

BOUNDARY PROBLEMS FOR DIRAC TYPE OPERATORS ON MANIFOLDS WITH MULTI-CYLINDRICAL END BOUNDARIES

PAUL LOYA AND JINSUNG PARK

ABSTRACT. The goal of this paper is to establish a geometric program to study elliptic pseudodifferential boundary problems which arise naturally under cutting and pasting of geometric and spectral invariants of Dirac type operators on manifolds with corners endowed with multi-cylindrical, or b -type, metrics and ‘ b -admissible’ partitioning hypersurfaces. We show that the Cauchy data space of a Dirac operator on such a manifold is Lagrangian for the self-adjoint case, the corresponding Calderón projector is a b -pseudodifferential operator of order 0, characterize Fredholmness, prove relative index formulæ, and solve the Bojarski conjecture.

1. INTRODUCTION

The purpose of this paper is to establish the *geometric* theory of elliptic pseudodifferential boundary problems for Dirac type operators on manifolds with *multi-cylindrical* end boundaries. The main impetus for this theory is to develop Fredholm and spectral theory and derive gluing formulas for the index, eta invariant, and ζ -determinant of Dirac type operators on such manifolds. In this paper, we focus on the Fredholm theory. We show that the Cauchy data space of such a Dirac operator is Lagrangian for the self-adjoint case, the corresponding Calderón projector is a b -pseudodifferential operator of order 0, and we prove relative index formulæ and solve the Bojarski conjecture for such manifolds. In the forthcoming articles [20], [21] we study the relative and gluing formulæ for the spectral invariants, respectively, for such manifolds.

We begin by describing geometrically our class of manifolds. An n -dimensional compact *manifold with corners* X is a compact topological space locally modelled on $[0, \infty)_x \times \mathbb{R}_y^{n-k}$, where k can run between 0 and n , such that X has only finitely many boundary hypersurfaces, say $\{H_0, H_1, \dots, H_r\}$, where each H_i is imbedded in the sense that near the hypersurface H_i , we have

$$(1.1) \quad X \cong [0, 1]_{x_i} \times H_i \quad , \quad i = 1, 2, \dots, r \quad , \quad H_i = \{x_i = 0\}.$$

We make the assumption that X is connected and H_0 is an *admissible* boundary hypersurface (this can be disconnected), which simply means that $H_0 \cap H_i \neq \emptyset$ for $i = 1, \dots, r$; see Figure 1 for examples of manifolds with corners and choices of H_0 . For future convenience in establishing gluing formulas we assume that near H_0 ,

$$(1.2) \quad X \cong [-1, 0]_u \times H_0 \quad , \quad H_0 = \{u = 0\}.$$

Date: September 29, 2005. file name: LoyaParkCald.tex.

Key words and phrases. Dirac operators; b -calculus; manifolds with corners.

2000 Mathematics Subject Classification. Primary: 58J28, 58J52.

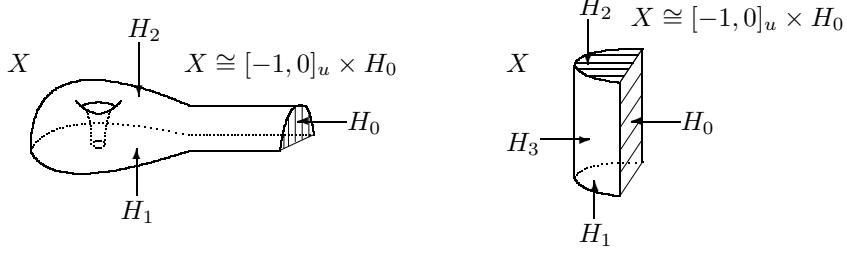


FIGURE 1. Two examples of three-dimensional manifolds with corners (each figure is solid). On the left, all the hypersurfaces are admissible, but we choose H_0 to be the right ‘cap’. On the right, H_0 and H_3 are admissible, but H_1 and H_2 are not.

How do these manifolds arise? They arise very naturally. One example is to take a (solid) soda can: the round portion is admissible and the top and bottom are not admissible. A related example is to take the soda can and stand it up, then slice it from top to bottom; each piece is a manifold with corners and the newly formed flat side is admissible as seen in the right picture in Figure 1 where we only show the left piece of the cut soda can.

We now put an *exact b-metric* g on X that geometrically pushes each hypersurface H_i , $i = 1, \dots, r$, out to infinity and which is smooth up to H_0 ; thus the ‘ b ’ refers only with respect to H_1, \dots, H_r . This means that the metric g is smooth up to H_0 and the metric degenerates up to each H_i , $i = 1, \dots, r$, as follows:

$$(1.3) \quad g = \sum_{i=1}^r \left(\frac{dx_i}{x_i} \right)^2 + h,$$

where h is a smooth symmetric two-form on X . Explicitly, pick a point $p \in X$, let us say $p \in H_0 \cap H_1 \cap \dots \cap H_k$, and p is in no other hypersurfaces; then we assume that there is a common decomposition stemming from (1.1) and (1.2) such that near the point p we can write

$$(1.4) \quad X \cong [-1, 0]_u \times [0, 1]_x^k \times Y,$$

where Y is the component of $H_0 \cap H_1 \cap \dots \cap H_k$ containing p . Then in this coordinate patch, g can be written as

$$(1.5) \quad g = a(u, x, y) du^2 + \left(\frac{dx_1}{x_1} \right)^2 + \dots + \left(\frac{dx_k}{x_k} \right)^2 + h(u, x, y),$$

where a is a smooth positive function (even smooth and positive when u or any x_i is zero) and h is a symmetric two-form on X that is smooth near p . There is a similar description of g in other patches away from H_0 .

This is the ‘compact viewpoint’; there is a ‘noncompact viewpoint’ in terms of attaching cylinders that might be easier to understand. Let us make the change of variables $t_i = \log x_i$. Observe that when $x_i = 1$, $t_i = 0$ and as $x_i \rightarrow 0$, $t_i \rightarrow -\infty$. Therefore, under this change of variables, the coordinates (1.4) and the metric (1.5) near the point p take the ‘multi-cylindrical’ forms:

$$X \cong [-1, 0]_u \times (-\infty, 0]_t^k \times Y,$$

where over this patch,

$$g = a'(u, t, y)du^2 + dt_1^2 + \cdots + dt_k^2 + h'(u, t, y).$$

Here, $a'(u, t, y) = a(u, e^t, y)$ and $h'(u, t, y) = h(u, e^t, y)$ where we define $e^t = (e^{t_1}, \dots, e^{t_k})$. One can work in this ‘noncompact viewpoint’ but we choose to work under the compactified viewpoint because from this perspective we have Melrose’s b -calculus machine [23] at our disposal.

We emphasize that b -metrics of the sort (1.3) arise very naturally under cutting: Take a solid soda can as we mentioned before and put an exact b -metric on it that geometrically pushes all its hypersurfaces to infinity. Now cut the can from top to bottom as before to get a piece like in the right picture in Figure 1. The resulting metric g is smooth up to H_0 but is still an exact b -metric up to H_1, H_2, H_3 . Thus, metrics of the sort (1.3) arise naturally in the context of cutting and pasting of geometric and spectral invariants of Dirac type operators on noncompact ‘multi-cylindrical end’ manifolds and noncompact partitioning hypersurfaces. We also remark that instead of considering metrics of the form (1.3) with dx_i/x_i ’s, we can consider metrics with dx_i/x_i^2 ’s. The resulting metric is called an exact cusp metric and very analogous results in this paper hold for such a metric. We choose to work with b -metrics only because in the sequel [21] to this paper we shall use certain analytic objects (the b -zeta determinant for instance) which are a little more natural to use in the b -setting.

Let E, F be Hermitian vector bundles over X and let $\mathcal{D} : C^\infty(X, E) \rightarrow C^\infty(X, F)$ be a Dirac type operator; that is, a Dirac type operator derived from the metric (1.3) (see Section 2 for a precise definition or Melrose’s anticipated [24]). We also assume that on the collar $[-1, 0]_u \times H_0$ of the admissible boundary hypersurface H_0 with $H_0 = \{u = 0\}$, the vector bundles E and F are isometric to $E_0 := E|_{H_0}$ and $F_0 := F|_{H_0}$, and

$$(1.6) \quad \mathcal{D} = G_u(\partial_u + D_u),$$

where $G_u : E_0 \rightarrow F_0$ is unitary and $D_u : C^\infty(H_0, E_0) \rightarrow C^\infty(H_0, E_0)$ is a Dirac type operator, and where both G_u and D_u are smooth up to $u = 0$ and restricting there to define a unitary map $G_0 : E_0 \rightarrow F_0$ and a formally *self-adjoint* Dirac type operator $D_0 : C^\infty(H_0, E_0) \rightarrow C^\infty(H_0, E_0)$.

We remark that boundary value problems on manifolds with cylindrical end boundaries, which are special cases of manifolds with multi-cylindrical ends boundaries, have been considered by Schrohe [28] and Schrohe and Schulze [29] in the context of Boutet de Monvel’s algebra [6] and by Mitrea and Nistor [27] generalizing the method of layer potentials to such manifolds. Grubb [10] (cf. also Grubb and Kokholm [11]) has studied boundary value problems for a class of noncompact manifolds that are Euclidean at ∞ .

We now state our main results. Given any real s , we define $H_b^s(X, E)$ as the natural b -Sobolev space and we define $H_b^\infty(X, E)$ as the intersection of all $H_b^s(X, E)$ and we denote by $\tilde{\Psi}_b^s(H_0, E_0)$ the space of b -pseudodifferential operators of order s in the ‘calculus with bounds’ (see Section 2); similar remarks hold for F . The *Cauchy data space* of \mathcal{D} is by definition

$$\mathcal{H}(\mathcal{D}) := \ker \mathcal{D}|_{H_0} = \{ \phi|_{H_0} \mid \phi \in H_b^\infty(X, E), \mathcal{D}\phi = 0 \} \subset H_b^\infty(H_0, E_0),$$

where $\ker \mathcal{D}$ denotes the kernel of \mathcal{D} on $H_b^\infty(X, E)$. For our first result, we extend the theory and application of the (orthogonalized) Calderón projector to our

category of manifolds. Here, the Calderón projector was introduced by Calderón [7] and Seeley [30], [31] (cf. also Hörmander [12] and Grubb [8]). The theory and application of the Calderón projector was extended to a class of manifolds which are Euclidean at ∞ by Grubb [10] (cf. also Grubb and Kokholm [11]).

Theorem 1.1. *The operator*

$$\mathcal{D} : H_b^\infty(X, E) \rightarrow H_b^\infty(X, F)$$

*is surjective, restriction to the boundary gives a canonical isomorphism between $\ker \mathcal{D}$ and $\mathcal{H}(\mathcal{D})$, and there exists a (unique) **orthogonalized** Calderón projector $\mathcal{C} \in \widetilde{\Psi}_b^0(H_0, E_0)$ whose image on $H_b^\infty(H_0, E_0)$ is exactly $\mathcal{H}(\mathcal{D})$. Moreover, the b -principal symbol of \mathcal{C} is the orthogonal projection onto the eigenspaces of the negative eigenvalues of the corresponding b -principal symbol of D_0 .*

We remark that \mathcal{D} is surjective for any real s and not just $s = \infty$. We also remark that we make no invertibility assumptions whatsoever on the Dirac type operator \mathcal{D} . This may sound striking because in the case of a manifold with corners with an exact b -metric (which pushes all its boundary hypersurfaces to ∞), it is well-known [19, Cor. 2.5] that a Dirac type operator is Fredholm if and only if *all* the induced Dirac type operators on the boundary hypersurfaces are invertible (see also Theorem 2.5 and Remark 6.3). Theorem 1.1 shows that with no invertibility assumptions whatsoever, if we cut the manifold forming an admissible face, then the Calderón projector is well-defined at that face, has the same properties as on a compact manifold, and even has the ‘nice’ structure of a b -operator. See Section 3 for an example where we compute the Calderón projector in a model case where none of the induced Dirac operators are invertible. This example also shows that, in general, the Calderón projector \mathcal{C} is *not* in the ‘small’ b -calculus.

Assume for the moment that $E = F$ and \mathcal{D} is formally self-adjoint. Then in particular, $G_0^2 = -\text{Id}$. One can check that

$$\Omega(v, w) := \langle G_0 \varphi, \psi \rangle_0 \quad \text{for } \varphi, \psi \in L_b^2(H_0, E_0)$$

is a Hermitian symplectic form on $L_b^2(H_0, E_0)$ where $\langle \cdot, \cdot \rangle_0$ is the L_b^2 inner product on H_0 . Our second result is

Theorem 1.2. *Assume $E = F$ and \mathcal{D} is formally self-adjoint. Then the closure of the Cauchy data space $\mathcal{H}(\mathcal{D})$ in $L_b^2(H_0, E_0)$ is Lagrangian with respect to Ω . In particular, with respect to the decomposition $L_b^2(H_0, E_0) = L_b^2(H_0, E_0^+) \oplus L_b^2(H_0, E_0^-)$, where E_0^\pm are the $(\pm i)$ -eigenspaces of G_0 , the **orthogonalized** Calderón projector \mathcal{C} takes the form*

$$\mathcal{C} = \frac{1}{2} \begin{pmatrix} \text{Id} & \kappa_0^{-1} \\ \kappa_0 & \text{Id} \end{pmatrix},$$

where $\kappa_0 : L_b^2(H_0, E_0^+) \rightarrow L_b^2(H_0, E_0^-)$ is a unitary operator.

We remark that Theorem 1.2 — the Lagrangian property of $\mathcal{H}(\mathcal{D})$ — holds without any product structures near H_0 . This is one of the important *geometric* properties of $\mathcal{H}(\mathcal{D})$, which plays the crucial rôle in the study of the spectral invariants for manifolds with multi-cylindrical end boundaries [20, 21]. In our next theorem, under the condition that \mathcal{D} is of product type near H_0 ; that is, in the decomposition (1.6) of \mathcal{D} near H_0 , both G_u and D_u are constant in u , we show that the orthogonalized Calderón projector \mathcal{C} is equal to the Calderón projector defined

from the Poisson operator of the invertible double of \mathcal{D} . Here, the invertible double is defined in Section 4 following the work of Wojciechowski [32].

Theorem 1.3. *Assume $E = F$ and \mathcal{D} is formally self-adjoint and assume that \mathcal{D} is of product-type near H_0 . Then the Calderón projector defined by the invertible double of \mathcal{D} equals the orthogonalized projector \mathcal{C} .*

Back to the general case, using the Calderón projector \mathcal{C} , we can characterize the Fredholm properties of \mathcal{D} with other projectors defining the boundary condition. Given an arbitrary projector \mathcal{P} on $L_b^2(H_0, E_0)$, we define

$$\mathcal{D}_{\mathcal{P}} : \text{dom}(\mathcal{D}_{\mathcal{P}}) \rightarrow L_b^2(X, F)$$

where

$$\text{dom}(\mathcal{D}_{\mathcal{P}}) := \{ \phi \in H_b^1(X, E) \mid \mathcal{P}(\phi|_{H_0}) = 0 \}.$$

Because this domain involves only $\ker \mathcal{P} = \text{ran}(\text{Id} - \mathcal{P})$ we can obtain the same domain by replacing the projector $\text{Id} - \mathcal{P}$ with its orthogonalization (cf. Remark 3.5 in [10]). For this reason, we consider only orthogonal projections and we define the *smooth self-adjoint Grassmanian* $Gr_{\infty}^*(\mathcal{D})$ as those orthogonal projections $\mathcal{P} \in \tilde{\Psi}_b^0(H_0, E_0)$ such that $\mathcal{P} - \mathcal{C} \in \tilde{\Psi}^{-\infty}(H_0, E_0)$, the space of Green operators (in the calculus with bounds — see the definition (2.12) in Section 2). This implies, in particular, that $\mathcal{P}\mathcal{C} : \text{ran}\mathcal{C} \rightarrow \text{ran}\mathcal{P}$ is Fredholm. The following theorem is the multi-cylindrical end version of reduction to the boundary.

Theorem 1.4. *For an arbitrary projection $\mathcal{P} \in Gr_{\infty}^*(\mathcal{D})$, the operator*

$$(1.7) \quad \mathcal{D}_{\mathcal{P}} : \text{dom}(\mathcal{D}_{\mathcal{P}}) \rightarrow L_b^2(X, F)$$

is Fredholm, and

$$\text{ind } \mathcal{D}_{\mathcal{P}} = \text{ind}(\mathcal{P}, \mathcal{C}),$$

where $\text{ind}(\mathcal{P}, \mathcal{C}) := \text{ind}(\mathcal{P}\mathcal{C} : \text{ran}\mathcal{C} \rightarrow \text{ran}\mathcal{P})$. In particular, for any $\mathcal{P} \in Gr_{\infty}^(\mathcal{D})$, the essential spectrum of the operator $\mathcal{D}_{\mathcal{P}}$ in (1.7) has a gap near 0 and for any two projections $\mathcal{P}_1, \mathcal{P}_2 \in Gr_{\infty}^*(\mathcal{D})$, we have*

$$(1.8) \quad \text{ind } \mathcal{D}_{\mathcal{P}_1} - \text{ind } \mathcal{D}_{\mathcal{P}_2} = \text{ind}(\mathcal{P}_1, \mathcal{P}_2).$$

As before, we remark that we make no invertibility assumptions whatsoever on \mathcal{D} . Again, at first sight, this theorem is quite unbelievable because of the strong invertibility assumptions needed for the boundary Dirac operators in order that the corresponding statements hold for Dirac operators on manifolds with corners with exact b -metrics (which push all their boundary hypersurfaces to ∞). Therefore, the results stated in Theorem 1.4 show the importance of the Calderón projector for the noncompact set-up. Moreover, the model case presented in Section 3 shows the failure of the *APS* spectral projector to be a pseudodifferential operator. This means that analytically it would be unfeasible to approach boundary problems via the *APS* spectral projector and thus illuminates the effectiveness of the Calderón method.

We now describe the *Bojarski conjecture*. Let M be a smooth manifold with corners with an exact b -metric and let $\mathcal{D} : C^{\infty}(M, E) \rightarrow C^{\infty}(M, F)$ be a Dirac type operator. Suppose that Y is a hypersurface inside M that divides M into two manifolds with corners

$$M = M_- \cup M_+,$$

where both M_{\pm} are of the form considered in Figure 1 such that $g|_{M_{\pm}}$ and $\mathcal{D}_{\pm} := \mathcal{D}|_{M_{\pm}}$ satisfy the conditions we have already discussed with respect to Y , which we assume is admissible for both M_{\pm} . The *Bojarski Conjecture* [4], later proved by Booß-Bavnbek and Wojciechowski [5, Th. 24.1] for compact smooth manifolds, gives a gluing formula for the index on M in terms of the Fredholm index of pairs of boundary conditions from M_{\pm} . Here is the multi-cylindrical end version.

Theorem 1.5. *Suppose that $\mathcal{D} : H_b^1(M, E) \rightarrow L_b^2(M, F)$ is Fredholm. Then for arbitrary projections $\mathcal{P}_{\pm} \in \text{Gr}_{\infty}^*(\mathcal{D}_{\pm})$, we have*

$$\text{ind } \mathcal{D} = \text{ind } \mathcal{D}_{\mathcal{P}_-} + \text{ind } \mathcal{D}_{\mathcal{P}_+} - \text{ind}(\mathcal{P}_-, \text{Id} - \mathcal{P}_+).$$

In Section 2, we give a self-contained, introductory presentation of b -pseudodifferential operators on manifolds with corners. In Section 3 we compute the Calderón projector explicitly in the ‘model’ case when $X = [0, 1]_u \times [0, \infty)^\ell$. We prove that the Calderón projector exists and is in the b -calculus with bounds and we compute its b -principal symbol. In Section 4, we construct the invertible double of a Dirac type operator over a manifold with multi-cylindrical end. In Sections 5 and 6, we prove Theorems 1.1, 1.2, 1.3, 1.4 and 1.5 using the results proved in the previous sections.

2. INTRODUCTION TO b -OPERATORS ON MANIFOLDS WITH CORNERS

We review the b -calculus on manifolds with corners. For more on these topics, see Melrose [23], Mazzeo [22], or the appendices of Melrose and Piazza [26], and Melrose and Nistor [25].

2.1. b -pseudodifferential operators. Let M be a compact manifold with corners with local expressions of the form (1.1) endowed with an exact b -metric of the form (1.3) (where exact means there are no admissible boundary hypersurfaces; *all* the boundary hypersurfaces are geometrically at ∞). In particular, near any point $p \in M$, as explained around (1.4) we can write

$$(2.1) \quad M \cong [0, 1]_v^\kappa \times Y,$$

where the v_i ’s represent those boundary defining functions x_j that vanish at p and Y is the component containing p where all the v_i ’s vanish. A *codimension κ face*, $\kappa \geq 1$, of M is a nonempty connected component of the intersection of κ hypersurfaces of M . In particular, a boundary hypersurface is just a codimension one face of M . The largest κ such that M has a (nonempty) codimension κ face is called the *codimension* of M .

We now describe the ‘small’ calculus. Let $\dot{C}^\infty(M)$ denote the space of smooth functions on M that vanish to infinite order at the boundary of M ; that is, in Taylor series at any x_i in a local patch such as (2.1). We first define the b -smoothing operators $\Psi_b^{-\infty}(M)$ (which are *not* genuine smoothing operators). These are operators R on $\dot{C}^\infty(M)$ described in local coordinates as follows. Let \mathcal{U} and \mathcal{U}' be coordinate patches on M of the form (2.1) where we take coordinates on the corresponding Y ’s for \mathcal{U} and \mathcal{U}' . We allow $\kappa = 0$ in (2.1) for either \mathcal{U} or \mathcal{U}' , which means that the coordinate patch is located in the interior of our manifold M . Let $v = (v_1, \dots, v_\ell)$ denote those boundary defining functions, if any, that are common to both coordinate patches \mathcal{U} and \mathcal{U}' , so that

$$(2.2) \quad \mathcal{U} = [0, 1]_v^\ell \times \mathcal{V} \quad , \quad \mathcal{U}' = [0, 1]_v^\ell \times \mathcal{V}',$$

where $\mathcal{V} \subset [0, \infty)^k \times \mathbb{R}^{n-\ell-k}$ and $\mathcal{V}' \subset [0, \infty)^{k'} \times \mathbb{R}^{n-\ell-k'}$. Let y denote the coordinates on \mathcal{V} and y' the coordinates on \mathcal{V}' . Then given any open set \mathcal{W} with compact closure in \mathcal{U}' , for any $\phi \in \dot{C}^\infty(M)$ having support in \mathcal{W} , the restriction of $R\phi$ to \mathcal{U} is of the form

$$(2.3) \quad R\phi = \int_{\mathcal{U}'} R\left(v, \frac{v}{v'}, y, y'\right) \phi(v', y') dg(v', y'),$$

where $v/v' := (v_1/v'_1, \dots, v_\ell/v'_\ell)$ and where $R(v, z, y, y')$ has the following regularity properties: It is smooth in all variables, vanishes to infinite order with all derivatives at any $z_i = 0$ or as $z_i \rightarrow \infty$, and vanishes to infinite order at any $y_i = 0$ or $y'_i = 0$ when the corresponding zero set represents a boundary hypersurface of M .

We now consider the general case of an operator $A \in \Psi_b^m(M)$, $m \in \mathbb{R}$, which is an operator on $\dot{C}^\infty(M)$ described in local coordinates as follows. Let \mathcal{U} and \mathcal{U}' be coordinate patches of the form (2.2) and let \mathcal{W} be an open set with compact closure in \mathcal{U}' .

- (I) If \mathcal{U} and \mathcal{U}' are disjoint, then given any $\phi \in \dot{C}^\infty(M)$ having support in \mathcal{W} , the restriction of $A\phi$ to \mathcal{U} is given by an operator $R \in \Psi_b^{-\infty}(M)$ as in (2.3).
- (II) Suppose now that $\mathcal{U} = \mathcal{U}' = [0, 1]_v^\ell \times \mathbb{R}_y^{n-\ell}$. Then there is a function $a(v, y, \xi)$, smooth in $(v, y) \in \mathcal{U}$ and a classical symbol of order m in ξ , such that given any $\phi \in \dot{C}^\infty(M)$ having support in \mathcal{W} , we have

$$(2.4) \quad A\phi = \int_{\mathbb{R}^n} v^{i\tau} e^{iy \cdot \eta} a(v, y, \tau, \eta) \hat{\phi}(\xi) d\tau d\eta,$$

where $\xi = (\tau, \eta)$, $v^{i\tau} = v_1^{i\tau_1} \dots v_\ell^{i\tau_\ell}$, and $\hat{\phi}(\xi)$ is the Mellin transform in v and the Fourier transform in y of ϕ ,

$$(2.5) \quad \hat{\phi}(\tau, \eta) = \int_{\mathcal{U}} v^{-i\tau} e^{-iy \cdot \eta} \phi(v, y) \frac{dv}{v} dy.$$

We also assume that $a(v, y, \xi)$ with all its derivatives extends to be an entire function of τ , and for $|\operatorname{Im} \tau|$ bounded by any fixed number, is a classical symbol of order m in (τ, η) as $|\operatorname{Re} \tau|, |\eta| \rightarrow \infty$; this is a “lacunary” type condition for $\tau \in \mathbb{C}^k$ as described by Hörmander [13, p. 114].

The space $\Psi_b^m(M)$ is called the *small calculus of b-pseudodifferential operators* of order m . It also turns out that any $A \in \Psi_b^m(M)$ defines a continuous map on $C^\infty(M)$. In the literature, b -operators are most often presented in terms of their Schwartz kernels. Observe that combining (2.4) and (2.5), we see that the Schwartz kernel of A on the product $M \times M$ near the diagonal is of the form

$$(2.6) \quad K_A = \int_{\mathbb{R}^n} \left(\frac{v}{v'}\right)^{i\tau} e^{i(y-y') \cdot \eta} a(v, y, \tau, \eta) d\tau d\eta \cdot \frac{dv'}{v'} dy',$$

where (v, y) are coordinates on the left factor of M and (v', y') are the same coordinates on the right factor of M . Introducing ‘logarithmic coordinates’ $w = (\log v_1, \dots, \log v_\ell, y)$, we can write this kernel as

$$(2.7) \quad K_A = \int_{\mathbb{R}^n} e^{i(w-w') \cdot \xi} a(v, y, \xi) d\xi \cdot dw',$$

which looks like the Schwartz kernel of a pseudodifferential operator that we are used to. However, the ‘singular’ presentation (2.6) has certain advantages; perhaps the major one is that it, quite remarkably, actually simplifies the proofs of the composition and mapping properties of these operators.

The space of b -pseudodifferential operators has many of the same properties as the common ones on compact boundaryless manifolds. For example, $\text{Diff}_b^m(M) \subset \Psi_b^m(M)$, where $\text{Diff}_b^m(M)$ is the space of totally characteristic differential operators, which are operators taking the following form over a patch $[0, 1]_v^\ell \times \mathbb{R}_y^{n-\ell}$:

$$(2.8) \quad P = \sum_{|\alpha|+|\beta| \leq m} a_{\alpha,\beta}(v, y) (v\partial_v)^\alpha \partial_y^\beta,$$

where the $a_{\alpha,\beta}$'s are smooth in $(v, y) \in [0, 1]_v^\ell \times \mathbb{R}_y^{n-\ell}$. The space of $\Psi_b^\bullet(M)$ is also a symbolically filtered $*$ -algebra of operators. Thus, it is closed under taking adjoints and compositions, and there is a ' b -' principal symbol map preserving these operations obtained by taking the leading homogeneous term of each symbol $a(v, y, \xi)$ in the local representation (2.7). For example, the b -principal symbol of P in (2.8) is

$${}^b\sigma_m(P)(v, y, \xi) = \sum_{|\alpha|+|\beta|=m} a_{\alpha,\beta}(v, y) (i\tau)^\alpha (i\eta)^\beta, \quad \xi = (\tau, \eta).$$

The b -principal symbol turns out to be a function on the b -cotangent bundle ${}^bT^*M$ of M minus the zero section [23, p. 30]. An operator A is said to be b -elliptic if its b -principal symbol is invertible. The space $L_b^2(M)$ consists of those functions on M that are square integrable with respect to dg and for any $m \in \mathbb{R}$, the Sobolev space $H_b^m(M)$ consists of those distributions ϕ on M having the property that $A\phi \in L_b^2(M)$ for all $A \in \Psi_b^m(M)$; one can check that $H_b^m(M)$ is just the usual Sobolev space in the interior of M and the Mellin transform based Sobolev space in the normal variables to the hypersurfaces of M . We define $H_b^\infty(M) = \bigcap_{m \in \mathbb{R}} H_b^m(M)$. Then any $A \in \Psi_b^m(M)$ defines a continuous linear map

$$A : H_b^s(M) \rightarrow H_b^{s-m}(M), \quad s \in \mathbb{R} \cup \{\infty\}.$$

Unfortunately, the 'small' space $\Psi_b^\bullet(M)$ is not spectrally invariant in the sense that this set is not closed under inversion, when inverses exist; however, inverses can be found in the 'larger' calculus with bounds, which we now describe. Let $\theta > 0$. We define $\Psi_b^{m,\theta}(M)$ as those operators A that satisfy (I) and (II) as before, but with the following modifications: In (I), we can write R as in (2.3):

$$(2.9) \quad R\phi = \int_{\mathcal{Q}'} R\left(v, \frac{v}{v'}, y, y'\right) \phi(v', y') dg(v', y'),$$

but now $R(v, z, y, y')$ has the following 'boundedness' properties: For each $i = 1, \dots, \ell$, we require that

$$R(v, z, y, y') = R_{1,i}(v, z, y, y') + v_i^{\theta+\varepsilon} R_{2,i}(v, z, y, y'),$$

where $R_{1,i}, R_{2,i}$ are smooth for v, y, y' in the interior of M and in $z > 0$, and there is an $\varepsilon > 0$ such that given any constant coefficient b -differential operators P in the variables z, y, y' and Q in the variables v, z, y, y' , the functions $\partial_v^\alpha P R_{1,i}$ (for any α) and $Q R_{2,i}$ are bounded for all v, z, y, y' , continuous at each $v_j = 0$, vanishes to order $z_j^{\theta+\varepsilon}$ and $z_j^{-\theta-\varepsilon}$ at each $z_j = 0$ and as $z_j \rightarrow \infty$, respectively, and vanishes to order $y_j^{\theta+\varepsilon}$ at any $y_j = 0$ and $(y_j')^{\theta+\varepsilon}$ at any $y_j' = 0$ if any of these sets represent boundary hypersurfaces of M . In (II), for each i , we require that

$$a(v, y, \xi) = a_{1,i}(v, y, \xi) + v_i^{\theta+\varepsilon} a_{2,i}(v, y, \xi),$$

where $a_{1,i}(v, y, \xi)$ is smooth in all variables and all b -derivatives of $a_{2,i}(v, y, \xi)$ are continuous (here ‘ b -’ refers only with respect to v), and $a_{1,i}$ and $a_{2,i}$ satisfy the lacunary condition for τ in the strip $|\operatorname{Im} \tau| \leq \theta + \varepsilon$.

This definition of the calculus with bounds can be found in [16, p. 88] or [17, p. 1262] from the ‘blown-up picture’. When θ is not important (as it will not be for this paper), we shall use the space

$$\tilde{\Psi}_b^m(M) = \bigcup_{\theta > 0} \Psi_b^{m, \theta}(M).$$

These spaces form the *calculus with bounds* and they too form an algebra in the sense that $\Psi_b^{m, \theta}(M) \circ \Psi_b^{m', \theta'}(M) \subset \Psi_b^{m+m', \theta'}(M)$, where $\theta' = \min\{\theta, \theta'\}$. They also define continuous maps between b -Sobolev spaces.

2.2. The normal operator and Fredholm properties. The normal operator governs the Fredholm b -pseudodifferential operators. Let Y be a codimension ℓ boundary face of M . Assume that Y is a component of $H_{i_1} \cap \cdots \cap H_{i_\ell}$, where $i_1 < \cdots < i_\ell$, so that $x_{i_1}, \dots, x_{i_\ell}$ are defining functions for M . Then near Y (cf. (2.1)) we have

$$(2.10) \quad M \cong [0, 1]_v^\ell \times Y, \quad v = (x_{i_1}, \dots, x_{i_\ell}).$$

Given $A \in \Psi_b^m(M)$, the normal operator of A at Y is defined as follows. Given a function $\psi \in C^\infty(Y)$, let $\phi \in C^\infty(M)$ be any smooth function such that $\phi|_Y = \psi$. The properties of the small calculus imply that given any fixed $\tau \in \mathbb{C}^\ell$, the function $v^{-i\tau} A(v^{i\tau} \phi)$ defines a smooth function on M ; in particular, this function is continuous at Y . Restricting this function to Y defines the *normal operator* of A at Y :

$$N_Y(A)(\tau)\psi := \left(v^{-i\tau} A(v^{i\tau} \phi) \right) \Big|_Y.$$

This operator does not depend on the choice of extension ϕ . Moreover, it readily follows that

$$(2.11) \quad N_Y(A^*)(\tau) = N_Y(A)(\bar{\tau})^*, \quad N_Y(AB)(\tau) = N_Y(A)(\tau)N_Y(B)(\tau)$$

for any $B \in \Psi_b^{m'}(M)$ and all $\tau \in \mathbb{C}^\ell$. For instance, to prove the composition property, let ϕ be an extension of ψ as before, and note that

$$\begin{aligned} N_Y(AB)(\tau)\psi &= \left(v^{-i\tau} ABv^{i\tau} \phi \right) \Big|_Y = \left(v^{-i\tau} Av^{i\tau} \left(v^{-i\tau} Bv^{i\tau} \phi \right) \right) \Big|_Y \\ &= N_Y(A)(\tau) \left(\left(v^{-i\tau} Bv^{i\tau} \phi \right) \Big|_Y \right) \\ &= N_Y(A)(\tau) (N_Y(B)(\tau)\phi). \end{aligned}$$

The following theorem characterizes Fredholm b -operators in terms of normal operators. The proof can be found in the appendix of [19].

Theorem 2.1. *If $A \in \Psi_b^m(M)$, $m \in \mathbb{R}^+$, then given any $s \in \mathbb{R}$ or $s = \infty$, the following are equivalent:*

- (1) $A : H_b^s(M) \rightarrow H_b^{s-m}(M)$ is Fredholm.
- (2) A is b -elliptic and $N_H(A)(\tau) : H_b^s(H) \rightarrow H_b^{s-m}(H)$ is invertible for all $\tau \in \mathbb{R}$ for each boundary hypersurface H .
- (3) A is b -elliptic and $N_Y(A)(\tau) : H_b^s(Y) \rightarrow H_b^{s-m}(Y)$ is invertible for all $\tau \in \mathbb{R}^\ell$ for each codimension ℓ face Y of M , for all $\ell \geq 1$.

If the statements (1), (2), or (3) hold for one s , then they automatically hold for any other s .

Because of this theorem, when we say $A \in \Psi_b^m(M)$ is Fredholm we can just fix $s = m$ so we can focus strictly on $A : H_b^m(M) \rightarrow L_b^2(M)$.

Finally, we remark that elements of $\tilde{\Psi}_b^{-\infty}(M)$ are *not* compact. The following theorem characterizes compact b -operators in terms of normal operators; see [19, Appendix] for the proof.

Theorem 2.2. *For $A \in \tilde{\Psi}_b^{-\infty}(M)$, the following are equivalent:*

- (1) $A : L_b^2(M) \rightarrow L_b^2(M)$ is compact.
- (2) $N_H(A)(\tau) \equiv 0$ for each boundary hypersurface H .
- (3) $N_Y(A)(\tau) \equiv 0$ for each codimension ℓ face Y of M , for all $\ell \geq 1$.

We define $\tilde{\Psi}^{-\infty}(M)$ — the (weak) *Green operators* (here we follow Schulze's terminology as explained in [15, Def. 3.6]) — as all elements of $\tilde{\Psi}_b^{-\infty}(M)$ that are compact. These operators can be described explicitly as follows. Let $\rho = x_1 x_2 \cdots x_r$ be the product of *all* the boundary defining functions of M . Then $\tilde{\Psi}^{-\infty}(M)$ consists of operators having Schwartz kernels of the form, for some $\theta > 0$:

$$(2.12) \quad K \in \tilde{\Psi}^{-\infty}(M) \iff K = \rho(p)^\theta \rho(q)^\theta R(p, q) dg(q)$$

where for any b -differential operators P_p and Q_q acting on the variables p and q , respectively, the function $P_p Q_q R(p, q)$ is bounded. Then K maps $H_b^s(M)$ to $\rho^\theta H_b^\infty(M)$ and defines a compact operator on $H_b^s(M)$ for any s . The b subscript in $\tilde{\Psi}_b^m(M)$ has to do with special pseudodifferential structure at the boundary of M , but operators in $\tilde{\Psi}^{-\infty}(M)$ vanish at the boundary, which accounts for the ‘missing’ b subscript for these compact operators.

The next theorem describes the generalized inverse of Fredholm operators.

Theorem 2.3. *If $A \in \Psi_b^m(M)$, $m \in \mathbb{R}^+$, is Fredholm, then its generalized inverse, $G : L_b^2(M) \rightarrow H_b^m(M)$, is in the full calculus: $G \in \tilde{\Psi}_b^{-m}(M)$. Here, the generalized inverse is defined by the equations*

$$A G = \text{Id} - \Pi_1, \quad G A = \text{Id} - \Pi_0,$$

where $\Pi_0, \Pi_1 \in \tilde{\Psi}^{-\infty}(M)$ are the orthogonal projections onto the null space of A in $H_b^m(M)$ and off the range of A in $L_b^2(M)$, respectively. Moreover, $\ker A \subset \rho^\theta H_b^\infty(M)$, and $\text{coker } A \cong \ker A^* \subset \rho^\theta H_b^\infty(M)$ for some $\theta > 0$.

In the following theorem, we give another characterization of Fredholmness which is quite useful in practice rather than in theory, and it will be exploited in a moment to prove the subsequent Theorem 2.5 for Dirac operators.

Theorem 2.4. *If $A \in \Psi_b^m(M)$, $m \in \mathbb{R}^+$, then given any $s \in \mathbb{R}$ or $s = \infty$, the following are equivalent:*

- (1) $A : H_b^s(M) \rightarrow H_b^{s-m}(M)$ is a Fredholm.
- (2) A is b -elliptic and $N_Y(A)(\tau) : H_b^s(Y) \rightarrow H_b^{s-m}(Y)$ has kernel (null space) 0 for all $\tau \in \mathbb{R}^\ell$ for each codimension ℓ face Y of M , for all $\ell \geq 1$.

If the statements (1) and (2) hold for one s , then they automatically hold for any other s .

Proof. By Theorem 2.1, it suffices to show that if A is b -elliptic and $N_Y(A)(\tau) : H_b^s(Y) \rightarrow H_b^{s-m}(Y)$ has kernel 0 for all $\tau \in \mathbb{R}^\ell$ for each codimension ℓ face Y of M , for all $\ell \geq 1$, then $N_Y(A)(\tau)$ must be invertible. We first note that being b -elliptic, A has a ‘small’ parametrix $B \in \Psi_b^{-m}(M)$ such that $AB = \text{Id} - R_1, BA = \text{Id} - R_2$ with $R_j \in \Psi_b^{-\infty}(M)$. Let Y be any boundary face of M . Then taking normal operators, we obtain

$$(2.13) \quad \begin{aligned} N_Y(A)(\tau) N_Y(B)(\tau) &= \text{Id} - N_Y(R_1)(\tau) \\ N_Y(B)(\tau) N_Y(A)(\tau) &= \text{Id} - N_Y(R_2)(\tau). \end{aligned}$$

Since the R_j ’s are of order $-\infty$, it follows that $N_Y(A)(\tau)$ is b -elliptic for all $\tau \in \mathbb{R}^\ell$ (with ℓ the codimension of Y in M), and as $|\tau| \rightarrow \infty$, $N_Y(R_j)(\tau) \rightarrow 0$ uniformly in the topology of $\Psi_b^{-\infty}(Y)$, and hence we can invert each operator on the right in (2.13) on L_b^2 for $|\tau|$ sufficiently large. In conclusion, $N_Y(A)(\tau)$ is a b -elliptic family that is invertible for $|\tau|$ sufficiently large.

We are now ready to prove our theorem by induction. Let ℓ_0 be the codimension of M . Consider the case when Y is a codimension ℓ_0 face of M , in which case Y is a compact boundaryless manifold. Then as we showed above, $N_Y(A)(\tau)$ is a family of elliptic (and hence Fredholm) operators on a closed manifold that is invertible for $|\tau|$ sufficiently large. Thus, $N_Y(A)(\tau)$ always has index zero. By assumption, $N_Y(A)(\tau)$ has kernel 0, so $N_Y(A)(\tau)$ must in fact be invertible. This proves that the claim is true for ℓ_0 . Let $\ell_0 \geq \ell_k + 1 > 1$ and assume that all the normal operators of A are invertible at all faces of M with codimension $\geq \ell_k + 1$. Let Y be a face of M with codimension ℓ_k . Then Y is a manifold with corners whose boundary hypersurfaces are codimension $\ell_k + 1$ faces of M . All its normal operators are invertible by induction hypothesis. Now as we showed above, $N_Y(A)(\tau)$ is a family of b -elliptic operators on Y that is invertible for $|\tau|$ sufficiently large, therefore by invertibility of its normal operators, $N_Y(A)(\tau)$ is a family of Fredholm operators with index zero. By assumption, this family has kernels 0, so $N_Y(A)(\tau)$ must in fact be invertible for all τ . This completes our proof. \square

Before speaking about Dirac type operators, we note that everything we have said for operators on functions works equally well for operators acting between sections of vector bundles with the obvious modifications.

2.3. Dirac type operators. We now define operators ‘of Dirac type’. An operator \mathcal{D} is called a (b -)Dirac type operator if $\mathcal{D} \in \text{Diff}_b^1(M, E, F)$, is b -elliptic with ${}^b\sigma_1(\mathcal{D})^2 =$ the metric g , and near each codimension ℓ face Y of M , in a decomposition (2.10), $M \cong [0, 1]_v^\ell \times Y$ near Y , we can write

$$(2.14) \quad \mathcal{D} = G_1 v_1 \partial_{v_1} + \cdots + G_\ell v_\ell \partial_{v_\ell} + B_Y + \mathcal{O}(v),$$

where $B_Y \in \text{Diff}_b^1(Y, E_Y, F_Y)$ with $E_Y := E|_Y$ and $F_Y := F|_Y$, $\mathcal{O}(v)$ is a first order b -differential operator that vanishes at Y , and where the maps $G_j : E_Y \rightarrow F_Y$ are unitary maps satisfying the relations

$$(2.15) \quad G_j^* G_k = -G_k^* G_j \quad (j \neq k) \quad , \quad G_j^* B_Y = B_Y^* G_j.$$

By definition of normal operator, we have

$$(2.16) \quad N_Y(\mathcal{D})(\tau) = G_1 i\tau_1 + \cdots + G_\ell i\tau_\ell + B_Y,$$

where the G_j ’s and B_Y satisfy (2.15); in fact, one can easily show that (2.16) implies (2.14) so we can define ‘of Dirac type’ using the normal operator expression

(2.16) instead of the expression (2.14). The operator B_Y is called the *induced* Dirac operator on Y . The word ‘induced’ follows from the fact that if we set $v_1 = \dots = v_\ell = 0$ in (2.16) we get exactly B_Y .

Theorem 2.5. *A Dirac type operator $\mathcal{D} \in \text{Diff}_b^1(M, E, F)$ is Fredholm if and only if for each boundary face Y of M , the induced Dirac type operator B_Y on Y has kernel 0 on $L_b^2(Y, E_Y)$.*

Proof. Let Y be a codimension ℓ face of M and fix any $0 \leq j \leq \ell$. We prove that $N_Y(\mathcal{D})(\tau)$ is invertible for all $\tau \in \mathbb{R}^\ell$ with $\tau_j \neq 0$. Indeed, observe that

$$N_Y(\mathcal{D})(\tau) = iG_1 \tau_1 + \dots + iG_\ell \tau_\ell + B_Y = G_j[i\tau_j + A_j(\tau)],$$

where

$$A_j(\tau) = \sum_{k \neq j} G_j^* G_k i\tau_k + G_j^* B_Y$$

Since $G_j^* G_k = -G_k^* G_j$ and $G_j^* B_Y = B_Y^* G_j$, the operator $A_j(\tau)$ is formally self-adjoint for all $\tau \in \mathbb{R}^\ell$. It follows that $N_Y(\mathcal{D})(\tau)$ is invertible for all $\tau \in \mathbb{R}^\ell$ with $\tau \neq 0$. It has kernel 0 at $\tau = 0$ if and only if B_Y has kernel 0. The Fredholm property of \mathcal{D} now follows Theorem 2.4. \square

Much of what we have studied on M works for our manifold with corners X with an admissible hypersurface H_0 . Take *any* manifold with corners M with an exact b -metric (pushing *all* its boundary hypersurfaces to ∞) that contains X as a smooth submanifold; e.g. take the manifold $M = \tilde{X}$ shown in Figure 3 of Section 4, which is obtained from X by essentially doubling it across H_0 . Then M is a manifold with corners of the type we’ve been studying, so $H_b^s(M)$, $\text{Diff}_b^m(M)$, and $\Psi_b^s(M)$ are defined. Now simply define $H_b^s(X)$, $\text{Diff}_b^m(X)$, and $\Psi_b^s(X)$ to be the restrictions to X of $H_b^s(M)$, $\text{Diff}_b^m(M)$, and $\Psi_b^s(M)$, respectively. These definitions are determined independently of the choice of extension M . We now describe Dirac type operators on X .

Let E, F be Hermitian vector bundles over X and let $\mathcal{D} : C^\infty(X, E) \rightarrow C^\infty(X, F)$ be a Dirac type operator; this means that \mathcal{D} is the restriction to X of a Dirac type operator on M . We assume that on the collar $[-1, 0]_u \times H_0$ of the admissible boundary hypersurface H_0 with $H_0 = \{u = 0\}$, the vector bundles E and F are isometric to $E_0 := E|_{H_0}$ and $F_0 := F|_{H_0}$, and

$$(2.17) \quad \mathcal{D} = G_u(\partial_u + D_u),$$

where $G_u : E_0 \rightarrow F_0$ is unitary and $D_u \in \text{Diff}_b^1(H_0, E_0)$ is a Dirac type operator on H_0 , and where both G_u and D_u are smooth up to $u = 0$ and restricting there to define a unitary map $G_0 : E_0 \rightarrow F_0$ and a formally *self-adjoint* Dirac type operator $D_0 \in \text{Diff}_b^1(H_0, E_0)$. Note that H_0 is an example of an ‘ M ’ as we have been considering in this section. We end this section with Green’s formula. Let $\mathcal{D}^* : C^\infty(X, F) \rightarrow C^\infty(X, E)$ denote the formal adjoint of \mathcal{D} . Then given $e \in H_b^1(X, E)$ and $f \in H_b^1(X, F)$, we have

$$(2.18) \quad \langle \mathcal{D}e, f \rangle_X - \langle e, \mathcal{D}^*f \rangle_X = \langle G_0 e_0, f_0 \rangle_0,$$

where $e_0 := e|_{H_0}$, $f_0 := f|_{H_0}$, and $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_0$ denote L_b^2 -inner products over X and H_0 , respectively. The proof of this formula is identical with the proof in the compact case with no changes.

3. AN ILLUMINATING EXAMPLE

In this section we compute the orthogonalized Calderón projector for a ‘model’ Dirac operator. This example illustrates the basic characteristics of Calderón projectors in the general case. Let $R > 0$ and set $X_R = [0, R]_u \times \overline{\mathbb{R}}_+^\ell$ where $\overline{\mathbb{R}}_+ = [0, \infty)$. In this case,

$$Y_R = \partial X_R = \left(\{0\} \times \overline{\mathbb{R}}_+^\ell \right) \sqcup \left(\{R\} \times \overline{\mathbb{R}}_+^\ell \right) \equiv \overline{\mathbb{R}}_+^\ell \sqcup \overline{\mathbb{R}}_+^\ell.$$

Consider the (b -) Dirac operator

$$\mathcal{D} = G_0 \partial_u + G_1 v_1 \partial_{v_1} + \cdots + G_\ell v_\ell \partial_{v_\ell} \in \text{Diff}_b^1(X_R, E),$$

where E is a Hermitian Clifford module (that is, we consider the trivial bundle $X_R \times E$ over X_R) and the G_j ’s are unitary matrices on E satisfying

$$(3.1) \quad G_j^2 = -\text{Id}, \quad G_j^* = -G_j, \quad G_j G_k = -G_k G_j \quad (j \neq k).$$

We can write our Dirac operator as

$$\mathcal{D} = G_0(\partial_u + B),$$

where

$$B = A_1 v_1 D_{v_1} + \cdots + A_\ell v_\ell D_{v_\ell} \in \text{Diff}_b^1(\overline{\mathbb{R}}_+^\ell, E),$$

with $A_j = iG_j G_0$ and $D_{v_j} = i^{-1} \partial_{v_j}$. The properties (3.1) of the G_j ’s imply that

$$(3.2) \quad A_j^2 = \text{Id}, \quad A_j^* = A_j, \quad A_j A_k = -A_k A_j, \quad \text{where } A_j = iG_j G_0.$$

These properties imply, in particular, that B is formally self-adjoint.

To find the orthogonalized Calderón projector we first determine the Cauchy data space of \mathcal{D} . To this end, let $\phi(u, v) \in H_b^\infty(X_R, E)$. We first write ϕ in terms of the Mellin transform

$$\phi(u, v) = \int_{\mathbb{R}^\ell} v^{i\tau} \hat{\phi}(u, \tau) d\tau,$$

where the ‘ $\hat{}$ ’ means Mellin transform with respect to v :

$$\hat{\phi}(u, \tau) = \int_{\overline{\mathbb{R}}_+^\ell} v^{-i\tau} \phi(u, v) \frac{dv}{v}.$$

Second, we note that

$$\mathcal{D}\phi = G_0(\partial_u + B) \int v^{i\tau} \hat{\phi}(u, \tau) d\tau = G_0 \int v^{i\tau} (\partial_u + A(\tau)) \hat{\phi}(u, \tau) d\tau,$$

where recalling that $B = A_1 v_1 D_{v_1} + \cdots + A_\ell v_\ell D_{v_\ell}$, we have

$$A(\tau) := {}^b\sigma_1(B)(\tau) = A_1 \tau_1 + \cdots + A_\ell \tau_\ell.$$

Thus, $\mathcal{D}\phi = 0$ if and only if $(\partial_u + A(\tau)) \hat{\phi}(u, \tau) = 0$, if and only if

$$\hat{\phi}(u, \tau) = e^{-uA(\tau)} \hat{\phi}(0, \tau).$$

With this formula in mind, for $\varphi \in H_b^\infty(\overline{\mathbb{R}}_+^\ell, E)$, we define the operator $e^{-uB}\varphi$ via the Mellin transform:

$$e^{-uB}\varphi := \int v^{i\tau} e^{-uA(\tau)} \hat{\varphi}(\tau) d\tau,$$

provided that the right-hand side exists. Then $\mathcal{D}\phi = 0$ if and only if

$$\phi(u, v) = e^{-uB}\varphi, \quad \varphi(v) = \phi(0, v).$$

We can now find the Cauchy data space. Recall that $X_R = [0, R] \times \overline{\mathbb{R}}_+^\ell$ and $Y_R = \partial X_R = \overline{\mathbb{R}}_+^\ell \sqcup \overline{\mathbb{R}}_+^\ell$. Then $H_b^\infty(Y_R, E) \equiv H_b^\infty(\overline{\mathbb{R}}_+^\ell, E) \oplus H_b^\infty(\overline{\mathbb{R}}_+^\ell, E)$ and with respect to this decomposition, we have

$$\phi|_{Y_R} = \begin{pmatrix} \phi|_{u=0} \\ \phi|_{u=R} \end{pmatrix} = \begin{pmatrix} \varphi \\ e^{-RB}\varphi \end{pmatrix}.$$

In conclusion, we have proved that if $\phi \in H_b^\infty(X_R, E)$ and $\mathcal{D}\phi = 0$, then $\phi|_{Y_R}$ is a column vector $(\varphi, e^{-RB}\varphi)$ with $\varphi, e^{-RB}\varphi \in H_b^\infty(\overline{\mathbb{R}}_+^\ell, E)$. The converse is straightforward (see the structure of $A(\tau)$ in (3.3) below.) Thus, we have proved

Theorem 3.1. *The Cauchy data space*

$$\mathcal{H}(\mathcal{D}) := \{ \phi|_{Y_R} \mid \phi \in H_b^\infty(X_R, E), \mathcal{D}\phi = 0 \} \subset H_b^\infty(\overline{\mathbb{R}}_+^\ell, E) \oplus H_b^\infty(\overline{\mathbb{R}}_+^\ell, E)$$

is given explicitly by

$$\mathcal{H}(\mathcal{D}) = \left\{ \begin{pmatrix} \text{Id} \\ e^{-RB} \end{pmatrix} \varphi \mid \varphi \in H_b^\infty(\overline{\mathbb{R}}_+^\ell, E), e^{-RB}\varphi \in H_b^\infty(\overline{\mathbb{R}}_+^\ell, E) \right\}.$$

In order to determine the properties of the orthogonalized Calderón projector, we first note that

$$(3.3) \quad A(\tau)^* = A(\tau) \quad , \quad A(\tau)^2 = |\tau|^2 \quad , \quad A(\tau)G_0 = -G_0A(\tau).$$

These properties follow directly from (3.2) or from the fact that $A(\tau) = {}^b\sigma_1(B)(\tau)$. It follows that $A(\tau)$ has eigenvalues $\pm|\tau|$ with the corresponding eigenspaces of the same dimension which G_0 intertwines. In particular, for any $u \in \mathbb{R}$,

$$(3.4) \quad e^{-uA(\tau)} \text{ acts as } e^{\mp u|\tau|} \text{ where } A(\tau) = \pm|\tau|.$$

From this fact, it is easy to show that for $\varphi \in H_b^\infty(\overline{\mathbb{R}}_+^\ell, E)$, we have $e^{-RB}\varphi \in H_b^\infty(\overline{\mathbb{R}}_+^\ell, E)$ if and only if for any k ,

$$(3.5) \quad \int e^{2R|\tau|} |\tau|^k |\hat{\varphi}(\tau)|^2 d\tau < \infty \quad \text{for all } k.$$

Now let us compute the orthogonalized Calderón projector:

Theorem 3.2. *The orthogonal projector onto the Cauchy data space $\mathcal{H}(\mathcal{D})$ is the pseudodifferential operator $\mathcal{C} \in \tilde{\Psi}_b^0(Y, E)$ given by (cf. (2.6))*

$$\mathcal{C} = \frac{1}{\text{Id} + e^{-2RB}} \begin{pmatrix} \text{Id} & e^{-RB} \\ e^{-RB} & e^{-2RB} \end{pmatrix} := \int_{\mathbb{R}^\ell} \left(\frac{v}{v'} \right)^{i\tau} c(\tau) d\tau \cdot \frac{dv'}{v'},$$

where

$$c(\tau) = \frac{1}{\text{Id} + e^{-2RA(\tau)}} \begin{pmatrix} \text{Id} & e^{-RA(\tau)} \\ e^{-RA(\tau)} & e^{-2RA(\tau)} \end{pmatrix}.$$

Moreover, the b -principal symbol of \mathcal{C} is given by

$${}^b\sigma_0(\mathcal{C})(\tau) = \begin{pmatrix} \Pi_+(A(\tau)) & 0 \\ 0 & \Pi_-(A(\tau)) \end{pmatrix},$$

where $\Pi_\pm(A(\tau))$ are the orthogonal projectors onto the $\pm|\tau|$ eigenspaces of the symbol $A(\tau) = {}^b\sigma_1(B)(\tau)$ of the tangential operator.

Proof. We first show that \mathcal{C} is a b -pseudodifferential operator and then we prove that \mathcal{C} does indeed project onto $\mathcal{H}(\mathcal{D})$. First, we note that $c(\tau)$ is smooth in all variables and we can write

$$c(\tau) = \begin{pmatrix} \frac{1}{\text{Id} + e^{-2RA(\tau)}} & \frac{1}{e^{RA(\tau)} + e^{-RA(\tau)}} \\ \frac{1}{e^{RA(\tau)} + e^{-RA(\tau)}} & \frac{1}{e^{2RA(\tau)} + \text{Id}} \end{pmatrix}.$$

From this expression, a straightforward verification from (3.4) shows that $c(\tau)$ is a classical symbol of order 0. Another straightforward verification shows that $c(\tau)$ satisfies the lacunary condition on a strip $|\text{Im } \tau| \leq \delta$ for some $\delta > 0$ (cf. the discussion after the proof). Second, we see that over the space where $A(\tau) = \pm|\tau|$, we have

$$\begin{aligned} c(\tau) &= \begin{pmatrix} \frac{1}{\text{Id} + e^{\mp 2R|\tau|}} & \frac{1}{e^{\pm R|\tau|} + e^{\mp R|\tau|}} \\ \frac{1}{e^{\pm R|\tau|} + e^{\mp R|\tau|}} & \frac{1}{e^{\pm 2R|\tau|} + \text{Id}} \end{pmatrix} \\ &= \begin{pmatrix} \Pi_+(A(\tau)) & 0 \\ 0 & \Pi_-(A(\tau)) \end{pmatrix} + \text{exponentially decreasing as } |\tau| \rightarrow \infty. \end{aligned}$$

This shows that

$${}^b\sigma_0(\mathcal{C})(\tau) = \begin{pmatrix} \Pi_+(A(\tau)) & 0 \\ 0 & \Pi_-(A(\tau)) \end{pmatrix}.$$

Third, by direct computation, we see that

$$\begin{aligned} c(\tau)^2 &= \frac{1}{(\text{Id} + e^{-2RA(\tau)})^2} \begin{pmatrix} \text{Id} & e^{-RA(\tau)} \\ e^{-RA(\tau)} & e^{-2RA(\tau)} \end{pmatrix} \begin{pmatrix} \text{Id} & e^{-RA(\tau)} \\ e^{-RA(\tau)} & e^{-2RA(\tau)} \end{pmatrix} \\ &= \frac{1}{\text{Id} + e^{-2RA(\tau)}} \begin{pmatrix} \text{Id} & e^{-RA(\tau)} \\ e^{-RA(\tau)} & e^{-2RA(\tau)} \end{pmatrix} = c(\tau). \end{aligned}$$

Since $c(\tau)$ is certainly self-adjoint, we conclude that

$$\mathcal{C}^2 = \mathcal{C} \quad \text{and} \quad \mathcal{C}^* = \mathcal{C}.$$

Thus, we have proven that $\mathcal{C} \in \tilde{\Psi}_b^0(Y, E)$ and is an orthogonal projection. It remains to show that \mathcal{C} defines the projection onto $\mathcal{H}(\mathcal{D})$. To this end, let $(\varphi, \psi) \in H_b^\infty(Y, E)$. Then by definition,

$$\begin{aligned} \mathcal{C}(\varphi, \psi) &= \int v^{i\tau} \frac{1}{\text{Id} + e^{-2RA(\tau)}} \begin{pmatrix} \text{Id} & e^{-RA(\tau)} \\ e^{-RA(\tau)} & e^{-2RA(\tau)} \end{pmatrix} \begin{pmatrix} \hat{\varphi}(\tau) \\ \hat{\psi}(\tau) \end{pmatrix} d\tau \\ &= \int e^{iv\tau} \begin{pmatrix} \text{Id} \\ e^{-RA(\tau)} \end{pmatrix} \hat{\omega}(\tau) d\tau \end{aligned}$$

where

$$\omega(v) = \int v^{i\tau} \frac{\hat{\varphi}(\tau) + e^{-RA(\tau)} \hat{\psi}(\tau)}{\text{Id} + e^{-2RA(\tau)}} d\tau.$$

Using (3.4), one sees that (3.5) is satisfied, so by Theorem 3.1, \mathcal{C} has image in the Cauchy data space $\mathcal{H}(\mathcal{D})$. Also, this computation shows that $\mathcal{C} = \text{Id}$ on $\mathcal{H}(\mathcal{D})$, so the image of \mathcal{C} is exactly $\mathcal{H}(\mathcal{D})$ and our proof is complete. \square

We remark that in general, \mathcal{C} is *not* in the small b -calculus. This is because the operator $(\text{Id} + e^{-2RA(\tau)})^{-1}$ will always have poles so the symbol $c(\tau)$ of \mathcal{C} is not entire. For example, let us consider the case $\ell = 1$ when $X_R = [0, R] \times \overline{\mathbb{R}}_+$. In

this case, we have $A(\tau) = \tau A_1$ where $A_1 = iG_1G_0$ satisfies $A_1^* = A_1$ and $A_1^2 = \text{Id}$. Thus, with respect to the decomposition into the ± 1 eigenspaces of A_1 , we have

$$\frac{1}{\text{Id} + e^{-2RA(\tau)}} = \frac{1}{\text{Id} + e^{\mp 2R\tau}} \quad \text{on the } \pm 1 \text{ eigenspaces of } A_1.$$

In particular, the function $\text{Id} + e^{\mp 2R\tau}$ vanishes when $\tau = \pm i\pi/(2R)$. Further, this equation shows that $c(\tau)$ satisfies the lacunary condition only on the strip $|\text{Im } \tau| < \pi/(2R)$. Thus, when $\ell = 1$, we can say precisely that $\mathcal{C} \in \Psi_b^{0,\theta}(Y, E)$ for any fixed $0 < \theta < \pi/(2R)$.

We end this section with the APS spectral projector. Back to the general case $X_R = [0, R] \times \mathbb{R}_+^\ell$. Since $\mathcal{D} = G_0(\partial_u + B)$, the induced Dirac operator on $Y = \partial X_R = \mathbb{R}_+^\ell \sqcup \mathbb{R}_+^\ell$ is

$$D_0 = \begin{pmatrix} -B & 0 \\ 0 & B \end{pmatrix}$$

Recalling that $B = A_1 v_1 D_{v_1} + \cdots + A_\ell v_\ell D_{v_\ell}$ we see that the nonpositive spectral projector of D_0 is the operator

$$\Pi_{APS} := \int_{\mathbb{R}^\ell} \left(\frac{v}{v'} \right)^{i\tau} \begin{pmatrix} \Pi_+(A(\tau)) & 0 \\ 0 & \Pi_-(A(\tau)) \end{pmatrix} d\tau \frac{dv'}{v'}.$$

Note that the *complete* symbol of Π_{APS} , which is just the b -principal symbol ${}^b\sigma_0(\mathcal{C})$ of the orthogonalized Calderón projector \mathcal{C} , is *not smooth* at $\tau = 0$, so Π_{APS} is *not* a pseudodifferential operator.

4. THE INVERTIBLE EXTENSION

In this section, we construct an extension of \mathcal{D} , which is an *invertible* Fredholm Dirac operator. This extension will be used to construct the Calderón projector and derive its properties.

Theorem 4.1. *There exists a smooth manifold \tilde{X} with corners that contains X as a submanifold and an invertible Fredholm Dirac type operator $\tilde{\mathcal{D}} \in \text{Diff}_b^1(\tilde{X}, \tilde{E}, \tilde{F})$ such that $\tilde{\mathcal{D}}|_X = \mathcal{D}$.*

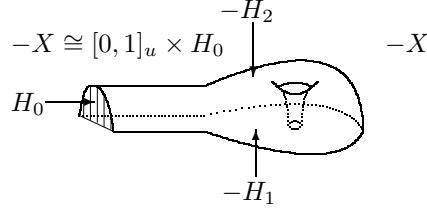
This theorem generalizes Wojciechowski's [32] (cf. Chapter 9 of [5]) result for smooth manifolds with boundary. Note that this theorem holds regardless of the dimension of X and with no invertibility assumptions on \mathcal{D} . To prove this theorem, we first reduce our theorem to the formally self-adjoint case. To do so, consider the vector bundle $E' := E \oplus F$ over X and the Dirac type operator

$$\mathcal{D}' := \begin{pmatrix} 0 & \mathcal{D}^* \\ \mathcal{D} & 0 \end{pmatrix} : C^\infty(X, E') \rightarrow C^\infty(X, E').$$

This operator is by construction formally self-adjoint and it still satisfies the collar decomposition property (1.6). If we prove our theorem for this operator, then we automatically get an invertible extension for \mathcal{D} . Thus, for the rest of this section, we assume that $\mathcal{D} : C^\infty(X, E) \rightarrow C^\infty(X, E)$ is formally self-adjoint.

Next, following Booß-Bavnbek and Wojciechowski, Chapter 9 of [5], we reduce our problem to the product case near H_0 . To this end, by (1.6) we know that over the collar neighborhood $[-1, 0] \times H_0$ of H_0 , we can write

$$\mathcal{D} = G_u(\partial_u + D_u),$$

FIGURE 2. $-X$ is the ‘mirror image’ of the original manifold X .

where G_u is a family of unitary maps on $E_0 := E|_{H_0}$, and D_u is a family of operators in $\text{Diff}_b^1(H_0, E_0)$ depending smoothly on u . Attaching the collar $[0, 2]_u \times H_0$ to H_0 we can form the manifold

$$X \sqcup ([0, 2]_u \times H_0),$$

then it is straightforward to extend \mathcal{D} to this manifold in such a way that \mathcal{D} is *independent* of u on $[1, 2]_u \times H_0$:

$$(4.1) \quad \mathcal{D} = G_0(\partial_u + D_0)$$

where G_0 is a unitary map on E_0 and $D_0 \in \text{Diff}_b^1(H_0, E_0)$ is formally self-adjoint. By proving Theorem 4.1 for this extended operator, we may assume that \mathcal{D} takes the product form (4.1) on the original neighborhood $N = [-1, 0] \times H_0$, which we now assume. Then using the fact that \mathcal{D} is of Dirac type and is formally self-adjoint, we observe that

$$(4.2) \quad G_0^2 = -\text{Id} \quad \text{and} \quad D_0 G_0 = -G_0 D_0.$$

With this product structure fixed, we proceed to construct the *invertible double* of the Dirac type operator \mathcal{D} .

4.1. The doubled manifold. We begin by defining the manifold $-X$ to be the same manifold X but on the collar $N = [-1, 0]_u \times Y$, we simply make the change of variables $u \mapsto -u$ so that near H_0 (see Figure 2)

$$-X \cong -N := [0, 1]_u \times H_0.$$

The minus sign just indicates that we always use the collar $-N$ near H_0 on $-X$ instead of N , which in the literature is sometimes stated as ‘ $-X$ is X with the reversed orientation’. Since H_0 is admissible, all the hypersurfaces intersect H_0 , therefore each of the hypersurfaces H_j with $j > 0$ are also reversed as seen in Figure 2. Of course, since $X \equiv -X$, the vector bundle E is still a vector bundle over $-X$ and \mathcal{D} defines an operator on $C^\infty(-X, E)$. The only difference is that on $-X$ we use the collar $-N$, and since \mathcal{D} is of the form (4.1) over N , under the change of variables $u \mapsto -u$, we have

$$(4.3) \quad \mathcal{D} = G_0(-\partial_u + D_0) \quad \text{over} \quad -N = [0, 1]_u \times H_0.$$

We now glue X to $-X$ via (see Figure 3)

$$(4.4) \quad \tilde{X} := X \sqcup_{H_0} (-X) \quad \text{and} \quad \tilde{E} := E \sqcup_{G_0} E,$$

which means that X and $-X$ are attached along H_0 and \tilde{E} is obtained by gluing E on X to E on $-X$ via the relation $e \sim G_0 e$ over H_0 . The C^∞ structure of \tilde{E} is

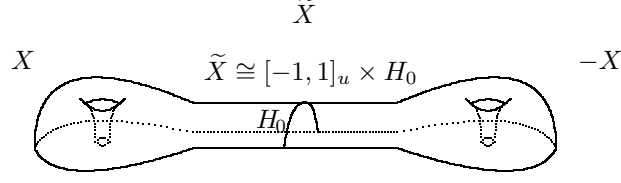


FIGURE 3. In this example, \tilde{X} is the double of the manifold on the left side in Figure 1. The hypersurfaces H_1 and H_2 automatically get doubled in the process.

defined as follows. Since $\tilde{E} \equiv E$ over X and $-X$ away from H_0 , we can focus near the gluing hypersurface H_0 . Since over N ,

$$(4.5) \quad E|_N \cong ([-1, 0] \times H_0, E_0),$$

over $-N = [0, 1] \times H_0$, we have

$$(4.6) \quad E|_{-N} \cong ([0, 1] \times H_0, E_0) \xrightarrow{G_0} ([0, 1] \times H_0, E_0),$$

where G_0 is an isomorphism on E_0 . Let $\tilde{N} := N \sqcup_{H_0} (-N)$. Then by definition of \tilde{E} , using (4.5) and (4.6) we get a bijection

$$\tilde{E}|_{\tilde{N}} := (E|_N) \sqcup_{G_0} (E|_{-N}) \longleftrightarrow ([-1, 1] \times H_0, E_0).$$

We define the C^∞ structure of \tilde{E} near H_0 by simply declaring this bijection to be a diffeomorphism. Now working out the definition of the C^∞ structure of \tilde{E} , we see that an element of $C^\infty(\tilde{X}, \tilde{E})$ can be identified as a pair

$$\phi = (\phi_1, \phi_2) \quad , \quad \text{where } \phi_1 \in C^\infty(X, E) \quad , \quad \phi_2 \in C^\infty(-X, E),$$

such that in the trivializations $E|_N \cong ([-1, 0]_u \times H_0, E_0)$ and $E|_{-N} \cong ([0, 1]_u \times H_0, E_0)$, the following section

$$(4.7) \quad \phi(u, y) := \begin{cases} \phi_1(u, y) & u < 0 \\ G_0 \phi_2(u, y) & u > 0 \end{cases}$$

extends to $u = 0$ to define a smooth section $\phi(u, y) \in C^\infty([-1, 1]_u \times H_0, E_0)$.

It is important to note that, by the admissibility of H_0 , if H_j with $j > 0$ is any boundary hypersurface, then the hypersurfaces H_j and $-H_j$ glue via (4.4) to form a hypersurface \tilde{H}_j in \tilde{X} such that (see Figure 3)

$$\tilde{H}_j = H_j \sqcup_{H_0 \cap H_j} (-H_j) \quad \text{and} \quad \tilde{E}|_{\tilde{H}_j} := E|_{H_j} \sqcup_{G_0} E|_{-H_j}.$$

Moreover, $\{\tilde{H}_j\}$ ($j > 0$) are *all* the boundary hypersurfaces of \tilde{X} and the boundary defining functions $\{x_j\}$ for $\{H_j\}$ induce corresponding boundary defining functions $\{\tilde{x}_j\}$ for the $\{\tilde{H}_j\}$, and the decomposition (1.1) continues to hold near \tilde{H}_i :

$$\tilde{X} \cong [0, 1]_{\tilde{x}_i} \times \tilde{H}_i \quad , \quad i = 1, 2, \dots, r \quad , \quad \tilde{H}_i = \{\tilde{x}_i = 0\}.$$

There are also decompositions of the sort (2.10): If \tilde{Y} is a component of $\tilde{H}_{i_1} \cap \dots \cap \tilde{H}_{i_\ell}$, where $1 \leq i_1 < \dots < i_\ell$, then near \tilde{Y} ,

$$(4.8) \quad \tilde{X} \cong [0, 1]_{\tilde{v}}^\ell \times \tilde{Y} \quad , \quad \tilde{v} = (\tilde{x}_{i_1}, \dots, \tilde{x}_{i_\ell}).$$

Note that \tilde{Y} is just the double of a component in $H_{i_1} \cap \cdots \cap H_{i_\ell}$. Finally, the metric g in (1.3) becomes an exact b -metric \tilde{g} on \tilde{X} that pushes *all* its faces $\{\tilde{H}_j\}$ to ∞ :

$$\tilde{g} = \sum_{i=1}^r \left(\frac{d\tilde{x}_i}{\tilde{x}_i} \right)^2 + \tilde{h},$$

where \tilde{h} is a smooth symmetric two-form on \tilde{X} .

4.2. The invertible double. We now define the doubled Dirac operator $\tilde{\mathcal{D}}$ acting on $C^\infty(\tilde{X}, \tilde{E})$ of the Dirac type operator \mathcal{D} . Given a section $\phi = (\phi_1, \phi_2) \in C^\infty(\tilde{X}, \tilde{E})$, we define

$$(4.9) \quad \tilde{\mathcal{D}}\phi := (\mathcal{D}\phi_1, -\mathcal{D}\phi_2).$$

We need to verify that $(\mathcal{D}\phi_1, -\mathcal{D}\phi_2) \in C^\infty(\tilde{X}, \tilde{E})$. To prove this, in view of (4.7), we need to show that $\mathcal{D}\phi_1$ and $-G_0\mathcal{D}\phi_2$ patch near H_0 to be smooth. To this end, we note that since $\mathcal{D} = G_0(\partial_u + D_0)$ over N , we have

$$\mathcal{D}\phi_1(u, y) = G_0(\partial_u + D_0)\phi(u, y).$$

From (4.2) we know that $G_0^2 = -\text{Id}$ and $D_0G_0 = -G_0D_0$. Thus, since $\mathcal{D} = G_0(-\partial_u + D_0)$ over $-N$, again recalling (4.7), we have

$$\begin{aligned} -G_0\mathcal{D}(\phi_2) &= -G_0 \cdot (G_0(-\partial_u + D_0))(-G_0\phi) = -(-\partial_u + D_0)G_0\phi \\ &= G_0(\partial_u + D_0)\phi. \end{aligned}$$

Comparing the formulas for $\mathcal{D}\phi_1(u, y)$ and $-G_0\mathcal{D}\phi_2(u, y)$, we see that

$$(4.10) \quad \tilde{\mathcal{D}}\phi = G_0(\partial_u + D_0)\phi(u, y) \quad \text{over} \quad \tilde{N} = [-1, 1]_u \times H_0.$$

Thus, $\tilde{\mathcal{D}} : C^\infty(\tilde{X}, \tilde{E}) \rightarrow C^\infty(\tilde{X}, \tilde{E})$ takes the product form (4.10) near H_0 . Since \mathcal{D} is a b -operator, the definition (4.9) shows that $\tilde{\mathcal{D}}$ is a b -differential operator away from H_0 and then the form (4.10) shows that $\tilde{\mathcal{D}}$ is a b -operator near H_0 . Hence, $\tilde{\mathcal{D}} \in \text{Diff}_b^1(\tilde{X}, \tilde{E})$. Note that \tilde{E} inherits an inner product from E , therefore $L_b^2(\tilde{X}, \tilde{E})$ has a naturally induced inner product. Since \mathcal{D} is by assumption formally self-adjoint, the definition (4.9) and the product form (4.10) shows that $\tilde{\mathcal{D}}$ is also formally self-adjoint.

4.3. Fredholmness of the double. We now show that $\tilde{\mathcal{D}}$ is of Dirac type and Fredholm. The definition (4.9) and the decomposition (4.10) show that ${}^b\sigma_1(\tilde{\mathcal{D}})^2 = \tilde{g}$, so it remains to verify the structure of $\tilde{\mathcal{D}}$ near the faces of \tilde{X} . Consider the decomposition of \tilde{X} near \tilde{Y} as in (4.8), and let $X \cong [0, 1]_v^\ell \times Y$ be the corresponding decomposition of X near Y . Then by (2.14), near Y we can write

$$\mathcal{D} = G_1 v_1 \partial_{v_1} + \cdots + G_\ell v_\ell \partial_{v_\ell} + B_Y + \mathcal{O}(v),$$

where $B_Y \in \text{Diff}_b^1(Y, E_Y)$ with $E_Y := E|_Y$, $\mathcal{O}(v)$ is a first order b -differential operator that vanishes at Y , and where the G_j 's are unitary maps on E_Y satisfying the relations

$$(4.11) \quad G_j^2 = -\text{Id} \quad , \quad G_j G_k = -G_k G_j \quad (j \neq k) \quad , \quad G_j B_Y = -B_Y G_j.$$

Note that these relations are slightly different from (2.15) because now \mathcal{D} is formally self-adjoint so we have more structure forced upon us. It follows that on the decomposition (4.8) near \tilde{Y} , $\tilde{X} \cong [0, 1]_{\tilde{v}}^\ell \times \tilde{Y}$, we have

$$\tilde{\mathcal{D}} = \tilde{G}_1 \tilde{v}_1 \partial_{\tilde{v}_1} + \cdots + \tilde{G}_\ell \tilde{v}_\ell \partial_{\tilde{v}_\ell} + \tilde{B}_{\tilde{Y}} + \mathcal{O}(\tilde{v}),$$

where the operators \tilde{G}_j and $\tilde{B}_{\tilde{Y}}$ are defined on sections on $\tilde{E}_{\tilde{Y}} := \tilde{E}|_{\tilde{Y}}$ over the double of Y by corresponding formulas to (4.9):

$$\tilde{G}_j \phi := (G_j \phi_1, -G_j \phi_2) \quad , \quad \tilde{B}_{\tilde{Y}} \phi := (B_Y \phi_1, -B_Y \phi_2).$$

We know that $\tilde{\mathcal{D}}$ maps smooth sections of \tilde{E} to itself, but it is not a priori automatic that \tilde{G}_j and $\tilde{B}_{\tilde{Y}}$ individually map smooth sections of $\tilde{E}_{\tilde{Y}}$ to itself. We prove this is so in the following lemma.

Lemma 4.2. *The operators \tilde{G}_j and $\tilde{B}_{\tilde{Y}}$ define operators on $C^\infty(\tilde{Y}, \tilde{E}_{\tilde{Y}})$ and they satisfy the properties (4.11) with ‘tildes’. In particular, $\tilde{\mathcal{D}}$ is a Dirac type operator over \tilde{X} .*

Proof. Taking the normal operator of $\tilde{\mathcal{D}}$, we see that

$$N_{\tilde{Y}}(\tilde{\mathcal{D}})(\tau) := (\tilde{v}^{-i\tau} \tilde{\mathcal{D}} \tilde{v}^{i\tau})|_{\tilde{Y}} = \tilde{G}_1 i\tau_1 + \cdots + \tilde{G}_\ell i\tau_\ell + \tilde{B}_{\tilde{Y}},$$

which by definition is a smooth family of b -operators on \tilde{Y} . In particular, setting $\tau = 0$ shows that $\tilde{B}_{\tilde{Y}}$ maps $C^\infty(\tilde{Y}, \tilde{E}_{\tilde{Y}})$ to itself. Then fixing j and setting $\tau_k = 0$ for $k \neq j$ implies that \tilde{G}_j maps $C^\infty(\tilde{Y}, \tilde{E}_{\tilde{Y}})$ to itself. The properties (4.11) are straightforward to verify; for instance,

$$\begin{aligned} \tilde{G}_k \tilde{B}_{\tilde{Y}} \phi &= \tilde{G}_k (B_Y \phi_1, -B_Y \phi_2) = (G_k B_Y \phi_1, G_k B_Y \phi_2) \\ &= -(B_Y G_k \phi_1, B_Y G_k \phi_2) \\ &= -\tilde{B}_{\tilde{Y}} (G_k \phi_1, -G_k \phi_2) = -\tilde{B}_{\tilde{Y}} \tilde{G}_k \phi. \end{aligned}$$

□

We now prove that $\tilde{\mathcal{D}}$ is Fredholm. By Theorem 2.5 we need to show that $\tilde{B}_{\tilde{Y}}$ has L_b^2 kernel 0. So assume that $\tilde{B}_{\tilde{Y}}(\phi_1, \phi_2) = (B_Y \phi_1, -B_Y \phi_2) = 0$. This implies that $B_Y \phi_1 = 0 = B_Y \phi_2$. Since $\mathcal{D} = G_0(\partial_u + D_0)$ near H_0 , we see that

$$B_Y = \mathcal{D}|_Y = G_Y(\partial_u + D_Y) \quad \text{over} \quad N \cap Y = [-1, 0]_u \times H_0 \cap Y,$$

where $G_Y = G_0|_Y$ and $D_Y = D_0|_Y$. Therefore we can apply Green’s formula (2.18) on Y to get

$$\begin{aligned} 0 &= 0 - 0 = \langle B_Y \phi_1, \phi_2 \rangle - \langle \phi_1, B_Y \phi_2 \rangle = \langle G_Y \phi_1|_Y, \phi_2|_Y \rangle_Y \\ &= -\langle \phi_2|_Y, \phi_2|_Y \rangle_Y = -\|\phi_2|_Y\|_Y^2, \end{aligned}$$

where we used that $\phi_1|_Y = G_Y \phi_2$. This implies that $\phi_2|_Y = 0$ and thus $\phi_1|_Y = G_Y \phi_2|_Y = 0$. Thus, with $B_Y \phi_1 = 0 = B_Y \phi_2$ and the unique continuation theorem, which also holds for our manifold [1, Cor. 1], we get $\phi_1 = 0 = \phi_2$. Thus, B_Y has L_b^2 kernel 0, so by Theorem 2.5, $\tilde{\mathcal{D}}$ is Fredholm. The same argument that we just did also shows that $\tilde{\mathcal{D}}$ has L_b^2 kernel 0. Thus, $\tilde{\mathcal{D}}$ is invertible and our proof of Theorem 4.1 is complete.

5. THE CALDERÓN PROJECTOR

In this section we prove Theorems 1.1, 1.2 and 1.3.

5.1. Proof of Theorem 1.1. First let us show that

$$\mathcal{D} : H_b^\infty(X, E) \rightarrow H_b^\infty(X, F)$$

is surjective. Let $\psi \in H_b^\infty(X, F)$ and choose any element $\tilde{\psi} \in H_b^\infty(\tilde{X}, \tilde{F})$ with $\tilde{\psi}|_X = \psi$ where \tilde{X} is the manifold defined in (4.4). Then set

$$\phi := (\tilde{\mathcal{D}}^{-1} \tilde{\psi})|_X,$$

where $\tilde{\mathcal{D}}$ is the invertible extension constructed in subsection 4.2. By definition, it is obvious that $\mathcal{D}\phi = \psi$. By the unique continuation principle [1, Cor. 1], the restriction of $\ker \mathcal{D}$ to the boundary gives a canonical isomorphism between $\ker \mathcal{D}$ and $\mathcal{H}(\mathcal{D})$. Now using the invertible extension $\tilde{\mathcal{D}}$, let us define

$$(5.1) \quad P := \lim_{\varepsilon \rightarrow 0^-} -\gamma_\varepsilon \tilde{\mathcal{D}}^{-1} \gamma_0^* G_0 : H_b^\infty(H_0, E_0) \rightarrow H_b^\infty(H_0, E_0),$$

where γ_ε is restriction to $\{\varepsilon\} \times H_0$ and γ_0^* is the adjoint map of γ_0 at $\{0\} \times H_0$. In the following lemma, we prove, in particular, that the limit in (5.1) exists.

Lemma 5.1. *$P \in \tilde{\Psi}_b^0(H_0, E_0)$ and is a projection (in general not orthogonal) whose image is $\mathcal{H}(\mathcal{D})$. Moreover, the b -principal symbol of P is the orthogonal projection onto the eigenspace of the negative eigenvalues of the b -principal symbol of D_0 .*

Proof. We prove this lemma in three steps.

Step I: We first prove that $P \in \tilde{\Psi}_b^0(H_0, E_0)$. To do so, we work on a neighborhood $[-1, 1]_u \times H_0$ near H_0 in \tilde{X} where, see (4.10),

$$\tilde{\mathcal{D}} = G_u(\partial_u + D_u),$$

where $G_u : E_0 \rightarrow F_0$ is unitary and $D_u \in \text{Diff}_b^1(H_0, E_0)$ is a Dirac type operator on H_0 , and where both G_u and D_u are smooth up to $u = 0$ and restricting there to define a unitary map $G_0 : E_0 \rightarrow F_0$ and a formally self-adjoint Dirac type operator $D_0 \in \text{Diff}_b^1(H_0, E_0)$. By Theorem 2.3, we know that $\tilde{\mathcal{D}}^{-1} \in \tilde{\Psi}_b^{-1}(\tilde{X}, \tilde{F}, \tilde{E})$, and the structure of the Schwartz kernel of $\tilde{\mathcal{D}}^{-1}$ is described in Section 2. To prove that $P \in \tilde{\Psi}_b^0(H_0, E_0)$, we shall analyze the limit defining P when $\tilde{\mathcal{D}}^{-1}$ is described in coordinate patches so that its Schwartz kernel takes the form (2.9), and when its Schwartz kernel takes the form (2.6).

Consider first the case (2.9). Let $\mathcal{U} = [0, 1]_v^\ell \times \mathcal{V}$ and $\mathcal{U}' = [0, 1]_v^\ell \times \mathcal{V}'$ be coordinate patches on H_0 , where $v = (v_1, \dots, v_\ell)$ denotes those boundary defining functions that are common to both coordinate patches \mathcal{U} and \mathcal{U}' , and where $\mathcal{V} \subset [0, \infty)^k \times \mathbb{R}^{n-1-\ell-k}$ and $\mathcal{V}' \subset [0, \infty)^{k'} \times \mathbb{R}^{n-1-\ell-k'}$. Let y denote the coordinates on \mathcal{V} and y' the coordinates on \mathcal{V}' . Then given any open set \mathcal{W} with compact closure in $(-1, 1)_u \times \mathcal{U}'$, for any $\phi \in \dot{C}^\infty(\tilde{X}, \tilde{F})$ having support in \mathcal{W} , according to (2.9), the restriction of $\tilde{\mathcal{D}}^{-1} \phi$ to $(-1, 1)_u \times \mathcal{U}$ is of the form

$$\tilde{\mathcal{D}}^{-1} \phi = \int_{(-1, 1)_{u'} \times \mathcal{U}'} R\left(v, \frac{v}{v'}, u, y, u', y'\right) \phi(u', v', y') dg(u', v', y'),$$

where $v/v' := (v_1/v'_1, \dots, v_\ell/v'_\ell)$ and where $R(v, z, u, y, u', y')$ satisfies the conditions listed after (2.9). Therefore, in such coordinates, we have

$$P\varphi = \lim_{\varepsilon \rightarrow 0^-} -\gamma_\varepsilon \tilde{\mathcal{D}}^{-1} \gamma_0^* G_0 \varphi = - \int_{\mathcal{U}'} R\left(v, \frac{v}{v'}, 0, y, 0, y'\right) G_0 \varphi(v', y') dg(0, v', y'),$$

which represents an element of $\tilde{\Psi}_b^{-\infty}(H_0, E_0)$. Consider now the case when the kernel of $\tilde{\mathcal{D}}^{-1}$ takes the form (2.6). In this case, let $\mathcal{U} = [0, 1]_v^\ell \times \mathbb{R}_y^{n-1-\ell}$. For a section $\phi(u, v, y)$ compactly supported in $(-1, 1)_u \times [0, 1]_v^\ell \times \mathbb{R}_y^{n-1-\ell}$, as in Equation (2.4), we can write

$$\tilde{\mathcal{D}}^{-1} \phi = \int_{\mathbb{R}^n} e^{iu\xi + iy \cdot \eta} v^{i\tau} a(u, v, y, \xi, \tau, \eta) \hat{\phi}(\xi, \tau, \eta) d\xi d\tau d\eta,$$

where $a(u, v, y, \xi, \tau, \eta)$ is the (complete) symbol of $\tilde{\mathcal{D}}^{-1}$ and with $\hat{\psi}$ denoting the Fourier transform of ψ in (u, y) and the Mellin transform in v . Thus, for a compactly supported φ on the cross section coordinate patch $[0, 1]^\ell \times \mathbb{R}^{n-1-\ell}$, we have

$$P\varphi = \lim_{\varepsilon \rightarrow 0^-} -\gamma_\varepsilon \tilde{\mathcal{D}}^{-1} (\gamma_0^* G_0 \varphi) = \int_{\mathbb{R}^{n-1}} v^{i\tau} e^{iy \cdot \eta} b(v, y, \tau, \eta) \hat{\varphi}(\tau, \eta) d\tau d\eta,$$

where

$$(5.2) \quad b(v, y, \tau, \eta) := - \lim_{u \rightarrow 0^-} \frac{1}{2\pi} \int_{\mathbb{R}} e^{iu\xi} a(u, v, y, \xi, \tau, \eta) G_0 d\xi,$$

provided, of course, that this limit exists. To prove that this limit exists, we note that by definition of the space $\tilde{\mathcal{D}}^{-1} \in \tilde{\Psi}_b^{-1}(\tilde{X}, \tilde{F}, \tilde{E})$, the symbol $a(u, v, y, \xi, \tau, \eta)$ is a symbol of order -1 with b -principal symbol equal to

$$(5.3) \quad a_1(u, v, y, \xi, \tau, \eta) = {}^b\sigma_1(\tilde{\mathcal{D}})(u, v, y, \xi, \tau, \eta)^{-1} = \frac{(-i\xi + \sigma(u, v, y, \tau, \eta))G_u^{-1}}{\xi^2 + |(\tau, \eta)|^2},$$

where $|\cdot|^2 = g(\cdot, \cdot)$ and σ is the b -principal symbol of D_u . Moreover, it follows from the explicit local parametrix construction of $\tilde{\mathcal{D}}$ (see e.g. [16, Th. 3.33] or [17, Th. 4.11]) that $a(u, v, y, \xi, \tau, \eta)$ is a *rational* symbol in ξ, τ, η in the sense that

$$a(u, v, y, \xi, \tau, \eta) \sim \sum_{j=1}^{\infty} a_j(u, v, y, \xi, \tau, \eta),$$

where

$$a_j(u, v, y, \xi, \tau, \eta) = \frac{p_j(u, v, y, \xi, \tau, \eta)}{(\xi^2 + |(\tau, \eta)|^2)^j}$$

with $p_j(u, v, y, \xi, \tau, \eta)$ a polynomial of degree j in (ξ, τ, η) . Note that by (5.3) $a_1(u, v, y, \xi, \tau, \eta)$ certainly satisfies this property and a straightforward induction argument shows that this holds for each j . Thus, we just have to understand

$$- \lim_{u \rightarrow 0^-} \frac{1}{2\pi} \int_{\mathbb{R}} e^{iu\xi} a_j(u, v, y, \xi, \tau, \eta) G_0 d\xi$$

for each j . To do so, we write $\xi^2 + |(\tau, \eta)|^2 = (\xi + i|(\tau, \eta)|)(\xi - i|(\tau, \eta)|)$ and use Cauchy's theorem to obtain

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{R}} e^{iu\xi} \frac{p_j(u, v, y, \xi, \tau, \eta)}{(\xi^2 + |(\tau, \eta)|^2)^j} d\xi &= \frac{i}{2\pi i} \int_{\mathbb{R}} e^{iu\xi} \frac{p_j(u, v, y, \xi, \tau, \eta)}{(\xi + i|(\tau, \eta)|)^j (\xi - i|(\tau, \eta)|)^j} d\xi \\ &= \frac{-i}{(j-1)!} \left(\frac{d}{d\xi} \right)^{j-1} \Big|_{\xi = -i|(\tau, \eta)|} \left(e^{iu\xi} \frac{p_j(u, v, y, \xi, \tau, \eta)}{(\xi - i|(\tau, \eta)|)^j} \right). \end{aligned}$$

Here, recalling that $u < 0$, we shifted the line $\mathbb{R} = \{\Im \xi = 0\}$ down to $\{\Im \xi = -\infty\}$ where the integral vanishes, and we picked up a pole at $\xi = -i|(\tau, \eta)|$. Therefore,

$$\begin{aligned} & - \lim_{u \rightarrow 0^-} \frac{1}{2\pi} \int_{\mathbb{R}} e^{iu\xi} a_j(u, v, y, \xi, \tau, \eta) G_0 d\xi \\ &= \frac{i}{(j-1)!} \left(\frac{d}{d\xi} \right)^{j-1} \Big|_{\xi = -i|(\tau, \eta)|} \left(\frac{p_j(u, v, y, \xi, \tau, \eta)}{(\xi - i|(\tau, \eta)|)^j} \right) G_0. \end{aligned}$$

It's easily checked that the right-hand side is homogeneous of degree $1 - j$ for each $j = 1, 2, 3, \dots$. It follows that the symbol $b(v, y, \tau, \eta)$ in (5.2) is a classical symbol of order 0 in (τ, η) and using (5.3), the b -principal symbol of $b(v, y, \tau, \eta)$ is equal to

$$\begin{aligned} \frac{p_1(0, v, y, -i|(\tau, \eta)|, \tau, \eta)}{-2|(\tau, \eta)|} G_0 &= \frac{(|(\tau, \eta)| - \sigma(0, v, y, \tau, \eta))}{2|(\tau, \eta)|} \\ &= \frac{1}{2} \left(\text{Id} - \frac{\sigma(0, v, y, \tau, \eta)}{|(\tau, \eta)|} \right). \end{aligned}$$

In particular, since $\sigma(0, v, y, \tau, \eta)^2 = |(\tau, \eta)|^2$, the leading order part of $b(v, y, \tau, \eta)$ is the orthogonal projection onto the eigenspace of the negative eigenvalues of the b -principal symbol of D_0 . Finally, it is straightforward to check that, from (5.2), a lacunary condition on $a(u, v, y, \xi, \tau, \eta)$ implies a lacunary condition on $b(v, y, \tau, \eta)$. In conclusion, we have shown that $P \in \tilde{\Psi}_b^0(H_0, E_0)$.

Step II: We now prove that $\text{ran } P = \mathcal{H}(\mathcal{D})$. The proof of this step is similar to Seeley [30, Th. 5]. We first show that $P = \text{Id}$ on $\mathcal{H}(\mathcal{D})$. Let $\varphi = \gamma_{0-}\phi$, where $\gamma_{0-} = \lim_{\varepsilon \rightarrow 0^-} \gamma_\varepsilon$, $\phi \in H_b^\infty(X, E)$, and $\mathcal{D}\phi = 0$, and define

$$\tilde{\phi} := \begin{cases} \phi & \text{on } X \\ 0 & \text{on } \tilde{X} \setminus X. \end{cases}$$

Since $\mathcal{D}\phi = 0$ and, according to (4.10), $\tilde{\mathcal{D}} = G_u(\partial_u + D_u)$ near H_0 , and the derivative of the Heaviside function is the delta distribution, it follows that

$$\tilde{\mathcal{D}}\tilde{\phi} = -\delta_0 \otimes G_0\varphi = -\gamma_0^* G_0\varphi,$$

since $\gamma_0^* = \delta_0 \otimes \cdot$ with δ_0 the delta distribution concentrated at the hypersurface $\{0\} \times H_0$. Thus, $\tilde{\phi} = -\tilde{\mathcal{D}}^{-1}\gamma_0^* G_0\varphi$, and so

$$P\varphi := -\gamma_{0-}(\tilde{\mathcal{D}}^{-1}\gamma_0^* G_0\varphi) = \gamma_{0-}(\tilde{\phi}) = \gamma_{0-}(\phi) = \varphi.$$

Hence, $P = \text{Id}$ on $\mathcal{H}(\mathcal{D})$. We now show that $P^2 = P$. Let $\varphi \in H_b^\infty(H_0, E_0)$. Then by definition of P , we have

$$P\varphi = \gamma_{0-}\phi = \varphi,$$

where $\phi := -(\tilde{\mathcal{D}}^{-1}\gamma_0^* G_0\varphi)|_X$. Note that our analysis in Step I shows that $\phi \in H_b^\infty(X, E)$. Also, we clearly have $\mathcal{D}\phi = 0$. Thus, $\gamma_{0-}\phi \in \mathcal{H}(\mathcal{D})$, so as we know that $P = \text{Id}$ on $\mathcal{H}(\mathcal{D})$, it follows that

$$P^2\varphi = P(P\varphi) = P(\gamma_{0-}\phi) = \gamma_{0-}\phi = P\varphi.$$

Our proof is now complete. \square

We now define

$$(5.4) \quad \mathcal{C} := PP^*[PP^* + (\text{Id} - P^*)(\text{Id} - P)]^{-1}.$$

Then, the right side is the orthogonal projection onto $\text{ran } P$ by Lemma 12.8 of [5] (see also [2]). Now by Lemma 5.1 and (5.4), we can see that the b -principal symbol of \mathcal{C} is also the orthogonal projection onto the eigenspace of the negative eigenvalues of the b -principal symbol of D_0 . To complete the proof of Theorem 1.1, it remains to show that $\mathcal{C} \in \tilde{\Psi}_b^0(H_0, E_0)$. To show this, let

$$A = PP^* + (\text{Id} - P^*)(\text{Id} - P).$$

By Lemma 5.1, the b -principal symbol of P is also a projection, so that

$${}^b\sigma_0(A) = {}^b\sigma_0(P) {}^b\sigma_0(P)^* + (\text{Id} - {}^b\sigma_0(P))(\text{Id} - {}^b\sigma_0(P)^*)$$

is always invertible, which means that A is b -elliptic. Moreover, by (2.11) any normal operator of A equals

$$N(A)(\tau) = N(P)(\tau)N(P)(\tau)^* + (\text{Id} - N(P)(\tau)^*)(\text{Id} - N(P)(\tau)),$$

and $N(P)(\tau)$ is a projection since $N(P)(\tau)^2 = N(P^2)(\tau) = N(P)(\tau)$. Thus, we can see that $N(A)(\tau)$ is invertible for all real τ . Thus, by Theorem 2.1 and 2.3, it follows that $A^{-1} \in \tilde{\Psi}_b^0(H_0, E_0)$. Finally, by the composition properties of b -pseudodifferential operator, we conclude that $\mathcal{C} = PP^*A^{-1}$ is also in $\tilde{\Psi}_b^0(H_0, E_0)$. This completes the proof of Theorem 1.1.

5.2. Proof of Theorem 1.2. We now assume that $E = F$ and \mathcal{D} is formally self-adjoint. This, in particular, implies that $G_0^2 = -\text{Id}$ and $G_0^* = -G_0$. Recall that

$$\Omega(\varphi, \psi) := \langle G_0\varphi, \psi \rangle_0 \quad \text{for } \varphi, \psi \in L_b^2(H_0, E_0)$$

is a Hermitian symplectic form on $L_b^2(H_0, E_0)$ where $\langle \cdot, \cdot \rangle_0$ is the L_b^2 inner product on H_0 . We begin with the following lemma.

Lemma 5.2. *The image $\text{ran } \mathcal{P}$ of an orthogonal projector \mathcal{P} on $L_b^2(H_0, E_0)$ is Lagrangian with respect to Ω if and only if*

$$G_0\mathcal{P} = (\text{Id} - \mathcal{P})G_0.$$

Proof. By definition, the subspace $\text{ran } \mathcal{P} \subset L_b^2(H_0, E_0)$ is Lagrangian with respect to Ω means that $\text{ran } \mathcal{P} = (\text{ran } \mathcal{P})_\Omega^\perp$ where

$$(5.5) \quad (\text{ran } \mathcal{P})_\Omega^\perp = \{ \varphi \in L_b^2(H_0, E_0) \mid \Omega(\varphi, \psi) = \langle G_0\varphi, \psi \rangle_0 = 0 \quad \forall \psi \in \text{ran } \mathcal{P} \}.$$

Therefore, the Lagrangian property of $\text{ran } \mathcal{P}$ is equivalent to

$$(5.6) \quad G_0 : \text{ran } \mathcal{P} \rightarrow (\text{ran } \mathcal{P})^\perp = \text{ran}(\text{Id} - \mathcal{P}) \quad \text{is an isomorphism.}$$

Now if $G_0\mathcal{P} = (\text{Id} - \mathcal{P})G_0$, then certainly (5.6) holds. Conversely, if (5.6) holds, then it follows that $G_0^* = -G_0 : \text{ran}(\text{Id} - \mathcal{P}) \rightarrow \text{ran } \mathcal{P}$, which implies that $G_0 : \text{ran}(\text{Id} - \mathcal{P}) \rightarrow \text{ran } \mathcal{P}$. Hence,

$$\begin{aligned} G_0\mathcal{P} &= (\text{Id} - \mathcal{P})G_0\mathcal{P} = (\text{Id} - \mathcal{P})G_0\mathcal{P} + 0 \\ &= (\text{Id} - \mathcal{P})G_0\mathcal{P} + (\text{Id} - \mathcal{P})G_0(\text{Id} - \mathcal{P}) \\ &= (\text{Id} - \mathcal{P})G_0(\mathcal{P} + \text{Id} - \mathcal{P}) = (\text{Id} - \mathcal{P})G_0. \end{aligned}$$

Thus, $G_0\mathcal{P} = (\text{Id} - \mathcal{P})G_0$ and our proof is now complete. \square

Thus, according to this lemma, to show that the closure of $\mathcal{H}(\mathcal{D})$ in $L_b^2(H_0, E_0)$, which for simplicity we again denote by $\mathcal{H}(\mathcal{D})$, is a Lagrangian subspace with respect to Ω , it is sufficient to show that

$$G_0\mathcal{C} = (\text{Id} - \mathcal{C})G_0, \quad \text{equivalently} \quad \mathcal{C}G_0 = G_0(\text{Id} - \mathcal{C})$$

since $\text{ran}(\mathcal{C}) = \mathcal{H}(\mathcal{D})$. To do so, we need the following extension of Grubb's result [10, Th. 7.5] to the category of manifolds with multi-cylindrical end boundaries.

Lemma 5.3. *We have*

$$(5.7) \quad P^*G_0 = G_0(\text{Id} - P).$$

Proof. First, we claim that for any $\phi \in H_b^\infty(X, E)$, we have

$$(5.8) \quad \phi = (r\tilde{\mathcal{D}}^{-1}e)\mathcal{D}\phi - (r\tilde{\mathcal{D}}^{-1}e)\gamma_0^*G_0\gamma_0\phi$$

where r is the restriction map from \tilde{X} to X , e is the extension map by 0 from X to \tilde{X} . This identity can be found in Booß-Bavnbek and Wojciechowski's book [5, Lem. 12.7] for the smooth closed manifold case. To prove this, we define

$$\tilde{\phi} := \begin{cases} \phi & \text{on } X \\ 0 & \text{on } \tilde{X} \setminus X. \end{cases}$$

Since $\tilde{\mathcal{D}} = G_u(\partial_u + D_u)$ near H_0 as described in (4.10), and the derivative of the Heaviside function is the delta distribution, it follows that

$$r\tilde{\mathcal{D}}\tilde{\phi} = \mathcal{D}\phi - \gamma_0^*G_0\gamma_0\phi.$$

Multiplying both sides by $r\tilde{\mathcal{D}}^{-1}e$ we get (5.8). In particular, applying γ_0 to both sides of (5.8), we obtain

$$(5.9) \quad P\gamma_0\phi = \gamma_0\phi - \gamma_0\tilde{\mathcal{D}}^{-1}e\mathcal{D}\phi.$$

Second, we follow the proof of Theorem 7.5 in [10]. Let us denote L^2 -pairings over X (H_0) by angular brackets $\langle \cdot, \cdot \rangle_X$ ($\langle \cdot, \cdot \rangle_0$) and distributional pairings by parentheses. Then, for $\phi \in H_b^\infty(X, E)$ and $\psi \in H_b^\infty(H_0, E_0)$, by (5.9) we have

$$(5.10) \quad \langle \gamma_0\tilde{\mathcal{D}}^{-1}e\mathcal{D}\phi, G_0\psi \rangle_0 = \langle (\gamma_0\phi - P\gamma_0\phi), G_0\psi \rangle_0 = \langle (\text{Id} - P)\gamma_0\phi, G_0\psi \rangle_0.$$

As distributions we can write the left-hand side of (5.10) as

$$\begin{aligned} \langle \gamma_0\tilde{\mathcal{D}}^{-1}e\mathcal{D}\phi, G_0\psi \rangle_0 &= (\gamma_0^*G_0\psi)(\tilde{\mathcal{D}}^{-1}e\mathcal{D}\phi) \\ &= (r\tilde{\mathcal{D}}^{-1}\gamma_0^*G_0\psi)(\mathcal{D}\phi) \\ &= \langle \mathcal{D}\phi, r\tilde{\mathcal{D}}^{-1}\gamma_0^*G_0\psi \rangle_X \\ &= \langle \mathcal{D}\phi, r\tilde{\mathcal{D}}^{-1}\gamma_0^*G_0\psi \rangle_X - \langle \phi, \mathcal{D}r\tilde{\mathcal{D}}^{-1}\gamma_0^*G_0\psi \rangle_X, \end{aligned}$$

since $\mathcal{D}r\tilde{\mathcal{D}}^{-1}\gamma_0^*G_0\psi = 0$ over X . By Green's formula, the right-hand side equals

$$\begin{aligned} \langle G_0\gamma_0\phi, \gamma_0\tilde{\mathcal{D}}^{-1}\gamma_0^*G_0\psi \rangle_0 &= -\langle G_0\gamma_0\phi, P\psi \rangle_0 \\ &= -\langle P^*G_0\gamma_0\phi, \psi \rangle_0 = -\langle G_0P^*G_0\gamma_0\phi, G_0\psi \rangle_0. \end{aligned}$$

Equating the far right term with the far right term in (5.10), we obtain $-G_0P^*G_0 = \text{Id} - P$, which, after multiplication by G_0 , proves our result. \square

As before, we put $A = PP^* + (\text{Id} - P^*)(\text{Id} - P)$. Then using the identity in (5.7), we obtain

$$\begin{aligned} AG_0 &= P(P^*G_0) + (\text{Id} - P^*)(\text{Id} - P)G_0 \\ &= P(G_0(\text{Id} - P)) + (\text{Id} - P^*)(G_0P^*) \\ &= (PG_0)(\text{Id} - P) + ((\text{Id} - P^*)G_0)P^* \\ &= (G_0(\text{Id} - P^*)(\text{Id} - P) + (G_0P)P^*) \\ &= G_0((\text{Id} - P^*)(\text{Id} - P) + PP^*) = G_0A. \end{aligned}$$

Thus, $AG_0 = G_0A$. Hence, $G_0A^{-1} = A^{-1}G_0$. Again using (5.7), we see that

$$\begin{aligned} \mathcal{C}G_0 &= PP^*A^{-1}G_0 = PP^*G_0A^{-1} = PG_0(\text{Id} - P)A^{-1} \\ (5.11) \quad &= G_0(\text{Id} - P^*)(\text{Id} - P)A^{-1}. \end{aligned}$$

Now

$$\begin{aligned} (\text{Id} - P^*)(\text{Id} - P)A^{-1} &= (PP^* + (\text{Id} - P^*)(\text{Id} - P))A^{-1} - PP^*A^{-1} \\ &= AA^{-1} - \mathcal{C} = \text{Id} - \mathcal{C}. \end{aligned}$$

Combining this and (5.11) proves

$$\mathcal{C}G_0 = G_0(\text{Id} - \mathcal{C}).$$

Hence by Lemma 5.2, $\mathcal{H}(\mathcal{D})$ is a Lagrangian subspace with respect to Ω . Now let us write

$$\mathcal{C} = \frac{1}{2} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

with respect to the decomposition $L_b^2(H_0, E_0) = L_b^2(H_0, E_0^+) \oplus L_b^2(H_0, E_0^-)$. Then using $\mathcal{C}^* = \mathcal{C}$ and $\mathcal{C}G_0 = G_0(\text{Id} - \mathcal{C})$ it is easy to check that $C_{11} = \text{Id}$, $C_{22} = \text{Id}$, $C_{21} : L_b^2(Y, E_Y^+) \rightarrow L_b^2(Y, E_Y^-)$ is unitary with $C_{12} = C_{21}^{-1}$. Therefore, setting $\kappa_0 := C_{12}$, we conclude that

$$\mathcal{C} = \frac{1}{2} \begin{pmatrix} \text{Id} & \kappa_0^{-1} \\ \kappa_0 & \text{Id} \end{pmatrix}.$$

This completes the proof of Theorem 1.2.

5.3. Proof of Theorem 1.3. Assume that $E = F$, \mathcal{D} is formally self-adjoint, and $\mathcal{D} = G_0(\partial_u + D_0)$ over $[-1, 0]_u \times H_0$. Let $\tilde{E} := E \sqcup_{G_0} E$ be the doubled vector bundle over the doubled manifold $\tilde{X} = X \sqcup_{H_0} (-X)$ constructed in Subsection 4.2 and let $\tilde{\mathcal{D}} := (\mathcal{D}, -\mathcal{D})$ be the corresponding invertible double. We define

$$P_{\pm} := \pm \lim_{\varepsilon \rightarrow 0^{\pm}} \gamma_{\varepsilon} \tilde{\mathcal{D}}^{-1} \gamma_0^* G_0 : H_b^{\infty}(H_0, E_0) \rightarrow H_b^{\infty}(H_0, E_0).$$

To be precise, P_{\pm} really defines a map on $H_b^{\infty}(H_0, \tilde{E}|_{H_0})$ but we make the following ‘left’ identification of $E_0 = E|_{H_0}$ with $\tilde{E}|_{H_0}$:

$$E_0 \longleftrightarrow \tilde{E}|_{H_0} := E_0 \sqcup_{G_0} E_0 \iff E_0 \ni \varphi \longleftrightarrow [(\varphi, G_0\varphi)] \in \tilde{E}|_{H_0}.$$

As a result of this ‘left’ identification, given any $\phi \in H_b^{\infty}(X, E)$ with $\phi|_{H_0} = \varphi \in H_b^{\infty}(H_0, E_0)$, considering ϕ as sections of \tilde{E} over X and $-X$, we have

$$(5.12) \quad \phi \in H_b^{\infty}(X, \tilde{E}) \implies \phi|_{H_0} = \varphi, \quad \phi \in H_b^{\infty}(-X, \tilde{E}) \implies \phi|_{H_0} = G_0\varphi,$$

the latter equality holds because restriction of ϕ from X equals φ on the left factor of $E_0 \sqcup_{G_0} E_0$, which is equivalent to $G_0\varphi$ on the right factor of $E_0 \sqcup_{G_0} E_0$. The ‘left’ identification is important because then the Cauchy data space of $\tilde{\mathcal{D}}$ taken from the left manifold X of \tilde{X} is exactly the same as the Cauchy data space of the original operator \mathcal{D} on X .

Lemma 5.4. P_- (P_+) is a projection whose image is exactly the Cauchy data space of the restriction of $\tilde{\mathcal{D}}$ to X ($-X$), and

$$(5.13) \quad P_- + P_+ = \text{Id}.$$

Proof. In Lemma 5.1 we showed that P_- is a projection whose image is exactly the Cauchy data space of the restriction of $\tilde{\mathcal{D}}$ to X and a similar proof establishes the corresponding claim for P_+ .

The proof of (5.13) is similar to Seeley’s proof [30, Th. 5] but now in the “ b -category”. Let $\phi \in H_b^\infty([-1, 1]_u \times H_0, \tilde{E})$ have compact support in $(-1, 1)_u \times H_0$ and let $\psi \in H_b^\infty(H_0, E_0)$. Let us denote distributional pairing by parentheses and, as usual, L^2 -pairings over H_0 by angular brackets $\langle \cdot, \cdot \rangle_0$. If $\tilde{\mathcal{K}} := \tilde{\mathcal{D}}^{-1}\gamma_0^*G_0$, then

$$(5.14) \quad \begin{aligned} \langle \gamma_0\phi, G_0\psi \rangle_0 &= (\gamma_0^*G_0\psi)(\phi) = (\tilde{\mathcal{D}}\tilde{\mathcal{K}}\psi)(\phi) = (\tilde{\mathcal{K}}\psi)(\tilde{\mathcal{D}}\phi) \\ &= \int_{-1}^1 \langle \tilde{\mathcal{D}}\phi, \tilde{\mathcal{K}}\psi \rangle_0 du, \end{aligned}$$

where we used the fact that from the proof of Lemma 5.1, $\tilde{\mathcal{K}}\psi = \tilde{\mathcal{D}}^{-1}\gamma_0^*G_0\psi$ has a left-hand limit at H_0 and a similar proof shows that it has a right-hand limit at H_0 , so the function $\tilde{\mathcal{K}}\psi$ is in H_b^∞ off H_0 with at most a jump discontinuity at H_0 . In particular, we can write

$$\int_{-1}^1 \langle \tilde{\mathcal{D}}\phi, \tilde{\mathcal{K}}\psi \rangle_0 du = \lim_{r \rightarrow 0^+} \int_{|u| > r} \langle \tilde{\mathcal{D}}\phi, \tilde{\mathcal{K}}\psi \rangle_0 du.$$

By (4.10) we know that $\tilde{\mathcal{D}} = G_0(\partial_u + D_u)$ over $[-1, 1]_u \times H_0$, so we can evaluate the right-hand integral as follows:

$$(5.15) \quad \begin{aligned} \int_{|u| > r} \langle \tilde{\mathcal{D}}\phi, \tilde{\mathcal{K}}\psi \rangle_0 du &= \int_{|u| > r} \langle (G_0(\partial_u + D_u))\phi, \tilde{\mathcal{K}}\psi \rangle_0 du \\ &= - \int_{|u| > r} \langle \partial_u\phi, G_0\tilde{\mathcal{K}}\psi \rangle_0 du + \int_{|u| > r} \langle \phi, (G_0D_u)\tilde{\mathcal{K}}\psi \rangle_0 du \\ &= - \int_{|u| > r} \partial_u \langle \phi, G_0\tilde{\mathcal{K}}\psi \rangle_0 du + \int_{|u| > r} \langle \phi, \tilde{\mathcal{D}}\tilde{\mathcal{K}}\psi \rangle_0 du \\ &= - \langle \gamma_{-r}\phi, G_0\gamma_{-r}\tilde{\mathcal{K}}\psi \rangle_0 + \langle \gamma_r\phi, G_0\gamma_r\tilde{\mathcal{K}}\psi \rangle_0, \end{aligned}$$

where we used that $\tilde{\mathcal{D}}\tilde{\mathcal{K}} = 0$ off H_0 . Taking $r \rightarrow 0^+$ in (5.15) and equating this with $\langle \gamma_0\phi, G_0\psi \rangle_0$ in (5.14), and using that $P_- := -\gamma_{0-}\tilde{\mathcal{K}}\psi$ and $P_+ := \gamma_{0+}\tilde{\mathcal{K}}\psi$, we conclude that

$$\langle \gamma_0\phi, G_0\psi \rangle_0 = \langle \gamma_0\phi, G_0P_-\psi \rangle_0 + \langle \gamma_0\phi, G_0P_+\psi \rangle_0.$$

Since ϕ and ψ were arbitrary, it follows that $\text{Id} = P_- + P_+$, and our proof is complete. \square

Now to prove Theorem 1.3, we need to demonstrate that $P_- = \mathcal{C}$; which reduces to proving that P_- is orthogonal. To this end, we first prove the following

Lemma 5.5. *We have $G_0 P_- = P_+ G_0$.*

Proof. We shall prove that $G_0 P_- = P_+ G_0$ first over $\text{ran } P_-$ and then over $\text{ran}(\text{Id} - P_-) = \text{ran } P_+$.

Let $\varphi \in \text{ran } P_- \subset H_b^\infty(H_0, E_0)$. Then $P_- \varphi = \varphi$ and there is a $\phi_1 \in C^\infty(X, E)$ such that $\mathcal{D}\phi_1 = 0$ and $\phi_1|_{H_0} = \varphi$. Let $\phi_2 \in C^\infty(-X, E)$ be the same section ϕ_1 , but considered on the reversed manifold $-X$. Since $\mathcal{D}\phi_1 = 0$, we have $\tilde{\mathcal{D}}\phi_2 = -\mathcal{D}\phi_2 = 0$, therefore $\phi_2|_{H_0} \in \text{ran } P_+$ or $P_+(\phi_2|_{H_0}) = \phi_2|_{H_0}$. In view of the identification (5.12), we have $\phi_2|_{H_0} = G_0 \varphi$, therefore

$$G_0(P_- \varphi) = G_0(\varphi) = \phi_2|_{H_0} = P_+(\phi_2|_{H_0}) = P_+(G_0 \varphi) = P_+ G_0 \varphi.$$

Now let $\varphi \in \text{ran}(\text{Id} - P_-) = \text{ran } P_+ \subset H_b^\infty(H_0, E_0)$. Then $P_- \varphi = 0$, so we just have to prove that $P_+ G_0 \varphi = 0$ too. Since $\varphi \in \text{ran } P_+$, we know that $P_+ \varphi = \varphi$ and there is a $\phi_2 \in C^\infty(-X, E)$ such that $\mathcal{D}\phi_2 = 0$ and $\phi_2|_{H_0} = \varphi$. Let $\phi_1 \in C^\infty(X, E)$ be the same section ϕ_2 , but considered on the left manifold X . Since $\mathcal{D}\phi_2 = 0$, we have $\tilde{\mathcal{D}}\phi_1 = \mathcal{D}\phi_1 = 0$, therefore $\phi_1|_{H_0} \in \text{ran } P_-$ or $P_-(\phi_1|_{H_0}) = \phi_1|_{H_0}$. Because $\phi_2|_{H_0} = \varphi$ from the right manifold $-X$, in view of the identification (5.12), from the left we must have $\phi_1|_{H_0} = -G_0 \varphi$. Hence, $P_-(-G_0 \varphi) = -G_0 \varphi$, and so

$$P_+ G_0 \varphi = P_+ P_-(-G_0 \varphi) = (\text{Id} - P_-)P_-(G_0 \varphi) = 0.$$

□

Now combining the identity (5.13) with Lemma 5.5, we obtain

$$G_0 P_- = (\text{Id} - P_-)G_0.$$

On the other hand, we also know that $G_0 P_-^* = (\text{Id} - P_-)G_0$ from (5.7). Hence, $P_- = P_-^*$ and our proof of Theorem 1.3 is now complete.

6. RELATIVE INDEX FORMULÆ AND THE BOJARSKI CONJECTURE

In this section we prove Theorems 1.4 and 1.5.

6.1. Proof of Theorem 1.4. The proof of Theorem 1.4 consists of three steps.

Step I: First, we consider the following general abstract situation. Let V_0, V_1, V_2 be topological vector spaces and let

$$T : V_1 \rightarrow V_2 \quad , \quad \gamma_0 : V_1 \rightarrow V_0$$

be continuous surjective linear transformations. Suppose there is a projection

$$\mathcal{C} : V_0 \rightarrow V_0$$

whose image is the *generalized Cauchy data space* of T :

$$\mathcal{H}(T) := \gamma_0 \ker T = \{ \gamma_0 \phi \mid \phi \in V_1, T\phi = 0 \} \subset V_0$$

and the *generalized unique continuation property* holds:

$$\psi \in \text{ran } \mathcal{C} \iff \exists! \phi \in V_1, T\phi = 0 \text{ and } \gamma_0 \phi = \psi.$$

Given a projection $\mathcal{P} : V_0 \rightarrow V_0$, consider the linear map

$$T_{\mathcal{P}} : \text{dom}(T_{\mathcal{P}}) \rightarrow V_2$$

where

$$\text{dom}(T_{\mathcal{P}}) := \{ \phi \in V_1 \mid \mathcal{P}(\gamma_0 \phi) = 0 \} \subset V_1.$$

Proposition 6.1. *For an arbitrary projection $\mathcal{P} : V_0 \rightarrow V_0$, the operator*

$$T_{\mathcal{P}} : \text{dom}(T_{\mathcal{P}}) \rightarrow V_2$$

is Fredholm; that is, has a finite-dimensional kernel and cokernel, if and only if

$$\mathcal{P}\mathcal{C} : \text{ran } \mathcal{C} \rightarrow \text{ran } \mathcal{P}$$

is Fredholm, in which case the following index formula holds:

$$\text{ind } T_{\mathcal{P}} = \text{ind}(\mathcal{P}, \mathcal{C}) := \text{ind}(\mathcal{P}\mathcal{C} : \text{ran } \mathcal{C} \rightarrow \text{ran } \mathcal{P}).$$

Proof. We just need to establish isomorphisms between kernels: $\ker T_{\mathcal{P}} \cong \ker \mathcal{P}\mathcal{C}$ and cokernels: $\text{coker } T_{\mathcal{P}} := V_2/\text{ran } T_{\mathcal{P}} \cong \text{ran } \mathcal{P}/\text{ran } \mathcal{P}\mathcal{C} =: \text{coker } \mathcal{P}\mathcal{C}$. This shows that $T_{\mathcal{P}}$ is Fredholm if and only if $\mathcal{P}\mathcal{C}$ is Fredholm. The first congruence is easy: Using that

$$\psi \in \text{ran } \mathcal{C} \iff \exists! \phi \in V_1, T\phi = 0 \text{ and } \gamma_0\phi = \psi$$

it follows that

$$\begin{aligned} \psi \in \text{ran } \mathcal{C}, \mathcal{P}\psi = 0 &\iff \exists! \phi, T\phi = 0 \text{ and } \gamma_0\phi = \psi \text{ and } \mathcal{P}\psi = 0 \\ &\iff \exists! \phi, T\phi = 0 \text{ and } \mathcal{P}\gamma_0\phi = 0 \text{ and } \gamma_0\phi = \psi \\ &\iff \exists! \phi \in \ker T_{\mathcal{P}} \text{ with } \gamma_0\phi = \psi. \end{aligned}$$

Therefore the map

$$\ker T_{\mathcal{P}} \ni \phi \mapsto \gamma_0\phi \in \ker \mathcal{P}\mathcal{C}$$

is an isomorphism.

It remains to prove that $\text{coker } T_{\mathcal{P}} \cong \text{coker } \mathcal{P}\mathcal{C}$. To prove this, we define a map

$$f : V_0 \rightarrow V_2/\text{ran } T_{\mathcal{P}} = \text{coker } T_{\mathcal{P}}$$

as follows. Let $\psi \in V_0$. Then there is a $\phi \in V_1$ such that $\gamma_0\phi = \psi$. We define

$$f(\psi) := [T\phi] \in V_2/\text{ran } T_{\mathcal{P}},$$

where $[\]$ denotes equivalence class. This is well-defined because if $\gamma_0\tilde{\phi} = \psi$ also, then $\mathcal{P}\gamma_0(\tilde{\phi} - \phi) = \mathcal{P}(0) = 0$, so $\tilde{\phi} - \phi \in \text{dom}(T_{\mathcal{P}})$, and thus,

$$[T\tilde{\phi}] = [T\phi + T(\tilde{\phi} - \phi)] = [T\phi].$$

Since we know that $T : V_1 \rightarrow V_2$ is surjective, f is also surjective. Observe that if $\mathcal{P}\psi = 0$, then with $\phi \in V_1$ such that $\gamma_0\phi = \psi$, we have $\phi \in \text{dom}(T_{\mathcal{P}})$, so $T\phi \in \text{ran } T_{\mathcal{P}}$ and thus $f(\psi) = [T\phi] = 0$. Hence,

$$f : \ker \mathcal{P} \rightarrow \text{ran } T_{\mathcal{P}}.$$

Therefore f descends to a (still surjective) map on the quotient:

$$\tilde{f} : V_0/\ker \mathcal{P} \rightarrow V_2/\text{ran } T_{\mathcal{P}}.$$

We are given that \mathcal{P} is a projection, so we have a canonical isomorphism

$$\text{ran } \mathcal{P} \cong V_0/\ker \mathcal{P}.$$

Indeed, the map $\text{ran } \mathcal{P} \ni \psi \mapsto [\psi] \in V_0/\ker \mathcal{P}$ is certainly one-to-one, and it is surjective because given any $\psi \in V_0$, we can write

$$\psi = \mathcal{P}\psi + (\text{Id} - \mathcal{P})\psi = \mathcal{P}\psi \quad \text{modulo } \ker \mathcal{P},$$

as $\ker \mathcal{P} = \text{ran}(\text{Id} - \mathcal{P})$ since \mathcal{P} is a projection. Thus, we obtain a surjective map

$$\tilde{f} : \text{ran } \mathcal{P} \rightarrow V_2/\text{ran } T_{\mathcal{P}}.$$

We claim that $\ker \tilde{f} = \text{ran } \mathcal{PC}$. Once we prove this, it follows that \tilde{f} descends to an *isomorphism* of vector spaces

$$\text{coker } \mathcal{PC} := \text{ran } \mathcal{P} / \text{ran } \mathcal{PC} \xrightarrow{\cong} V_2 / \text{ran } T_{\mathcal{P}} =: \text{coker } T_{\mathcal{P}},$$

which completes our proof. We first show that $\ker \tilde{f} \subset \text{ran } \mathcal{PC}$. An element of $\ker \tilde{f}$ is an element $\psi \in \text{ran } \mathcal{P}$ where we choose $\phi \in V_1$ such that $\gamma_0 \phi = \psi$ and $T\phi \in \text{ran } T_{\mathcal{P}}$. The inclusion $T\phi \in \text{ran } T_{\mathcal{P}}$ means that

$$\exists \tilde{\phi} \in V_1, \mathcal{P}\gamma_0 \tilde{\phi} = 0, T\tilde{\phi} = T\phi.$$

In particular, $T(\phi - \tilde{\phi}) = 0$, so $\gamma_0(\phi - \tilde{\phi}) = \psi - \gamma_0 \tilde{\phi} \in \text{ran } \mathcal{C}$. Thus,

$$\psi = \mathcal{P}\psi - 0 = \mathcal{P}\psi - \mathcal{P}\gamma_0 \tilde{\phi} = \mathcal{P}(\psi - \gamma_0 \tilde{\phi}) \in \text{ran } \mathcal{PC}.$$

The proof that $\text{ran } \mathcal{PC} \subset \ker \tilde{f}$ is similar: An element of $\text{ran } \mathcal{PC}$ is an element $\psi \in \text{ran } \mathcal{P}$ such that $\psi \in \text{ran } \mathcal{C}$ also. The inclusion $\psi \in \text{ran } \mathcal{C}$ means that

$$\exists! \phi \in V_1, \gamma_0 \phi = \psi, T\phi = 0.$$

Choose $\tilde{\phi} \in V_1$ with $\gamma_0 \tilde{\phi} = \psi$. Then $\mathcal{P}\gamma_0(\tilde{\phi} - \phi) = \mathcal{P}(\psi - \psi) = 0$, so $\tilde{\phi} - \phi \in \text{dom}(T_{\mathcal{P}})$ and thus,

$$\tilde{f}(\psi) = [T\tilde{\phi}] = [T\phi + T(\tilde{\phi} - \phi)] = [T\phi] = [0] = 0.$$

□

Step II: Now we apply Proposition 6.1 to

$$\mathcal{D} : H_b^\infty(X, E) \rightarrow H_b^\infty(X, F), \quad \gamma_0 : H_b^\infty(H_0, E_0) \rightarrow H_b^\infty(H_0, E_0),$$

where γ_0 is restriction to H_0 , which is obviously surjective. The surjective property of \mathcal{D} follows from Theorem 1.1. Also, we know that

$$\mathcal{C} : H_b^\infty(H_0, E_0) \rightarrow H_b^\infty(H_0, E_0)$$

has image equal to the Cauchy data space of \mathcal{D} :

$$\mathcal{H}(\mathcal{D}) := \gamma_0 \ker \mathcal{D} = \{ \gamma_0 \phi \mid \phi \in H_b^\infty(X, E), \mathcal{D}\phi = 0 \} \subset H_b^\infty(H_0, E_0)$$

and

$$\psi \in \text{ran } \mathcal{C} \iff \exists! \phi \in H_b^\infty(X, E), \mathcal{D}\phi = 0 \text{ and } \gamma_0 \phi = \psi.$$

The uniqueness here follows from the unique continuation principle, which also holds for our manifold [1, Cor. 1]. Therefore, we have satisfied all the conditions of Proposition 6.1, so we conclude that

Proposition 6.2. *For an arbitrary projection $\mathcal{P} \in \text{Gr}_\infty^*(\mathcal{D})$, the operator*

$$(6.1) \quad \mathcal{D}_{\mathcal{P}} : \text{dom}(\mathcal{D}_{\mathcal{P}})_\infty := \{ \phi \in H_b^\infty(X, E) \mid \mathcal{P}(\gamma_0 \phi) = 0 \} \rightarrow H_b^\infty(X, F)$$

is Fredholm; that is, $\mathcal{D}_{\mathcal{P}}$ has a finite-dimensional kernel and cokernel if and only if $\mathcal{PC} : \text{ran } \mathcal{C} \rightarrow \text{ran } \mathcal{P}$ is Fredholm.

Remark 6.3. In applications of Proposition 6.1, the surjective condition of T is very crucial as its proof shows. Here is one case where we can clearly see this point. Let us consider the half infinite cylinder $X = [0, \infty) \times Y$ where Y is compact boundaryless manifold and a Dirac type operator

$$\mathcal{D} = G(\partial_u + D_Y) : H_b^\infty(X, E) \rightarrow H_b^\infty(X, F)$$

where D_Y is a Dirac type operator over Y . We assume that D_Y is *non-invertible*. Then it is easy to see that the Calderón projection \mathcal{C} at $\{0\} \times Y$ is the spectral projection $\Pi_{>}$ onto the positive eigenspaces of D_Y . We choose an augmented

APS spectral projection $P_\sigma = \Pi_{>} + \frac{1-\sigma}{2}\Pi_0$ where σ is an involution over $\ker D_Y$ such that $G\sigma = -\sigma G$ and Π_0 is the orthogonal projection onto $\ker D_Y$. Then $P_\sigma \Pi_{>} : \text{ran}(\Pi_{>}) \rightarrow \text{ran}(P_\sigma)$ is certainly a Fredholm operator yet it is well-known, and easy to prove, that \mathcal{D}_{P_σ} is *not* Fredholm since D_Y is not invertible. Therefore, this example shows that Proposition 6.1 does not hold for this simple case. The reason is that the operator $\mathcal{D} : H_b^\infty(X, E) \rightarrow H_b^\infty(X, F)$ is *not* surjective when D_Y is not invertible. For instance, given $\varphi \in \ker D_Y$, it is easy to check that $\psi := \frac{1}{1+u} G \varphi(y) \in H_b^\infty(X, F)$ is not in the image of \mathcal{D} . In order to establish a Fredholm theory for \mathcal{D} we need to ‘perturb’ it as explained in Section 3 of [18].

Step III: For $\mathcal{P} \in Gr_\infty^*(\mathcal{D})$, let us consider the L_b^2 domain:

$$(6.2) \quad \text{dom}(\mathcal{D}_{\mathcal{P}}) := \{ \phi \in H_b^1(X, E) \mid \mathcal{P}(\gamma_0 \phi) = 0 \},$$

where $\gamma_0 : H_b^1(X, E) \rightarrow H_b^{1/2}(H_0, E_0)$ is restriction to the boundary, and

$$(6.3) \quad \mathcal{D}_{\mathcal{P}} : \text{dom}(\mathcal{D}_{\mathcal{P}}) \rightarrow L_b^2(X, F)$$

is regarded as an unbounded operator on L_b^2 . To complete the proof of Theorem 1.4, we shall prove that $\mathcal{D}_{\mathcal{P}}$ in (6.3) is L_b^2 -Fredholm and its index is the same as the index of $\mathcal{D}_{\mathcal{P}}$ in (6.1):

$$\text{ind} \left(\mathcal{D}_{\mathcal{P}} : \text{dom}(\mathcal{D}_{\mathcal{P}}) \subset H_b^1 \rightarrow L_b^2 \right) = \text{ind} \left(\mathcal{D}_{\mathcal{P}} : \text{dom}(\mathcal{D}_{\mathcal{P}})_\infty \subset H_b^\infty \rightarrow H_b^\infty \right).$$

To establish these properties, we first prove

Lemma 6.4. *There is an operator Q having the property that for any $s > 1/2$, $Q : H_b^s(X, F) \oplus H_b^{s+1/2}(H_0, E_0) \rightarrow H_b^{s+1}(X, E)$ is continuous and*

$$(6.4) \quad Q \circ \begin{pmatrix} \mathcal{D} \\ \mathcal{P}\gamma_0 \end{pmatrix} = \text{Id} + \mathcal{S} : H_b^s(X, E) \rightarrow H_b^s(X, E),$$

where $\mathcal{S} : H_b^s(X, E) \rightarrow H_b^\infty(X, E)$ is a compact operator. In particular $\text{ran } \mathcal{D}_{\mathcal{P}}$ is closed and $\ker \mathcal{D}_{\mathcal{P}} \subset H_b^\infty(X, E)$.

Proof. We define Q as the 1×2 matrix

$$Q := [(\text{Id} + \mathcal{K}(\text{Id} - \mathcal{C})\gamma_0)(r \tilde{\mathcal{D}}^{-1}e), \mathcal{K}\mathcal{P}],$$

where $\tilde{\mathcal{D}}$ is an invertible extension of \mathcal{D} , r is the restriction map from \tilde{X} to X , e is the extension map by 0 from X to \tilde{X} , and

$$\mathcal{K} := -r \tilde{\mathcal{D}}^{-1} \gamma_0^* G_0 : H_b^s(H_0, E_0) \rightarrow H_b^{s+1/2}(X, E)$$

is the Poisson operator of \mathcal{D} . Recall that $\text{ran } P = \text{ran } \mathcal{C}$ where $P = \gamma_0 \mathcal{K}$ by (5.1). (Note that if \mathcal{D} is of product type near H_0 and $\tilde{\mathcal{D}}$ is the invertible double, then $P = \mathcal{C}$ by Theorem 1.3, but $P \neq \mathcal{C}$ in general). By (5.8), we have $(r \tilde{\mathcal{D}}^{-1}e)\mathcal{D} = \text{Id} - \mathcal{K}\gamma_0$, and using this we obtain

$$\begin{aligned} Q \circ \begin{pmatrix} \mathcal{D} \\ \mathcal{P}\gamma_0 \end{pmatrix} &= [(\text{Id} + \mathcal{K}(\text{Id} - \mathcal{C})\gamma_0)(r \tilde{\mathcal{D}}^{-1}e), \mathcal{K}\mathcal{P}] \begin{pmatrix} \mathcal{D} \\ \mathcal{P}\gamma_0 \end{pmatrix} \\ &= (\text{Id} + \mathcal{K}(\text{Id} - \mathcal{C})\gamma_0)(r \tilde{\mathcal{D}}^{-1}e)\mathcal{D} + \mathcal{K}\mathcal{P}\gamma_0 \\ &= \text{Id} - \mathcal{K}\gamma_0 + \mathcal{K}(\text{Id} - \mathcal{C})\gamma_0 - \mathcal{K}(\text{Id} - \mathcal{C})\gamma_0 \mathcal{K}\gamma_0 + \mathcal{K}\mathcal{P}\gamma_0 \\ &= \text{Id} - \mathcal{K}\gamma_0 + \mathcal{K}\gamma_0 - \mathcal{K}\mathcal{C}\gamma_0 - \mathcal{K}(\text{Id} - \mathcal{C})\mathcal{P}\gamma_0 + \mathcal{K}\mathcal{P}\gamma_0 \\ &= \text{Id} - \mathcal{K}\mathcal{C}\gamma_0 + \mathcal{K}\mathcal{P}\gamma_0 = \text{Id} + \mathcal{S}, \end{aligned}$$

where $\mathcal{S} = \mathcal{K}(\mathcal{P} - \mathcal{C})\gamma_0$. By assumption, $\mathcal{P} - \mathcal{C} \in \tilde{\Psi}^{-\infty}(H_0, E_0)$, from which the compactness and regularity of \mathcal{S} follows. Note that the regularity property of \mathcal{S} implies that $\ker \mathcal{D}_{\mathcal{P}} \subset H_b^\infty(X, E)$.

It remains to prove that $\text{ran } \mathcal{D}_{\mathcal{P}}$ is closed. To do so, let $\{\phi_n\}$ be a sequence in $\text{dom}(\mathcal{D}_{\mathcal{P}}) \subset H_b^1(X, E)$ with $\psi_n := \mathcal{D}_{\mathcal{P}}\phi_n \rightarrow \psi \in L_b^2(X, F)$. We need to find a $\phi \in \text{dom}(\mathcal{D}_{\mathcal{P}})$ with $\mathcal{D}_{\mathcal{P}}\phi = \psi$. To this end, observe that since \mathcal{S} is compact we may assume that $\mathcal{S}\phi_n$ converges to an element of $H_b^\infty(X, E)$. Then applying ϕ_n to both sides of (6.4) and using that $\phi_n \in \text{dom}(\mathcal{D}_{\mathcal{P}})$ so that $\mathcal{P}\gamma_0\phi_n = 0$ and that $Q : L_b^2(X, F) \oplus 0 \rightarrow H_b^1(X, E)$ is continuous, we see that

$$\phi_n = Q \begin{pmatrix} \psi_n \\ 0 \end{pmatrix} - \mathcal{S}\phi_n$$

converges in the topology of $H_b^1(X, E)$, say to an element ϕ . Then

$$\mathcal{D}_{\mathcal{P}}\phi = \lim_{n \rightarrow \infty} \mathcal{D}_{\mathcal{P}}\phi_n = \psi$$

and, since $\mathcal{P}\gamma_0\phi_n = 0$ for each n , we have $\mathcal{P}\gamma_0\phi = \lim_{n \rightarrow \infty} \mathcal{P}\gamma_0\phi_n = 0$. This proves that $\text{ran } \mathcal{D}_{\mathcal{P}}$ is closed and completes our proof. \square

The following proposition completes the proof of Theorem 1.4 except for the relative index formula, which we take care of in a moment.

Proposition 6.5. *The Calderón projector of $\mathcal{D}^* : C^\infty(X, F) \rightarrow C^\infty(X, E)$ is the projector $\mathcal{P}' := G_0(\text{Id} - \mathcal{C})G_0^*$ with $\mathcal{P}' \in \text{Gr}_\infty^*(\mathcal{D}^*)$, and the L_b^2 adjoint of the map $\mathcal{D}_{\mathcal{P}}$ in (6.3) is the map*

$$(\mathcal{D}^*)_{\mathcal{P}'} : \text{dom}((\mathcal{D}^*)_{\mathcal{P}'}) \rightarrow L_b^2(X, E),$$

where

$$\text{dom}((\mathcal{D}^*)_{\mathcal{P}'}) := \{ \phi \in H_b^1(X, F) \mid \mathcal{P}'(\gamma_0\phi) = 0 \}.$$

Proof. The fact that the Calderón projector of \mathcal{D}^* is $G_0(\text{Id} - \mathcal{C})G_0^*$ follows by applying Theorem 1.2 to the formally self-adjoint Dirac type operator

$$\begin{pmatrix} 0 & \mathcal{D}^* \\ \mathcal{D} & 0 \end{pmatrix} : C^\infty(X, E') \rightarrow C^\infty(X, E') \quad , \quad E' = E \oplus F.$$

To see that the L_b^2 adjoint of the map $\mathcal{D}_{\mathcal{P}}$ is $(\mathcal{D}^*)_{\mathcal{P}'}$, let $\varphi \in H_b^\infty(H_0, E_0)$, $f \in H_b^\infty(X, F)$, and choose $e \in H_b^\infty(X, E)$ with $e|_{H_0} = (\text{Id} - \mathcal{P})\varphi$. Then $e \in \text{dom}(\mathcal{D}_{\mathcal{P}})$ and by Green's formula, as a distributional pairing, we see that

$$\begin{aligned} ((\text{Id} - \mathcal{P})G_0^*\gamma_0 f)(\varphi) &= \langle \varphi, (\text{Id} - \mathcal{P})G_0^*f \rangle_0 = \langle G_0(\text{Id} - \mathcal{P})\varphi, f \rangle_0 \\ (6.5) \qquad \qquad \qquad &= \langle \mathcal{D}e, f \rangle_X - \langle e, \mathcal{D}^*f \rangle_X. \end{aligned}$$

Now let us assume that $f \in \text{dom}((\mathcal{D}^*)_{\mathcal{P}'})$. Then this equality implies that f is in the domain of the L_b^2 adjoint of $\mathcal{D}_{\mathcal{P}}$. The hard part is the other direction: Now assume that f is in the domain of the L_b^2 adjoint of $\mathcal{D}_{\mathcal{P}}$; we need to show that $f \in \text{dom}((\mathcal{D}^*)_{\mathcal{P}'})$. By definition of the L_b^2 adjoint, we know that $f \in L_b^2(X, F)$ and $\mathcal{D}^*f \in L_b^2(X, E)$. Since $f \in L_b^2(X, F)$, the same proof as in the compact case shows that $\gamma_0 f \in H_b^{-1/2}(H_0, F_0)$. In particular, $(\text{Id} - \mathcal{P})G_0^*\gamma_0 f \in H_b^{-1/2}(H_0, E_0)$ defines a distribution acting on $H_b^\infty(H_0, E_0)$, so by the above Green's formula equality (6.5), we see that this distribution must be zero. It remains to prove that $f \in H_b^1(X, F)$.

To see this, we apply Lemma 6.4 to \mathcal{D}^* , to obtain an operator Q' such that for any $s > 1/2$, $Q' : H_b^s(X, E) \oplus H_b^{s+1/2}(H_0, F_0) \rightarrow H_b^{s+1}(X, F)$ is continuous and

$$Q' \circ \begin{pmatrix} \mathcal{D}^* \\ \mathcal{P}'\gamma_0 \end{pmatrix} = \text{Id} + \mathcal{S}' : H_b^s(X, E) \rightarrow H_b^s(X, E),$$

where \mathcal{S}' is a compact regularizing operator. We can apply this formula in the distributional sense to f , in which case, using that $\mathcal{D}^*f \in L_b^2(X, F)$ and, as a distribution, $\mathcal{P}'\gamma_0 f = 0$, we see that

$$f = Q' \circ \begin{pmatrix} \mathcal{D}^*f \\ 0 \end{pmatrix} - \mathcal{S}'f \in H_b^1(X, F).$$

Our proof is now complete. \square

Finally, the relative index formula (1.8) follows directly from the following lemma.

Lemma 6.6. *For $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \in Gr_\infty^*(\mathcal{D})$, $\mathcal{P}_1\mathcal{P}_2 : \text{ran } \mathcal{P}_2 \rightarrow \text{ran } \mathcal{P}_1$ is Fredholm. Its index $\text{ind}(\mathcal{P}_1, \mathcal{P}_2) := \text{ind}(\mathcal{P}_1\mathcal{P}_2 : \text{ran } \mathcal{P}_2 \rightarrow \text{ran } \mathcal{P}_1)$ satisfies*

$$\text{ind}(\mathcal{P}_1, \mathcal{P}_2) = -\text{ind}(\mathcal{P}_2, \mathcal{P}_1) = \text{ind}(\text{Id} - \mathcal{P}_2, \text{Id} - \mathcal{P}_1)$$

and the ‘logarithm property’

$$\text{ind}(\mathcal{P}_1, \mathcal{P}_3) = \text{ind}(\mathcal{P}_1, \mathcal{P}_2) + \text{ind}(\mathcal{P}_2, \mathcal{P}_3).$$

Proof. Since $\text{ind } \mathcal{P}_1\mathcal{P}_2 = -\text{ind}(\mathcal{P}_1\mathcal{P}_2)^* = \text{ind } \mathcal{P}_2\mathcal{P}_1$, the equality $\text{ind}(\mathcal{P}_1, \mathcal{P}_2) = -\text{ind}(\mathcal{P}_2, \mathcal{P}_1)$ is obvious. To prove that $\text{ind}(\mathcal{P}_2, \mathcal{P}_1) = \text{ind}(\text{Id} - \mathcal{P}_1, \text{Id} - \mathcal{P}_2)$, we simply compute:

$$(6.6) \quad \ker \mathcal{P}_2 \mathcal{P}_1 = \{\phi \mid (\text{Id} - \mathcal{P}_1)\phi = 0, \mathcal{P}_2\phi = 0\}$$

and

$$\begin{aligned} \text{coker } [\mathcal{P}_2 \mathcal{P}_1 : \text{ran}(\mathcal{P}_1) \rightarrow \text{ran}(\mathcal{P}_2)] &\cong \ker [(\mathcal{P}_2 \mathcal{P}_1)^* : \text{ran}(\mathcal{P}_2) \rightarrow \text{ran}(\mathcal{P}_1)] \\ &= \ker [\mathcal{P}_1 \mathcal{P}_2 : \text{ran}(\mathcal{P}_2) \rightarrow \text{ran}(\mathcal{P}_1)] \\ (6.7) \quad &= \{\phi \mid (\text{Id} - \mathcal{P}_2)\phi = 0, \mathcal{P}_1\phi = 0\}. \end{aligned}$$

Replacing \mathcal{P}_2 with $\text{Id} - \mathcal{P}_1$ and \mathcal{P}_1 with $\text{Id} - \mathcal{P}_2$ in (6.6) and (6.7), we obtain

$$\ker(\text{Id} - \mathcal{P}_1)(\text{Id} - \mathcal{P}_2) = \{\phi \mid \mathcal{P}_2\phi = 0, (\text{Id} - \mathcal{P}_1)\phi = 0\}$$

and

$$\text{coker}(\text{Id} - \mathcal{P}_1)(\text{Id} - \mathcal{P}_2) = \{\phi \mid \mathcal{P}_1\phi = 0, (\text{Id} - \mathcal{P}_2)\phi = 0\}.$$

Comparing these spaces with (6.6) and (6.7), we see that $\text{ind}(\mathcal{P}_2, \mathcal{P}_1) = \text{ind}(\text{Id} - \mathcal{P}_1, \text{Id} - \mathcal{P}_2)$.

To prove the logarithm property, just note that by the logarithm property of the usual index, we have

$$\text{ind}(\mathcal{P}_1, \mathcal{P}_2) + \text{ind}(\mathcal{P}_2, \mathcal{P}_3) = \text{ind} [\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3 : \text{ran}(\mathcal{P}_3) \rightarrow \text{ran}(\mathcal{P}_1)].$$

Hence, it remains to show that $\text{ind}(\mathcal{P}_1\mathcal{P}_3) = \text{ind}(\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3)$. To see this, we write

$$\mathcal{P}_1\mathcal{P}_3 = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3 + \mathcal{P}_1(\text{Id} - \mathcal{P}_2)\mathcal{P}_3.$$

Since $\mathcal{P}_1 = \mathcal{P}_2$ modulo compact, it follows that modulo compact, we have

$$\mathcal{P}_1(\text{Id} - \mathcal{P}_2)\mathcal{P}_3 = \mathcal{P}_2(\text{Id} - \mathcal{P}_2)\mathcal{P}_3 = 0.$$

Thus, $\mathcal{P}_1\mathcal{P}_3$ and $\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3$ differ by a compact operator, so $\text{ind}(\mathcal{P}_1\mathcal{P}_3) = \text{ind}(\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3)$, and our proof is complete. \square

6.2. Proof of Theorem 1.5. Let M be a manifold with corners with an exact b -metric g and let $\mathcal{D} : C^\infty(M, E) \rightarrow C^\infty(M, F)$ be a Dirac operator. Assume that M is decomposed into two submanifolds M_\pm with corners along a hypersurface Y :

$$M = M_- \cup M_+ \quad , \quad \partial M_- = \partial M_+ = Y$$

where $g|_{M_\pm}$ satisfies the conditions for manifold with multi-cylindrical end boundaries, in particular, Y is admissible for both M_\pm . Now let us prove Theorem 1.5: If

$$\mathcal{D} : H_b^1(M, E) \rightarrow L_b^2(M, F)$$

is Fredholm, then for arbitrary projections $\mathcal{P}_\pm \in Gr_\infty^*(\mathcal{D}_\pm)$, we have

$$\text{ind } \mathcal{D} = \text{ind } \mathcal{D}_{\mathcal{P}_-} + \text{ind } \mathcal{D}_{\mathcal{P}_+} - \text{ind}(\mathcal{P}_-, \text{Id} - \mathcal{P}_+).$$

Before proceeding with the proof, we need a classical result about *Fredholm pairs*. Let \mathcal{H}_1 and \mathcal{H}_2 be closed subspaces in a Hilbert space \mathcal{H} . Then the pair $(\mathcal{H}_1, \mathcal{H}_2)$ is called as *Fredholm pair* of subspaces if

$$\dim(\mathcal{H}_1 \cap \mathcal{H}_2) < \infty \quad \text{and} \quad \dim(\mathcal{H}_1^\perp \cap \mathcal{H}_2^\perp) < \infty,$$

in which case, we define the *index* of the pair $(\mathcal{H}_1, \mathcal{H}_2)$ to be the integer

$$\text{ind}(\mathcal{H}_1, \mathcal{H}_2) := \dim(\mathcal{H}_1 \cap \mathcal{H}_2) - \dim(\mathcal{H}_1^\perp \cap \mathcal{H}_2^\perp).$$

Now we recall the following classic result from [5, p. 263].

Lemma 6.7. *Let \mathcal{H}_i , $i = 1, 2$, be closed subspaces in a Hilbert space \mathcal{H} and let P_i be the orthogonal projection onto \mathcal{H}_i . Then \mathcal{H}_1 and \mathcal{H}_2 form a Fredholm pair in \mathcal{H} if and only if*

$$(\text{Id} - P_2) P_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_2^\perp$$

is Fredholm, in which case,

$$\text{ind}(\mathcal{H}_1, \mathcal{H}_2) = \text{ind}(\text{Id} - P_2, P_1).$$

Proof. We simply compute:

$$\ker(\text{Id} - P_2) P_1 = \{\phi \in \mathcal{H}_1 \mid (\text{Id} - P_2)\phi = 0\} = \mathcal{H}_1 \cap \mathcal{H}_2$$

and

$$\begin{aligned} \text{coker} \left[(\text{Id} - P_2) P_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_2^\perp \right] &\cong \ker \left[((\text{Id} - P_2) P_1)^* : \mathcal{H}_2^\perp \rightarrow \mathcal{H}_1 \right] \\ &= \ker \left[P_1 (\text{Id} - P_2) : \mathcal{H}_2^\perp \rightarrow \mathcal{H}_1 \right] \\ &= \{\phi \in \mathcal{H}_2^\perp \mid P_1 \phi = 0\} \\ &= \mathcal{H}_1^\perp \cap \mathcal{H}_2^\perp. \end{aligned}$$

\square

Now to prove the Bojarski conjecture let us first prove

$$\text{ind } \mathcal{D} = \text{ind}(\text{Id} - \mathcal{C}_-, \mathcal{C}_+) = \text{ind}(\text{Id} - \mathcal{C}_+, \mathcal{C}_-).$$

By the unique continuation property [1, Cor. 1], it is obvious that

$$\ker \mathcal{D} \cong \text{ran}(\mathcal{C}_-) \cap \text{ran}(\mathcal{C}_+) \quad , \quad \text{coker } \mathcal{D} \cong \ker \mathcal{D}^* \cong \text{ran}(\mathcal{C}_-^*) \cap \text{ran}(\mathcal{C}_+^*)$$

where \mathcal{C}_\pm^* denotes the orthogonal Calderón projection for the restriction of \mathcal{D}^* to M_\pm . Applying Proposition 6.5, we obtain

$$\begin{aligned}\operatorname{ran}(\mathcal{C}_-^*) \cap \operatorname{ran}(\mathcal{C}_+^*) &= \operatorname{ran}(G_0(\operatorname{Id} - \mathcal{C}_-)G_0^*) \cap \operatorname{ran}(G_0(\operatorname{Id} - \mathcal{C}_+)G_0^*) \\ &\cong \operatorname{ran}(\operatorname{Id} - \mathcal{C}_-) \cap \operatorname{ran}(\operatorname{Id} - \mathcal{C}_+) \\ &\cong \operatorname{ran}(\mathcal{C}_-)^{\perp} \cap \operatorname{ran}(\mathcal{C}_+)^{\perp}.\end{aligned}$$

Thus, by Lemma 6.7,

$$\begin{aligned}\operatorname{ind} \mathcal{D} &= \dim \ker \mathcal{D} - \dim \operatorname{coker} \mathcal{D} \\ &= \dim (\operatorname{ran}(\mathcal{C}_-) \cap \operatorname{ran}(\mathcal{C}_+)) - \dim (\operatorname{ran}(\mathcal{C}_-)^{\perp} \cap \operatorname{ran}(\mathcal{C}_+)^{\perp}) \\ &= \operatorname{ind}(\operatorname{Id} - \mathcal{C}_-, \mathcal{C}_+).\end{aligned}$$

Finally, for arbitrary projections $\mathcal{P}_\pm \in Gr_\infty^*(\mathcal{D}_\pm)$, we have

$$\begin{aligned}\operatorname{ind} \mathcal{D} &= \operatorname{ind}(\operatorname{Id} - \mathcal{C}_-, \mathcal{C}_+) \\ &= \operatorname{ind}(\operatorname{Id} - \mathcal{C}_-, \mathcal{P}_+) + \operatorname{ind}(\mathcal{P}_+, \mathcal{C}_+) \\ &= -\operatorname{ind}(\mathcal{C}_-, \operatorname{Id} - \mathcal{P}_+) + \operatorname{ind}(\mathcal{P}_+, \mathcal{C}_+) \\ &= -\operatorname{ind}(\mathcal{C}_-, \mathcal{P}_-) - \operatorname{ind}(\mathcal{P}_-, \operatorname{Id} - \mathcal{P}_+) + \operatorname{ind}(\mathcal{P}_+, \mathcal{C}_+) \\ &= \operatorname{ind}(\mathcal{P}_-, \mathcal{C}_-) + \operatorname{ind}(\mathcal{P}_+, \mathcal{C}_+) - \operatorname{ind}(\mathcal{P}_-, \operatorname{Id} - \mathcal{P}_+) \\ &= \operatorname{ind} \mathcal{D}_{\mathcal{P}_-} + \operatorname{ind} \mathcal{D}_{\mathcal{P}_+} - \operatorname{ind}(\mathcal{P}_-, \operatorname{Id} - \mathcal{P}_+).\end{aligned}$$

REFERENCES

1. C. Bär: ‘Zero sets of solutions to semilinear elliptic systems of first order’, *Invent. Math.* **138**, 183–202, 1999.
2. M. Sh. Birman and M. Z. Solomyak: ‘On subspaces that admit a pseudodifferential projector’, *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.*, no. vyp. 1, 18–25, 133, 1982.
3. D. Bleecker and B. Booß-Bavnbek: ‘Spectral invariants of operators of Dirac type on partitioned manifolds’, *Aspects of Boundary Problems in Analysis and Geometry*, Birkhäuser, Boston, 1–130, 2004.
4. B. Bojarski: ‘The abstract linear conjugation problem and Fredholm pairs of subspaces’, *In Memoriam I.N. Vekua*, Tbilisi Univ, Tbilisi, 45–60, 1979 (Russian).
5. B. Booß-Bavnbek and K. P. Wojciechowski: ‘Elliptic Boundary Problems for Dirac Operators’, Birkhäuser, Boston, 1993.
6. L. Boutet de Monvel: ‘Boundary problems for pseudo-differential operators’ *Acta Math.* **126**, no. 1-2, 11–51, 1971.
7. A. -P. Calderón: ‘Boundary value problems for elliptic equations’, *Outlines Joint Sympos. Partial Differential Equations (Novosibirsk, 1963)*, Acad. Sci. USSR Siberian Branch, Moscow, pp. 303–304, 1963.
8. G. Grubb: ‘Boundary problems for systems of partial differential operators of mixed order’, *J. Funct. Anal.* **26**, 131–165, 1977.
9. ———, *Functional calculus of pseudodifferential boundary problems*, second ed., Progress in Math., Birkhäuser, Boston, 1996.
10. ———: ‘Trace expansions for pseudodifferential boundary problems for Dirac-type operators and more general systems’, *Ark. Math.* **37**, 45–86, 1999.
11. G. Grubb and N. Kokholm: ‘A global calculus of parameter-dependent pseudodifferential boundary problems in L_p Sobolev spaces’, *Acta Math.* **171**, no. 2, 165–229, 1993.
12. L. Hörmander: ‘Pseudo-differential operators and non-elliptic boundary problems’, *Ann. Math.* **83**, 129–209, 1966.
13. ———: *The analysis of linear partial differential operators. III*, second ed., Springer-Verlag, Berlin, 1985.
14. R. Lauter: ‘On representations of ψ^* -algebras and C^* -algebras of b -pseudodifferential operators on manifolds with corners’, *J. Math. Sci. (New York)* **98**, no. 6, 684–705, 2000.

15. R. Lauter and J. Seiler, *Pseudodifferential analysis on manifolds with boundary—a comparison of b -calculus and cone algebra*, Approaches to singular analysis (Berlin, 1999), Birkhäuser, Basel, 2001, pp. 131–166.
16. P. Loya: ‘The structure of the resolvent of elliptic pseudodifferential operators’, *J. Funct. Anal.* **184**, no. 1, 77–135, 2001.
17. ———: ‘Tempered operators and the heat kernel and complex powers of elliptic pseudodifferential operators’, *Comm. Partial Differential Equations* **26**, no. 7 & 8, 1253–1321, 2001.
18. ———: ‘Dirac operators, Boundary Value Problems, and the b -Calculus’, *Contemp. Math.* **366**, 241–280, 2005.
19. P. Loya and R. Melrose: ‘Fredholm perturbations of Dirac operators on manifolds with corners’, *Preprint*, 2003.
20. P. Loya and J. Park: ‘The spectral invariants and Krein’s spectral shift function for Dirac operators on manifolds with multi-cylindrical end boundaries’, *Preprint*, 2004.
21. ———: ‘Gluing formulæ for the spectral invariants of Dirac operators on noncompact manifolds’, *In preparation*.
22. R. Mazzeo: ‘Elliptic theory of differential edge operators. I’, *Comm. Partial Differential Equations* **16**, no. 10, 1615–1664, 1991.
23. R. B. Melrose: *The Atiyah-Patodi-Singer Index Theorem*, A.K. Peters, Wellesley, 1993.
24. ———: *Differential analysis on manifolds with corners*, *In preparation*.
25. R.B. Melrose and V. Nistor: ‘ K -theory of C^* -algebras of b -pseudodifferential operators’, *Geom. Funct. Anal.* **8**, no. 1, 88–122, 1998.
26. R.B. Melrose and P. Piazza: ‘Analytic K -theory on manifolds with corners’, *Adv. Math.* **92**, no. 1, 1–26, 1992.
27. M. Mitrea and V. Nistor: ‘A note on boundary value problems on manifolds with cylindrical ends’, *Aspects of Boundary Problems in Analysis and Geometry*, Birkhäuser, Boston, 472–494, 2004.
28. E. Schrohe: ‘Fréchet algebra techniques for boundary value problems on noncompact manifolds: Fredholm criteria and functional calculus via spectral invariance’, *Math. Nachr.* **199**, 145–185, 1999.
29. E. Schrohe and B.-W. Schulze: ‘Boundary value problems in Boutet de Monvel’s algebra for manifolds with conical singularities. II’, *Boundary value problems, Schrödinger operators, deformation quantization*, Math. Top., vol. 8, Akademie Verlag, Berlin, 70–205, 1995.
30. R. T. Seeley: ‘Singular integrals and boundary value problems’, *Amer. J. Math.* **88**, 781–809, 1966.
31. ———: ‘Topics in pseudo-differential operators’, *Pseudo-Diff. Operators (C.I.M.E., Stresa, 1968)*, 167–305, 1969.
32. K.P. Wojciechowski: ‘Elliptic operators and relative K -homology groups on manifolds with boundary’, *C. R. Math. Rep. Acad. Sci. Canada* **7**, no. 2, 149–154, 1985.

DEPARTMENT OF MATHEMATICS, BINGHAMTON UNIVERSITY, VESTAL PARKWAY EAST, BINGHAMTON, NY 13902, U.S.A.

E-mail address: paul@math.binghamton.edu

SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, 207-43, CHEONGNYANGNI 2-DONG, DONGDAEMUN-GU, SEOUL 130-722, KOREA

E-mail address: jinsung@kias.re.kr