ADIABATIC DECOMPOSITION OF THE ζ-DETERMINANT AND DIRICHLET TO NEUMANN OPERATOR

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Abstract. We discuss the adiabatic decomposition formula of the ζ-determinant of a Laplace type operator on a closed manifold. We also analyze the adiabatic behavior of the ζ-determinant of a Dirichlet to Neumann operator. This analysis makes it possible to compare the adiabatic decomposition formula with the Meyer-Vietoris type formula for the ζ-determinant proved by Burghelea, Friedlander and Kappeler. As a byproduct of this comparison, we obtain the exact value of the local constant which appears in their formula for the case of Dirichlet boundary condition.

1. Introduction and Statement of the Results

In this paper we continue our study of the adiabatic decomposition of the ζ-determinant of the Laplace type operator. In [12], [13] the decomposition formula of the ζ-determinant of Dirac Laplacian was given in terms of the non-local Atiyah-Patodi-Singer boundary condition. Here we discuss a formula which involves the *Laplace type operator* and the *Dirichlet boundary condition*.

Let $\Delta : C^\infty(M,E) \to C^\infty(M,E)$ denote a Laplace type operator acting on sections of a vector bundle E over a closed manifold M of dimension n. The operator Δ is a self-adjoint operator with discrete spectrum $\{\lambda_k\}_{k\in\mathbf{N}}$. Let us decompose M into two sub-manifolds M_1, M_2 with common boundary Y,

$$M = M_1 \cup M_2 \ , \quad M_1 \cap M_2 = Y = \partial M_1 = \partial M_2 \ . \tag{1.1}$$

The ζ-function $\zeta_\Delta(s)$ is defined by

$$\zeta_\Delta(s) = \sum_{\lambda_k \neq 0} \lambda_k^{-s} \ ,$$

which is a holomorphic function in the half-plane $\Re(s) > \frac{n}{2}$ and extends to a meromorphic function on the whole complex plane with $s = 0$ as a regular point. The ζ-determinant of Δ is defined by

$$\log \det{}_\zeta \Delta = -\frac{d}{ds}\zeta_\Delta(s)\Big|_{s=0} \ . \tag{1.2}$$

The derivative of $\zeta_\Delta(s)$ at $s = 0$ can be represented in the following way

$$\frac{d}{ds}\zeta_\Delta(s)\Big|_{s=0} = \lim_{s\to 0}(\kappa(s) - \frac{a'_{n/2}}{s}) + \gamma a'_{n/2} \ . \tag{1.3}$$

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Here γ denotes Euler's constant and $a'_{n/2} := a_{n/2} - \dim\ker(\Delta)$ where $a_{n/2}$ is constant term in the following asymptotic expansion near $t = 0$,

$$\mathrm{Tr}\,(e^{-t\Delta}) \ \sim\ t^{-\frac{n}{2}} \sum_{k=0} a_k t^{\frac{k}{2}} \quad .$$

The function $\kappa(s)$ is defined as the integral

$$\kappa(s) = \int_0^\infty t^{s-1} \big(\mathrm{Tr}\,(e^{-t\Delta}) - \dim\ker(\Delta) \big)\ dt \tag{1.4}$$

for $\Re(s) > \frac{n}{2}$. It has a meromorphic extension to the whole complex plane and it can be represented as

$$\kappa(s) = \frac{a'_{n/2}}{s} + h(s)$$

in a neighborhood of $s = 0$, where $h(s)$ is a holomorphic function of s . The value of the function $h(s)$ at $s = 0$ is not a local invariant, and this fact implies the non-locality of the ζ-determinant. This is the main reason, that there is no straightforward decomposition formula for the ζ-determinant of the operator Δ onto contributions coming from M_1 and M_2 (see [10], [11] and [13] for more detailed discussion).

We assume that there is a bicollar neighborhood $N \cong [-1, 1] \times Y$ of Y in M such that the Riemannian structure on M and the Hermitian structure on E are products of the corresponding structures over $[-1, 1]$ and Y when restricted to N . We also assume that the operator Δ restricted to N has the following form

$$\Delta = -\partial_u^2 + \Delta_Y \quad . \tag{1.5}$$

Here u denotes the normal variable and Δ_Y is a u-independent Laplace type operator on Y .

We replace the bicollar N by $N_R = [-R, R] \times Y$ to obtain a new closed manifold M_R and extend the vector bundle E to M_R in an obvious way. We use formula (1.5) to extend Δ to the Laplace operator Δ_R on M_R . We decompose M_R into $M_{1,R}$ and $M_{2,R}$ by cutting M_R at $\{0\} \times Y$. We denote by $\Delta_{i,R}$ the operator $\Delta_R|_{M_{i,R}}$ subject to the Dirichlet boundary condition. The operator $\Delta_{i,R}$ is a self-adjoint operator with discrete spectrum and smooth eigensections. The ζ-determinant of $\Delta_{i,R}$ is defined as $\det_\zeta \Delta_R$ and it enjoys all the nice properties of the ζ-determinant of the Laplacian on a closed manifold. The concern of this paper is to investigate the *adiabatic decomposition* of $\det_\zeta \Delta_R$, that is, the limit of

$$\frac{\det_\zeta \Delta_R}{\det_\zeta \Delta_{1,R} \cdot \det_\zeta \Delta_{2,R}} \qquad \text{as} \quad R \to \infty \quad . \tag{1.6}$$

The case of the invertible tangential operator Δ_Y was described in [9], [10] and [11]. The invertibility assumption on Δ_Y implies that we have only finitely many eigenvalues of Δ_R converging to 0 as $R \to \infty$. This allows us to discard the large time contribution to the ζ-determinant of Δ_R under the adiabatic process and the adiabatic decomposition of the ζ-determinant easily follows from a standard application of the *Duhamel principle*.

The non-invertible case was studied in [13]. The decomposition formula introduced in [13] uses Atiyah-Patodi-Singer boundary conditions. The new feature of the non-invertible tangential operator is the presence of infinitely many eigenvalues approaching 0 as $R \to \infty$. The behavior of these eigenvalues can be understood in terms of suitable scattering operators described in [8]. We used this description of *small* eigenvalues in the proof of our decomposition formula (see [13], see also announcement [12]). Since the presented results in [12], [13] hold only for the Dirac type operator, we need some modifications to deal with the Laplace case in this paper.

To avoid delicate analytical issues we make one more assumption. Let us recall the classification of the eigenvalues of a Dirac type operator $\mathcal{D}_R$ over M_R. The operator $\mathcal{D}_R$ has finitely many eigenvalues $\{\lambda_k(R)\}$, which decay exponentially as $R \to \infty$, meaning that there exists positive constants c_1 and c_2 such that

$$|\lambda_k(R)| < c_1 e^{-c_2 R} \ .$$

We called them *e-values* in [13]. There are also infinite families of eigenvalues, which decay like R^{-1}, of $\mathcal{D}_R$ and the restrictions of $\mathcal{D}_R$ to $M_{i,R}$ with generalized APS spectral boundary conditions. We called those eigenvalues *s-values* in [13]. Finally, we have infinitely many eigenvalues bounded away from 0 . By our definition, the set of zero eigenvalues is a subset of the set of *e-values* and it is known that the set of *e-values* is stable under the adiabatic process although the set of zero eigenvalues is not. Up to now, no analysis has been known to deal with *e-values*. In order to avoid analytical difficulties related to exponentially small eigenvalues, throughout this paper we assume the following condition:

(1.7) There are no eigenvalues of Δ_R exponentially decaying to 0 as $R \to \infty$.

Hence, this condition means that all the eigenvalues of $\Delta_R, \Delta_{i,R}$ converging to 0 are *s-values* decaying like R^{-2}. There are many natural Laplace type operators satisfying the condition (1.7). For example, let $\Delta^k_{\rho,R}$ denote the Hodge Laplacian over M_R acting on the space of k-forms twisted by the flat vector bundle defined by a unitary representation ρ of $\pi_1(M_R)$. Then, as in Section 4 of [3], one can show that there are no eigenvalues of $\Delta^k_{\rho,R}$ exponentially decaying to 0 as $R \to \infty$ if $\Delta^k_{\rho,0}$ has no zero eigenvalues.

Let $M_{i,\infty}$ denote the manifold M_i with the half infinite cylinder attached and $\Delta_{i,\infty}$ denote the Laplace operators on $M_{i,\infty}$ determined by Δ_i . The operator $\Delta_{i,\infty}$ defines a scattering matrix $C_i(0) : \ker(\Delta_Y) \to \ker(\Delta_Y)$, which is an involution over $\ker(\Delta_Y)$. The following theorem is the first main result of this paper,

Theorem 1.1. *Let us assume that* Δ_R *satisfies* (1.7). *Then we have*

$$\lim_{R\to\infty} R^{h_Y} \frac{\det_\zeta \Delta_R}{\det_\zeta \Delta_{1,R} \cdot \det_\zeta \Delta_{2,R}} = 2^{-h_Y} \sqrt{\det^*_\zeta \Delta_Y} \cdot \det\left(\frac{\mathrm{Id} - C_{12}}{2}\right) \quad , \tag{1.8}$$

where $h_Y := \dim\ker(\Delta_Y)$, $C_{12} := C_1(0)\circ C_2(0)$ *is a unitary operator and* $\det^*_\zeta \Delta_Y$ *denotes the* ζ*-determinant of the operator* Δ_Y *restricted to the orthogonal complement of* $\ker(\Delta_Y)$.

Remark 1.2. The condition (1.7) implies that the operator C_{12} is a unitary operator with no unity eigenvalues (see Remark 2.8). It follows that $\det\left(\frac{\mathrm{Id}-C_{12}}{2}\right)$ is a positive real number. The operators $\Delta_{i,R}$ are Laplacians subject to the Dirichlet conditions so that all their eigenvalues satisfy (1.7) by a standard application of the mini-max principle. The formula (1.8) in Theorem 1.1 has been used in [1] where the adiabatic surgery formula of the determinant line bundle is investigated. The related decomposition formula for the analytic torsion was also worked out by Hassell in [3]. He proved the analytic surgery formula of the analytic torsion using the *b-calculus*. We also refer to the work of Hassell-Mazzeo-Melrose [4] where the analytic surgery problem is investigated extensively.

Our proof of Theorem 1.1 is modelled on a proof given in [13], with necessary modifications since we are dealing with a different type of boundary conditions. The main modification is a revised relation between *s-values* and the scattering matrix $C_i(0)$. This is the main achievement of the first part of this paper, which consists of the following two sections.

In the second part, we study the adiabatic limit of the ζ-determinant of certain operator $\mathcal{R}_R$ appearing in the formula of Burghelea, Friedlander and Kappeler [2] (in short, BFK from now on). The BFK formula can be formulated in our situation as follows,

$$\frac{\det_\zeta \Delta_R}{\det_\zeta \Delta_{1,R} \cdot \det_\zeta \Delta_{2,R}} = C(Y) \det_\zeta \mathcal{R}_R \qquad \text{for any } R \quad , \tag{1.9}$$

where $C(Y)$ is a locally computable constant and $\mathcal{R}_R$ is defined as the sum of the Dirichlet to Neumann operators over the decomposed manifolds $M_{i,R}$. It is well known that $\mathcal{R}_R$ is a nonnegative pseudo-differential operator of order 1. In particular, under the condition (1.7), $\mathcal{R}_R$ is a positive operator for any R.

Remark 1.3. The BFK constant $C(Y)$ is locally computable from symbols of Δ_R^{-1} over Y, so that $C(Y)$ may depend on the intrinsic data over Y as well as the extrinsic data out of Y like the normal derivatives of the symbol of Δ_R^{-1} at Y. However, under the assumption of the product structure near Y , the constant $C(Y)$ depends on only the intrinsic data over Y , in particular $C(Y)$ does not change under the adiabatic process.

In Section 4, we study the adiabatic limit of $\det_\zeta \mathcal{R}_R$. Here we consider the case of the non-invertible tangential operator Δ_Y , as a result, the adiabatic limit of $\det_\zeta \mathcal{R}_R$ contains the contribution determined by Δ_Y as well as the scattering data. The following theorem is the main result for this,

Theorem 1.4. *Let us assume* (1.7)*. Then we have the following formula*

$$\lim_{R\to\infty} R^{h_Y} \cdot \det_\zeta \mathcal{R}_R = 2^{\zeta_{\Delta_Y}(0)} \det{}^*_\zeta \sqrt{\Delta_Y} \cdot \det \Big(\frac{\mathrm{Id} - C_{12}}{2}\Big) \ . \tag{1.10}$$

Now we can use Theorem 1.1, the BFK formula (1.9) and Theorem 1.4 to obtain the local invariant $C(Y)$ as a byproduct of our main theorems.

Corollary 1.5. *The BFK constant $C(Y)$ in the case of Dirichlet boundary condition is equal to*

$$C(Y) = 2^{-\zeta_{\Delta_Y}(0) - h_Y} \ . \tag{1.11}$$

This result is also proved in [5] independently using the local computation of symbols of $\mathcal{R}_R$.

In Section 5 we discuss the proof of the technical result which was used in Section 4 in the computation of the adiabatic limit of the ζ-determinant of $\mathcal{R}_R$. Our approach is based on the representation of the inverse of Δ_R in terms of the heat kernel $e^{-t\Delta_R}$, which enables us to apply the heat kernel analysis and some results proved in the first part of the paper.

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2. Small eigenvalues and scattering matrices

In this section we study the relation between the *s-values* of the operators Δ_R, $\Delta_{i,R}$ and the scattering matrices $C_i(\lambda)$ determined by the operators $\Delta_{i,\infty}$ on $M_{i,\infty}$. This analysis is necessary in order to determine the large time contribution in the adiabatic decomposition formula. The corresponding result for Dirac Laplacians was formulated and proved in [13]. Here we treat the case of a general Laplace type operator and we need to rework some of the details of the analysis presented in [13].

Now let ψ be an element of $\ker(\Delta_Y)$ and λ denote a sufficiently small real number. The couple (ψ, λ) determines a generalized eigensection $E(\psi, \lambda) \in C^\infty(M_{1,\infty}, E)$ of the operator $\Delta_{1,\infty}$ such that

$$\Delta_{1,\infty} E(\psi, \lambda) = \lambda^2 E(\psi, \lambda) \quad .$$

The function $\lambda \to E(\psi, \lambda)$ has a meromorphic extension to a certain subset of $\mathbb{C}$, in particular, this function is analytic function on the interval $(-\delta, \delta)$ for sufficiently small $\delta > 0$. The generalized eigensection $E(\psi, \lambda)$ has the following expression on the cylinder $[0, \infty)_u \times Y$,

$$E(\psi, \lambda) = e^{-i\lambda u}\psi + e^{i\lambda u} C_1(\lambda)\psi + \hat{E}(\psi, \lambda) \quad , \tag{2.1}$$

where $\hat{E}(\psi, \lambda)$ is a smooth L^2-section orthogonal to $\ker(\Delta_Y)$ and $\hat{E}(\psi, \lambda)|_{u=R}$ and $\partial_u \hat{E}(\psi, \lambda)|_{u=R}$ are exponentially decaying as $R \to \infty$. The scattering matrix

$$C_1(\lambda) : \ker(\Delta_Y) \to \ker(\Delta_Y)$$

is a unitary operator. The analyticity of $E(\psi, \lambda)$ implies that $\{C_1(\lambda)\}_{\lambda \in (-\delta, \delta)}$ is an analytic family of linear operators. The operator $C_1(\lambda)$ satisfies the following functional equation

$$C_1(\lambda)\ C_1(-\lambda) = \mathrm{Id} \quad . \tag{2.2}$$

In particular, $C_1(0)^2 = \mathrm{Id}$, hence $C_1(0)$ is an involution over $\ker(\Delta_Y)$.

Let Φ_R be a normalized eigensection of $\Delta_{1,R}$ for the Dirichlet boundary problem, which corresponds to the *s-value* $\lambda^2 = \lambda(R)^2$ with $|\lambda| \le R^{-\kappa}$ for some fixed κ with $0 < \kappa \le 1$. That is,

$$\Delta_{1,R}\Phi_R = \lambda^2 \Phi_R \quad , \quad \Phi_R|_{\{R\} \times Y} = 0 \qquad \text{and} \qquad ||\Phi_R|| = 1 \quad . \tag{2.3}$$

The section Φ_R can be represented in the following way on $[0, R]_u \times Y \subset M_{1,R}$

$$\Phi_R = e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2 + \hat{\Phi}_R$$

where $\psi_i \in \ker(\Delta_Y)$ and $\hat{\Phi}_R$ is orthogonal to $\ker(\Delta_Y)$.

We introduce $F := \Phi_R - E(\psi_1, \lambda)|_{M_{1,R}}$ where λ is the positive square root of λ^2 . Green's theorem gives

$$\begin{aligned} 0 &= \langle \Delta_{1,R} F, F \rangle_{M_{1,R}} - \langle F, \Delta_{1,R} F \rangle_{M_{1,R}} \\ &= -\int_{\partial M_{1,R}} \langle \partial_u F|_{u=R}, F|_{u=R} \rangle \, dy + \int_{\partial M_{1,R}} \langle F|_{u=R}, \partial_u F|_{u=R} \rangle \, dy \quad , \end{aligned} \tag{2.4}$$

and we can obtain the following equalities

$$\begin{aligned} & 2\lambda\, i \, \| \, C_1(\lambda)\psi_1 - \psi_2 \, \|^2 \\ &= - \ \langle \ \partial_u(\, \hat{\Phi}_R - \hat{E}(\psi_1, \lambda) \,)|_{u=R} \ , \ (\, \hat{\Phi}_R - \hat{E}(\psi_1, \lambda) \,)|_{u=R} \ \rangle \\ &\quad + \ \langle \ (\, (\hat{\Phi}_R - \hat{E}(\psi_1, \lambda) \,)|_{u=R} \ , \ \partial_u(\, \hat{\Phi}_R - \hat{E}(\psi_1, \lambda) \,)|_{u=R} \ \rangle \\ &= - \ \langle \ \partial_u(\, \hat{\Phi}_R - \hat{E}(\psi_1, \lambda) \,)|_{u=R} \ , \ -\hat{E}(\psi_1, \lambda)|_{u=R} \ \rangle \\ &\quad + \ \langle \ -\hat{E}(\psi_1, \lambda)|_{u=R} \ , \ \partial_u(\, \hat{\Phi}_R - \hat{E}(\psi_1, \lambda) \,)|_{u=R} \ \rangle \quad . \end{aligned} \tag{2.5}$$

The following lemma will be used to show that the right side of (2.5) is exponentially small as $R \to \infty$,

Lemma 2.1. *For $R \gg 0$, there exists a constant C independent of R such that*

$$|| \, \partial_u \hat{\Phi}_R|_{u=R} \, ||_Y \leq C.$$

Proof. We have the representation of $\hat{\Phi}_R$ on the cylinder $[0,R]_u \times Y \subset M_{1,R}$,

$$\hat{\Phi}_R(u,y) \;=\; \sum_{k=h_Y+1}^{\infty} (a_k(R) e^{\sqrt{\mu_k^2-\lambda^2}u} + b_k(R) e^{-\sqrt{\mu_k^2-\lambda^2}u}) \phi_k$$

where $\{\mu_k^2, \phi_k\}$ is the spectral resolution of Δ_Y , such that $\{\phi_k\}_{k=1}^{h_Y}$ is an orthonormal basis of $\ker(\Delta_Y)$. The normalized condition for Φ_R implies the inequality

$$\sum_{k=h_Y+1}^{\infty} \int_0^R | \; a_k(R) e^{\sqrt{\mu_k^2-\lambda^2}u} + b_k(R) e^{-\sqrt{\mu_k^2-\lambda^2}u} \; |^2 \; du \;\; \leq 1$$

which leads to

$$1 \;\geq\; \sum_{k=h_Y+1}^{\infty} \Bigg(\frac{1}{2\sqrt{\mu_k^2-\lambda^2}} \big(\; |a_k(R)|^2 (e^{2\sqrt{\mu_k^2-\lambda^2}R} - 1) + |b_k(R)|^2 (1 - e^{-2\sqrt{\mu_k^2-\lambda^2}R}) \; \big) + \; 2\Re(a_k(R) b_k(R)) R \Bigg) \; .$$

The boundary condition put the following constraint on the coefficients $a_k(R)$, $b_k(R)$

$$a_k(R) e^{\sqrt{\mu_k^2-\lambda^2}R} + b_k(R) e^{-\sqrt{\mu_k^2-\lambda^2}R} \;=\; 0 \;\; .$$

As a result, if $R \gg 0$, the following estimate holds,

$$\begin{aligned} (2.6) \qquad 1 \;&\geq\; \sum_{k=h_Y+1}^{\infty} \frac{|a_k(R)|^2 e^{2\sqrt{\mu_k^2-\lambda^2}R}}{4\sqrt{\mu_k^2-\lambda^2}} \Bigg(\; 1 + e^{2\sqrt{\mu_k^2-\lambda^2}R} - 8\sqrt{\mu_k^2-\lambda^2}R \; \Bigg) \\ &\geq\; \sum_{k=h_Y+1}^{\infty} (\mu_k^2 - \lambda^2)|a_k(R)|^2 \; e^{2\sqrt{\mu_k^2-\lambda^2}R} \Bigg(\frac{1 + e^{\sqrt{\mu_k^2-\lambda^2}R}}{4(\mu_k^2-\lambda^2)^{3/2}} \Bigg) \\ &\geq\; \sum_{k=h_Y+1}^{\infty} (\mu_k^2 - \lambda^2)|a_k(R)|^2 \; e^{2\sqrt{\mu_k^2-\lambda^2}R} \;\; . \end{aligned}$$

On the other hand, we can see that

$$(2.7) \qquad || \; \partial_u \hat{\Phi}_R|_{u=R} \; ||_Y^2 \;=\; 4 \sum_{k=h_Y+1}^{\infty} (\mu_k^2 - \lambda^2)|a_k(R)|^2 e^{2\sqrt{\mu_k^2-\lambda^2}R} \;\; .$$

By (2.6) and (2.7), there is a constant C independent of R such that

$$\| \; \partial_u \hat{\Phi}_R|_{u=R} \; \|_Y \leq C \;\; .$$

□

Now Lemma 2.1 and the fact that $\hat{E}(\psi,\lambda)|_{u=R}$ and $\partial_u\hat{E}(\psi,\lambda)|_{u=R}$ are exponentially decaying as $R\to\infty$ imply

$$\| C_1(\lambda)\psi_1-\psi_2 \|^2 \le c_1\lambda^{-1}e^{-c_2R}\le e^{-c_3R} \tag{2.8}$$

for some positive constants c_1,c_2,c_3 . The second inequality follows from the condition (1.7). Now the Dirichlet boundary condition at $u=R$ of

$$\Phi_R=e^{-i\lambda u}\psi_1+e^{i\lambda u}\psi_2+\hat{\Phi}_R$$

provides us with the following equality,

$$\psi_2=-e^{-2i\lambda R}\psi_1.$$

From this equality and the estimate (2.8), we get the following inequality,

$$\| e^{2i\lambda R}C_1(\lambda)\psi_1+\psi_1 \| \le e^{-cR} \quad . \tag{2.9}$$

Recall that $\{C_1(\lambda)\}_{\lambda\in(-\delta,\delta)}$ is an analytic family of the operators. Analytic perturbation theory guarantees the existence of the real analytic functions $\alpha_j(\lambda)$ of $\lambda\in(-\delta,\delta)$, such that $\exp(i\alpha_j(\lambda))$ are the corresponding eigenvalues of $C_1(\lambda)$ for $\lambda\in(-\delta,\delta)$. Hence, from (2.9), we can obtain

$$|\, e^{i(2\lambda R+\alpha_j(\lambda))}+1\,| \le e^{-cR} \quad .$$

This immediately implies

Proposition 2.2. *For $R\gg 0$, the positive square root $\lambda(R)$ of* s-value $\lambda(R)^2$ *of $\Delta_{1,R}$ with $\lambda(R)\le R^{-\kappa}$ $(0<\kappa\le 1)$ satisfies*

$$2R\lambda(R)+\alpha_j(\lambda(R))=(2k+1)\pi+O(e^{-cR}) \tag{2.10}$$

for an integer k with $0<(2k+1)\pi-\alpha_j(\lambda(R))\le R^{1-\kappa}$, where $\exp(i\alpha_j(\lambda))$ is an eigenvalue of the unitary operator $C_1(\lambda):\ker(\Delta_Y)\to\ker(\Delta_Y)$.

Now, we consider equation (2.10) when $k=0$. The function $\alpha_j(\lambda)$ is a real analytic function of λ , hence we have

$$2R\lambda(R)+\alpha_{j0}+\alpha_{j1}\lambda(R)+\alpha_{j2}\lambda(R)^2+\cdots=\pi+O(e^{-cR}) \tag{2.11}$$

for some constants α_{jk}'s. The operator $C_1(0)$ is an involution, so $\alpha_{j0}=0$ or $\alpha_{j0}=\pi$. It is not difficult to show that, if we assume $\alpha_{j0}=\pi$, then λ decays exponentially. However, the operator $\Delta_{1,R}$ does not have the exponentially decaying eigenvalues, therefore $\alpha_{j0}=0$. Now we proved

Proposition 2.3. *For $R\gg 0$, the positive square root $\lambda(R)$ of $s-value$ $\lambda(R)^2$ of $\Delta_{1,R}$ with $\lambda(R)\le R^{-\kappa}$ $(0<\kappa\le 1)$ satisfies*

$$2R\lambda(R)=(2k+1)\pi+O(R^{-\kappa}) \qquad \textit{or} \qquad 2R\lambda(R)=2k\pi+O(R^{-\kappa}) \tag{2.12}$$

where $0<(2k+1)\pi\le R^{1-\kappa}$ or $0<2k\pi\le R^{1-\kappa}$.

Now one can easily prove that the similar result as in Proposition 2.3 holds for $\Delta_{2,R}$ simply repeating the previous argument with the scattering matrix $C_2(\lambda) : \ker(\Delta_Y) \to \ker(\Delta_Y)$.

We are going to formulate Proposition 2.3 and the corresponding result for $\Delta_{2,R}$ in terms of certain model operator over S^1. Let $U : W \to W$ denote a unitary operator acting on a d-dimensional vector space W with eigenvalues $e^{i\alpha_j}$ for $j = 1, \cdots, d$. We define the operator $\Delta(U)$,

$$\Delta(U) := -\frac{1}{4}\frac{d^2}{du^2} : C^\infty(S^1, E_U) \to C^\infty(S^1, E_U)$$

where E_U is the flat vector bundle over $S^1 = \mathbb{R}/\mathbb{Z}$ defined by the holonomy U . The spectrum of $\Delta(U)$ is equal to

$$\{\ (\pi k + \frac{1}{2}\alpha_j)^2 \mid k \in \mathbb{Z}\ ,\ j = 1, \cdots, d\ \}\ . \tag{2.13}$$

We also have

$$\det{}_\zeta \Delta(U) = 4^d \prod_{j=1}^{d} \sin^2(\frac{\alpha_j}{2}) \tag{2.14}$$

if $\alpha_j \neq 2k\pi$ ($k \in \mathbb{Z}$) for $j = 1, \cdots, d$ (see for instance [7]). Putting $\overline{C}_i := -C_i(0)$, by definition, the operator $\Delta(\overline{C}_i)$ has a nontrivial kernel which is determined by (1)-eigenspace of $\overline{C}_i$. We denote by h_i the dimension of this space.

Proposition 2.4. *For any family of eigenvalues $\lambda(R)^2$ of $\Delta_{i,R}$ converging to zero as $R \to \infty$, there exists the eigenvalue λ_k^2 of $\Delta(\overline{C}_i)$ with $\lambda_k > 0$ so that for $R \gg 0$,*

$$R^2\lambda(R)^2 = \lambda_k^2 + O(R^{1-2\kappa})\ , \tag{2.15}$$

and there is R_1 depending on R with $|R_1^{1-\kappa} - R^{1-\kappa}| \leq \frac{\pi}{2}$ such that (2.15) defines one to one correspondence between the eigenvalues of $\Delta_{i,R}$ with $0 < \lambda(R)^2 \leq R^{-2\kappa}$ and the eigenvalues of $\Delta(\overline{C}_i)$ with $0 < \lambda_k^2 \leq R_1^{2-2\kappa}$ and $\lambda_k > 0$.

Proof. The equality (2.15) follows from Proposition 2.3, the corresponding result for $\Delta_{2,R}$ and the definition of $\Delta(\overline{C}_i)$. For the second statement, by definitions, it is obvious that (2.15) defines an injective map from the eigenvalues of $\Delta_{i,R}$ with $0 < \lambda(R)^2 \leq R^{-2\kappa}$ to the eigenvalues of $\Delta(\overline{C}_i)$ with $0 < \lambda_k^2 \leq R^{2-2\kappa}$ and $\lambda_k > 0$. To define R_1 with the desired property, let us decompose $M_{i,R}$ into M_i and the cylindrical part of length R. Then the restrictions of $\Delta_{i,R}$ onto these decomposed parts provide us with the Laplace type operators imposing the Dirichlet boundary conditions. By the mini-max principle, for $R \gg 0$, the number of eigenvalues $\leq R^{-2\kappa}$ of $\Delta_{i,R}$ is same as the number of eigenvalues $\leq R^{-2\kappa}$ of the operator over the cylindrical part since there are no such small eigenvalues of the operator over M_i. By the explicit computation over the cylinder of length R, the eigenvalues of the operator over the cylinder of length R are given by h_Y-copies of $k^2\pi^2R^{-2}$ with $k \in \mathbb{N}$. Therefore, the number of eigenvalues $\leq R^{-2\kappa}$ of the operator over the cylindrical part is given by $h_Y[\pi^{-1}R^{1-\kappa}]$. Using (2.13), we can choose R_1 such that $|R_1^{1-\kappa} - R^{1-\kappa}| \le \frac{\pi}{2}$ and $h_Y[\pi^{-1}R^{1-\kappa}]$ is same as the number of the eigenvalues of $\Delta(\overline{C}_i)$ with $\lambda_k^2 \le R_1^{2-2\kappa}$ and $\lambda_k > 0$. This completes the proof. □

Now we split

$$\mathrm{Tr}\,(e^{-tR^2\Delta_{i,R}}) = \mathrm{Tr}_{\,1,R}(e^{-tR^2\Delta_{i,R}}) + \mathrm{Tr}_{\,2,R}(e^{-tR^2\Delta_{i,R}}) \quad ,$$

where $\mathrm{Tr}_{\,1,R}(\cdot)$, $\mathrm{Tr}_{\,2,R}(\cdot)$ denote the parts of the traces restricted to the nonzero eigenvalues $> R^{\frac{1}{2}}$ or $\le R^{\frac{1}{2}}$ of $R^2\Delta_{i,R}$ respectively. Similarly, we split

$$\mathrm{Tr}\,(e^{-t\Delta(\overline{C}_i)}) - h_i = \mathrm{Tr}_{\,1,R}(e^{-t\Delta(\overline{C}_i)}) + \mathrm{Tr}_{\,2,R}(e^{-t\Delta(\overline{C}_i)})$$

where $\mathrm{Tr}_{\,1,R}(\cdot)$, $\mathrm{Tr}_{\,2,R}(\cdot)$ denote the parts of the traces restricted to the nonzero eigenvalues $> R_1^{\frac{1}{2}}$ or $\le R_1^{\frac{1}{2}}$ of $\Delta(\overline{C}_i)$ respectively. Now we have the estimate for $\mathrm{Tr}_{\,2,R}(\cdot)$ in the following proposition.

Proposition 2.5. *For $R \gg 0$, there exist positive constants c_1, c_2 such that*

$$|\ \mathrm{Tr}_{\,2,R}(e^{-tR^2\Delta_{i,R}}) - \frac{1}{2}[\,\mathrm{Tr}_{\,2,R}(e^{-t\Delta(\overline{C}_i)}) - h_i\,]\ | \ \le\ c_1\, R^{-\frac{1}{4}}\, t\, e^{-c_2 t} \quad .$$

Proof. We apply Proposition 2.4 for fixed $\kappa = \frac{3}{4}$ and obtain that for any eigenvalue $\lambda(R)^2$ of $\Delta_{i,R}$ with $|\lambda(R)| \le R^{-\frac{3}{4}}$, there exists a function $\alpha(R)$ such that

$$R^2\lambda(R)^2 = \lambda_j^2 + \alpha(R), \qquad |\alpha(R)| \le c\, R^{-\frac{1}{2}}$$

if R is sufficiently large. We use the elementary inequality $|e^{-\lambda} - 1| \le |\lambda| e^{|\lambda|}$ to get

$$\begin{aligned} |e^{-tR^2\lambda(R)^2} - e^{-t\lambda_j^2}| &= |e^{-t\lambda_j^2}(e^{-t[R^2\lambda(R)^2-\lambda_j^2]} - 1)| \\ &\le c\, R^{-\frac{1}{2}}\, t\, e^{-(\lambda_j^2-\alpha(R))t} \le c\, R^{-\frac{1}{2}}\, t\, e^{-\frac{1}{2}\lambda_j^2 t} \quad . \end{aligned}$$

Let us fix a sufficiently large R. We take the sum over finitely many nonzero eigenvalues $\lambda(R)^2$ of $\Delta_{i,R}$ with $\lambda(R)^2 \le R^{-\frac{3}{2}}$, and obtain

$$|\ \mathrm{Tr}_{\,2,R}(e^{-tR^2\Delta_{i,R}}) - \frac{1}{2}[\,\mathrm{Tr}_{\,2,R}(e^{-t\Delta(\overline{C}_i)}) - h_i\,]\ | \le c\, R^{-\frac{1}{2}}\, t \sum_{\lambda_j^2 \le R_1^{\frac{1}{2}}} e^{-\frac{1}{2}\lambda_j^2 t} \quad .$$

The operator $\Delta(\overline{C}_i)$ is a Laplace type operator over S^1 , hence the number of eigenvalues λ_j^2 with $\lambda_j^2 \le R_1^{\frac{1}{2}}$ can be estimated by $R_1^{\frac{1}{4}}$. Since $|R_1^{\frac{1}{4}} - R^{\frac{1}{4}}| \le \frac{\pi}{2}$, we have

$$c\, R^{-\frac{1}{2}}\, t \sum_{\lambda_j^2 \le R_1^{\frac{1}{2}}} e^{-\frac{1}{2}\lambda_j^2 t} \le c'\, R^{-\frac{1}{4}}\, t\, e^{-\frac{1}{2}\lambda_1^2 t}$$

where λ_1^2 denotes the first non-zero eigenvalue of $\Delta(\overline{C}_i)$. This completes the proof. □

Now we shall prove the corresponding result for the *s-values* of Δ_R over M_R. Let Ψ_R denote (a normalized) eigensection of Δ_R corresponding to *s-value* λ^2, that is, $\Delta_R \Psi_R = \lambda^2 \Psi_R$ and $\|\Psi_R\| = 1$. Over the cylindrical part $[-R, R]_u \times Y$ in M_R, the eigensection Ψ_R corresponding to *s-value* λ^2 of Δ_R has the following form,

$$\Psi_R = e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2 + \hat{\Psi}_R \tag{2.16}$$

where $\psi_i \in \ker(\Delta_Y)$ and $\hat{\Psi}_R$ is orthogonal to $\ker(\Delta_Y)$. We first need the following lemma, where $\{0\} \times Y$ denotes the cutting hypersurface in M_R.

Lemma 2.6. *We have the following estimates*

$$||\hat{\Psi}_R|_{u=0}||_Y \le c_1 e^{-c_2 R} \quad , \quad ||\partial_u \hat{\Psi}_R|_{u=0}||_Y \le c_1 e^{-c_2 R}$$

where c_1, c_2 are positive constants independent of R .

Proof. The section $\hat{\Psi}_R$ has the following form on $[-R, R]_u \times Y \subset M_R$,

$$\hat{\Psi}_R(u, y) \ = \ \sum_{k=h_Y+1}^{\infty} (a_k(R) e^{\sqrt{\mu_k^2 - \lambda^2} u} + b_k(R) e^{-\sqrt{\mu_k^2 - \lambda^2} u}) \phi_k \quad .$$

The normalization condition on the eigensection implies

$$\sum_{k=h_Y+1}^{\infty} \int_{-R}^{R} |\ a_k(R) e^{\sqrt{\mu_k^2 - \lambda^2} u} + b_k(R) e^{-\sqrt{\mu_k^2 - \lambda^2} u}\ |^2 \ du \ \le 1 \ ,$$

and now we have the following estimates for sufficiently large R

$$\begin{aligned}
1 \ \ge \ & \sum_{k=h_Y+1}^{\infty} \Bigg(\frac{1}{2\sqrt{\mu_k^2 - \lambda^2}} [\ |a_k(R)|^2 (e^{2\sqrt{\mu_k^2 - \lambda^2} R} - e^{-2\sqrt{\mu_k^2 - \lambda^2} R}) \\
& \qquad + |b_k(R)|^2 (e^{2\sqrt{\mu_k^2 - \lambda^2} R} - e^{-2\sqrt{\mu_k^2 - \lambda^2} R})\ ] + \ 4\Re(a_k(R) b_k(R)) R \Bigg) \\
\ge \ & \sum_{k=h_Y+1}^{\infty} \frac{1}{4\sqrt{\mu_k^2 - \lambda^2}} (\ \ |a_k(R)|^2 e^{2\sqrt{\mu_k^2 - \lambda^2} R} \\
& \qquad + |b_k(R)|^2 e^{2\sqrt{\mu_k^2 - \lambda^2} R} - 16 |a_k(R) b_k(R)| R\ ) \\
\ge \ & \sum_{k=h_Y+1}^{\infty} \frac{1}{8\sqrt{\mu_k^2 - \lambda^2}} (\ \ |a_k(R)|^2 e^{2\sqrt{\mu_k^2 - \lambda^2} R} + |b_k(R)|^2 e^{2\sqrt{\mu_k^2 - \lambda^2} R}\ ) \quad .
\end{aligned}$$

This immediately implies

$$\sum_{k=h_Y+1}^{\infty} |a_k(R)|^2 + |b_k(R)|^2 \ \le \ c_1 e^{-\sqrt{\mu_{h_Y+1}^2 - \lambda^2} R} \ \le \ c_1 e^{-c_2 R}$$

for some positive constants c_1, c_2. Hence, the first estimate is proved and the proof of the second estimate follows in the same way. □

Changing variable $v = u + R$, we regard that the cylindrical part is given by $[0, 2R]_v \times Y$. In particular, we have the new expression for Ψ_R from (2.16),

$$\Psi_R = e^{-i\lambda v}\phi_1^1 + e^{i\lambda v}\phi_2^1 + \hat{\Psi}_R$$

where $\phi_1^1 = e^{i\lambda R}\psi_1$, $\phi_2^1 = e^{-i\lambda R}\psi_2$. Now repeating the argument which leads us to (2.8), we obtain

$$\| C_1(\lambda)\phi_1^1 - \phi_2^1 \| \le e^{-cR} \tag{2.17}$$

for a positive constant c. Note that here we used the condition (1.7) and Lemma 2.6. Now we want to get the corresponding estimate involving the scattering matrix $C_2(\lambda)$. For this, we change the variable by $v = u - R$ and regard the cylindrical part as $[-2R, 0]_v \times Y$. Then we have the corresponding expression for Ψ_R,

$$\Psi_R = e^{-i\lambda v}\phi_1^2 + e^{i\lambda v}\phi_2^2 + \hat{\Psi}_R$$

where $\phi_1^2 = e^{-i\lambda R}\psi_1$, $\phi_2^2 = e^{i\lambda R}\psi_2$. We again repeat the previous argument to obtain

$$\| C_2(\lambda)\phi_2^2 - \phi_1^2 \| \le e^{-cR} \tag{2.18}$$

for a positive constant c. Here $C_2(\lambda)$ is the scattering matrix defined from the generalized eigensection attached to (λ, ϕ_2^2). By definition, we have

$$\phi_1^1 = e^{2i\lambda R}\phi_1^2 \quad , \quad \phi_2^1 = e^{-2i\lambda R}\phi_2^2 \quad . \tag{2.19}$$

Now, combining (2.17), (2.18) and (2.19), we get

$$\| e^{4i\lambda R} C_1(\lambda) \circ C_2(\lambda)\phi_2^1 - \phi_2^1 \| \le e^{-cR} \quad . \tag{2.20}$$

As before, $C_1(\lambda) \circ C_2(\lambda)$ is an analytic family for $\lambda \in (-\delta, \delta)$ for sufficiently small $\delta > 0$. Then there exist the analytic functions $\alpha_j(\lambda)$ for $\lambda \in (-\delta, \delta)$ such that $\exp((i\alpha_j(\lambda))$ are the eigenvalues of the unitary operator $C_{12}(\lambda) := C_1(\lambda) \circ C_2(\lambda)$ on $\ker(\Delta_Y)$. Hence the equality (2.20) implies

$$| \, e^{i(4\lambda R + \alpha_j(\lambda))} - 1 \, | \le e^{-cR} \quad .$$

Therefore we obtain

Proposition 2.7. *For $R \gg 0$, the positive square root $\lambda(R)$ of* s-value $\lambda(R)^2$ *of Δ_R with $\lambda(R) \le R^{-\kappa}$ satisfies*

$$4R\lambda(R) + \alpha_j(\lambda(R)) = 2k\pi + O(e^{-cR}) \tag{2.21}$$

for an integer k with $0 < 2k\pi - \alpha_j(\lambda(R)) \le 4R^{1-\kappa}$, where $\exp(i\alpha_j(\lambda))$ *is the eigenvalue of the unitary operator $C_{12}(\lambda)$ on* $\ker(\Delta_Y)$.

Remark 2.8. Note that the spectrum of the unitary operator $C_{12} := C_{12}(0)$ acting on $\ker(\Delta_Y)$ consists of m eigenvalues of -1 (such that $h_Y - m \ge 0$ is an even number) and $\{ \, e^{i\alpha_j(0)}, e^{-i\alpha_j(0)} \mid j = 1, \cdots, \frac{h_Y - m}{2} \, \}$ where $\alpha_j(0)$ is not equal to $k\pi$ for $k \in \mathbb{Z}$. This follows from the argument presented around (2.11) and the condition (1.7).

Now we follow the way to prove Proposition 2.4 and obtain

Proposition 2.9. *For any family of eigenvalues $\lambda(R)^2$ of Δ_R converging to zero as $R \to \infty$, there exists the eigenvalue λ_k^2 of $\Delta(C_{12})$ with $\lambda_k > 0$ so that for $R \gg 0$,*

$$4R^2\lambda(R)^2 = \lambda_k^2 + O(R^{1-2\kappa}) \ , \tag{2.22}$$

and there is R_1 depending on R with $|R_1^{1-\kappa} - R^{1-\kappa}| \leq \frac{\pi}{4}$ such that (2.22) defines one to one correspondence between the eigenvalues of Δ_R with $0 < \lambda(R)^2 \leq R^{-2\kappa}$ and the eigenvalues of $\Delta(C_{12})$ with $0 < \lambda_k^2 \leq 4R_1^{2-2\kappa}$ and $\lambda_k > 0$.

We split

$$\mathrm{Tr}\,(e^{-tR^2\Delta_R}) = \mathrm{Tr}_{\,1,R}(e^{-tR^2\Delta_R}) + \mathrm{Tr}_{\,2,R}(e^{-tR^2\Delta_R}) \ ,$$

where $\mathrm{Tr}_{\,1,R}(\cdot)$, $\mathrm{Tr}_{\,2,R}(\cdot)$ denote the parts of the traces restricted to the nonzero eigenvalues $> R^{\frac{1}{2}}$ or $\leq R^{\frac{1}{2}}$ of $R^2\Delta_R$ respectively. Similarly we split

$$\mathrm{Tr}\,(e^{-t\frac{1}{4}\Delta(C_{12})}) = \mathrm{Tr}_{\,1,R}(e^{-t\frac{1}{4}\Delta(C_{12})}) + \mathrm{Tr}_{\,2,R}(e^{-t\frac{1}{4}\Delta(C_{12})})$$

where $\mathrm{Tr}_{\,1,R}(\cdot)$, $\mathrm{Tr}_{\,2,R}(\cdot)$ denote the parts of the traces restricted to the nonzero eigenvalues $> R_1^{\frac{1}{2}}$ or $\leq R_1^{\frac{1}{2}}$ of $\frac{1}{4}\Delta(C_{12})$ respectively. As in Proposition 2.5, we can prove the following proposition.

Proposition 2.10. *For $R \gg 0$, there exist positive constants c_1, c_2 such that*

$$|\ \mathrm{Tr}_{\,2,R}(e^{-tR^2\Delta_R}) - \frac{1}{2}\mathrm{Tr}_{\,2,R}(e^{-t\frac{1}{4}\Delta(C_{12})})\ | \ \leq \ c_1\, R^{-\frac{1}{4}}\, t\, e^{-c_2 t} \quad .$$

3. Proof of Theorem 1.1

In this section we present a proof of Theorem 1.1. Since the analysis of *s-values* is done in Section 2, now we can proceed by a standard way as in [12] and [13].

We define relative ζ-function $\zeta_{\mathrm{rel}}^R(s)$,

$$\zeta_{\mathrm{rel}}^R(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\mathrm{Tr}\,(e^{-t\Delta_R} - e^{-t\Delta_{1,R}} - e^{-t\Delta_{2,R}})\ dt \ , \tag{3.1}$$

and we decompose $\zeta_{\mathrm{rel}}^R(s)$ into two parts

$$\zeta_{\mathrm{s}}^R(s) = \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} (\cdot)\ dt \ \ , \ \ \zeta_{\mathrm{l}}^R(s) = \frac{1}{\Gamma(s)} \int_{R^{2-\varepsilon}}^\infty (\cdot)\ dt$$

where $\varepsilon > 0$ is a fixed sufficiently small number. The derivatives of $\zeta_{\mathrm{s}}^R(s)$ and $\zeta_{\mathrm{l}}^R(s)$ at $s = 0$ give the small and large time contributions to our formula. First, we prove

Lemma 3.1. *There exist positive constants c_1 and c_2 such that*

$$| \operatorname{Tr}(e^{-t\Delta_R} - e^{-t\Delta_{1,R}} - e^{-t\Delta_{2,R}}) - \frac{1}{2}\operatorname{Tr}(e^{-t\Delta_Y}) | \le c_1 e^{-c_2 \frac{R^2}{t}} \quad .$$

Proof. By the standard application of Duhamel principle as in [10], [13], the estimate of $\operatorname{Tr}(e^{-t\Delta_R} - e^{-t\Delta_{1,R}} - e^{-t\Delta_{2,R}})$ follows from the estimate of the parametrices of the heat kernels $e^{-t\Delta_R}$, $e^{-t\Delta_{i,R}}$. These parametrices are constructed from the heat kernels on the closed manifold M_R and heat kernels of the boundary problems on the half infinite cylinders. The interior contributions cancel each other out up to the error term of the size $O(e^{-c\frac{R^2}{t}})$ for a positive constant c and only the boundary contribution is left. This boundary term is equal to

$$\int_{-R}^{R} \frac{1}{\sqrt{4\pi t}} \operatorname{Tr}(e^{-t\Delta_Y})\, du - 2\int_0^R \frac{1}{\sqrt{4\pi t}}\{1 - e^{-\frac{u^2}{t}}\}\operatorname{Tr}(e^{-t\Delta_Y})\, du$$
$$= 2\int_0^R \frac{1}{\sqrt{4\pi t}} e^{-\frac{u^2}{t}} \operatorname{Tr}(e^{-t\Delta_Y})\, du = \frac{1}{\sqrt{\pi}}\int_0^{\frac{R}{\sqrt{t}}} e^{-v^2}\operatorname{Tr}(e^{-t\Delta_Y})\, dv$$
$$= \frac{1}{2}\operatorname{Tr}(e^{-t\Delta_Y}) + O(e^{-\frac{R^2}{t}}) \quad .$$

This completes the proof. □

Now we can determine the small time part in (3.1).

Proposition 3.2. *We have*

$$\lim_{R\to\infty} [\, (\zeta_s^R)'(0) - \frac{h_Y}{2}(\gamma + (2-\epsilon)\log R)\,] = \frac{1}{2}\zeta'_{\Delta_Y}(0) \quad ,$$

where

$$\zeta_{\Delta_Y}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\big(\operatorname{Tr}(e^{-t\Delta_Y}) - h_Y\big)\; dt \quad .$$

Proof. By Lemma 3.1, the function

$$f_R(s) = \frac{1}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}\big(\operatorname{Tr}(e^{-t\Delta_R} - e^{-t\Delta_{1,R}} - e^{-t\Delta_{2,R}}) - \frac{1}{2}\operatorname{Tr}(e^{-t\Delta_Y})\big)\; dt$$

is a holomorphic function of s on the whole complex plane. Moreover, the following equalities hold

$$\lim_{R\to\infty} f_R(0) = 0 \quad , \quad \lim_{R\to\infty} \frac{d}{ds} f_R(s)\Big|_{s=0} = 0 \; .$$

Combining these facts with the following equality

$$\frac{d}{ds}\Big|_{s=0}\left(\frac{h_Y}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}\; dt\right) = h_Y(\gamma + (2-\varepsilon)\log R) \quad , \tag{3.2}$$

completes the proof. □

To deal with the large time part, we need the following lemma.

Lemma 3.3. *For $R \gg 0$, there exists a positive constant c_1 such that*

$$\int_{R^{-\epsilon}}^{\infty} t^{-1} \mathrm{Tr}_{\,1,R}(e^{-tR^2\Delta_{i,R}})\, dt \le c_1\, e^{-R^{\frac{1}{2}-\varepsilon}}$$

and the similar estimates hold for $\mathrm{Tr}_{\,1,R}(e^{-tR^2\Delta_R})$, $\mathrm{Tr}_{\,1,R}(e^{-t\Delta(\overline{C}_i)}) - h_i$ *and* $\mathrm{Tr}_{\,1,R}(e^{-t\frac{1}{4}\Delta(C_{12})})$.

Proof. Let $\lambda_{k_0}^2(R)$ denote the smallest *large* eigenvalue of $\Delta_{i,R}$ such that $\lambda_{k_0}^2(R) > R^{-\frac{3}{2}}$. Then, if $R \gg 0$ we have

$$\begin{aligned}
\mathrm{Tr}_{\,1,R}(e^{-tR^2\Delta_{i,R}}) &= \sum_{\lambda_k^2 > R^{-\frac{3}{2}}} e^{-tR^2\lambda_k^2} = \sum_{\lambda_k^2 > R^{-\frac{3}{2}}} e^{-(tR^2-1)\lambda_k^2} e^{-\lambda_k^2} \\
&\le e^{-(tR^2-1)\lambda_{k_0}^2} \sum_{\lambda_k^2 > R^{-\frac{3}{2}}} e^{-\lambda_k^2} \le e^{-(tR^2-1)\lambda_{k_0}^2} \mathrm{Tr}\,(e^{-\Delta_{i,R}}) \\
&\le c_2\, R\, e^{-(tR^2-1)R^{-\frac{3}{2}}} \le c_3\, R\, e^{-R^{\frac{1}{2}}t}
\end{aligned}$$

for positive constants c_2, c_3. We have used here the obvious estimate

$$\mathrm{Tr}\,(e^{-\Delta_{i,R}}) \le c\,\mathrm{vol}(M_{i,R}) \le c' R$$

for positive constants c, c'. Now we have

$$\begin{aligned}
\int_{R^{-\varepsilon}}^{\infty} t^{-1} \mathrm{Tr}_{\,1,R}(e^{-tR^2\Delta_{i,R}})\, dt &\le \int_{R^{-\varepsilon}}^{\infty} t^{-1} c_3\, R\, e^{-tR^{\frac{1}{2}}}\, dt \\
&\le c_3 R \int_{R^{\frac{1}{2}-\varepsilon}}^{\infty} e^{-v} dv \le c_1\, e^{-R^{\frac{1}{2}-\varepsilon}} \quad .
\end{aligned}$$

This completes the proof of the first estimate and the other cases can be proved in the same way.

□

Now we can express the large time part in terms of the model operators.

Proposition 3.4.

$$\lim_{R\to\infty} \int_{R^{2-\varepsilon}}^{\infty} t^{-1} \mathrm{Tr}\,(e^{-t\Delta_R} - e^{-t\Delta_{1,R}} - e^{-t\Delta_{2,R}})\; dt + \frac{h_Y}{2}(\gamma - \varepsilon\cdot \log R)$$

$$= \frac{1}{2}\frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\mathrm{Tr}\,(e^{-\frac{t}{4}\Delta(C_{12})} - e^{-t\Delta(\overline{C}_1)} - e^{-t\Delta(\overline{C}_2)}) + h_Y \right)\, dt \quad .$$

Proof. First, let us observe that Remark 2.8 and the relation $C_i(0)^2 = \mathrm{Id}$ imply $h_Y = h_1 + h_2$. Using this and the change of variable $t \to R^{-2}t$, one can obtain following equality from Proposition 2.5, 2.10 and Lemma 3.3,

$$\lim_{R\to\infty}\Big(\int_{R^{2-\varepsilon}}^{\infty} t^{-1}\mathrm{Tr}\,(e^{-t\Delta_R}-e^{-t\Delta_{1,R}}-e^{-t\Delta_{2,R}})\ dt - \frac{1}{2}\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_{R^{-\varepsilon}}^{\infty} t^{s-1}\ [\mathrm{Tr}\,(e^{-\frac{t}{4}\Delta(C_{12})} - e^{-t\Delta(\overline{C}_1)} - e^{-t\Delta(\overline{C}_2)}) + h_Y]\ dt\Big) = 0\ .$$

Note that near $t=0$,

$$|\ \mathrm{Tr}\,(e^{-\frac{t}{4}\Delta(C_{12})}-e^{-t\Delta(\overline{C}_1)}-e^{-t\Delta(\overline{C}_2)})\,|\le c\sqrt{t}$$

for a positive constant c. By this estimate, one can easily show

$$\lim_{R\to\infty}\Big(h_Y(\gamma-\varepsilon\cdot\log\,R)-\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^{R^{-\varepsilon}} t^{s-1}\ [\mathrm{Tr}\,(e^{-\frac{t}{4}\Delta(C_{12})} - e^{-t\Delta(\overline{C}_1)} - e^{-t\Delta(\overline{C}_2)}) + h_Y]\ dt\Big) = 0\ .$$

These complete the proof. □

Proposition 3.2 and 3.4 combined together lead to the following equality

$$\lim_{R\to\infty}\Big((\zeta_{\mathrm{s}}^R)'(0)-\frac{h_Y}{2}(\gamma+(2-\varepsilon)\cdot\log R)+(\zeta_{\mathrm{l}}^R)'(0)+\frac{h_Y}{2}(\gamma-\varepsilon\cdot\log R)\Big) = \frac{1}{2}\Big(\ \zeta'_{\Delta_Y}(0)+\zeta'_{\frac{1}{4}\Delta(C_{12})}(0)-\zeta'_{\Delta(\overline{C}_1)}(0)-\zeta'_{\Delta(\overline{C}_2)}(0)\Big)\ . \tag{3.3}$$

Now the following proposition gives the exact value of the large time contribution,

Proposition 3.5. *We have*

$$\det{}_\zeta\frac{1}{4}\Delta(C_{12}) = 2^{2h_Y}\det(\frac{\mathrm{Id}-C_{12}}{2})^2\quad,\quad \det{}^*_\zeta\Delta(\overline{C}_i)=2^{2h_Y}\ .$$

Proof. The first equality follows directly from (2.14). For the second one, the zeta function of $\Delta(\overline{C}_i)$ is given by

$$\zeta_{\Delta(\overline{C}_i)}(s)=h_i\,2\pi^{-2s}\sum_{k=1}^{\infty}k^{-2s}+(h_Y-h_i)\,2\pi^{-2s}\sum_{k=0}^{\infty}(k+\frac{1}{2})^{-2s}$$

where h_i is the dimension of $(+1)$-eigenspace of $\overline{C}_i$. Then the derivative of $\zeta_{\Delta(\overline{C}_i)}(s)$ at $s=0$ is equal to $-h_Y\log 4$. This completes the proof of the second one. □

Finally we obtain Theorem 1.1 using the equality (3.3) and Proposition 3.5.

4. The adiabatic limit of $\det_\zeta \mathcal{R}_R$

In this section we study the behavior of $\det_\zeta \mathcal{R}_R$ when $R \to \infty$.

Let us describe the construction of $\mathcal{R}_R$. It is defined as the composition of the following maps

$$C^\infty(Y, E|_Y) \xrightarrow{I_g} C^\infty(Y, E|_Y) \oplus C^\infty(Y, E|_Y) \xrightarrow{\mathcal{K}_\mathcal{R}} C^\infty(\overline{M}_R, E)$$
$$\xrightarrow{\gamma_1} C^\infty(Y, E|_Y) \oplus C^\infty(Y, E|_Y) \xrightarrow{I_f} C^\infty(Y, E|_Y).$$

Here $I_g(\phi) := (\phi, \phi)$ and $\mathcal{K}_R$ is the Poisson operator of the operator $\Delta_{1,R} \sqcup \Delta_{2,R}$ over a manifold $\overline{M}_R := M_{1,R} \sqcup M_{2,R}$. For (Φ_1, Φ_2) where Φ_i is a section over $M_{i,R}$, the map γ_1 is given by $\gamma_1(s) := (\partial_u\big|_{Y_1}\Phi_1, \partial_u\big|_{Y_2}\Phi_2)$ and $I_f(\phi, \psi) := \phi - \psi$. It is well known that the operator

$$\mathcal{R}_R := I_f\, \gamma_1\, \mathcal{K}_R\, I_g \ : \ C^\infty(Y, E|_Y) \to C^\infty(Y, E|_Y)$$

is an elliptic, nonnegative, pseudo-differential operator of order 1 . By definition, the operator $\mathcal{R}_R$ can be written as

$$\mathcal{R}_R \ = \ \mathcal{N}_{1,R} + \mathcal{N}_{2,R}$$

where $\mathcal{N}_{i,R}$ is the Dirichlet to Neumann operator for $\Delta_R|_{M_{i,R}}$.

A careful analysis of the small eigenvalues enables us to compute the scattering contribution to the adiabatic limit of the ζ-determinant of $\mathcal{R}_R$. Let us recall that $\{\mu_k^2, \phi_k\}_{k\in\mathbb{N}}$ denotes the spectral resolution of the operator Δ_Y with $h_Y = \dim\ker(\Delta_Y)$. The equality (2.2) implies

$$C_i(0)C_i'(0) = C_i'(0)C_i(0) \quad ,$$

hence we may choose ϕ_k (for $1 \le k \le h_Y$) so that ϕ_k is a normalized eigensection for both operators $C_i(0)$ and $C_i'(0)$. Now, we have

Proposition 4.1. *For any couple (ϕ_m, ϕ_n) with $1 \le m, n \le h_Y$,*

$$\langle \mathcal{N}_{i,R}\phi_m, \phi_n\rangle \ = \ \begin{cases} \frac{1}{R}(1 - \frac{\alpha}{2R})^{-1} & \text{if} \quad m = n \ , \quad C_i(0)\phi_m = -\phi_m \quad , \\ O(e^{-cR}) & \text{if} \quad m \neq n \quad \text{or} \quad C_i(0)\phi_m = \phi_m \end{cases}$$

where $C_i'(0)\phi_n = i\alpha\phi_n$, that is, $i\alpha$ is the eigenvalue of $C_i'(0)$ and c is a positive constant .

Proof. We present a proof for the case of $i = 1$. The case for $i = 2$ can be proved in the same way. Let Φ_R denote a solution of the problem

$$\Delta_{M_{1,R}}\Phi_R = 0 \quad \text{and} \quad \Phi_R|_Y = \phi_m \quad ,$$

hence

$$\partial_u\Phi_R|_{u=R} = \mathcal{N}_{1,R}\phi_m \ . \tag{4.1}$$

To simplify notation in the proof we skip the indices m in ϕ_m and R in Φ_R . Let us define

$$\Phi(\phi, \lambda) := e^{-i\lambda R}\Phi \quad ,$$

for a small positive λ. For such a λ and $\psi := \phi_n \in \ker(\Delta_Y)$, there exists the generalized eigensection $E(\psi, \lambda)$ over $M_{1,\infty}$, which has the following form on the cylinder $[0,\infty)_u \times Y \subset M_{1,\infty}$,

$$E(\psi,\lambda) = e^{-i\lambda u}\psi + e^{i\lambda u}C_1(\lambda)\psi + \hat{E}(\psi,\lambda)$$

where $\hat{E}(\psi,\lambda)$ is a L^2-section. We also define

$$G = G(\phi,\psi,\lambda) := E(\psi,\lambda)|_{M_{1,R}} - \Phi(\phi,\lambda) \quad .$$

An auxiliary section $G(\phi,\psi,\lambda)$ has the following properties

$$\Delta_{1,R}G(\phi,\psi,\lambda) = \lambda^2 E(\psi,\lambda) \quad ,$$

$$G|_{u=R} = e^{-i\lambda R}\psi + e^{i\lambda R}C_1(\lambda)\psi - e^{-i\lambda R}\phi + O(e^{-cR}) \quad ,$$

$$\partial_u G|_{u=R} = -i\lambda e^{-i\lambda R}\psi + i\lambda e^{i\lambda R}C_1(\lambda)\psi - e^{-i\lambda R}\mathcal{N}_{1,R}\phi + O(e^{-cR}) \quad .$$

Green's formula for G reads as

$$\begin{aligned}(4.2)\quad & \langle \Delta_{1,R}G, G\rangle_{M_{1,R}} - \langle G, \Delta_{1,R}G\rangle_{M_{1,R}} \\ &= -\langle \partial_u G|_{\{R\}\times Y}, G|_{\{R\}\times Y}\rangle_{\{R\}\times Y} + \langle G|_{\{R\}\times Y}, \partial_u G|_{\{R\}\times Y}\rangle_{\{R\}\times Y} \quad .\end{aligned}$$

The equation (4.2) can be rewritten as follows

$$\begin{aligned} & \lambda^2(\langle \Phi, E\rangle_{M_{1,R}} - \langle E,\Phi\rangle_{M_{1,R}}) \\ = \ & e^{-2i\lambda R}\langle \mathcal{N}_{1,R}\phi, C_1(\lambda)\psi\rangle_Y - e^{2i\lambda R}\langle C_1(\lambda)\psi, \mathcal{N}_{1,R}\phi\rangle_Y \\ & + i\lambda e^{-2i\lambda R}\langle \phi, C_1(\lambda)\psi\rangle_Y + i\lambda e^{2i\lambda R}\langle C_1(\lambda)\psi,\phi\rangle_Y \\ & + \langle \mathcal{N}_{1,R}\phi,\psi\rangle_Y - \langle \psi, \mathcal{N}_{1,R}\phi\rangle_Y - \langle \mathcal{N}_{1,R}\phi,\phi\rangle_Y + \langle \phi, \mathcal{N}_{1,R}\phi\rangle_Y \\ & - i\lambda\langle\phi,\psi\rangle_Y - i\lambda\langle\psi,\phi\rangle_Y + O(e^{-cR}) \quad . \end{aligned} \tag{4.3}$$

We differentiate both sides of the equality (4.3) at $\lambda = 0$ and obtain

$$\begin{aligned} & -2iR(\langle \mathcal{N}_{1,R}\phi, C_1(0)\psi\rangle_Y + \langle C_1(0)\psi, \mathcal{N}_{1,R}\phi\rangle_Y) \\ & + \langle \mathcal{N}_{1,R}\phi, C_1'(0)\psi\rangle_Y - \langle C_1'(0)\psi, \mathcal{N}_{1,R}\phi\rangle_Y \\ & + i(\langle \phi, C_1(0)\psi\rangle_Y + \langle C_1(0)\psi,\phi\rangle_Y) - i\langle\phi,\psi\rangle_Y - i\langle\psi,\phi\rangle_Y = O(e^{-cR}) \quad . \end{aligned} \tag{4.4}$$

Proposition 4.1 follows easily from (4.4). Let us consider for instance the case of

$$\phi = \psi = \phi_n \in \ker(C_1(0)+1) \subset \ker(\Delta_Y) \quad .$$

Then, the equation (4.4) is now

$$(2iR - i\alpha)(\langle \mathcal{N}_{1,R}\phi,\phi\rangle_Y + \langle\phi, \mathcal{N}_{1,R}\phi\rangle_Y) = 4i + O(e^{-cR}) \quad ,$$

and this gives the following formula,

$$\langle \mathcal{N}_{1,R}\phi,\phi\rangle_Y + \langle\phi,\mathcal{N}_{1,R}\phi\rangle_Y = \frac{2}{R}(1 - \frac{\alpha}{2R})^{-1} + O(e^{-cR}) \quad . \tag{4.5}$$

□

Let us also observe the following fact, which is an immediate corollary of Proposition 4.1.

Corollary 4.2. *We have*

$$\langle \mathcal{R}_R\phi, \phi\rangle \;=\; O(e^{-cR}) \quad \textit{for} \quad \phi \in \ker(C_1(0)-1)\cap\ker(C_2(0)-1)$$

for a positive constant c.

Remark 4.3. Corollary 4.2 and an elementary application of the mini-max principle show that, in general, the operator $\mathcal{R}_R$ may have exponentially decaying eigenvalues. Moreover, the number of these eigenvalues is equal to

$$\dim\big(\ \ker(C_1(0)-1)\cap\ker(C_2(0)-1)\,\big) \ \ .$$

On the other hand, the condition (1.7) and Remark 2.8 imply

$$\ker(C_1(0)-1)\cap\ker(C_2(0)-1) = \{0\}\ \ , \tag{4.6}$$

hence it excludes the existence of exponentially small eigenvalues of $\mathcal{R}_R$ under the condition (1.7). A simple example where (4.6) holds is the Dirac Laplacian over the double of a manifold with boundary. It is easy to observe that in this case we have $C_1(0) = -C_2(0)$ and there is no exponentially small eigenvalues of $\mathcal{R}_R$.

Proposition 4.1 suggests the introduction of the operator $L(R)$ on $\ker(\Delta_Y)$,

$$L(R) \;=\; \frac{1}{R}\left(\frac{\mathrm{Id}-C_1(0)}{2}+\frac{\mathrm{Id}-C_2(0)}{2}\right) \quad .$$

Proposition 4.4. *Assume that* $\ker(C_1(0)-\mathrm{Id})\cap\ker(C_2(0)-\mathrm{Id}) = \{0\}$ *. Then we have*

$$\det L(R) \;=\; R^{-h_Y}\det\big(\frac{\mathrm{Id}-C_{12}}{2}\big) \tag{4.7}$$

where $C_{12} := C_1(0)\circ C_2(0)$*.*

Proof. First of all, the assumption implies that the direct sum of the ranges of the projections $\frac{\mathrm{Id}-C_1(0)}{2}$, $\frac{\mathrm{Id}-C_2(0)}{2}$ spans the space $\ker(\Delta_Y)$. It also follows from the definition that we have a formula

$$\det L(R) = R^{-h_Y}\det\left(\frac{\mathrm{Id}-C_1(0)}{2}+\frac{\mathrm{Id}-C_2(0)}{2}\right) \quad .$$

Now, we use the fact that

$$\frac{\mathrm{Id}-C_2(0)}{2} = \left(\frac{\mathrm{Id}-C_1(0)C_2(0)}{2}\right)^{-1}\frac{\mathrm{Id}+C_1(0)}{2}\left(\frac{\mathrm{Id}-C_1(0)C_2(0)}{2}\right)\ , \tag{4.8}$$

hence, essentially our concern is the determinant of the operator acting on $\mathbb{C}^{h_Y}$ with the form

$$P + g^{-1}(\mathrm{Id}-P)g \ \ ,$$

putting $P = \frac{\mathrm{Id}-C_1(0)}{2}$ and $g = \frac{\mathrm{Id}-C_1(0)C_2(0)}{2}$. We write

$$P + g^{-1}(\mathrm{Id}-P)g = g^{-1}(gP + (\mathrm{Id}-P)g) \ \ .$$

The second operator on the right side can be represented in the following form

$$gP + (\mathrm{Id} - P)g = \begin{pmatrix} PgP & 0 \\ 2(\mathrm{Id} - P)gP & (\mathrm{Id} - P)g(\mathrm{Id} - P) \end{pmatrix} \tag{4.9}$$

with respect to range $(P) \oplus$ range $(\mathrm{Id} - P)$. The corresponding decomposition for the operator $P - g^{-1}(\mathrm{Id} - P)g$ is

$$g^{-1}\begin{pmatrix} PgP & 0 \\ 0 & -(\mathrm{Id} - P)g(\mathrm{Id} - P) \end{pmatrix} \quad .$$

This shows that

$$\begin{aligned} \det\left(\frac{\mathrm{Id} - C_1(0)}{2} + \frac{\mathrm{Id} - C_2(0)}{2}\right) &= (-1)^{h_2} \det\left(\frac{\mathrm{Id} - C_1(0)}{2} - \frac{\mathrm{Id} - C_2(0)}{2}\right) \\ &= (-1)^{h_2} \det\left(\frac{\mathrm{Id} - C_{12}}{2}\right) \det C_2(0) = \det\left(\frac{\mathrm{Id} - C_{12}}{2}\right) \quad . \end{aligned}$$

□

Proof of Theorem 1.4: Let P^0 and $P^{\perp}$ denote orthogonal projections onto the subspaces $\ker(\Delta_Y)$ and $\ker(\Delta_Y)^{\perp}$. For any trace class operator L acting on $L^2(Y, E|_Y)$, we define

$$\mathrm{Tr}^{\,0}(L) := \mathrm{Tr}\,(P^0 L P^0) \quad , \quad \mathrm{Tr}^{\,\perp}(L) := \mathrm{Tr}\,(P^{\perp} L P^{\perp}) \quad .$$

We decompose $\mathrm{Tr}\,(e^{-t\mathcal{R}_R})$ into $\mathrm{Tr}^{\,0}(e^{-t\mathcal{R}_R})$ and $\mathrm{Tr}^{\,\perp}(e^{-t\mathcal{R}_R})$. By Proposition 4.1, it is easy to see that the part $\mathrm{Tr}^{\,0}(e^{-t\mathcal{R}_R})$ contributes by $\det L(R)$ up to the error of the size $O(R^{-h_Y-1})$. By Proposition 4.4, this is $R^{-h_Y} \det(\frac{\mathrm{Id} - C_{12}}{2})$ up to the error of the size $O(R^{-h_Y-1})$.

Now let us see the contribution from $\mathrm{Tr}^{\,\perp}(e^{-t\mathcal{R}_R})$. Let us consider

$$\begin{aligned} &\frac{i}{2\pi}\int_{\Gamma} \lambda^{-s}\ \mathrm{Tr}^{\,\perp}((\mathcal{R}_R - \lambda)^{-1} - (2\sqrt{\Delta_Y} - \lambda)^{-1})\ d\lambda \\ &= (-1)^k k! \frac{i}{2\pi}\int_{\Gamma} (s-1)^{-1} \cdots (s-k)^{-k} \lambda^{-s+k} \\ &\qquad\qquad \mathrm{Tr}^{\,\perp}((\mathcal{R}_R - \lambda)^{-(k+1)} - (2\sqrt{\Delta_Y} - \lambda)^{-(k+1)})\ d\lambda \end{aligned}$$

for sufficiently large k. Here, Γ is a curve surrounding $\{0\} \cup \mathbb{R}^-$ in $\mathbb{C}$. Let us remark that $\mathcal{R}_R - 2\sqrt{\Delta_Y}$ is a smoothing operator. We refer the proof of this fact to [14]. Now the integrand on the right side can be estimated as

$$\begin{aligned} &|\ \mathrm{Tr}^{\,\perp}((\mathcal{R}_R - \lambda)^{-(k+1)} - (2\sqrt{\Delta_Y} - \lambda)^{-(k+1)})\ | \\ &\le \frac{C}{|\lambda|^k + 1}\ |\ \mathrm{Tr}^{\,\perp}(\mathcal{R}_R^{-1} - (2\sqrt{\Delta_Y})^{-1})\ | \end{aligned}$$

for a positive constant C. Here $(2\sqrt{\Delta_Y})^{-1}$ denotes the inverse of $2\sqrt{\Delta_Y}$ over $\ker(\Delta_Y)^{\perp}$. Now, we use Proposition 5.1 proved in Section 5, to show that the concerned integrand converges to 0 uniformly for every s in the compact neighborhood of 0 as $R \to \infty$. Hence its derivative at $s = 0$ converges to 0 as $R \to \infty$. This completes the proof of Theorem 1.4 if we use

$$\det{}^*_\zeta(2\sqrt{\Delta_Y}) = 2^{\zeta_\Delta(0)}\det{}^*_\zeta\sqrt{\Delta_Y}\ . \tag{4.10}$$

Proof of Corollary 1.5: Let us now come back to the BFK formula (1.9),

$$\frac{\det_\zeta \Delta_R}{\det_\zeta \Delta_{1,R} \cdot \det_\zeta \Delta_{2,R}} \ = \ C(Y)\det_\zeta \mathcal{R}_R \ \ .$$

We can use Theorem 1.1 and Theorem 1.4 to find the exact value of the local constant $C(Y)$. Let us recall that $C(Y)$ does not depend on the adiabatic process. Now, we have

$$2^{-h_Y}\sqrt{\det{}^*_\zeta\Delta_Y}\cdot\det\big(\frac{\mathrm{Id}-C_{12}}{2}\big) = \lim_{R\to\infty} R^{h_Y}\frac{\det_\zeta \Delta_R}{\det_\zeta \Delta_{1,R} \cdot \det_\zeta \Delta_{2,R}}$$

$$= C(Y)\ \lim_{R\to\infty} R^{h_Y}\det_\zeta \mathcal{R}_R = C(Y)\ 2^{\zeta_{\Delta_Y}(0)}\det{}^*_\zeta\sqrt{\Delta_Y}\cdot\det\big(\frac{\mathrm{Id}-C_{12}}{2}\big)\ \ .$$

From this and the equality $\sqrt{\det{}^*_\zeta\Delta_Y} = \det{}^*_\zeta\sqrt{\Delta_Y}$, we conclude

$$C(Y) = 2^{-\zeta_{\Delta_Y}(0)-h_Y}\ \ .$$

5. Proof of technical proposition

In this section we present the proof of the following proposition,

Proposition 5.1. *For $R \gg 0$, there exist positive constants c_1 and c_2 such that*

$$|\ \mathrm{Tr}^{\perp}(\mathcal{R}_R^{-1} - (2\sqrt{\Delta_Y})^{-1})\ |\ \le c_1 e^{-c_2 R^{\frac{1}{2}}}\ \ .$$

Instead of using $2\sqrt{\Delta_Y}$, we compare the operator $\mathcal{R}_R$ with the model operator $\mathcal{R}_R^c$ on the cylinder defined as follows. We introduce the cylinder $N_R = [-R, R] \times Y$ with the Laplacian $\Delta_R^c = -\partial_u^2 + \Delta_Y$ subject to the Dirichlet boundary conditions at $\{\pm R\} \times Y$. Now, we cut N_R at $u = 0$ and get the operator $\mathcal{R}_R^c$ in an obvious way. An explicit computation shows that the operator $\mathcal{R}_R^c$ converges to $2\sqrt{\Delta_Y}$ exponentially on the space $\ker(\Delta_Y)^\perp$, more precisely

$$|\ \mathrm{Tr}^{\perp}(\mathcal{R}_R^c - 2\sqrt{\Delta_Y})\ |\ \le\ c_3 e^{-c_4 R}$$

for some positive constants c_3, c_4 . Therefore, it is sufficient to show

$$|\ \mathrm{Tr}^{\perp}(\mathcal{R}_R^{-1} - (\mathcal{R}_R^c)^{-1})\ |\ \le c_1 e^{-c_2 R^{\frac{1}{2}}}\ \ . \tag{5.1}$$

In order to prove (5.1), we recall the following formula for $\mathcal{R}_R^{-1}$ established in [2], [6],

$$\mathcal{R}_R^{-1}\ =\ \gamma\Delta_R^{-1}\gamma^*$$

where γ is the restriction map to $\{0\} \times Y$ and γ^* is the adjoint of γ. We combine this equality with

$$\Delta_R^{-1} \;=\; \int_0^\infty e^{-t\Delta_R}\; dt \quad , \tag{5.2}$$

in order to reduce our problem to the heat kernel estimates. We decompose the left side of (5.2) into two parts as follows

$$\int_0^\infty e^{-t\Delta_R}\; dt \;=\; \int_0^{R^{2-\varepsilon}} e^{-t\Delta_R}\; dt \;+\; \int_{R^{2-\varepsilon}}^\infty e^{-t\Delta_R}\; dt \quad .$$

We will consider the large and small time contributions separately in the following lemmas.

Lemma 5.2. *For $R \gg 0$, there are positive constants c_1, c_2 such that*

$$|\operatorname{Tr}^{\perp}\Big(\int_{R^{2-\varepsilon}}^\infty \gamma\, e^{-t\Delta_R}\gamma^*\; dt\Big)\,| \le c_1 e^{-c_2 R^{1-\varepsilon}}$$

and the same estimate holds for Δ_R^c .

Proof. We note that

$$\gamma\, e^{-t\Delta_R}\,\gamma^* = \sum_k e^{-t\lambda_k^2}\Phi_k(x)|_{u=0} \otimes \Phi_k^*(y)|_{u=0} \tag{5.3}$$

where $\{\lambda_k^2, \Phi_k\}$ is a spectral resolution of the operator Δ_R. We split the restriction of the eigensection Φ_k to $\{0\}\times Y$ into Φ_k^0 the part in $\ker(\Delta_Y)$ and $\hat{\Phi}_k$ the remaining part. We employ an argument similar to the proof of Lemma 2.6 to obtain

$$\|\hat{\Phi}_k\| \;\le\; c_1 e^{-\sqrt{\mu_{h_Y+1}^2 - \lambda_k^2}R} \quad . \tag{5.4}$$

Here, we note that the right side of (5.4) has to be changed into the constant c_1 if $\lambda_k > \mu_{h_Y+1}$, and the constant c_1 is independent of k . We need to discuss only the contribution determined by $\hat{\Phi}_k$ since we are concerning only on $\operatorname{Tr}^{\perp}(\cdot)$. We split this contribution in (5.3) into two parts, that is, the sums over all eigenvalues $R^{-1} \le \lambda_k^2$ and $\lambda_k^2 < R^{-1}$.

In order to discuss the sum over the eigenvalues smaller than R^{-1}, we use (5.4) and the fact that each eigenvalue of Δ_R is bounded from below by $\frac{c}{R^{2+\frac{\varepsilon}{2}}}$ (since there is no exponentially small eigenvalues). Then we have

$$\int_{R^{2-\varepsilon}}^\infty \Bigg(\sum_{\lambda_k^2 < R^{-1}} e^{-t\lambda_k^2}\|\hat{\Phi}_k\|^2 \Bigg)\; dt \;\le\; c_1 e^{-c_2 R} \int_{R^{2-\varepsilon}}^\infty \Bigg(\sum_{\lambda_k^2 < R^{-1}} e^{-t\lambda_k^2} \Bigg)\; dt \tag{5.5}$$
$$\le c_1 e^{-c_2 R}\operatorname{Tr}(e^{-\Delta_R}) \int_{R^{2-\varepsilon}}^\infty e^{-(t-1)R^{-(2+\frac{\varepsilon}{2})}}\; dt \le\; c_3 e^{-c_4 R}$$

for positive constants c_1, c_2, c_3, c_4. We have used here the obvious estimate

$$\operatorname{Tr}(e^{-\Delta_R}) \le c_5 \operatorname{vol}(M_R) \le c_6 R \quad .$$

The sum over the eigenvalues $R^{-1} \leq \lambda_k^2$ can be estimated as

$$\int_{R^{2-\varepsilon}}^{\infty} \Big(\sum_{R^{-1}\leq\lambda_k^2} e^{-t\lambda_k^2} \|\hat{\Phi}_k\|^2 \Big)\, dt \;\leq\; c_1^2 \int_{R^{2-\varepsilon}}^{\infty} \Big(\sum_{R^{-1}\leq\lambda_k^2} e^{-t\lambda_k^2} \Big)\, dt \tag{5.6}$$

$$\leq\; c_1^2 \cdot \operatorname{Tr}(e^{-\Delta_R}) \int_{R^{2-\varepsilon}}^{\infty} e^{-\frac{t-1}{R}}\, dt \;\leq\; c_7 R \cdot \int_{R^{2-\varepsilon}}^{\infty} e^{-\frac{t-1}{R}}\, dt \;\leq\; c_8 e^{-R^{1-\varepsilon}} \quad .$$

The first claim follows from (5.5) and (5.6). In the same way, we can show that the same estimate holds for the operator Δ_R^c. □

Lemma 5.3. *For $R \gg 0$, there are positive constants c_1, c_2 such that*

$$|\operatorname{Tr}^{\perp}\Big(\int_0^{R^{2-\varepsilon}} \gamma\,(e^{-t\Delta_R} - e^{-t\Delta_R^c})\,\gamma^*\; dt\Big)| \leq c_1 e^{-c_2 R^{\varepsilon}} \quad . \tag{5.7}$$

Proof. It is sufficient to show that the following term has the claimed bound,

$$\int_0^{R^{2-\varepsilon}} \int_Y \|\gamma\,(e^{-t\Delta_R}(x,x) - e^{-t\Delta_R^c}(x,x))\,\gamma^*\|\, dy\; dt.$$

For this, we apply *Finite Propagation Speed Property for the Wave Operator* to compare Δ_R over M_R with Δ_R^c over N_R where we identify the parts $N_{\frac{R}{2}}$ of these in an obvious way. Then we obtain the estimate

$$\|\mathcal{E}_R(t;x,y) - \mathcal{E}_R^c(t;x,y)\| \leq c_3 e^{-c_4 \frac{R^2}{t}}$$

where $\mathcal{E}_R(t;x,y)$, $\mathcal{E}_R^c(t;x,y)$ are heat kernels of Δ_R, Δ_R^c respectively and $x, y \in N_{\frac{R}{2}}$. Therefore, the following estimate holds

$$\|\gamma\,(e^{-t\Delta_R} - e^{-t\Delta_R^c})\,\gamma^*\| \leq c_3 e^{-c_4 \frac{R^2}{t}} \quad . \tag{5.8}$$

We combine (5.8) with the following inequality

$$c_3 \int_0^{R^{2-\varepsilon}} e^{-c_4 \frac{R^2}{t}} dt \;\leq\; c_1 e^{-c_2 R^{\varepsilon}} \quad .$$

This completes the proof.

□

Putting $\varepsilon = \frac{1}{2}$, Lemma 5.2 and 5.3 complete the proof of Proposition 5.1.

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