\rho_\ell)|_{\partial(M_{1,R})}\rangle_{\partial(M_{1,R})}$$
$$-(\rho_k-\rho_\ell)^{-1}\langle GE(\phi_k^2,\rho_k)|_{\partial(M_{2,R})},E(\phi_\ell^2,\rho_\ell)|_{\partial(M_{2,R})}\rangle_{\partial(M_{2,R})}\ .$$

Now using (3.18), the restrictions of constant terms over $\partial(M_{i,R})$ cancel each other out and the remaining terms are $O(e^{-cR})$. Hence, in this case, the left side of (3.21) is $O(e^{-cR})$, so all the terms in (3.20) are $O(e^{-cR})$. When $\rho_k=\rho_\ell$, note that $\langle \phi_k^i, \phi_\ell^i \rangle = 0$ for $i = 1, 2$, so applying Proposition 3.1, we can see that all the terms are $O(e^{-cR})$ except the following terms,

$$\langle C_1(-\rho_k)C_1'(\rho_k)\phi_k^1, \phi_\ell^1 \rangle + \langle C_2(-\rho_k)C_2'(\rho_k)\phi_k^2, \phi_\ell^2 \rangle \ . \tag{3.22}$$

Using the conditions in (3.17) for ϕ_k^i, ϕ_ℓ^i and the relation

$$e^{4i\rho_k R}C_2(\rho_k)\psi_k = C_1(-\rho_k)\psi_k \ , \quad e^{4i\rho_k R}C_2(\rho_k)\psi_\ell = C_1(-\rho_k)\psi_\ell \ , \tag{3.23}$$

one can show that the terms in (3.22) equal

$$\langle e^{4i\rho_k R}C_1'(\rho_k)C_2(\rho_k)\psi_k, \psi_\ell \rangle + \langle e^{4i\rho_k R}C_1(\rho_k)C_2'(\rho_k)\psi_k, \psi_\ell \rangle \ . \tag{3.24}$$

Now we choose a family of sections $\psi_k(t)$ with $\psi_k(0) = \psi_k$ for $t \in (-\epsilon, \epsilon)$ such that

$$a(t)C_1(\rho_k + t)C_2(\rho_k + t)\psi_k(t) = \psi_k(t)$$

where $a(0) = e^{4i\rho_k R}$. Taking the derivative of this at $t = 0$, we obtain

$$\begin{aligned} &e^{4i\rho_k R}C_1'(\rho_k)C_2(\rho_k)\psi_k + e^{4i\rho_k R}C_1(\rho_k)C_2'(\rho_k)\psi_k \\ &\qquad = -a'(0)C_1(\rho_k)C_2(\rho_k)\psi_k - e^{4i\rho_k R}C_1(\rho_k)C_2(\rho_k)\psi_k'(0) + \psi_k'(0) \ . \end{aligned}$$

Using this, (3.23) and $\langle \psi_k, \psi_\ell \rangle = 0$, we can see that (3.24) equals

$$\begin{aligned} &\langle -e^{4i\rho_k R}C_1(\rho_k)C_2(\rho_k)\psi_k'(0) + \psi_k'(0), \psi_\ell \rangle \\ &\qquad = \langle \psi_k'(0), \psi_\ell \rangle - \langle \psi_k'(0), e^{-4i\rho_k R}C_2(-\rho_k)C_1(-\rho_k)\psi_\ell \rangle = 0 \ . \end{aligned}$$

Hence, in the case of $\rho_k = \rho_l$, all the terms in (3.20) are $O(e^{-cR})$. This completes the proof of the first claim recalling $\Psi_k = \tilde{\Psi}_k / ||\tilde{\Psi}_k||$.

For the second claim, let us recall that the eigenspaces of the *e-values* are spanned by the sections defined by gluing (as in (3.19)) the elements in $\ker_{L^2} \mathcal{D}_{i,\infty}$ for $i = 1, 2$ or the extended L^2-solutions of $\mathcal{D}_{i,\infty}$ whose limiting values lying in $L_1 \cap L_2$. By our assumption, this space is the same as $\ker \mathcal{D}_R$. For a section Ψ given by gluing elements in $\ker_{L^2} \mathcal{D}_{i,\infty}$, the claim follows easily by applying Green's formula as above. For a section Ψ given by gluing the extended L^2-solutions of $\mathcal{D}_{i,\infty}$ whose limiting values lying in $L_1 \cap L_2$, we use Theorem 3.8, which implies that such a Ψ is actually given by (3.19) for the couple (ψ_k, ρ_k) with $\rho_k = 0$. Hence, the claim for this case can be proved as in the previous case of $\rho_k \neq \rho_\ell$. This completes the proof of the second claim.

□

In general, the map $C_{12} := C_{12}(0) : (\ker B)_- \to (\ker B)_-$ does not equal the identity map, but it is not difficult to see that

$$C_1(0) \circ C_2(0)\phi = \phi \quad \text{if and only if} \quad \phi \in (L_1 \cap L_2) \oplus (GL_1 \cap GL_2) \ .$$

Putting

$$I_+ = 1 + iG : \ker B \to (\ker B)_- \ ,$$

we can see that $I_+(L_1 \cap L_2)$ and $I_+(GL_1 \cap GL_2)$ are the same subspace in $(\ker B)_-$.

Proposition 3.7. *The map C_{12} equals the identity map when restricted to the subspace $I_+(L_1 \cap L_2)$ and the multiplicity of the eigenvalue $(+1)$ of the operator C_{12} is* $\dim(L_1 \cap L_2) = \dim(I_+(L_1 \cap L_2))$.

Proof. Using the following diagram

$$\begin{array}{ccc} L_1 \cap L_2 & \xrightarrow{I_+} & (\ker B)_- \\ {\scriptstyle C_1(0)\circ C_2(0)}\downarrow & & \downarrow{\scriptstyle C_1(0)\circ C_2(0)} \\ L_1 \cap L_2 & \xrightarrow{I_+} & (\ker B)_- \end{array} ,$$

we can easily see that the first claim holds. To complete the proof, it is sufficient to show that if $C_1(0) \circ C_2(0)\phi = \phi$, then $\phi \in (L_1 \cap L_2) \oplus (GL_1 \cap GL_2)$. For this, choose $\phi_+ \in L_1$, then $C_1(0) \circ C_2(0)\phi_+ = \phi_+$ implies $\phi_+ = C_1(0)\phi_+ = C_2(0)\phi_+$ since $C_1(0)^2 = \mathrm{Id}$. Hence, this means that $\phi_+ \in L_2$, so $\phi_+ \in L_1 \cap L_2$. Repeating the same argument, if $\phi_- \in GL_1$ and $C_1(0)\circ C_2(0)\phi_- = \phi_-$, then $\phi_- \in GL_1 \cap GL_2$. Since $\ker B = L_1 \oplus GL_1$, this completes the proof. □

Now let us consider the eigenvalues $\lambda(R)$, which correspond to $\alpha_j(0) = 0$ and $k = 0$ in the following equality equivalent to (3.13),

$$4\lambda R + \alpha_j(\lambda) = 2\pi k + O(e^{-cR}) \quad .$$

It is easy to see that such eigenvalues must be *e-values*. Hence, by Lemma 3.7 this provides another proof of the following result, originally shown in [5].

Theorem 3.8. *The space of eigensections corresponding to* e-values*, which are not determined by* $\ker_{L^2}(\mathcal{D}_{i,\infty})$ *for* $i = 1, 2$*, is given by the space* $L_1 \cap L_2$*.*

Now let us consider the following Dirac type operator

$$\mathcal{D}_R = d_\rho + d_\rho^* \ : \ \oplus_{i=0}^n \Omega^i(M_R, V_\rho) \ \to \ \oplus_{i=0}^n \Omega^i(M_R, V_\rho) \tag{3.25}$$

where V_ρ denotes the flat vector bundle defined by a unitary representation ρ of $\pi_1(M_R)$. The dimension of $\ker \mathcal{D}_R$ is constant with respect to R since $\ker \mathcal{D}_R$ is the space of the twisted harmonic forms over M_R and this space is always isomorphic to de Rham cohomology $H^*(M_R, V_\rho)$ by the Hodge theorem. Moreover, one can show that all the *e-values* of the operator $\mathcal{D}_R$ in (3.25) are the zero eigenvalues using the argument in Section 4 of [11].

Proposition 3.9. *For the operator* $\mathcal{D}_R$ *in* (3.25)*, all the* e-values *of* $\mathcal{D}_R$ *are the zero eigenvalues.*

Proof. First let us observe that $\mathcal{D}_{i,\infty}$ is self adjoint, so $\ker_{L^2} \mathcal{D}_{i,\infty} = \ker_{L^2} \mathcal{D}^2_{i,\infty}$ and L_i is also the limiting value of extended L^2-solutions of $\mathcal{D}^2_{i,\infty}$. Let $\Delta^q_{i,\infty}$ be the restriction of $\mathcal{D}^2_{i,\infty}$ to $\Omega^q(M_R, V_\rho)$ and

$$h^q_M := \dim \ker_{L^2} \Delta^q_{1,\infty} + \dim \ker_{L^2} \Delta^q_{2,\infty} + \dim L^q_1 \cap L^q_2 \tag{3.26}$$

where L^q_i is the limiting values of the extended L^2-solutions of $\Delta^q_{i,\infty}$. Then, it is sufficient to show that $\beta^q := \dim(\ker \mathcal{D}^2_R \cap \Omega^q(M_R, V_\rho)) \geq h^q_M$ since $\beta_q \leq h^q_M$ by definition. For this, we use the following Mayer-Vietoris sequence

$$\ldots \to H^{q-1}(Y) \to H^q(M_R) \to H^q_a(M_{1,R}) \oplus H^q_a(M_{2,R}) \to H^q(Y) \to \ldots \tag{3.27}$$

where $H^q_a(M_{i,R})$ denotes the absolute cohomology. (Here, for simplicity, the bundle V_ρ is dropped in the notation.) The space $\oplus_{q=0}^n H^q_a(M_{i,R})$ can be identified with the kernel of the operator $\mathcal{D}_{i,R}$ with the absolute boundary condition. In more detail, the operator $\mathcal{D}_R = d_\rho + d_\rho^*$ has the following form over N_R,

$$\mathcal{D}_R = d_\rho + d_\rho^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\partial_u + \begin{pmatrix} 0 & d_Y + d_Y^* \\ d_Y + d_Y^* & 0 \end{pmatrix} \right) \tag{3.28}$$

with respect to

$$\Omega^*(N_R) \cong (\Omega^*(Y) \oplus \Omega^*(Y)) \otimes C^\infty([-R, R]_u). \tag{3.29}$$

Here d_Y, d_Y^* denote the restricted operator to Y of d_ρ and its adjoint respectively. The operator $\mathcal{D}_{i,R}$ has the same form near the boundary and with respect to (3.29). A section Ψ in $\Omega^*(M_{i,R})$ over the cylinder near the boundary Y has the following form,

$$\Psi = \Psi_0 + \Psi_1 \wedge du$$

where Ψ_i has no factor du. Then the absolute boundary condition for $\mathcal{D}_{i,R}$ is given by $\Psi_1 = 0$. Similarly the relative boundary condition for $\mathcal{D}_{i,R}$ is given by $\Psi_0 = 0$. We denote by $\mathcal{D}_{i,R}^a$, $\mathcal{D}_{i,R}^r$ the resulting operators. Now let us recall that the Cauchy data spaces $\mathcal{H}(\mathcal{D}_{i,R})$ of $\mathcal{D}_{i,R}$ are Lagrangian subspaces in $\Omega^*(Y) \oplus \Omega^*(Y)$ with respect to the symplectic form $\langle G\,,\,\rangle$ where $G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\langle\,,\,\rangle$ are the inner product over $\Omega^*(Y) \oplus \Omega^*(Y)$. Then, this implies that $\mathcal{H}_0(\mathcal{D}_{i,R}) := \mathcal{H}(\mathcal{D}_{i,R}) \cap (H^*(Y) \oplus H^*(Y))$ are also Lagrangian subspaces in $H^*(Y) \oplus H^*(Y)$. Moreover, the space $\mathcal{H}_0(\mathcal{D}_{i,R})$ has the following decomposition,

$$\mathcal{H}_0(\mathcal{D}_{i,R}) = A_i \oplus R_i$$

where A_i, R_i the spaces spanned by the boundary values of $\ker \mathcal{D}_{i,R}^a, \ker \mathcal{D}_{i,R}^r$ in $H^*(Y) \oplus H^*(Y)$. Decomposing $A_i = \oplus_{q=0}^{n-1} A_i^q$, $R_i = \oplus_{q=0}^{n-1} R_i^q$ where A_i^q, R_i^q are spaces of q-form parts, the Lagrangian subspace property of $\mathcal{H}_0(\mathcal{D}_{i,R})$ in $H^*(Y) \oplus H^*(Y)$ implies

$$H^q(Y) \cong A_i^q \oplus R_i^q \ . \tag{3.30}$$

By the exactness of (3.27), we also have

$$\begin{aligned} H^q(M_R) &\cong \operatorname{Im}(H^{q-1}(Y) \to H^q(M_R)) \oplus \operatorname{Im}(H^q(M_R) \to H_a^q(M_{1,R}) \oplus H_a^q(M_{2,R})) \\ &\cong (\operatorname{Im} k^{q-1})^\perp \oplus \ker k^q \end{aligned}$$

where k^q is the boundary map from $H_a^q(M_{1,R}) \oplus H_a^q(M_{2,R})$ to $H^q(Y)$. Now we summarize the consequences of the previous considerations. First, by (3.30), we have

$$(\operatorname{Im} k^{q-1})^\perp = (A_1^{q-1} + A_2^{q-1})^\perp = R_1^{q-1} \cap R_2^{q-1} \ .$$

Second, we note that $\ker k^q$ contains the harmonic sections whose boundary values are lying in $A_1^q \cap A_2^q$ and the harmonic sections that can be extended as L^2-solutions of $\Delta_{i,\infty}^q$. Hence,

$$\dim \ker k^q \geq \dim(A_1^q \cap A_2^q) + \dim \ker_{L^2} \Delta_{1,\infty}^q + \dim \ker_{L^2} \Delta_{2,\infty}^q \ .$$

By these facts and the following equality

$$\dim(L_1^q \cap L_2^q) = \dim(A_1^q \cap A_2^q) + \dim(R_1^{q-1} \cap R_2^{q-1}),$$

we can conclude that $\beta^q \geq h^q_M$ recalling (3.26). This completes the proof. $\square$

Let Ψ_R be a normalized eigensection of $(\mathcal{D}_{1,R})_{P_1}$, which corresponds to the *s-value* $\lambda = \lambda(R)$ with $|\lambda(R)| \leq R^{-\kappa}$ where κ is the fixed constant such that $0 < \kappa < 1$. Then we have

$$\mathcal{D}_{1,R}\Psi_R = \lambda\Psi_R \ \ , \ \ \|\Psi_R\| = 1 \ \ , \ \ P_1(\Psi_R|_{\{v=R\}\times Y}) = 0 \ \ . \tag{3.31}$$

The section Ψ_R can be represented in the following way on $[0, R]_v \times Y \subset M_{1,R}$

$$\Psi_R = e^{-i\lambda v}\psi_1 + e^{i\lambda v}\psi_2 + \hat{\Psi}_R \tag{3.32}$$

where $\psi_1 \in (\ker B)_+, \psi_2 \in (\ker B)_-$ and $\hat{\Psi}_R$ is a smooth L^2-section orthogonal to $\ker B$. The next result corresponds to Proposition 3.3, which can be proved in the same way as Proposition 8.14 of [17],

Proposition 3.10. *The zero-eigenmode $e^{-i\lambda v}\psi_1 + e^{i\lambda v}\psi_2$ of the eigensection Ψ_R of* s-value $\lambda(R)$ *of $(\mathcal{D}_{1,R})_{P_1}$ is a non-trivial.*

Now we define

$$\begin{aligned} I_\pm &= 1 \pm iG : \ker B \to (\ker B)_\mp \ \ , \\ I_{\sigma_1} &= I_-|_{\ker(\sigma_1+1)} : \ker(\sigma_1 + 1) \to (\ker B)_+ \ \ , \\ P_{\sigma_1} &= \frac{1}{2}(1 - \sigma_1) : \ker B \to \ker(\sigma_1 + 1) \end{aligned}$$

and

$$S_{\sigma_1}(\lambda) = -P_{\sigma_1} \circ C_1(\lambda) \circ I_{\sigma_1} : \ker(\sigma_1 + 1) \to \ker(\sigma_1 + 1) \ \ .$$

For ψ_1 in (3.32), by Lemma 3.4, there exists a unique $\phi \in \ker(\sigma_1 + 1)$ such that $\psi_1 = \phi - iG\phi$. As in the derivation of (3.7), we compare Ψ_R with $E(\phi, \lambda)$ and using Proposition 3.10 we obtain

$$\|C_1(\lambda)\psi_1 - \psi_2\| \leq e^{-cR} \ \ . \tag{3.33}$$

By the boundary condition in (3.31), we have

$$e^{-2i\lambda R}P_{\sigma_1}(\psi_1) = -P_{\sigma_1}(\psi_2) \ \ .$$

Combining this equation and (3.33), we derive

$$\|e^{2i\lambda R}S_{\sigma_1}(\lambda)\phi - \phi\| \leq e^{-cR}$$

for $\phi \in \ker(\sigma_1 + 1)$. We also define

$$\begin{aligned} I_{\sigma_2} &= I_+|_{\ker(\sigma_2+1)} : \ker(\sigma_2 + 1) \to (\ker B)_- \ \ , \\ P_{\sigma_2} &= \frac{1}{2}(1 - \sigma_2) : \ker B \to \ker(\sigma_2 + 1) \end{aligned}$$

and

$$S_{\sigma_2}(\lambda) := -P_{\sigma_2} \circ C_2(\lambda) \circ I_{\sigma_2} : \ker(\sigma_2 + 1) \to \ker(\sigma_2 + 1) \ \ ,$$

where $C_2(\lambda)$ is the scattering matrix defined from the generalized eigensection over $M_{2,\infty}$. By the same way as above we can derive

$$\|e^{2i\lambda R}S_{\sigma_2}(\lambda)\phi - \phi\| \leq e^{-cR}$$

for $\phi \in \ker(\sigma_2 + 1)$. Now we introduce

$$\Omega_i(R) := \{ \rho \in \mathbb{R} - \{0\} \mid \det(e^{2i\rho R} S_{\sigma_i}(\rho) - \mathrm{Id}) = 0 \ , \ |\rho| \le R^{-\kappa} \}$$

for $i = 1, 2$. We repeat the argument used in [17] to prove the corresponding result for *s-values* of $(\mathcal{D}_{i,R})_{P_i}$ noting all the argument for the involution $C_i(0)$ in [17] holds for the involution σ_i, and we obtain

Theorem 3.11. *For $i = 1, 2$, let $\lambda_1(R) \le \lambda_2(R) \le \ldots \le \lambda_{p(R)}(R)$ be the nonzero eigenvalues, counted to multiplicity, of $(\mathcal{D}_{i,R})_{P_i}$ satisfying $|\lambda_k(R)| \le R^{-\kappa}$, and let $\rho_1(R) \le \rho_2(R) \le \ldots \le \rho_{m(R)}(R)$ be the nonzero element , counted to multiplicity, of $\Omega_i(R)$. Then there exist R_0 and $c > 0$, independent of R, such that for $R \ge R_0$, $p(R) = m(R)$ and*

$$|\lambda_k(R) - \rho_k(R)| \le e^{-cR} \quad \textit{for} \quad k = 1, \ldots, p(R) \ .$$

We now have the following proposition

Proposition 3.12. *There is a natural isomorphism*

$$\ker(\mathcal{D}_{i,R})_{P_i} \cong \ker_{L^2} \mathcal{D}_{i,\infty} \oplus \ker(\sigma_i - 1) \cap L_i$$

for $i = 1, 2$.

Proof. Let $\Psi \in \ker(\mathcal{D}_{1,R})_{P_1}$. Then the section Ψ satisfies $G(\partial_v + B)\Psi = 0$ on the cylinder $[0, R]_v \times Y$, and it has the following representation when restricted to this cylinder

$$\Psi = \phi_0 + \sum_{\mu_j > 0} c_j e^{-\mu_j v} \phi_j$$

where $(\sigma_i - 1)(\phi_0) = 0$. We use this expansion to extend Ψ to a smooth section $\tilde{\Psi}$ on $M_{1,\infty}$ satisfying $\mathcal{D}_{1,\infty}\tilde{\Psi} = 0$. This means that $\tilde{\Psi}$ belongs to the space of the extended L^2-solutions of $\mathcal{D}_{1,\infty}$. Hence ϕ_0 is an element of L_1 . Let $E(\phi_0, \lambda)$ be the generalized eigensection attached to ϕ_0. Then $\tilde{\Psi} - \frac{1}{2}E(\phi_0, 0)$ is square integrable and $\mathcal{D}_{1,\infty}(\tilde{\Psi} - \frac{1}{2}E(\phi_0, 0)) = 0$, and the map

$$\Psi \longrightarrow (\tilde{\Psi} - \frac{1}{2}E(\phi_0, 0), \phi_0)$$

gives the expected isomorphism. □

The restriction of $S_{\sigma_i} := S_{\sigma_i}(0)$ to $\ker(\sigma_i + 1) \cap \ker(C_i(0) + 1)$ is equal to the identity map and

$$\dim(\ker(\sigma_i + 1) \cap \ker(C_i(0) + 1)) = \dim(\ker(\sigma_i - 1) \cap \ker(C_i(0) - 1)) \ .$$

It follows from Proposition 3.12 that the number of $(+1)$–eigenspace of $S_{\sigma_i} := S_{\sigma_i}(0)$ is equal to the dimension of the subspace of $\ker(\mathcal{D}_{i,R})_{P_i}$ complementary to the subspace $\ker_{L^2} \mathcal{D}_{i,\infty}$ for $i = 1, 2$.

Now we define our model operator. Let $C : W \to W$ denote a unitary operator acting on a d-dimensional vector space W with eigenvalues $e^{i\alpha_j}$ for $j = 1, \ldots, d$. We introduce the operator $D(C)$,

$$D(C) := -i\frac{1}{2}\frac{d}{du} : C^\infty(\mathbb{S}^1, E_C) \to C^\infty(\mathbb{S}^1, E_C) \tag{3.34}$$

where E_C is the flat vector bundle over $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ defined by the holonomy $\overline{C}$, the complex conjugate of C . The spectrum of $D(C)$ is equal to

$$\{\ \pi k - \frac{1}{2}\alpha_j \mid k \in \mathbb{Z}, j = 1, \ldots, d\ \}\ . \tag{3.35}$$

The operators C_{12} , S_{σ_1} and S_{σ_2} are the unitary operators acting on finite dimensional vector spaces. Hence we can define self-adjoint, elliptic operators $D(C_{12}), D(S_{\sigma_1}), D(S_{\sigma_2})$ on $\mathbb{S}^1$.

Theorem 3.13. *Assume that all the* e-values *of $\mathcal{D}_R$ are zero eigenvalues. Let $\lambda_1(R) \le \lambda_2(R) \le \ldots \le \lambda_{p(R)}(R)$ be the nonzero eigenvalues, counted to multiplicity, of $\mathcal{D}_R$ satisfying $|\lambda_k(R)| \le R^{-\kappa}$, and let $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_{n(R)}$ be the nonzero eigenvalues, counted to multiplicity, of $D(C_{12})$ satisfying $|\lambda_k| \le 2R^{1-\kappa}$. Then there exist R_0 and $C > 0$, independent of R, such that for $R \ge R_0$, $p(R) = n(R)$ and*

$$|2R\lambda_k(R) - \lambda_k| \le C\, R^{-\kappa} \quad \textit{for} \quad k = 1, \ldots, p(R)\ .$$

The similar statement holds for $(\mathcal{D}_{i,R})_{P_i}$ and $D(S_{\sigma_i})$ with the relation

$$|R\lambda_k(R) - \lambda_k| \le C\, R^{-\kappa} \quad \textit{for} \quad k = 1, \ldots, p_i(R)$$

where $p_i(R)$ is the number of s-values *of $(\mathcal{D}_{i,R})_{P_i}$ with $|\lambda_k(R)| \le R^{-\kappa}$.*

Proof. First we introduce

$$\Omega^*(R) := \{\ \rho \in \mathbb{R} - \{0\} \mid \det(e^{4i\rho R}C_{12} - \mathrm{Id}) = 0\ ,\ |\rho| \le R^{-\kappa}\ \}\ .$$

By definition, this set consists of the nonzero solution $\rho^*_{j,k}$ of

$$4\lambda R + \alpha_j(0) = 2\pi k \quad \text{with} \quad |\lambda| \le R^{-\kappa} \tag{3.36}$$

where $e^{i\alpha_j(0)}$ for $j = 1, \ldots, \frac{h_Y}{2}$ are the eigenvalues of $C_{12} = C_{12}(0)$. Now, for an element $\rho_{j,k}$ in $\Omega(R)$ defined in (3.12), one can show (as near (3.14)) that if $R \gg 0$ there is the corresponding solution $\rho^*_{j,k}$ of

$$4\lambda R + \alpha_j(0) = 2\pi k \quad \text{with} \quad |\lambda| \le R^{-\kappa} + R^{-1-\kappa}\ ,$$

noting $|\alpha_j(\lambda) - \alpha_j(0)| \le cR^{-\kappa}$ for a positive constant c. Since $|\rho_{j,k} - \rho^*_{j,k}| \le cR^{-1-\kappa}$, this gives a one to one correspondence from $\Omega(R)$ to $\Omega^*(R_0)$ with $R_0^{-\kappa} = R^{-\kappa} + R^{-1-\kappa}$ for $R \gg 0$. Now, let us observe that for any pair of $\rho^*_{j,k} \ne \rho^*_{j',k'}$ in $\Omega^*(R)$, $|\rho^*_{j,k} - \rho^*_{j',k'}| \ge a_0 R^{-1}$ for a positive constant a_0. Hence, for $R \gg 0$, this implies that $\Omega^*(R) = \Omega^*(R_0)$ with $R_0^{-\kappa} = R^{-\kappa} + R^{-1-\kappa}$. In conclusion, there is a one to one correspondence between $\Omega(R)$ and $\Omega^*(R)$ for $R \gg 0$ with the relation

$$|\rho_k - \rho^*_k| \le cR^{-1-\kappa}$$

where $\rho_1 \le \ldots \le \rho_{m(R)}$ ($\rho^*_1 \le \ldots \le \rho^*_{n(R)}$) denotes the elements, counted to multiplicity, of $\Omega(R)$ ($\Omega^*(R)$). For $\rho^* \in \Omega^*(R)$, the map $\rho^* \to 2R\rho^*$ gives a one to one correspondence from $\Omega^*(R)$ to the subset of the eigenvalues λ_k of $D(C_{12})$ with $|\lambda_k| \le 2R^{1-\kappa}$. Now, applying Theorem 3.5 completes the proof for *s-values* for $\mathcal{D}_R$. The case of $(\mathcal{D}_{i,R})_{P_i}$ can be proved in the same way.

□

4. Large time contribution

In this section, we prove the following proposition

Proposition 4.1.

$$\lim_{R\to\infty}\int_{R^{2-\varepsilon}}^{\infty} t^{-1}[\mathrm{Tr}\,(e^{-t\mathcal{D}_R^2}-e^{-t(\mathcal{D}_{1,R})^2_{P_1}}-e^{-t(\mathcal{D}_{2,R})^2_{P_2}})-h]\;dt-h(\gamma-\varepsilon\cdot\log R)$$
$$=\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}[\mathrm{Tr}\,(e^{-t\frac{1}{4}D(C_{12})^2}-e^{-tD(S_{\sigma_1})^2}-e^{-tD(S_{\sigma_2})^2})-h]\;dt$$

where $h=\dim(L_1\cap L_2)-\dim(L_1\cap\ker(\sigma_1-1))-\dim(L_2\cap\ker(\sigma_2-1))$.

Recalling

$$\frac{d}{ds}\Big|_{s=0}\zeta_l^R(s)=\int_{R^{2-\varepsilon}}^{\infty} t^{-1}[\,\mathrm{Tr}\,(e^{-t\mathcal{D}_R^2}-e^{-t(\mathcal{D}_{1,R})^2_{P_1}}-e^{-t(\mathcal{D}_{2,R})^2_{P_2}})-h\,]\;dt\ \ ,$$

Proposition 4.1 immediately implies that the large time contribution to the adiabatic decomposition formula for the ζ-determinant is equal to

$$\frac{\det_\zeta\frac{1}{4}D(C_{12})^2}{\det_\zeta D(S_{\sigma_1})^2\cdot\det_\zeta D(S_{\sigma_2})^2}\ \ .$$

We start with the following result,

Proposition 4.2. *The following equality holds,*

$$\lim_{R\to\infty}\Big(\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^{R^{-\varepsilon}} t^{s-1}[\mathrm{Tr}\,(e^{-t\frac{1}{4}D(C_{12})^2}$$
$$-e^{-tD(S_{\sigma_1})^2}-e^{-tD(S_{\sigma_2})^2})-h]\,dt+h(\gamma-\varepsilon\cdot\log R)\Big)=0\ \ .$$

Proof. Recalling the definition of $D(C)$ in (3.34), we can see that if $\mathcal{L}$ is one of $D(C_{12})^2$, $D(S_{\sigma_1})^2$, $D(S_{\sigma_1})^2$, then

$$\mathrm{Tr}\,(e^{-t\mathcal{L}})\sim\sqrt{\frac{\pi}{t}}\,\frac{h_Y}{2}+O(\sqrt{t})\quad\text{near}\quad t=0$$

since $h_Y=2\dim(\ker B)_-=2\dim\ker(\sigma_i+1)$. Hence there exists a constant c_1 such that

$$|\mathrm{Tr}\,(e^{-t\frac{1}{4}D(C_{12})^2}-e^{-tD(S_{\sigma_1})^2}-e^{-tD(S_{\sigma_2})^2})|<c_1\sqrt{t}\quad\text{near}\quad t=0\ . \tag{4.1}$$

This allows us to estimate

$$\Big|\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^{R^{-\varepsilon}} t^{s-1}\mathrm{Tr}\,(e^{-t\frac{1}{4}D(C_{12})^2}-e^{-tD(S_{\sigma_1})^2}-e^{-tD(S_{\sigma_2})^2})\;dt\Big|$$
$$\le c_2\cdot\int_0^{R^{-\varepsilon}}\frac{dt}{\sqrt{t}}=2c_2\cdot R^{-\frac{\varepsilon}{2}}\ \ .$$

Combining this with

$$\frac{d}{ds}\Big|_{s=0}\frac{h}{\Gamma(s)}\int_0^{R^{-\varepsilon}} t^{s-1}dt=h(\gamma-\varepsilon\cdot\log R)$$

completes the proof. □

It follows from Proposition 4.2 that Proposition 4.1 is equivalent to the following equation

$$
\begin{aligned}
(4.2)\quad &\lim_{R\to\infty}\int_{R^{2-\varepsilon}}^{\infty} t^{-1}[\,\mathrm{Tr}\,(e^{-t\mathcal{D}_R^2}-e^{-t(\mathcal{D}_{1,R})^2_{P_1}}-e^{-t(\mathcal{D}_{2,R})^2_{P_2}})-h\,]\,dt\\
&=\lim_{R\to\infty}\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_{R^{-\varepsilon}}^{\infty} t^{s-1}[\,\mathrm{Tr}\,(e^{-t\frac14 D(C_{12})^2}-e^{-tD(S_{\sigma_1})^2}-e^{-tD(S_{\sigma_2})^2})-h\,]\,dt.
\end{aligned}
$$

Now using change variables we obtain

$$
\begin{aligned}
&\int_{R^{2-\varepsilon}}^{\infty} t^{-1}[\,\mathrm{Tr}\,(e^{-t\mathcal{D}_R^2}-e^{-t(\mathcal{D}_{1,R})^2_{P_1}}-e^{-t(\mathcal{D}_{2,R})^2_{P_2}})-h\,]\,dt\\
&=\int_{R^{-\varepsilon}}^{\infty} t^{-1}[\,\mathrm{Tr}\,(e^{-tR^2\mathcal{D}_R^2}-e^{-tR^2(\mathcal{D}_{1,R})^2_{P_1}}-e^{-tR^2(\mathcal{D}_{2,R})^2_{P_2}})-h\,]\,dt\quad .
\end{aligned}
$$

Then the equality (4.2) is equivalent to

$$
\begin{aligned}
(4.3)\quad &\lim_{R\to\infty}\int_{R^{-\varepsilon}}^{\infty} t^{-1}[\,\mathrm{Tr}\,(e^{-tR^2\mathcal{D}_R^2}-e^{-tR^2(\mathcal{D}_{1,R})^2_{P_1}}-e^{-tR^2(\mathcal{D}_{2,R})^2_{P_2}})-h\,]\,dt\\
&=\lim_{R\to\infty}\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_{R^{-\varepsilon}}^{\infty} t^{s-1}[\,\mathrm{Tr}\,(e^{-t\frac14 D(C_{12})^2}-e^{-tD(S_{\sigma_1})^2}-e^{-tD(S_{\sigma_2})^2})-h\,]\,dt.
\end{aligned}
$$

Now we split

$$
\mathrm{Tr}\,(e^{-tR^2\mathcal{D}_R^2}-e^{-tR^2(\mathcal{D}_{1,R})^2_{P_1}}-e^{-tR^2(\mathcal{D}_{2,R})^2_{P_2}})-h
$$

into two parts

$$
\begin{aligned}
&\mathrm{Tr}\,_{I,R}(e^{-tR^2\mathcal{D}_R^2}-e^{-tR^2(\mathcal{D}_{1,R})^2_{P_1}}-e^{-tR^2(\mathcal{D}_{2,R})^2_{P_2}})\quad ,\\
&\mathrm{Tr}\,_{II,R}(e^{-tR^2\mathcal{D}_R^2}-e^{-tR^2(\mathcal{D}_{1,R})^2_{P_1}}-e^{-tR^2(\mathcal{D}_{2,R})^2_{P_2}})
\end{aligned}
$$

where $\mathrm{Tr}\,_{I,R}(\cdot)$ ($\mathrm{Tr}\,_{II,R}(\cdot)$) is the part of the trace restricted to the eigenvalues of $R^2\mathcal{D}_R^2$, $R^2(\mathcal{D}_{1,R})^2_{P_1}$, $R^2(\mathcal{D}_{2,R})^2_{P_2}$ which are larger (smaller or equal to) than $R^{\frac12}$. The next proposition shows that $\mathrm{Tr}\,_{I,R}(\cdot)$ can be neglected as $R\to\infty$,

Proposition 4.3. *We have the following estimate*

$$
\int_{R^{-\varepsilon}}^{\infty} t^{-1}\mathrm{Tr}\,_{I,R}(e^{-tR^2\mathcal{D}_R^2}-e^{-tR^2(\mathcal{D}_{1,R})^2_{P_1}}-e^{-tR^2(\mathcal{D}_{2,R})^2_{P_2}})\,dt\le c_1e^{-c_2R^{\frac12-\varepsilon}}
$$

for some positive constants c_1, c_2.

Proof. Let $\lambda_{k_0}(R)^2$ denote the smallest *large* eigenvalue of $\mathcal{D}_R^2$, that is, the smallest one with $\lambda_{k_0}(R)^2 > R^{-\frac{3}{2}}$. We now estimate $\mathrm{Tr}_{I,R}(e^{-tR^2\mathcal{D}_R^2})$ as follows;

$$\begin{aligned}\mathrm{Tr}_{I,R}(e^{-tR^2\mathcal{D}_R^2}) &= \sum_{\lambda_k^2 > R^{-\frac{3}{2}}} e^{-tR^2\lambda_k^2} = \sum_{\lambda_k^2 > R^{-\frac{3}{2}}} e^{-(tR^2-1)\lambda_k^2} e^{-\lambda_k^2} \\ &\le e^{-(tR^2-1)\lambda_{k_0}^2} \sum_{\lambda_k^2 > R^{-\frac{3}{2}}} e^{-\lambda_k^2} \le e^{-(tR^2-1)\lambda_{k_0}^2}\,\mathrm{Tr}\,(e^{-\mathcal{D}_R^2}) \\ &\le b_1 R e^{-(tR^2-1)R^{-\frac{3}{2}}} \le b_1 R e^{-b_2 t\sqrt{R}} \quad ,\end{aligned}$$

for some positive constants b_1, b_2 . Hence we have

$$\begin{aligned}\int_{R^{-\varepsilon}}^{\infty} t^{-1}\mathrm{Tr}_{I,R}(e^{-tR^2\mathcal{D}_R^2})\, dt &\le \int_{R^{-\varepsilon}}^{\infty} t^{-1} b_1 R e^{-b_2 t\sqrt{R}}\, dt \\ &\le \frac{b_1}{b_2} R^{\frac{1}{2}+\varepsilon} \int_{b_2 R^{\frac{1}{2}-\varepsilon}}^{\infty} e^{-v} dv \le b_3 e^{-b_4 R^{\frac{1}{2}-\varepsilon}} \quad .\end{aligned}$$

The trace $\mathrm{Tr}_{I,R}(e^{-tR^2(\mathcal{D}_{i,R})^2_{P_i}})$ for $i = 1, 2$ can be estimated in the same way. This completes the proof. $\square$

We also split $\mathrm{Tr}\,(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) - h$ into two parts

$$\begin{aligned}&\mathrm{Tr}_{I,R}(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) \quad , \\ &\mathrm{Tr}_{II,R}(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2})\end{aligned}$$

where $\mathrm{Tr}_{I,R}(\cdot)$ ($\mathrm{Tr}_{II,R}(\cdot)$) is taken over the nonzero eigenvalues of $\frac{1}{4}D(C_{12})^2$, $D(S_{\sigma_1})^2$, $D(S_{\sigma_2})^2$ which are larger (smaller or equal to) than $R^{\frac{1}{2}}$. The following proposition corresponds to Proposition 4.3 and its proof is essentially the same as the proof of Proposition 4.3.

Proposition 4.4. *There exist positive constants c_1, c_2 such that*

$$\int_{R^{-\varepsilon}}^{\infty} t^{-1}\mathrm{Tr}_{I,R}(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) \le c_1 e^{-c_2 R^{\frac{1}{2}-\varepsilon}} \quad .$$

By Propositions 4.3 and 4.4, we can see that the equality (4.3) is equivalent to

$$\begin{aligned}\lim_{R\to\infty}\Bigg(&\int_{R^{-\varepsilon}}^{\infty} t^{-1}\mathrm{Tr}_{II,R}(e^{-tR^2\mathcal{D}_R^2} - e^{-tR^2(\mathcal{D}_{1,R})^2_{P_1}} - e^{-tR^2(\mathcal{D}_{2,R})^2_{P_2}})\, dt \\ &- \int_{R^{-\varepsilon}}^{\infty} t^{-1}\mathrm{Tr}_{II,R}(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2})\, dt\Bigg) = 0 \quad .\end{aligned} \tag{4.4}$$

The equation (4.4) is a consequence of the next result

Proposition 4.5. *For sufficiently large R, there exist positive constants c_1, c_2 independent of R and t, such that*

$$\begin{aligned} |\ \mathrm{Tr}_{II,R}(e^{-tR^2\mathcal{D}_R^2}) - \mathrm{Tr}_{II,R}(e^{-t\frac{1}{4}D(C_{12})^2})\ | \ &\leq\ c_1 t R^{-\frac{1}{4}} e^{-c_2 t} \quad , \\ |\ \mathrm{Tr}_{II,R}(e^{-tR^2(\mathcal{D}_{i,R})^2_{P_i}}) - \mathrm{Tr}_{II,R}(e^{-tD(S_{\sigma_i})^2})\ | \ &\leq\ c_1 t R^{-\frac{1}{4}} e^{-c_2 t} \end{aligned}$$

for any $t > 0$.

Proof. We use the analysis of *s-values* developed in Section 3. We fix $\kappa = \frac{3}{4}$. It follows from Theorem 3.13 that, for any eigenvalue $\lambda(R)$ of $\mathcal{D}_R$ with $|\lambda(R)| \leq R^{-\frac{3}{4}}$ there exists an analytic function $\alpha(\lambda)$ such that

$$R\lambda(R) = \lambda_j + \frac{1}{4}\lambda(R)\alpha(\lambda(R)) + O(e^{-cR})$$

where λ_j is an eigenvalue of $\frac{1}{2}D(C_{12})$ with $|\lambda_j| \leq R^{\frac{1}{4}}$. Therefore, there exist functions $c(R), d(R)$ and a constant $C > 0$ such that

$$R^2\lambda(R)^2 = \lambda_j^2 + \lambda_j \frac{c(R)}{R^{\frac{3}{4}}} + \frac{d(R)}{R^{\frac{3}{2}}} \quad \text{with} \quad |c(R)| \leq C \ , \quad |d(R)| \leq C \quad ,$$

for any sufficiently large R . We use the elementary inequality $|e^{-\lambda} - 1| \leq |\lambda| e^{|\lambda|}$ to get

$$\begin{aligned} |e^{-tR^2\lambda(R)^2} - e^{-t\lambda_j^2}| &= |e^{-t\lambda_j^2}(e^{-t[R^2\lambda(R)^2 - \lambda_j^2]} - 1)| \\ &\leq \Big(\frac{|\lambda_j c(R)|}{R^{\frac{3}{4}}} + \frac{|d(R)|}{R^{\frac{3}{2}}}\Big)\, t\, e^{-(\lambda_j^2 - \frac{|\lambda_j c(R)|}{R^{\frac{3}{4}}} - \frac{|d(R)|}{R^{\frac{3}{2}}})t} \leq \frac{C}{R^{\frac{1}{2}}}\, t\, e^{-\frac{1}{2}\lambda_j^2 t} \end{aligned}$$

for $R \gg 0$. In the last inequality we used the fact that $|\lambda_j| \leq R^{\frac{1}{4}}$. Let us fix a sufficiently large R. We take the sum over finitely many eigenvalues λ_j^2 of $\frac{1}{4}D(C_{12})^2$ with $\lambda_j^2 \leq R^{\frac{1}{2}}$, and obtain

$$|\mathrm{Tr}_{II,R}(e^{-tR^2\mathcal{D}_R^2}) - \mathrm{Tr}_{II,R}(e^{-t\frac{1}{4}D(C_{12})^2})| \leq C\frac{t}{R^{\frac{1}{2}}} \sum_{\lambda_j^2 \leq R^{\frac{1}{2}}} e^{-\frac{1}{2}\lambda_j^2 t} \quad .$$

The operator $\frac{1}{4}D(C_{12})^2$ is a Laplace type operator over S^1 , hence the number of eigenvalues λ_j^2 of $\frac{1}{4}D(C_{12})^2$ with $\lambda_j^2 \leq R^{\frac{1}{2}}$ can be estimated by $R^{\frac{1}{4}}$. Therefore, we have

$$C\frac{t}{R^{\frac{1}{2}}} \sum_{\lambda_j^2 \leq R^{\frac{1}{2}}} e^{-\frac{1}{2}\lambda_j^2 t} \leq c_1 \frac{t}{R^{\frac{1}{2}}} R^{\frac{1}{4}} e^{-\frac{1}{2}\lambda_1^2 t}$$

where λ_1^2 denotes the first nonzero eigenvalue of $\frac{1}{4}D(C_{12})^2$. Note that c_1 and λ_1^2 are independent of R. This proves the first claim putting $c_2 = \frac{1}{2}\lambda_1^2$. The proof of the second claim goes in the same way. □

The proof of Proposition 4.1 is now complete.

Proof of Theorem 1.1: Now Proposition 2.5 and Proposition 4.1 give us the following equality,

$$\begin{aligned}
&\lim_{R\to\infty}\Big(\zeta_s^{R'}(0)+h(\gamma+(2-\varepsilon)\cdot\log R)+\zeta_l^{R'}(0)-h(\gamma-\varepsilon\cdot\log R)\Big)\\
=&\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}[\,\mathrm{Tr}\,(e^{-t\frac{1}{4}D(C_{12})^2}-e^{-tD(S_{\sigma_1})^2}-e^{-tD(S_{\sigma_2})^2})-h\,]\;dt\\
&+\zeta_{B^2}(0)\cdot\log 2\quad .
\end{aligned}$$

By an elementary computation (for instance, see Proposition 2.2 in [15]), we can derive

$$\begin{aligned}
\det{}_\zeta\frac{1}{4}D(C_{12})^2 &= 2^{h_Y+2h_M}\det{}^*\Big(\frac{2\,\mathrm{Id}-C_{12}-C_{12}^{-1}}{4}\Big)\quad ,\\
\det{}_\zeta D(S_{\sigma_i})^2 &= 2^{h_Y}\det{}^*\Big(\frac{2\,\mathrm{Id}-S_{\sigma_i}-S_{\sigma_i}^{-1}}{4}\Big)\quad .
\end{aligned}\tag{4.5}$$

Combining all these equalities provides us with the final formula (1.10) in Theorem 1.1.

5. A proof of the decomposition formula of the η-invariant

In this section we offer a new proof of the decomposition formula for the η-invariant. This formula has been proved by several authors (see [4], [9], [12], [18], [23], [3], [13], [16]) and the proof we discuss in this section is not the simplest one. Still we believe that it is worthy to present the *scattering approach* to the decomposition of the η-invariant. The key in our proof is to show that the *scattering data* provides us with the contribution given by the boundary conditions in the decomposition formula for the η-invariant.

Let us remind the reader that the η-function of a Dirac operator $\mathcal{D}$ on a closed manifold M, introduced in [1], is defined as

$$\eta_{\mathcal{D}}(s)=\sum_{\lambda_k\neq 0}\mathrm{sign}\,(\lambda_k)|\lambda_k|^{-s}\quad ,$$

where the sum is taken over all nonzero eigenvalues of $\mathcal{D}$. The η-function is well-defined for $\Re(s)$ large and it has a meromorphic extension to the whole complex plane and $s=0$ is a regular point, hence $\eta_{\mathcal{D}}(0)$ is well-defined. Following [1] we introduce the η-invariant of $\mathcal{D}$ as

$$\eta(\mathcal{D})=\frac{1}{2}\cdot(\eta_{\mathcal{D}}(0)+\dim\ker\mathcal{D})\quad .\tag{5.1}$$

Now, let us assume that we have a decomposition of a closed odd-dimensional manifold M to $M_1\cup M_2$ in the way described in the introduction. For $\mathcal{D}_i:=\mathcal{D}|_{M_i}$, we impose the boundary conditions given by the generalized APS spectral projections P_i defined in (1.6). Then the η-function of $(\mathcal{D}_i)_{P_i}$ is also well-defined and it has the same properties as the η-function of the Dirac operator on a closed manifold, in particular, the η-function of $(\mathcal{D}_i)_{P_i}$ is regular at $s=0$. Hence, we can define the η-invariant of $(\mathcal{D}_i)_{P_i}$ as in (5.1). The following result was proved by several authors as we remarked above,

Theorem 5.1. *The following formula holds,*

$$\eta(\mathcal{D}) = \eta((\mathcal{D}_1)_{P_1}) + \eta((\mathcal{D}_2)_{P_2}) + \eta(\mathcal{D};\sigma_1,\sigma_2) \mod \mathbb{Z} \ , \tag{5.2}$$

where $\eta(\mathcal{D};\sigma_1,\sigma_2)$ denotes the η-invariant of the operator $\mathcal{D} = G(\partial_u + B)$ over $N \cong [-1,1]\times Y$, subject to the boundary condition P_2 at $u=-1$ and P_1 at $u=1$.

For the involution σ_i which defines P_i in (1.6), let us observe that

$$U = \sigma_1\sigma_2 : \ker B \to \ker B$$

is the unitary operator, such that $UG = GU$, $\det U = 1$ and $U^* = \sigma_1 U \sigma_1$. It follows that the spectrum of U is invariant under complex conjugation. Moreover, the maps $U_\pm = U|_{(\ker B)_\pm} : (\ker B)_\pm \to (\ker B)_\pm$ are well-defined. The following result proved in Section 2 of [14] was the key ingredient in the proof of Theorem 5.1.

Proposition 5.2. *We have the following formulas*

$$\eta(\mathcal{D};\sigma_1,\sigma_2) = -\frac{1}{2\pi i}\log\det(-U_+) \mod \mathbb{Z} \ . \tag{5.3}$$

One way to prove the decomposition formula (5.2) is to use the *adiabatic analysis* we developed in the proof of Theorem 1.1. This analysis easily gives us the following theorem,

Theorem 5.3. *The following formula for the η-invariant holds,*

$$\begin{aligned}\eta(\mathcal{D}) - \eta((\mathcal{D}_1)_{P_1}) - \eta((\mathcal{D}_2)_{P_2})& \\ = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2}))& \mod \mathbb{Z} \ .\end{aligned} \tag{5.4}$$

Proof. We repeat the corresponding argument to derive Theorem 1.1 for the η-invariant to obtain the expected formula

$$\begin{aligned}\lim_{R\to\infty}\{\eta(\mathcal{D}_R) - \eta((\mathcal{D}_{1,R})_{P_1}) - \eta((\mathcal{D}_{2,R})_{P_2})\}& \\ = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2}))& \mod \mathbb{Z}.\end{aligned}$$

Now, we use the fact that $\eta(\mathcal{D}_R), \eta((\mathcal{D}_{i,R})_{P_i})$ are independent of R modulo integer (see Proposition 2.16 of [17]) to complete the proof. □

Now we need to show

$$\eta(\mathcal{D};\sigma_1,\sigma_2) = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \mod \mathbb{Z} \ .$$

For this, we observe the followings: The scattering matrix $C_i = C_i(0)$ can be represented in the following way,

$$C_i = \begin{pmatrix} 0 & C(i)_- \\ C(i)_+ & 0 \end{pmatrix} \quad \text{where} \quad C(i)_\pm C(i)_\mp = \mathrm{Id} \ ,$$

with respect to the decomposition $\ker B = (\ker B)_+ \oplus (\ker B)_-$. We see that

$$C_{12} = C(1)_+ C(2)_- : (\ker B)_- \to (\ker B)_- \ . \tag{5.5}$$

Similar formulas hold for the involutions σ_i and we have

$$S_{\sigma_1} = -P_{\sigma_1} \circ C_1 \circ I_{\sigma_1} = -\frac{1}{2}\begin{pmatrix} \mathrm{Id} & -\sigma(1)_- \\ -\sigma(1)_+ & \mathrm{Id} \end{pmatrix}\begin{pmatrix} 0 & C(1)_- \\ C(1)_+ & 0 \end{pmatrix}\begin{pmatrix} 2\mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \sigma(1)_- C(1)_+ & 0 \\ -C(1)_+ & 0 \end{pmatrix} .$$

We can also see that each element of $\ker(\sigma_1 + 1)$ is represented in the form $\begin{pmatrix} f \\ -\sigma(1)_+ f \end{pmatrix}$ for some $f \in (\ker B)_+$. This allows us to represent the map S_{σ_1} over $\ker(\sigma_1 + 1)$ as

$$S_{\sigma_1}\begin{pmatrix} f \\ -\sigma(1)_+ f \end{pmatrix} = \begin{pmatrix} \sigma(1)_- C(1)_+ & 0 \\ -C(1)_+ & 0 \end{pmatrix}\begin{pmatrix} f \\ -\sigma(1)_+ f \end{pmatrix} = \begin{pmatrix} \sigma(1)_- C(1)_+ f \\ -\sigma(1)_+ \sigma(1)_- C(1)_+ f \end{pmatrix} .$$

Therefore, from the spectral point of view, the operator S_{σ_1} is equal to the operator

$$\sigma(1)_- C(1)_+ : (\ker B)_+ \to (\ker B)_+ ,$$

or equivalently to the operator

$$C(1)_+ \sigma(1)_- : (\ker B)_- \to (\ker B)_- .$$

The corresponding analysis for the operator S_{σ_2} implies that S_{σ_2} is equivalent to

$$C(2)_- \sigma(2)_+ : (\ker B)_+ \to (\ker B)_+ \quad \text{or} \quad \sigma(2)_+ C(2)_- : (\ker B)_- \to (\ker B)_- .$$

Combining (5.5) and these, we obtain

$$\frac{\det(C_{12})}{\det(S_{\sigma_1})\det(S_{\sigma_2})} = \det(\sigma(1)_+ \sigma(2)_-) . \tag{5.6}$$

For the operator $D(C)$ on S^1 defined by a unitary map C in (3.34),

$$\eta(D(C)) = -\frac{1}{2\pi i}\log\det(-\overline{C}) \quad \mod \mathbb{Z} . \tag{5.7}$$

(see Theorem 2.1 and Lemma 2.3 in [14]). If we combine (5.6) and (5.7), we have

$$\eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) = -\frac{1}{2\pi i}\log\det(-\overline{\sigma(1)_+ \sigma(2)_-}) \quad \mod \mathbb{Z} .$$

Noting $\det(-\overline{\sigma(1)_+ \sigma(2)_-}) = \det(-\sigma(1)_- \sigma(2)_+)$, this and Proposition 5.2 end the proof of the following theorem,

Theorem 5.4.

$$\eta(\mathcal{D}; \sigma_1, \sigma_2) = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \quad \mod \mathbb{Z} . \tag{5.8}$$

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