ETA INVARIANT AND SELBERG ZETA FUNCTION OF ODD TYPE OVER CONVEX CO-COMPACT HYPERBOLIC MANIFOLDS

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Abstract. We show meromorphic extension and give a complete description of the divisors of a Selberg zeta function of odd type $Z^o_{\Gamma,\Sigma}(\lambda)$ associated to the spinor bundle Σ on an odd dimensional convex co-compact hyperbolic manifold $\Gamma\backslash\mathbb{H}^{2n+1}$. As a byproduct we do a full analysis of the spectral and scattering theory of the Dirac operator on asymptotically hyperbolic manifolds. We show that there is a natural eta invariant $\eta(D)$ associated to the Dirac operator D over a convex co-compact hyperbolic manifold $\Gamma\backslash\mathbb{H}^{2n+1}$ and that $\exp(\pi i\eta(D)) = Z^o_{\Gamma,\Sigma}(0)$, thus extending Millson's formula to this setting. Under some assumption on the exponent of convergence of Poincaré series for the group Γ, we also define an eta invariant for the odd signature operator, and we show that for Schottky 3-dimensional hyperbolic manifolds it gives the argument of a holomorphic function which appears in the Zograf factorization formula relating two natural Kähler potentials for Weil-Petersson metric on Schottky space.

1. Introduction

The eta invariant is a measure of the asymmetry of the spectrum of self-adjoint elliptic operators which has been introduced by Atiyah, Patodi and Singer as the boundary term in the index formula for compact manifolds with boundary [1]. For an elliptic self-adjoint pseudodifferential operator D of positive order acting on a bundle over a closed manifold, it is defined as the value at $s = 0$ of the meromorphic function

$$\eta(D, s) := \mathrm{Tr}\left(D(D^2)^{-\frac{s+1}{2}}\right) = \frac{1}{\Gamma((s+1)/2)}\int_0^\infty t^{s-\frac{1}{2}}\,\mathrm{Tr}\left(De^{-tD^2}\right)dt,$$

which admits a meromorphic continuation from $\Re(s) \gg 0$ to $s \in \mathbb{C}$ and is regular at $s = 0$. In heuristic terms, $\eta(D) := \eta(D, 0)$ computes the asymmetry $\mathrm{Tr}(D|D|^{-1})$.

By applying Selberg's trace formula, Millson [27] proved that for any $(4m-1)$-dimensional closed hyperbolic manifold $X_\Gamma := \Gamma\backslash\mathbb{H}^{4m-1}$, the eta invariant $\eta(A)$ of the odd signature operator A on odd forms $\Lambda^{\mathrm{odd}} = \oplus_{p=0}^{2m}\Lambda^{2p-1}$ can be expressed in terms of the geodesic flow on the unit sphere bundle of X_Γ. Millson defined a Selberg zeta function of odd type by

$$Z^o_{\Gamma,\Lambda}(\lambda) := \exp\left(-\sum_{\gamma\in\mathcal{P}}\sum_{k=1}^{\infty}\frac{\chi_+(R(\gamma)^k) - \chi_-(R(\gamma)^k)}{|\det(\mathrm{Id} - P(\gamma)^k)|^{\frac{1}{2}}}\frac{e^{-\lambda k\ell(\gamma)}}{k}\right) \tag{1.1}$$

where $\mathcal{P}$ denotes the set of primitive closed geodesics in X_Γ, $R(\gamma) \in \mathrm{SO}(4m-2)$ is the holonomy along a geodesic γ, $\chi_\pm$ denotes the character associated to the two irreducible representations of $\mathrm{SO}(4m-2)$ corresponding to the $\pm i$ eigenspace of $\star$ acting on Λ^{2m-1}, $P(\gamma)$ is the linear Poincaré map along γ, and $\ell(\gamma)$ is the length of the closed geodesic γ. Then he showed that $Z^o_{\Gamma,\Lambda}(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$, its only zeros and poles occur on the line $\Re(\lambda) = 0$ with order given in terms of the multiplicity of the eigenvalues of A, and the following remarkable identity holds:

$$e^{\pi i\eta(A)} = Z^o_{\Gamma,\Lambda}(0).$$

The same result has been extended to compact locally symmetric manifolds of higher rank by Moscovici-Stanton [28]. It is somehow believed that central values of Ruelle or Selberg type dynamical zeta functions have some kind of topological meaning and this identity, as well as Fried's identity [8], provide striking examples.

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It is a natural question to try to extend this identity and to study the meromorphic extension and the zeros and poles of the zeta function $Z^o_{\Gamma,\Lambda}(\lambda)$ on non-compact hyperbolic manifolds. The first step in this direction has been done by the third author in [29] for cofinite hyperbolic quotients, where the functional equation satisfied by the Selberg zeta function of odd type holds with extra contributions from the cusps, in the guise of the determinant of the scattering matrix. In the present work, we carry out this program for *convex co-compact* manifolds, i.e., geometrically finite hyperbolic manifolds with infinite volume and no cusps. For particular 3-dimensional Schottky groups, our results have interesting connections to Teichmüller theory, as we explain below.

The proof of the meromorphic extension of any reasonable dynamical zeta function on co-compact hyperbolic manifolds is contained in the work of Fried [9] using transfer operator techniques in dynamics, but as explained by Patterson-Perry [30], it extends to the convex co-compact setting in a natural way. However there is no general description of the zeros and poles, while we know in the co-compact and cofinite cases that these are related to spectral and topological data since the work of Selberg [32]. There are now some rather recent works of Patterson-Perry [30] and Bunke-Olbrich [3] which give a complete description of the zeros and poles of the original Selberg zeta function on convex co-compact hyperbolic (real) manifolds. The case of Selberg zeta functions attached to homogeneous vector bundles is not yet completely described.

In this paper we study mainly the Dirac operator acting on the spinor bundle Σ over a convex co-compact hyperbolic manifold $X_\Gamma := \Gamma\backslash\mathbb{H}^{2n+1}$. The basic quantity associated with Γ for this case is its exponent δ_Γ defined to be the smallest number such that

$$\sum_{\gamma\in\Gamma}\exp(-\lambda r_\gamma) < \infty \tag{1.2}$$

for all $\lambda > \delta_\Gamma$. Here r_γ denotes the hyperbolic distance $d_{\mathbb{H}^{2n+1}}(m,\gamma m)$ for a fixed point $m \in \mathbb{H}^{d+1}$. For $\lambda > \delta_\Gamma - n$, we define the Selberg zeta function of odd type $Z^o_{\Gamma,\Sigma}(\lambda)$ associated to the spinor bundle Σ exactly like in (1.1) except that $R(\gamma)$ denotes now the holonomy in the spinor bundle Σ along γ, and $\chi_\pm$ denotes the character of the two irreducible representations of $\mathrm{Spin}(2n)$ corresponding to the $\pm i$ eigenspaces of the Clifford multiplication $\mathrm{cl}(T_\gamma)$ with the tangent vector field T_γ to γ. Like for the hyperbolic space $\mathbb{H}^{2n+1}$, the Dirac operator D acting on the spinor bundle Σ on a convex co-compact hyperbolic manifold X_Γ has continuous spectrum the real line $\mathbb{R}$, and one can define its resolvent for $\Re(\lambda) > 0$ in two ways

$$R_+(\lambda) := (D + i\lambda)^{-1}, \qquad R_-(\lambda) := (D - i\lambda)^{-1}$$

as analytic families of bounded operators acting on $L^2(X_\Gamma;\Sigma)$. We then first show

Theorem 1.1. *The Selberg zeta function of odd type $Z^o_{\Gamma,\Sigma}(\lambda)$ associated to the spinor bundle Σ on an odd dimensional spin convex co-compact hyperbolic manifold $X_\Gamma = \Gamma\backslash\mathbb{H}^{2n+1}$ has a meromorphic extension to $\mathbb{C}$ and it is analytic in a neighborhood of the right half plane $\{\Re(\lambda) \geq 0\}$. The resolvents $R_\pm(\lambda)$ of the Dirac operator have meromorphic continuation to $\lambda \in \mathbb{C}$ when considered as operators mapping $C_0^\infty(X_\Gamma,\Sigma)$ to its dual $C^{-\infty}(X_\Gamma,\Sigma^*)$, and the poles have finite rank polar part. A point $\lambda_0 \in \{\Re(\lambda) < 0\}$ is a zero or pole of $Z^o_{\Gamma,\Sigma}(\lambda)$ if and only if the meromorphic extension of $R_+(\lambda)$ or of $R_-(\lambda)$ has a pole at λ_0, in which case the order of λ_0 as a zero or pole of $Z^o_{\Gamma,\Sigma}(\lambda)$ (with the positive sign convention for zeros) is given by*

$$\operatorname{rank}\operatorname{Res}_{\lambda_0} R_-(\lambda) - \operatorname{rank}\operatorname{Res}_{\lambda_0} R_+(\lambda).$$

We stress that our approach is closer to that of Patterson-Perry than that of Bunke-Olbrich. In so far as analysis is concerned, we deal with a much more general geometric setting in arbitrary dimensions and we prove various results which were previously known for the Laplacian on functions. We consider *asymptotically hyperbolic manifolds* (AH in short). These are complete Riemannian manifolds (X, g) which compactify smoothly to compact manifolds with boundary $\bar{X}$, whose metric near the boundary is of the form $g = \bar{g}/x^2$ where $\bar{g}$ is a smooth metric on $\bar{X}$ and x is any boundary defining function of $\partial\bar{X}$ in $\bar{X}$, and finally such that $|dx|_{\bar{g}} = 1$ at $\partial\bar{X}$, a condition which is equivalent to assuming that the curvature tends to -1 at the boundary. Convex co-compact hyperbolic manifolds are special cases of AH manifolds. Using the machinery of Mazzeo-Melrose

[25], we show that the spectrum of D on AH manifolds is absolutely continuous and given by $\mathbb{R}$, that the resolvents $R_{\pm}(\lambda)$ defined above have meromorphic extensions to $\lambda \in \mathbb{C}$, and we define the *scattering operator* $S(\lambda) : C^\infty(\partial\bar{X}, \Sigma) \to C^\infty(\partial\bar{X}, \Sigma)$ by considering asymptotic profiles of generalized eigenspinors on the continuous spectrum. The family $S(\lambda)$ extends to a meromorphic family of elliptic pseudo-differential operators acting on the boundary with the same principal symbol as $D_{h_0}|D_{h_0}|^{2\lambda-1}$ (up to a multiplicative constant), where D_{h_0} is the Dirac operator induced by the metric $\bar{g}|_{T\partial\bar{X}}$. The scattering operator is a fundamental object in the analysis of Selberg zeta function for convex co-compact manifolds, and we study it thoroughly in this work. We also show in a follow-up note [15] that the construction and properties of the scattering operator have some nice applications, for instance the invertibility of $S(\lambda)$ except at discrete λ's implies that the index of $D^+_{h_0}$ vanishes (the so-called cobordism invariance of the index), and the operator $\frac{1}{2}(\mathrm{Id} - S(0))$ is a complementary Calderón projector of the Dirac operator $\bar{D}$ corresponding to $\bar{g}$, providing a natural way of constructing the Calderón projector without extending $\bar{D}$ or doubling the manifold $\bar{X}$.

For the second result, we prove that Millson's formula holds for the Dirac operator on odd dimensional spin convex co-compact hyperbolic manifolds, and also for the signature operator under some condition on δ_Γ.

Theorem 1.2. *Let $X_\Gamma = \Gamma\backslash\mathbb{H}^{2n+1}$ be an odd dimensional spin convex co-compact hyperbolic manifold. Then the function $\mathrm{tr}(De^{-tD^2})(m) \in C^\infty(X_\Gamma)$ is in $L^1(X_\Gamma)$, where tr denotes the local trace on the spinor bundle. The eta invariant $\eta(D)$ can be defined as a convergent integral by*

$$\eta(D) := \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \Big(\int_{X_\Gamma} \mathrm{tr}(De^{-tD^2})(m)\,\mathrm{dv}(m) \Big)\, dt, \tag{1.3}$$

and the following equality holds

$$e^{\pi i \eta(D)} = Z^o_{\Gamma,\Sigma}(0). \tag{1.4}$$

If $2n+1 = 4m-1$ and the exponent of convergence of Poincaré series δ_Γ is strictly less than $n = 2m-1$, then the eta invariant $\eta(A)$ can also be defined replacing D by the odd signature operator A and Σ by the bundle of forms $\Lambda^{\mathrm{odd}} = \oplus_{p=0}^{2m}\Lambda^{2p-1}$ in (1.3), moreover we also have $e^{\pi i\eta(A)} = Z^o_{\Gamma,\Lambda}(0)$.

The assumption about δ_Γ for the equality $e^{\pi i\eta(A)} = Z^o_{\Gamma,\Lambda}(0)$ is rather a technical condition than a serious problem. Most of the analysis we do here for Dirac operator D goes through without significant difficulties to the signature operator A, but it appears to be slightly more involved essentially due to the fact that the continuous spectrum of A has two layers corresponding to closed and co-closed forms. The complete analysis for forms in all dimensions will be included elsewhere.

To conclude this introduction and to motivate the eta invariant $\eta(A)$ of the odd signature operator A, we describe the particular case of Schottky 3-dimensional manifolds with $\delta_\Gamma < 1$, where the eta invariant $\eta(A)$ can be considered as a function on the *Schottky space* $\mathfrak{S}_g$. Here the Schottky space $\mathfrak{S}_g$ is the space of marked normalized Schottky groups with g generators. It is a complex manifold of dimension $3g-3$, covering the Riemann moduli space $\mathfrak{M}_g$ and with universal cover the Teichmüller space $\mathfrak{T}_g$. It describes the deformation space of the 3-dimensional hyperbolic Schottky manifolds $X_\Gamma = \Gamma\backslash\mathbb{H}^3$. Like $\mathfrak{T}_g$, the Schottky space $\mathfrak{S}_g$ has a natural Kähler metric, the Weil-Petersson metric. In [33, 34], Takhtajan-Zograf constructed two Kähler potentials of the Weil-Petersson metric on $\mathfrak{S}_g$, that is,

$$\partial\overline{\partial}S = \partial\overline{\partial}\Big(-12\pi\log\frac{\mathrm{Det}\Delta}{\det\mathrm{Im}\,\tau}\Big) = 2i\,\omega_{WP}$$

where ∂ and $\overline{\partial}$ are the $(1,0)$ and $(0,1)$ components of the de Rham differential d on $\mathfrak{S}_g$ respectively, and ω_{WP} is the symplectic form of the Weil-Petersson metric; here S is the so-called *classical Liouville action*, $\mathrm{Det}\Delta$ and τ denote the ζ-regularized determinant of the Laplacian Δ of hyperbolic metric and the period matrix respectively over the Riemann surface corresponding to an inverse image in $\mathfrak{T}_g$ of a point in $\mathfrak{S}_g$. Let us remark that $\mathrm{Det}\Delta$ and $\det \mathrm{Im}\,\tau$ descend to well-defined functions on $\mathfrak{S}_g$. We show that

Theorem 1.3. *The function F defined on $\mathfrak{S}_g^0 := \{\Gamma \in \mathfrak{S}_g; \delta_\Gamma < 1\}$ by*

$$F := \frac{\mathrm{Det}\Delta}{\det \mathrm{Im}\,\tau} \exp\left(\frac{S}{12\pi} - i\pi\eta(A)\right)$$

is holomorphic. In particular, the eta invariant $\eta(A)$ is a pluriharmonic function on $\mathfrak{S}_g^0$.

The condition $\delta_\Gamma < 1$ in Theorem 1.3 simplifies the proof at several stages. But, one can expect that a similar result still holds over the whole Schottky space $\mathfrak{S}_g$. This extension problem will be discussed elsewhere.

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2. The Dirac operator on real hyperbolic space

2.1. **Dirac operators over hyperbolic spaces.** The $(d+1)$-dimensional real hyperbolic space is the manifold

$$\mathbb{H}^{d+1} = \left\{ x \in \mathbb{R}^{d+2} \,|\, x_0^2 + x_1^2 + \ldots + x_d^2 - x_{d+1}^2 = -1,\ x_{d+1} > 0 \right\}$$

equipped with the metric of curvature -1. The orientation preserving isometries of $\mathbb{H}^{d+1}$ form the group $\mathrm{SO}_0(d+1,1)$, which is the identity connected component of $\mathrm{SO}(d+1,1)$. The isotropy subgroup of the base point $(0,\ldots,0,1)$ is isomorphic to $\mathrm{SO}(d+1)$. Hence the real hyperbolic space $\mathbb{H}^{d+1}$ can be identified with the symmetric space $\mathrm{SO}_0(d+1,1)/\mathrm{SO}(d+1)$. Since $G = \mathrm{Spin}(d+1,1)$, $K = \mathrm{Spin}(d+1)$ are double coverings of $\mathrm{SO}_0(d+1,1)$, $\mathrm{SO}(d+1)$ respectively, we see that $\mathrm{SO}_0(d+1,1)/\mathrm{SO}(d+1) = G/K$ and we use the identification $\mathbb{H}^{d+1} \cong G/K$ for our purpose. We denote the Lie algebras of G, K by $\mathfrak{g} = \mathfrak{spin}(d+1,1)$, $\mathfrak{k} = \mathfrak{spin}(d+1)$ respectively. The Cartan involution θ on $\mathfrak{g}$ gives us the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}, \mathfrak{p}$ are the $1, -1$ eigenspaces of θ respectively. The subspace $\mathfrak{p}$ can be identified with the tangent space $T_o(G/K) \cong \mathfrak{g}/\mathfrak{k}$ at $o = eK \in G/K$ where e denotes the identity element in G. The invariant metric of curvature -1 over $\mathbb{H}^{d+1}$ is given by the normalized Cartan-Killing form

$$\langle X, Y\rangle := -\frac{1}{2d} C(X, \theta Y) \tag{2.1}$$

where the Killing form is defined by $C(X,Y) = \mathrm{tr}(\mathrm{ad}\,X \circ \mathrm{ad}\,Y)$ for $X, Y \in \mathfrak{g}$.

Let $\mathfrak{a}$ be a fixed maximal abelian subspace of $\mathfrak{p}$. Then the dimension of $\mathfrak{a}$ is 1. Let $M = \mathrm{Spin}(d)$ be the centralizer of $A = \exp(\mathfrak{a})$ in K with Lie algebra $\mathfrak{m}$. We put β to be the positive restricted root of $(\mathfrak{g}, \mathfrak{a})$. Let ρ denote the half sum of the positive roots of $(\mathfrak{g}, \mathfrak{a})$, that is, $\rho = \frac{d}{2}\beta$. From now on, we use the identification

$$\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C} \qquad \text{by} \quad \lambda\beta \longrightarrow \lambda. \tag{2.2}$$

Let $\mathfrak{n}$ be the positive root space of β and $N = \exp(\mathfrak{n}) \subset G$. The Iwasawa decomposition is given by $G = KAN$. Throughout this paper we use the following Haar measure on G,

$$dg = a^{2\rho} dk\, da\, dn = a^{-2\rho} dn\, da\, dk \tag{2.3}$$

where $g = kan$ is the Iwasawa decomposition and $a^{2\rho} = \exp(2\rho(\log a))$. Here dk is the Haar measure over K with $\int_K dk = 1$, da is the Euclidean Lebesgue measure on A given by the identification $A \cong \mathbb{R}$ via $a_r = \exp(rH)$ with $H \in \mathfrak{a}$, $\beta(H) = 1$, and dn is the Euclidean Lebesgue measure on N induced by the normalized Cartan-Killing form $\langle\cdot,\cdot\rangle$ given in (2.1).

The spinor bundle $\Sigma(\mathbb{H}^{d+1})$ can be identified with the associated homogeneous vector bundle over $\mathbb{H}^{d+1} = G/K$ with the spin representation τ_d of $K \cong \mathrm{Spin}(d+1)$ acting on $V_{\tau_d} = \mathbb{C}^{2^{[d+1/2]}}$, that is,

$$\Sigma(\mathbb{H}^{d+1}) = G \times_{\tau_d} V_{\tau_d} \longrightarrow \mathbb{H}^{d+1} = G/K. \tag{2.4}$$

Here points of $G \times_{\tau_d} V_{\tau_d}$ are given by equivalence classes $[g, v]$ of pairs (g, v) under $(gk, v) \sim (g, \tau_d(k)v)$. Hence the sections of $G \times_{\tau_d} V_{\tau_d}$ from G/K consist of functions $f : G \to V_{\tau_d}$ with the K-equivariant condition,

$$f(gk) = \tau_d(k)^{-1} f(g)$$

for $g \in G$, $k \in K$. Recall that τ_d is irreducible if $d+1$ is odd, while it splits into 2 irreducible representations if $d+1$ is even.

Let us denote by

$$\nabla : C^\infty(\mathbb{H}^{d+1}; \Sigma(\mathbb{H}^{d+1})) \longrightarrow C^\infty(\mathbb{H}^{d+1}; T^*(\mathbb{H}^{d+1}) \otimes \Sigma(\mathbb{H}^{d+1}))$$

the covariant derivative induced by the lift of the Levi-Civita connection to the spinor bundle $\Sigma(\mathbb{H}^{d+1})$, and by $\mathrm{cl} : T_m(\mathbb{H}^{d+1}) \to \mathrm{End}\,\Sigma_m(\mathbb{H}^{d+1})$ the Clifford multiplication. Then the Dirac operator $D_{\mathbb{H}^{d+1}}$ acting on $C_0^\infty(\mathbb{H}^{d+1}; \Sigma(\mathbb{H}^{d+1}))$ is defined by

$$D_{\mathbb{H}^{d+1}} f(m) = \sum_{j=1}^{d+1} \mathrm{cl}(e_j) \nabla_{e_j} f(m) \qquad \text{for} \quad f \in C_0^\infty(\mathbb{H}^{d+1}; \Sigma(\mathbb{H}^{d+1}))$$

where $(e_j)_{j=1}^{d+1}$ denotes an orthonormal frame of $T_m(\mathbb{H}^{d+1})$. The Dirac operator $D_{\mathbb{H}^{d+1}}$ is an essentially self-adjoint, elliptic and G-invariant differential operator of first order, and we use the same notation for its self adjoint extension to $L^2(\mathbb{H}^{d+1}; \Sigma(\mathbb{H}^{d+1}))$. It is well known that the spectrum of $D_{\mathbb{H}^{d+1}}$ on $L^2(\mathbb{H}^{d+1}; \Sigma(\mathbb{H}^{d+1}))$ consists only of the absolutely continuous spectrum $\mathbb{R}$ (for instance, by Cor. 4.11 in [6]).

2.2. **The resolvent on $\mathbb{H}^{d+1}$.** Let us define the resolvent of D^2 in the half-plane $\{\Re(\lambda) > 0\}$ by

$$R_{\mathbb{H}^{d+1}}(\lambda) := (D^2_{\mathbb{H}^{d+1}} + \lambda^2)^{-1}$$

which maps $L^2(\mathbb{H}^{d+1}; \Sigma(\mathbb{H}^{d+1}))$ to itself. Recall the hypergeometric function $F(a, b, c, z)$ defined by

$$F(a, b, c; z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+k)} \frac{z^k}{k!} \qquad \text{for} \quad |z| < 1.$$

Then we have from the work of Camporesi [6, Th. 6.2 and 6.3],

Proposition 2.1. [Camporesi] *For $\Re(\lambda) > 0$, the respective Schwartz kernels of $R_{\mathbb{H}^{d+1}}(\lambda)$ and $D_{\mathbb{H}^{d+1}} R_{\mathbb{H}^{d+1}}(\lambda)$ are given by*

$$\begin{aligned} R_{\mathbb{H}^{d+1}}(\lambda; m, m') = {} & 2^{-(d+1)} \pi^{-\frac{d+1}{2}} \frac{\Gamma(\frac{d+1}{2} + \lambda)\Gamma(\lambda)}{\Gamma(2\lambda + 1)} (\cosh(r/2))^{-d-2\lambda} \\ & F\left(\frac{d+1}{2} + \lambda, \lambda, 2\lambda + 1; \cosh^{-2}(r/2)\right) U(m, m'), \end{aligned} \tag{2.5}$$

(2.6)

$$\begin{aligned} D_{\mathbb{H}^{d+1}} R_{\mathbb{H}^{d+1}}(\lambda; m, m') = & - 2^{-(d+1)} \pi^{-\frac{d+1}{2}} \frac{\Gamma(\frac{d+1}{2} + \lambda)\Gamma(\lambda + 1)}{\Gamma(2\lambda + 1)} (\cosh(r/2))^{-(d+1)-2\lambda} \sinh(r/2) \\ & F\left(\frac{d+1}{2} + \lambda, \lambda + 1, 2\lambda + 1; \cosh^{-2}(r/2)\right) \mathrm{cl}(v_{m,m'}) U(m, m') \end{aligned}$$

where $r = d_{\mathbb{H}^{d+1}}(m, m')$ for $m, m' \in \mathbb{H}^{d+1}$, $v_{m,m'}$ is the unit tangent vector at m to the geodesic from m' to m and $U(m, m')$ is the parallel transport from m' to m along the geodesic between them. Moreover $R_{\mathbb{H}^{d+1}}(\lambda)$ has an analytic continuation in $\mathbb{C} \setminus \{0\}$ with a simple pole at $\lambda = 0$ and $D_{\mathbb{H}^{d+1}} R_{\mathbb{H}^{d+1}}(\lambda)$ admits an analytic continuation to $\lambda \in \mathbb{C}$ (thus with no pole), as distributions on $\mathbb{H}^{d+1} \times \mathbb{H}^{d+1}$.

Remark. If one denotes $R_{\mathbb{H}^{d+1}}(\lambda; m; m') = Q_\lambda(r)U(m, m')$ where $r = d_{\mathbb{H}^{d+1}}(m, m')$, the function $Q_\lambda(r)$ satisfies that $Q_\lambda(r) - Q_{-\lambda}(r)$ is smooth in r near $r = 0$. This can be checked using functional equations of hypergeometric functions but actually follows directly from elliptic regularity since $Q_\lambda(r) - Q_{-\lambda}(r)$ (since the difference of resolvents too) solves an elliptic ODE. The kernel $D\Pi(\lambda; m, m')$ of $D(R_{\mathbb{H}^{d+1}}(\lambda) - R_{\mathbb{H}^{d+1}}(-\lambda))$ is then also smooth near the diagonal $m = m'$ and following the proof of [6, Th 6.3], we see that it can be written under the form

$$D\Pi(\lambda; m, m') = -\frac{1}{2}\sinh(r)\frac{\partial_{\cosh^{-2}(r/2)}\Big((\cosh^{-2}(\frac{r}{2}))^{-\frac{d}{2}}H_\lambda(\cosh^{-2}(\frac{r}{2}))\Big)}{\cosh(\frac{r}{2})^{d+4}}\mathrm{cl}(v_{m,m'})U(m, m')$$

where $H_\lambda(\cosh^{-2}(r/2)) := Q_\lambda(r) - Q_{-\lambda}(r)$ with $H_\lambda(u)$ smooth near $u = 1$. Then we deduce that on the diagonal $D\Pi(\lambda; m, m) = 0$.

2.3. **Dirac operators over convex co-compact hyperbolic manifolds.** Let Γ denote a convex co-compact torsion-free discrete subgroup of $G = \mathrm{Spin}(d + 1, 1)$ such that its co-volume $\mathrm{Vol}(\Gamma\backslash G) = \infty$. Hence

$$X_\Gamma := \Gamma\backslash G/K$$

is a $(d+1)$-dimensional convex co-compact hyperbolic manifold of infinite volume, which is a spin manifold by construction. The boundary $\partial\mathbb{H}^{d+1}$, which can be identified with K/M, admits a Γ-invariant decomposition into $\Omega(\Gamma) \cup \Lambda(\Gamma)$ where $\Omega(\Gamma) \neq \varnothing$ is open and Γ acts freely and co-compactly on $\mathbb{H}^{d+1} \cup \Omega(\Gamma)$. Hence X_Γ can be compactified by adjoining the geodesic boundary $\Gamma\backslash\Omega(\Gamma)$.

By the identification (2.4) of the spinor bundle $\Sigma(\mathbb{H}^{d+1})$ with the homogeneous vector bundle $G \times_{\tau_d} V_{\tau_d}$, we can also identify the spinor bundle $\Sigma(X_\Gamma)$ over X_Γ with the locally homogeneous vector bundle $\Gamma\backslash\big(G \times_{\tau_d} V_{\tau_d}\big)$. Here Γ acts on $G \times_{\tau_d} V_{\tau_d}$ by $\gamma[g, v] = [\gamma g, v]$ for $\gamma \in \Gamma$. We can also push down the Dirac operator $D_{\mathbb{H}^{d+1}}$ to X_Γ, which we denote by D. We also use the same notation for its unbounded self-adjoint extension in $L^2(X_\Gamma; \Sigma(X_\Gamma))$, that is,

$$D : \ L^2(X_\Gamma; \Sigma(X_\Gamma)) \longrightarrow L^2(X_\Gamma; \Sigma(X_\Gamma)).$$

By Corollary 3.4 below (cf. Cor. 7.9 and Th. 11.2 in [4]), the Dirac operator D over $L^2(X_\Gamma; \Sigma(X_\Gamma))$ has no discrete spectrum and only absolutely continuous spectrum $\mathbb{R}$.

3. Resolvent of Dirac operator on Asymptotically Hyperbolic manifolds

In this section, we analyze the resolvent $R(\lambda)$ of D^2 on an asymptotically hyperbolic manifold (AH in short) of dimension $(d + 1)$. An asymptotically hyperbolic manifold is a complete non-compact Riemannian manifold (X, g) which compactifies in a smooth manifold with boundary $\bar{X}$ and there is a diffeomorphism ψ (called *product decomposition*) from a collar neighbourhood $[0, \epsilon)_x \times \partial\bar{X}$ of the boundary to a neighbourhood of $\partial\bar{X}$ in $\bar{X}$ so that

$$\psi^* g = \frac{dx^2 + h_x}{x^2} \tag{3.1}$$

for some one-parameter family of metrics h_x on the boundary $\partial\bar{X}$ depending smoothly on $x \in [0, \epsilon)$. By abuse of notations, we will write x for $\psi_* x$, and x is then a boundary defining function in $\bar{X}$ near $\partial\bar{X}$, satisfying $|dx|_{x^2 g} = 1$. A boundary defining functions satisfying $|dx|_{x^2 g} = 1$ near the boundary is called *geodesic boundary defining function*, and it yields a diffeomorphism ψ like in (3.1) by taking the flow of the gradient $\nabla^{x^2 g} x$ starting at the boundary. Following the terminology of [12], we shall say that

(3.2) the metric is *even* if the Taylor expansion of h_x at $x = 0$ contains only even powers of x.

This property does not depend on the choice of the diffeomorphism ψ but only on g, see [12, Lemma 2.1]. It is well known that convex co-compact quotients $X_\Gamma = \Gamma\backslash\mathbb{H}^{d+1}$ are even AH manifolds (see [25]). Note that the metric h_0 is not canonical since it depends on the choice of ψ, but its conformal class $[h_0]$ is canonical with respect to g.

3.1. **0-structures, spinor bundle and Dirac operator.** Following the ideas of Mazzeo-Melrose [25] (and refer to this paper for more details), there is a natural structure associated to AH manifolds, this is encoded in the Lie algebra $\mathcal{V}_0(\bar{X})$ of smooth vector fields vanishing at the boundary, whose local basis over $C^\infty(\bar{X})$ is given near the boundary $\partial\bar{X}$ by the vector fields $(x\partial_x, x\partial_{y_1}, \dots, x\partial_{y_d})$ if $(x, y_1, \dots, y_d)$ is a local chart near a point $p \in \partial\bar{X}$ and x is a smooth boundary defining function in $\bar{X}$. The algebra is also the space of smooth section of a bundle ${}^0T\bar{X}$ with local basis near p given by $(x\partial_x, x\partial_{y_1}, \dots, x\partial_{y_d})$ and its dual space is denoted ${}^0T^*\bar{X}$, with local basis $(dx/x, dy_1/x, \dots, dy_d/x)$. The metric g is a smooth section of the bundle of positive definite symmetric form $S^2_+({}^0T^*\bar{X})$ of ${}^0T^*\bar{X}$.

Let us define $\bar{g} := x^2 g$ where x is a boundary defining function appearing in (3.1). If $(\bar{X}, \bar{g})$ is orientable, there exists an $\mathrm{SO}(d+1)$-bundle ${}_oF(\bar{X}) \to \bar{X}$ over $\bar{X}$, but also an $\mathrm{SO}(d+1)$-bundle ${}^0_oF(\bar{X}) \to \bar{X}$ defined using the 0-tangent bundle ${}^0T\bar{X}$ and the metric g smooth on it. If $(\bar{X}, \bar{g})$ admits a spin structure, then there exists a 0-spin structure on (X, g) in the sense that there is a $\mathrm{Spin}(d+1)$-bundle ${}^0_sF(\bar{X}) \to \bar{X}$ which double covers ${}^0_oF(\bar{X})$ and is compatible with it in the usual sense. This corresponds to a rescaling of the spin structure related to $(\bar{X}, \bar{g})$. The 0-spinor bundle ${}^0\Sigma(\bar{X})$ can then be defined as a bundle associated to the $\mathrm{Spin}(d+1)$ principal bundle ${}^0_sF(\bar{X})$, with fiber at $p \in \bar{X}$

$$ {}^0\Sigma_p(\bar{X}) = {}^0_sF_p \times_{\tau_d} V_{\tau_d}. $$

The vector field $x\partial_x := x\nabla^{\bar{g}}(x)$ in the collar neighbourhood is unit normal to all hypersurfaces $\{x = \text{constant}\}$. The 0-spinor bundle on $\bar{X}$ splits near the boundary under the form

$$ {}^0\Sigma = {}^0\Sigma_+ \oplus {}^0\Sigma_-, \quad \text{where } {}^0\Sigma_\pm := \ker(\mathrm{cl}(x\partial_x) \mp i), $$

note that this splitting is dependent of the choice of the geodesic boundary defining function x except at the boundary $\partial\bar{X}$ where it yields an independent splitting of the spinor (since the one-jet of $x\nabla^{x^2g}x$ is independent of x at $\partial\bar{X}$). To avoid confusions later (and emphasize the fact that it is only depending on the conformal class $(\partial\bar{X}, [h_0])$), we shall define $\mathrm{cl}(\nu)$ the linear map on ${}^0\Sigma|_{\partial\bar{X}}$ by

$$ \mathrm{cl}(\nu)\psi := \mathrm{cl}(x\partial_x)\psi. $$

At the boundary, ${}^0\Sigma|_{\partial\bar{X}}$ is diffeomorphic to the spinor bundle $\Sigma(\partial\bar{X})$ on $(\partial\bar{X}, h_0)$, this is not canonical since it depends on h_0 and thus on the choice of x, however the splitting above is. Notice also that in even dimension $d+1 = 2m$, the splitting ${}^0\Sigma_+ \oplus {}^0\Sigma_-$ near the boundary is not the usual splitting of the spinor bundle into positive and negative spinors, i.e., into the ± 1 eigenspaces of the involution $\omega := i^m \mathrm{cl}(e_1)\dots\mathrm{cl}(e_{2m})$, where $(e_i)_i$ is any orthonormal oriented local basis of ${}^0T\bar{X}$. The Dirac operator near the boundary has the form

$$ D = x^{\frac{d}{2}} \left(\mathrm{cl}(x\partial_x)x\partial_x + P\right) x^{-\frac{d}{2}}, \quad P \in \mathrm{Diff}^1_0(\bar{X}; {}^0\Sigma) \tag{3.3} $$

where P is a first order differential operator in tangential derivatives, which anticommutes with $\mathrm{cl}(x\partial_x)$ and such that $P = xD_{h_0} + O(x^2)$ where D_{h_0} is the Dirac operator on $\partial\bar{X}$ equipped with the metric h_0. If the metric g is even, it is easy to see that locally near any point y' of the boundary, if $(x\partial_x, xY_1, \dots, xY_d)$ is an orthonormal frame near y' and (x, y) are coordinates on $[0, \epsilon) \times \partial\bar{X}$ there, then P is of the form

$$ P = \sum_{i=1}^{d} P_i(x^2, y; \nabla^{\bar{g}}_{xY_i}) $$

for some differential operators P_i of order 1 and with smooth coefficients in (x^2, y). This can be checked for instance by using the conformal change formula $D = x^{\frac{d}{2}+1}\bar{D}x^{-\frac{d}{2}}$ where $\bar{D}$ is the Dirac operator for the metric $\bar{g} = x^2 g$ which is smooth in the coordinates $(\mathbf{x} = x^2, y)$ down to $\mathbf{x} = 0$. From these properties, it is straightforward to check that if g is even, then for x geodesic boundary defining function fixed, D preserves the space $\mathcal{A}_\pm \subset C^\infty(\bar{X}; {}^0\Sigma)$ of smooth spinors which have expansion at the boundary of the form

$$ \sigma \sim_{x\to 0} \sum_{j=0}^{\infty} x^j \psi_j, \text{ with } \psi_{2j} \in \Sigma_\pm(\partial\bar{X}) \text{ and } \psi_{2j+1} \in \Sigma_\mp(\partial\bar{X}). \tag{3.4} $$

3.2. **The stretched product.** Following Mazzeo-Melrose [25], we define the stretched product $\bar{X} \times_0 \bar{X}$ as the blow-up $[\bar{X} \times \bar{X}, \Delta_\partial]$ of $\bar{X} \times \bar{X}$ around the diagonal in the boundary $\Delta_\partial := \{(y,y) \in \partial\bar{X} \times \partial\bar{X}\}$. The blow-up is a smooth manifold with codimension 2 corners, and 3 boundary hypersurfaces, the left boundary denoted lb, the right boundary denoted rb and the new face, called 'front face' and denoted ff, obtained from the blow-up. The blow-down map is denoted $\beta : \bar{X} \times_0 \bar{X} \to \bar{X} \times \bar{X}$ and maps int(lb) to $\partial\bar{X} \times X$, int(rb) to $X \times \partial\bar{X}$ and ff to Δ_∂. The face ff is a bundle over $\Delta_\partial \simeq \partial\bar{X}$ with fibers a quarter of d-dimensional sphere. Let us use the boundary defining function x in (3.1), which induces $x := \pi_L^* x$ and $x' := \pi_R^* x$ as boundary defining functions of $\bar{X} \times \bar{X}$ where π_L, π_R are the left and right projection $\bar{X} \times \bar{X} \to \bar{X}$. The fibre ff_p of the front face ff (with $p = (y', y') \in \partial\bar{X} \times \partial\bar{X}$) is, by definition of blow-up, given by the quotient

$$\mathrm{ff}_p = \Big(\Big(N_p(\Delta_\partial, \partial\bar{X} \times \partial\bar{X}) \times (\mathbb{R}^+\partial_x) \times (\mathbb{R}^+\partial_{x'})\Big) \setminus \{0\}\Big) \Big/ \{(w,t,u) \sim s(w',t',u'), s > 0\} \tag{3.5}$$

where in general $N(M, Y)$ denotes the normal bundle of a submanifold M in a manifold Y. Since $T_{y'}\partial\bar{X}$ is canonically isomorphic to $N_p(\Delta_\partial, \partial\bar{X} \times \partial\bar{X})$ by $z \in T_{y'}\partial\bar{X} \to (z, -z) \in T_p(\partial\bar{X} \times \partial\bar{X})$, $h_0(y')$ induces a metric on $N_p(\Delta_\partial, \partial\bar{X} \times \partial\bar{X})$. Then ff_p is clearly identified with the quarter of sphere

$$\mathrm{ff}_p \simeq \{w + t\partial_x + u\partial_{x'} \in N_p(\Delta_\partial, \partial\bar{X} \times \partial\bar{X}) \times (\mathbb{R}^+\partial_x) \times (\mathbb{R}^+\partial_{x'}), t^2 + u^2 + |w|^2_{h_0(y')} = 1\}.$$

In projective coordinates $(s := t/u, z := w/u) \in (0,\infty) \times \mathbb{R}^d$, the interior of the front face fiber ff_p is diffeomorphic to $\mathbb{H}^{d+1}$. In the same way we define the blow-up $\bar{X} \times_0 \partial\bar{X}$ of $\bar{X} \times \partial\bar{X}$ around Δ_∂ and the blow-up $\partial\bar{X} \times_0 \partial\bar{X}$ of $\partial\bar{X} \times \partial\bar{X}$ around Δ_∂. The first one is canonically diffeomorphic to the face rb of $\bar{X} \times_0 \bar{X}$ while the second one is canonically diffeomorphic to lb $\cap$ rb.

The manifold $\bar{X} \times \bar{X}$ carries the bundle

$$\mathcal{E} = {}^0\Sigma(\bar{X}) \boxtimes {}^0\Sigma^*(\bar{X})$$

which on the diagonal is isomorphic to $\mathrm{End}({}^0\Sigma)$. This bundle lifts under β to a bundle over $\bar{X} \times_0 \bar{X}$, still denoted by $\mathcal{E}$, whose fiber at the front face ff is given by ${}^0\Sigma_{y'}(\bar{X}) \boxtimes {}^0\Sigma^*_{y'}(\bar{X})$ everywhere on the fiber ff_p (here $p = (y', y') \in \Delta_\partial$) if ${}^0\Sigma_{y'}(\bar{X})$ is the fiber of ${}^0\Sigma(\bar{X})$ at the point $y' \in \partial\bar{X}$.

On a manifold with corners M with a smooth bundle $E \to M$, let us denote by $\dot{C}^\infty(M; E)$ the space of smooth section of E which vanish to all order at the (topological) boundary and let $C^{-\infty}(M; E^*)$ be its dual, the elements of which are called extendible distributions. Then β^* is an isomorphism between $\dot{C}^\infty(\bar{X} \times \bar{X}; \mathcal{E})$ and $\dot{C}^\infty(\bar{X} \times_0 \bar{X}; \mathcal{E})$ and also between their duals, meaning that distributions on $\bar{X} \times \bar{X}$ can be as well considered on the stretched product. In what follows, we consider the Schwartz kernel $K_A \in C^{-\infty}(\bar{X} \times \bar{X}; \mathcal{E})$ of an operator $A : \dot{C}^\infty(\bar{X}; {}^0\Sigma) \to C^{-\infty}(\bar{X}; {}^0\Sigma)$ defined by

$$\langle A\psi, \phi\rangle = \langle K_A, \phi \boxtimes \psi\rangle$$

where $\langle ., .\rangle$ is the duality pairing using the volume density of the metric. By abuse of notations we will write $A(m, m')$ for $K_A(m, m')$ and the bundle $\mathcal{E}$ at the diagonal will be identified to $\mathrm{End}({}^0\Sigma)$.

3.3. **Pseudo-differential operators.** We define the space $\Psi_0^{m,\alpha,\beta}(\bar{X}; {}^0\Sigma)$ for $m \in \mathbb{R}$, $\alpha, \beta \in \mathbb{C}$ as in [22, 25], and refer the reader to these references for more details. An operator A is in $\Psi_0^{m,\alpha,\beta}(\bar{X}; {}^0\Sigma)$ if its Schwartz kernel K_A lifts to $\bar{X} \times_0 \bar{X}$ to a distribution $\beta^*(K_A)$ which can be decomposed as a sum $K_A^1 + K_A^2$ with $K_A^1 \in \rho_{\mathrm{lb}}^\alpha \rho_{\mathrm{rb}}^\beta C^\infty(\bar{X} \times_0 \bar{X}; \mathcal{E})$ and $K_A^2 \in I^m_{\mathrm{cl}}(\bar{X} \times_0 \bar{X}, \Delta; \mathcal{E})$ where $I^m_{\mathrm{cl}}(\bar{X} \times_0 \bar{X}, \Delta; \mathcal{E})$ denotes the space of distribution on $\bar{X} \times_0 \bar{X}$ classically conormal to the lifted diagonal $\Delta := \overline{\beta^*(\{(m,m) \in X \times X\})}$ of order m and vanishing to infinite order at the left and right boundaries lb $\cup$ rb.

3.4. **Microlocal structure of the resolvent on $\mathbb{H}^{d+1}$.** We want to describe the resolvent kernel as a conormal distribution on a compactification of $\mathbb{H}^{d+1} \times \mathbb{H}^{d+1}$, in order to show later that a similar result holds for convex co-compact quotients and more generally asymptotically hyperbolic manifolds. Here we let $\overline{\mathbb{H}^{d+1}}$ be the natural compactification of $\mathbb{H}^{d+1}$, i.e., the unit ball in $\mathbb{R}^{d+1}$.

Lemma 3.1. *The analytically extended resolvent* $R_{\pm}^{\mathbb{H}^{d+1}}(\lambda) := (D_{\mathbb{H}^{d+1}} \pm i\lambda)^{-1}$ *of the Dirac operator* $D_{\mathbb{H}^{d+1}}$ *on* $\mathbb{H}^{d+1}$ *is in the space* $\Psi_0^{-1,\lambda+\frac{d}{2},\lambda+\frac{d}{2}}(\overline{\mathbb{H}^{d+1}}; {}^0\Sigma)$.

Proof. The proof is very similar to the case of the Laplacian on functions dealt with in [25], since we have $\cosh(r/2)^{-2} \in \rho_{\rm lb}\rho_{\rm rb}C^\infty(\overline{\mathbb{H}^{d+1}} \times_0 \overline{\mathbb{H}^{d+1}} \setminus \Delta)$ and is a smooth function of the distance, and since, by the remark after Lemma A.2, the lift of $U(m,m)$ by β^* is smooth on $\overline{\mathbb{H}^{d+1}} \times_0 \overline{\mathbb{H}^{d+1}}$, combining the formulae (2.5), (2.6) proves our claim, except for the singularity at the diagonal Δ. The conormal diagonal singularity can be easily seen by applying the first step of the parametrix of Mazzeo-Melrose (we refer to the proof of Proposition 3.2 below) near the diagonal, indeed the construction shows that there exists $Q_{\pm}^0(\lambda) \in \Psi_0^{-1,\infty,\infty}(\overline{\mathbb{H}^{d+1}}; {}^0\Sigma)$ such that $(D_{\mathbb{H}^{d+1}} \pm i\lambda)(R_{\pm}^{\mathbb{H}^{d+1}}(\lambda) - Q_{\pm}^0(\lambda))$ and $(R_{\pm}^{\mathbb{H}^{d+1}}(\lambda) - Q_{\pm}^0(\lambda))(D_{\mathbb{H}^{d+1}} \pm i\lambda)$ have a smooth kernel in a neighbourhood of Δ down to the front face ff, and so by 0-elliptic regularity $R_{\pm}^{\mathbb{H}^{d+1}}(\lambda) - Q_{\pm}^0(\lambda)$ is smooth near Δ down to ff. □

Note that the resolvent can be also considered as a convolution kernel on $\mathbb{H}^{d+1}$ with a conormal singularity at the center $0 \in \mathbb{H}^{d+1}$.

3.5. **The parametrix construction of Mazzeo-Melrose.** We can construct the resolvent $R_{\pm}(\lambda) := (D \pm i\lambda)^{-1}$ through a pseudo-differential parametrix, following Mazzeo-Melrose [25] or Mazzeo [22]. We will not give the full details since this is a straightforward application of the paper [25] and the analysis of the resolvent $R_{\pm}^{\mathbb{H}^{d+1}}(\lambda)$ on the model space $\mathbb{H}^{d+1}$. This will be done in 3 steps. If E, F are smooth bundles over $\bar{X}$, we will say that a family of operator $A(\lambda) : \dot{C}^\infty(\bar{X}; E) \to C^{-\infty}(\bar{X}; F)$ depending meromorphically on a parameter $\lambda \in \mathbb{C}$ is *finite meromorphic* if the polar part of $A(\lambda)$ at any pole is a finite rank operator.

Proposition 3.2. *Let* (X, g) *be a spin asymptotically hyperbolic manifold and* D *be a Dirac operator over* X. *Then the resolvent* $R_{\pm}(\lambda) = (D \pm i\lambda)^{-1}$ *extends from* $\{\Re(\lambda) > 0\}$ *to* $\mathbb{C} \setminus (-\mathbb{N}/2)$ *as a finite meromorphic family of operators in* $\Psi_0^{-1,\lambda+\frac{d}{2},\lambda+\frac{d}{2}}(\bar{X}; {}^0\Sigma)$. *Moreover* $R_{\pm}(\lambda)$ *maps* $\dot{C}^\infty(\bar{X}; {}^0\Sigma)$ *to* $x^{\lambda+\frac{d}{2}}C^\infty(\bar{X}; {}^0\Sigma)$ *and for all* $\sigma \in \dot{C}^\infty(\bar{X}; {}^0\Sigma)$, *we have* $[x^{-\lambda-\frac{d}{2}}R_{\pm}(\lambda)\sigma]|_{x=0} \in C^\infty(\partial\bar{X}; {}^0\Sigma_{\mp})$.

Proof. The proof goes along the lines of the construction of Mazzeo-Melrose [25], but we also use arguments of Epstein-Melrose-Mendoza [7] which somehow simplify it. Since there is no real novelty, we do not give the full details but only the important steps and additional arguments to our case which are needed. First, we construct an operator $Q_{\pm}^0(\lambda) \in \Psi_0^{-1,\infty,\infty}(\bar{X}; {}^0\Sigma)$ supported near the interior diagonal such that $(D \pm i\lambda)Q_{\pm}^0(\lambda) = \mathrm{Id} - K_{\pm}^0(\lambda)$ with $K_{\pm}^0(\lambda) \in \Psi_0^{-\infty,\infty,\infty}(\bar{X}; {}^0\Sigma)$, thus a smooth kernel on $\bar{X} \times_0 \bar{X}$ and whose support actually does not intersect the right and left boundary. Note that this can be done thanks to the ellipticity of D and it can be chosen analytic in λ, moreover notice also that $Q_{\pm}^0(\lambda)(D \pm i\lambda) - \mathrm{Id} \in \Psi_0^{-\infty,\infty,\infty}(\bar{X}; {}^0\Sigma)$ by standards arguments of pseudo-differential calculus. The error $K_{\pm}^0(\lambda)$ is a priori not compact on any weighted space $x^sL^2(X; {}^0\Sigma)$ so this parametrix is not sufficient for our purpose. To be compact on such a space, it would be enough to have vanishing of the error on the front face $K_{\pm}^0(\lambda)|_{\rm ff} = 0$.

Next we need to solve away the term at the front face ff, i.e., $K_{\pm}^0(\lambda)|_{\rm ff}$. We can use the normal operator of D: the normal operator $N_{y'}(D)$ of D at $y' \in \partial\bar{X}$ is an operator acting on the space $X_{y'} := \{z \in {}^0T_{y'}\bar{X}, \frac{dx}{x}(z) > 0\}$ obtained by freezing coefficients of D at $y' \in \partial\bar{X}$, when considered as polynomial in the 0-vector fields $x\partial_x, x\partial_y$. Here the spinor bundle over $X_{y'}$ is trivial, i.e., it is given by $X_{y'} \times {}^0\Sigma_{y'}(\bar{X})$ where ${}^0\Sigma_{y'}(\bar{X})$ is the fiber of ${}^0\Sigma$ at the boundary point $y' \in \partial\bar{X}$. The half space $X_{y'}$ equipped with the metric g frozen at y' (the metric here is considered as a symmetric tensor on the 0-cotangent space ${}^0T^*\bar{X}$) is isometric to $\mathbb{H}^{d+1}$ and the operator $N_{y'}(D)$ corresponds to the Dirac operator on $\mathbb{H}^{d+1}$ using this isometry. Moreover this space is also canonically identified to the interior of the front face fibre ${\rm ff}_p$ with basis point $p = (y', y') \in \Delta_\partial$. One has from [25] that the composition $(D \pm i\lambda)G_{\pm}$ for $G_{\pm} \in \Psi_0^{-\infty,\alpha,\beta}(\bar{X}; {}^0\Sigma)$ is in the calculus $\Psi_0(\bar{X})$ and the restriction at the front face fiber ${\rm ff}_p$ is given by

$$((D \pm i\lambda)G_{\pm})|_{{\rm ff}_p} = N_{y'}(D \pm i\lambda).G_{\pm}|_{{\rm ff}_p}$$

which is understood as the action of the differential operator $N_{y'}(D \pm i\lambda)$ on the conormal distribution $G_\pm|_{\mathrm{ff}_p}$ on $\mathrm{ff}_p \simeq X_{y'}$. Thus to solve away the error term at ff, it suffices to find an operator $Q^1_\pm(\lambda)$ in the calculus such that

$$N_{y'}(D \pm i\lambda).Q^1_\pm(\lambda)|_{\mathrm{ff}_p} = K^0_\pm(\lambda)|_{\mathrm{ff}_p}$$

for all $y' \in \partial\bar{X}$. This can be done smoothly in y' by taking $Q^1_\pm(\lambda)|_{\mathrm{ff}_p} := R^{X_{y'}}_\pm(\lambda)(K^0_\pm(\lambda)|_{\mathrm{ff}_p})$ where $R^{X_{y'}}_\pm(\lambda)$ is the analytically extended resolvent of $N_{y'}(D \pm i\lambda) \simeq (D_{\mathbb{H}^{d+1}} \pm i\lambda)$ on $X_{y'} \times {}^0\Sigma_{y'}(\bar{X}) \simeq \mathbb{H}^{d+1} \times {}^0\Sigma(\mathbb{H}^{d+1})$, and then defining $Q^1_\pm(\lambda)$ to be a distribution on $\bar{X} \times_0 \bar{X}$ whose restriction to each fiber ff_p is $R^{X_{y'}}_\pm(\lambda)(K^0_\pm(\lambda)|_{\mathrm{ff}_p})$. As we studied above, the resolvent $R^{\mathbb{H}^{d+1}}_\pm(\lambda)$ is analytic in λ and it maps $\dot{C}^\infty(\mathbb{H}^{d+1}; {}^0\Sigma)$ to $\rho^{\lambda+\frac{d}{2}} C^\infty(\overline{\mathbb{H}^{d+1}}; {}^0\Sigma)$ if ρ is a boundary defining function of the compactification of $\mathbb{H}^{d+1}$, moreover the leading asymptotic term is of the form $\rho^{\lambda+\frac{d}{2}}\psi_\mp$ for some $\psi_\mp \in C^\infty(\partial\overline{\mathbb{H}^{d+1}}; {}^0\Sigma_\mp)$. Thus, the composition $R^{X_{y'}}_\pm(\lambda)(K^0_\pm(\lambda)|_{\mathrm{ff}_p})$ is a conormal distribution in the class $(\rho_{\mathrm{lb}}\rho_{\mathrm{rb}})^{\lambda+\frac{d}{2}} C^\infty(\mathrm{ff}_p; \mathrm{End}({}^0\Sigma_{y'}))$ and it is then possible to find $Q^1_\pm(\lambda) \in \Psi_0^{-\infty,\lambda+\frac{d}{2},\lambda+\frac{d}{2}}(\bar{X}; {}^0\Sigma)$ with the correct restriction at ff. Let $P_\pm$ denote the canonical projection $P_\pm : {}^0\Sigma(\partial\bar{X}) \to {}^0\Sigma_\pm(\partial\bar{X})$. The restriction of a conormal kernel in $\Psi_0^{-\infty,0,\beta}(\bar{X}; {}^0\Sigma)$ at lb can be considered as a section $C^{-\infty}(\partial\bar{X} \times \bar{X}; \mathcal{E})$ conormal to all boundary faces. From the mapping property of $R^{\mathbb{H}^{d+1}}_\pm(\lambda)$ just discussed, it is possible to choose $Q^1_\pm(\lambda)$ such that $P_\pm[\rho_{\mathrm{lb}}^{-\lambda-\frac{d}{2}} Q^1_\pm(\lambda)]|_{\mathrm{lb}} = 0$, which will be important for the next step. Then we get $(D \pm i\lambda)(Q^0_\pm(\lambda) + Q^1_\pm(\lambda)) = \mathrm{Id} - K^1_\pm(\lambda)$ where $K^1_\pm(\lambda) \in \rho_{\mathrm{ff}}\Psi_0^{-\infty,\lambda+\frac{d}{2},\lambda+\frac{d}{2}}(\bar{X}; {}^0\Sigma)$.

The final terms in the parametrix are those at the left boundary, solved away through the indicial equation for $z \in \mathbb{C}$: for all $\psi_\pm \in C^\infty(\partial\bar{X}; {}^0\Sigma_\pm)$

$$(D \pm i\lambda)x^{\frac{d}{2}+z}(\psi_+ + \psi_-) = i(z \pm \lambda)x^{\frac{d}{2}+z}\psi_+ + i(-z \pm \lambda)x^{\frac{d}{2}+z}\psi_- + O(x^{\frac{d}{2}+z+1}), \tag{3.6}$$

which is an easy consequence of (3.3). Lifting D as acting on the left variable on the space $\bar{X} \times_0 \bar{X}$, it satisfies the same type of indicial equation: if $G \in \Psi_0^{-\infty,\alpha+\frac{d}{2},\beta+\frac{d}{2}}(\bar{X}; {}^0\Sigma)$ for some $\alpha, \beta \in \mathbb{C}$, then $(D \pm i\lambda)G \in \Psi_0^{-\infty,\alpha+\frac{d}{2},\beta+\frac{d}{2}}(\bar{X}; {}^0\Sigma)$ and the leading term at lb is

$$[i(\alpha \pm \lambda)\rho_{\mathrm{lb}}^{\alpha+\frac{d}{2}} P_+[(\rho_{\mathrm{lb}}^{-\alpha-\frac{d}{2}} G)|_{\mathrm{lb}}] + i(-\alpha \pm \lambda)\rho_{\mathrm{lb}}^{\alpha+\frac{d}{2}} P_-[(\rho_{\mathrm{lb}}^{-\alpha-\frac{d}{2}} G)|_{\mathrm{lb}}] \tag{3.7}$$

where the restriction at lb is considered as a section $C^{-\infty}(\partial\bar{X} \times \bar{X}; \mathcal{E})$ (conormal to all boundary faces). Then since for $\alpha = \lambda$, the term (3.7) vanishes if $P_\pm(\rho_{\mathrm{lb}}^{-\lambda-\frac{d}{2}} G)|_{\mathrm{lb}} = 0$, one clearly has $K^1_\pm(\lambda) \in \rho_{\mathrm{ff}}\Psi_0^{-\infty,\lambda+\frac{d}{2}+1,\lambda+\frac{d}{2}}(\bar{X}; {}^0\Sigma)$ thanks to the choice of $Q^1_\pm(\lambda)$ and now, since $\alpha = \lambda + j$ for $j \in \mathbb{N}$ is not solution of the indicial equation above when $\lambda \notin -\mathbb{N}/2$, it is possible by induction and using Borel Lemma to construct a term $Q^2_\pm(\lambda) \in \rho_{\mathrm{ff}}\Psi_0^{-\infty,\lambda+\frac{d}{2}+1,\lambda+\frac{d}{2}}(\bar{X}; {}^0\Sigma)$, holomorphic in $\mathbb{C} \setminus (-\mathbb{N}/2)$ such that $(D \pm i\lambda)Q^2_\pm(\lambda) = K^1_\pm(\lambda) - K^2_\pm(\lambda)$ for some operator $K^2_\pm(\lambda) \in \rho_{\mathrm{ff}}\Psi_0^{-\infty,\infty,\lambda+\frac{d}{2}}(\bar{X}; {}^0\Sigma)$.

By [24, Prop. 3.29], the error term $K^2_\pm(\lambda)$ is now already compact on $\rho^z L^2$ for all $z \in [0, \infty)$ such that $\Re(\lambda) + z > 0$. We can now improve the parametrix by using exactly the same arguments as in the proof of Theorem 14.5 of Epstein-Melrose-Mendoza [7] for complex asymptotically hyperbolic manifolds: take a kernel $Q^3_\pm(\lambda)$ which matches to infinite order on $\bar{X} \times \bar{X}$ with the formal Neumann series composition

$$(Q^0_\pm(\lambda) + Q^1_\pm(\lambda) + Q^2_\pm(\lambda)) \sum_{j=0}^{\infty} (K^2_\pm(\lambda))^j,$$

and the error term $K^3_\pm(\lambda)$ will now be in $\rho_{\mathrm{ff}}^\infty \Psi_0^{-\infty,\infty,\lambda+\frac{d}{2}}(\bar{X}; {}^0\Sigma)$. The compositions above are still in the calculus by Mazzeo's composition theorem [24, Th. 3.15], but with a larger index set at the right boundary and front face than for $\sum_{i=0}^{2} Q^i_\pm(\lambda)$ (see the proof of Theorem 4.15 of [7] where this is explained in full details). Now fix λ_0 such that $\Re(\lambda_0) > 0$ where $R_\pm(\lambda_0)$ is bounded on $L^2(X)$. Then we use a standard argument, we can add a finite rank term $Q^4_\pm(\lambda) = Q^4_\pm(\lambda_0) \in \rho_{\mathrm{ff}}^\infty \Psi_0^{-\infty,\infty,\infty}(\bar{X}; {}^0\Sigma)$ to $Q^3_\pm(\lambda)$ in case $(\mathrm{Id} - K^3_\pm(\lambda_0))$ has non empty null space in $\rho^z L^2$, so that $\mathrm{Id} - K^4_\pm(\lambda_0)$ is invertible if $K^4_\pm(\lambda) := \mathrm{Id} - (D \pm i\lambda)(Q^3_\pm(\lambda) + Q^4_\pm(\lambda))$. The operator

$Q_{\pm}(\lambda) := Q^3_{\pm}(\lambda) + Q^4_{\pm}(\lambda)$ is bounded from $\rho^z L^2$ to $\rho^{-z}L^2$ if $\Re(\lambda) + z > 0$ by [24, Th. 3.25], and so Fredholm theorem proves that $R_{\pm}(\lambda) = Q_{\pm}(\lambda)(\mathrm{Id} - K^4_{\pm}(\lambda))^{-1}$ on the weighted space $\rho^z L^2$ for $z \in [0, \infty)$ such that $\Re(\lambda) + z > 0$. Finally, writing $(\mathrm{Id} - K^4_{\pm}(\lambda))^{-1} = \mathrm{Id} + T_{\pm}(\lambda)$, we see that $T_{\pm}(\lambda) = K^4_{\pm}(\lambda) + K^4_{\pm}(\lambda)(\mathrm{Id} - K^4_{\pm}(\lambda))^{-1}K^4_{\pm}(\lambda)$. We claim that $R_{\pm}(\lambda) \in \Psi_0^{-1,\lambda+\frac{d}{2},\lambda+\frac{d}{2}}(\bar{X}; {}^0\Sigma)$: since there are no new arguments needed, we do not give details here and refer the reader to the proof of Theorem 4.15 in [7], we notice though that one of the points explained in [7] is to check that the additional exponents in the index sets obtained using the composition theorem of Mazzeo in the Neumann series of $K^2_{\pm}(\lambda)$ are actually absent in $R_{\pm}(\lambda)$, this is based on the adjointness properties of the resolvent for what concerns the right boundary index set and on the properties of the normal operator for the front face index set (this however requires to add a last term to the parametrix). The mapping property of $R_{\pm}(\lambda)$ acting on $\dot{C}^\infty(\bar{X}; {}^0\Sigma)$ follows again from Mazzeo [24, Prop. 3.28], and the fact that for all $\sigma \in \dot{C}^\infty(\bar{X}; {}^0\Sigma)$ we have $R_{\pm}(\lambda)\sigma = x^{\lambda+\frac{d}{2}}\psi_{\mp} + O(x^{\lambda+\frac{d}{2}+1})$ for some $\psi_{\mp} \in C^\infty(\partial\bar{X}; {}^0\Sigma_{\mp})$ is a straightforward consequence of the indicial equation (3.6). □

Let us now discuss the nature of the spectrum of D. We start by an application of Green formula, usually called *boundary pairing property* (compare to [11, Prop. 3.2])

Lemma 3.3. *Let $\Re(\lambda) = 0$ and $\sigma_i = x^{\frac{d}{2}-\lambda}\sigma_i^- + x^{\frac{d}{2}+\lambda}\sigma_i^+$ for $i = 1, 2$ and $\sigma_i^{\pm} \in C^\infty(\bar{X}; {}^0\Sigma)$, then if $(D^2 + \lambda^2)\sigma_i = r_i \in \dot{C}^\infty(\bar{X}; {}^0\Sigma)$, one has*

$$\int_X (\langle\sigma_1, r_2\rangle - \langle r_1, \sigma_2\rangle)\mathrm{dv}_g = 2\lambda \int_{\partial\bar{X}} \langle\sigma_1^-|_{\partial\bar{X}}, \sigma_2^-|_{\partial\bar{X}}\rangle - \langle\sigma_1^+|_{\partial\bar{X}}, \sigma_2^+|_{\partial\bar{X}}\rangle \mathrm{dv}_{h_0}$$

where $\langle\ ,\ \rangle$ denotes the scalar product with respect to g on X and to h_0 on $\partial\bar{X}$.

Proof. The proof of this Lemma is straightforward by using integration by parts in $\{x \geq \epsilon\}$ and letting $\epsilon \to 0$. □

As a corollary of the resolvent extension and this Lemma, we obtain the following

Corollary 3.4. *On a spin asymptotically hyperbolic manifold, the extended resolvent $R_{\pm}(\lambda) = (D \pm i\lambda)^{-1}$ is holomorphic on the imaginary line $i\mathbb{R}$, consequently the spectrum of D is $\mathbb{R}$ and absolutely continuous.*

Proof. In view of the meromorphy of $R_{\pm}(\lambda)$, it clearly suffices from Stone's formula to prove that $R_{\pm}(\lambda)$ has no pole on the imaginary line. Assume λ_0 is such a pole with order p, then the most singular coefficient of the Laurent expansion is a finite rank operator whose range is made of generalized eigenspinors σ solving $(D \pm i\lambda_0)\sigma = 0$ and $\sigma \in x^{\lambda_0+\frac{d}{2}}C^\infty(\bar{X}; {}^0\Sigma)$. In particular it satisfies $(D^2 + \lambda_0^2)\sigma = 0$ and by applying Lemma 3.3 with $\sigma_1 = \sigma_2 = \sigma$ we see that $(x^{-\lambda_0-\frac{d}{2}}\sigma)|_{x=0} = 0$ and so $\sigma \in x^{\lambda_0+\frac{d}{2}+1}C^\infty(\bar{X}; {}^0\Sigma)$. Now from the indicial equation (3.6), this implies $\sigma \in \dot{C}^\infty(\bar{X}; {}^0\Sigma)$ if $\lambda_0 \neq 0$. Then Mazzeo's unique continuation theorem [23] says that for a class of operators including D^2, there is no eigenfunction vanishing to infinite order at the boundary except $\sigma \equiv 0$, we deduce that $\sigma = 0$ and thus by induction this shows that the polar part of Laurent expansion of $R(\lambda)$ at λ_0 is 0. Now there remains the case $\lambda_0 = 0$. First from self-adjointness of D, we easily get

$$\lambda||\sigma||_{L^2} \leq ||(D \pm i\lambda)\sigma||_{L^2} \tag{3.8}$$

for all $\lambda > 0$ and σ in the L^2-Sobolev space $H^1(X; {}^0\Sigma)$ of order 1, and this implies that $R_{\pm}(\lambda)$ has a pole of order at most 1 at $\lambda = 0$, i.e., one has $R_{\pm}(\lambda) = A_{\pm}\lambda^{-1} + B_{\pm}(\lambda)$ for some $B_{\pm}(\lambda)$ holomorphic (these can be considered as operators from $\dot{C}^\infty(\bar{X}; {}^0\Sigma)$ to its dual). By (3.8), we also see by taking $\lambda \to 0$ that $||A_{\pm}\sigma||_{L^2} \leq ||\sigma||_{L^2}$ for all $\sigma \in \dot{C}^\infty(\bar{X}; {}^0\Sigma)$ and so $A_{\pm}$ is bounded on L^2 and also maps into $\ker(D)$ by $(D \pm i\lambda)R_{\pm}(\lambda) = \mathrm{Id}$. Now in view of the structure of the kernel of $R_{\pm}(\lambda)$, it is not hard to check (e.g. see [13]) that the elements in the range of $A_{\pm}$ are harmonic spinors of the form $\sigma \in x^{\frac{d}{2}}C^\infty(\bar{X}; {}^0\Sigma)$, which can only be L^2 if the leading asymptotic $(x^{-\frac{d}{2}}\sigma)|_{x=0} = 0$, i.e., if $\sigma \in x^{\frac{d}{2}+1}C^\infty(\bar{X}; {}^0\Sigma)$. Using again the indicial equation, (3.6), we deduce that $\sigma \in \dot{C}^\infty(\bar{X}; {}^0\Sigma)$ and thus $\sigma = 0$ by Mazzeo's unique continuation theorem, so $A_{\pm} = 0$. □

Another corollary of Proposition 3.2 is

Corollary 3.5. *The resolvent* $R(\lambda) := (D^2 + \lambda^2)^{-1}$ *extends meromorphically to* $\lambda \in \mathbb{C} \setminus (-\mathbb{N}/2)$ *with poles of finite multiplicity, except at* $\lambda = 0$ *where it has a simple infinite rank pole with residue* $(2i)^{-1}(R_-(0) - R_+(0))$*. Moreover* $R(\lambda)$ *is an operator in* $\Psi_0^{-2,\lambda+\frac{d}{2},\lambda+\frac{d}{2}}(\bar{X}; {}^0\Sigma)$ *and for any* $\sigma \in \dot{C}^\infty(\bar{X}; {}^0\Sigma)$*, one has* $R(\lambda)\sigma \in x^{\lambda+\frac{d}{2}} C^\infty(\bar{X}; {}^0\Sigma)$.

Proof. : The extension and the structure of $R(\lambda)$ are a consequence of Proposition 3.2 since $R(\lambda) = (2i\lambda)^{-1}(R_-(\lambda) - R_+(\lambda))$. As for the mapping property, this is a consequence of mapping properties of operators in $\Psi_0^{*,*,*}(\bar{X})$ in [24]. The question of the simple pole at $\lambda = 0$ is also clear since $R_\pm(\lambda)$ are holomorphic at $\lambda = 0$. It remains to show that the residue Π_0 is infinite rank. One way to prove it is to consider the asymptotic of $\Pi_0\phi$ if $\phi \in C_0^\infty(\bar{X}; {}^0\Sigma)$. First, both $R_-(0)$ and $R_+(0)$ have infinite rank since $DR_\pm(0) = \mathrm{Id}$ on $C_0^\infty(\bar{X}; {}^0\Sigma)$, but moreover if ϕ is smooth compactly supported, $R_\pm(0)\phi$ has an asymptotic of the form $x^{\frac{d}{2}}\psi_\pm$ at the boundary where $\psi_\pm \in C^\infty(\partial\bar{X}; {}^0\Sigma_\mp)$ according to Proposition 3.2. If $\psi_\pm = 0$, then $R_\pm(0)\phi = O(x^{\frac{d}{2}+1})$, and by the indicial equation (3.6) it must vanish to infinite order at $\partial\bar{X}$, which by Mazzeo's unique continuation theorem implies that $R_\pm(0)\phi = 0$, a contradiction. This then shows that the range of $R_+(0)$ on $C_0^\infty(\bar{X}; {}^0\Sigma)$ does not intersect the range of $R_-(0)$ acting on the same space, concluding the proof. □

Remark. By self-adjointness of D^2, one deduces easily that $R(\lambda)^* = R(\bar{\lambda})$, or in terms of kernels

$$R(\lambda; m, m')^* = R(\bar{\lambda}; m', m) \quad \forall m, m' \in X, \quad m \neq m',$$

here $A^* \in \Sigma_{m'} \boxtimes \Sigma_m$ means the adjoint of $A \in \Sigma_m \boxtimes \Sigma_{m'}$ if we identify the dual Σ_m^* with Σ_m via the Hermitian product induced by the metric g.

3.6. **Another parametrix construction when the curvature is constant near ∞.** When (X, g) is asymptotically hyperbolic with constant curvature outside a compact set (which is the case of a convex co-compact quotient $X_\Gamma = \Gamma\backslash\mathbb{H}^{d+1}$), one may use a simplified construction similar to that of Guillopé-Zworski [17] for the Laplacian on functions.

Indeed, there exists a covering of a neighbourhood of $\partial\bar{X}$ by open sets (U_j) with isometries

$$\iota_j : (U_j, g) \to (B, g_{\mathbb{H}^{d+1}}),$$

$$\text{where } B := \{(x_0, y_0) \in (0,\infty) \times \mathbb{R}^d, x_0^2 + |y_0|^2 < 1\} \quad \text{and} \quad g_{\mathbb{H}^{d+1}} = \frac{dx_0^2 + |dy_0|^2}{x_0^2}.$$

We denote by $\bar{B} := \{(x_0, y_0) \in [0,\infty) \times \mathbb{R}^d, x_0^2 + |y_0|^2 < 1\}$ a half-ball in $\mathbb{H}^{d+1}$ and $\partial\bar{B} := \bar{B} \cap \{x_0 = 0\}$. We shall also use the notation ι_j for the restriction $\iota_j|_{U_j \cap \partial\bar{X}}$.

Note that the function x_0 in B is not pulled-back to a boundary defining function putting the metric g under the form $g = (dx^2 + h_x)/x^2$, but we have $(x/\iota_j^* x_0)|_{\partial\bar{X}} = \iota_j^*\eta_j$ for some functions $\eta_j \in C^\infty(\partial\bar{B})$. Through ι_j, the spinor bundle on $\mathbb{H}^{d+1}$ pulls-back to the spinor bundle ${}^0\Sigma(\bar{X})|_{U_j}$ but the splitting induced by $\mathrm{cl}(x_0\partial_{x_0})$ does not correspond to the splitting ${}^0\Sigma_+ \oplus {}^0\Sigma_-$, except at the boundary $x_0 = 0$, since the eigenspaces of $\mathrm{cl}(x_0\partial_{x_0})$ are the eigenspaces of $\mathrm{cl}({\iota_j}_*(x\partial_x))$ when restricted to the boundary.

The Dirac operator $D_{\mathbb{H}^{d+1}}$ pulls back to D in U_j, consequently one may choose a partition of unity $\iota_j^*\chi_j^1$ near $\partial\bar{X}$ associated to $(U_j)_j$, that is $\sum_j \iota_j^*\chi_j^1 = \chi$ where $\chi \in C^\infty(\bar{X})$ is equal to 1 near $\partial\bar{X}$. Take now $\chi_j^2 \in C_0^\infty(\bar{B})$ some functions which are equal to 1 on the support of χ_j^1. Let us define ϕ_j^i on $\partial\bar{B}$ by $\chi_j^i(0, y_0) = \phi_j^i(y_0)$ so that $\sum_j \iota_j^*\phi_j^1 = 1$ on $\partial\bar{X}$ and $\phi_j^2\phi_j^1 = \phi_j^1$.

The first parametrix we can use for $(D^2 + \lambda^2)^{-1}$ is

$$R_0(\lambda) = \sum_j \iota_j^*\chi_j^2 R_{\mathbb{H}^{d+1}}(\lambda)\chi_j^1 {\iota_j}_* + Q_0(\lambda) \tag{3.9}$$

where $Q_0(\lambda) \in \Psi_0^{-2;\infty,\infty}(\bar{X}; {}^0\Sigma)$ is holomorphic, compactly supported and solves $(D^2+\lambda^2)Q_0(\lambda) = (1-\chi)\mathrm{Id} + K_0(\lambda)$ for some $K_0(\lambda) \in C_0^\infty(X \times X; \mathcal{E})$. Here ι_j^* denotes the pull back on sections of the spinor bundle and $\iota_{j_*} := (\iota_j^{-1})^*$. We obtain

$$(D^2+\lambda^2)R_0(\lambda) = \mathrm{Id} + \sum_j \iota_j^*[D^2_{\mathbb{H}^{d+1}}, \chi_j^2]R_{\mathbb{H}^{d+1}}(\lambda)\chi_j^1\iota_{j_*} + K_0(\lambda).$$

The last term $K_0(\lambda)$ is clearly compact on all weighted spaces $x^N L^2(X; {}^0\Sigma)$ while the first one is not. Since on $\mathbb{H}^{d+1}$ one has

$$D^2_{\mathbb{H}^{d+1}} = x_0^{\frac{d}{2}}\left(-(x_0\partial_{x_0})^2\mathrm{Id} + x_0^2 D_{\mathbb{R}^d} + ix_0AD_{\mathbb{R}^d}\right)x_0^{-\frac{d}{2}}, \quad A := \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = -i\mathrm{cl}(x_0\partial_{x_0}),$$

in the splitting induced by $\mathrm{cl}(x_0\partial_{x_0})$. The operator $[D^2_{\mathbb{H}^{d+1}}, \chi_j^2]$ can be written as follows:

$$\begin{aligned}[D^2_{\mathbb{H}^{d+1}}, \chi_j^2] &= dx_0(\partial_{x_0}\chi_j^2(x_0,y_0))\mathrm{Id}\\ &\quad - x_0^2(\partial^2_{x_0}\chi_j^2(x_0,y_0))\mathrm{Id} + x_0^2[D_{\mathbb{R}^d}, \chi_j^2(x_0,y_0)] + ix_0[AD_{\mathbb{R}^d}, \chi_j^2(x_0,y_0)].\end{aligned}$$

Using the fact that $(\nabla\chi_j^2)\chi_j^1 = 0$ and the expression of $R_{\mathbb{H}^{d+1}}(\lambda)$, we deduce that

$$[D^2_{\mathbb{H}^{d+1}}, \chi_j^2]R_{\mathbb{H}^{d+1}}(\lambda)\chi_j^1 \in x_0^{\lambda+\frac{d}{2}+1}{x_0'}^{\lambda+\frac{d}{2}}C^\infty(\bar{B}\times\bar{B}; \mathcal{E})$$

where (x_0, y_0, x_0', y_0') are the natural coordinates on $B\times B$. This error term can be solved away using the indicial equation explained above for the general AH case and one can thus construct, for all $N\in\mathbb{N}$, an operator $R_N(\lambda) \in x^{\lambda+\frac{d}{2}+1}x'^{\lambda}C^\infty(\bar{X}\times\bar{X}; \mathcal{E})$ such that

$$(D^2+\lambda^2)(R_0(\lambda)+R_N(\lambda)) = \mathrm{Id} + K_N(\lambda), \quad K_N(\lambda) \in x^{\lambda+\frac{d}{2}+N}x'^{\lambda+\frac{d}{2}}C^\infty(\bar{X}\times\bar{X}; \mathcal{E})$$

and $K_N(\lambda)$ is compact on $x^{N'}L^2(X; {}^0\Sigma)$ if $0 < N' < N$ and $\Re(\lambda) > -N+N'$ and $\Re(\lambda) > -N'$. All these terms are holomorphic in λ except possibly at $-\mathbb{N}/2$ where first order poles come from the indicial equation and at $\lambda = 0$ where $R_{\mathbb{H}^{d+1}}(\lambda)$ has an infinite rank pole. As above for the general case, we can take an asymptotic series using Borel Lemma, which gives an operator $R_\infty(\lambda) \in x^{\lambda+\frac{d}{2}+1}x'^{\lambda+\frac{d}{2}}C^\infty(\bar{X}\times\bar{X}; \mathcal{E})$, holomorphic in $\lambda \notin -\mathbb{N}_0/2$ so that $(D^2+\lambda^2)(R_0(\lambda) + R_\infty(\lambda)) = \mathrm{Id} + K_\infty(\lambda)$ for some $K_\infty(\lambda) \in x^\infty x'^{\lambda+\frac{d}{2}}C^\infty(\bar{X}\times\bar{X}; \mathcal{E})$. And again, as in the proof of Proposition 3.2, up to the addition of a residual finite rank term for $R_\infty(\lambda)$, we can assume that there is λ_0 with $\Re(\lambda_0) > 0$ such that $\mathrm{Id} + K_\infty(\lambda_0)$ is invertible on $x^N L^2(X)$ for all $N > 0$. The extended resolvent of $D^2+\lambda^2$ is thus given by

$$R(\lambda) = (R_0(\lambda) + R_\infty(\lambda))(\mathrm{Id} + K_\infty(\lambda))^{-1},$$

it is finite meromorphic in $\mathbb{C}\backslash(-\mathbb{N}_0/2)$. Moreover standard arguments show that $(\mathrm{Id}+K_\infty(\lambda))^{-1} = \mathrm{Id} + S_\infty(\lambda)$ for some $S_\infty(\lambda) \in x^\infty x'^{\lambda+\frac{d}{2}}C^\infty(\bar{X}\times\bar{X}; \mathcal{E})$ and so

$$R(\lambda) - R_0(\lambda)(\mathrm{Id} + S_\infty(\lambda)) \in x^{\lambda+\frac{d}{2}+1}x'^{\lambda+\frac{d}{2}}C^\infty(\bar{X}\times\bar{X}; \mathcal{E}).$$

Using the composition results of Mazzeo [24, Th. 3.15], we get $R_0(\lambda)S_\infty(\lambda) \in (xx')^{\lambda+\frac{d}{2}}C^\infty(\bar{X}\times\bar{X}; \mathcal{E})$ so

$$R(\lambda) - R_0(\lambda) \in x^{\lambda+\frac{d}{2}}x'^{\lambda+\frac{d}{2}}C^\infty(\bar{X}\times\bar{X}; \mathcal{E}). \tag{3.10}$$

Similarly, using the remark following Corollary 3.5, we deduce that the kernel

$$R(\lambda; m, m') - R_0(\bar{\lambda}; m', m)^* \in x^{\lambda+\frac{d}{2}}x'^{\lambda+\frac{d}{2}}C^\infty(\bar{X}\times\bar{X}; \mathcal{E}) \tag{3.11}$$

where $R_0(\bar{\lambda}; m', m)^*$ is given, by symmetry of $R_{\mathbb{H}^{d+1}}(\lambda)$, by

$$R_0(\bar{\lambda}; m', m)^* = \sum_j (\iota_j^*\chi_j^1 R_{\mathbb{H}^{d+1}}(\lambda)\chi_j^2\iota_{j_*})(m, m') + Q_0(\bar{\lambda})^*. \tag{3.12}$$

The expressions (3.10), (3.11) and (3.12) will be very useful in what follows for obtaining an explicit formula of the scattering operator modulo a smoothing term.

4. Scattering and Eisenstein series

4.1. **Definitions and properties.** Similarly to the Laplacian on functions, we can define Eisenstein series and scattering operator for Dirac operator. The Eisenstein series $E(\lambda)$ is an operator mapping $C^\infty(\partial\bar{X};{}^0\Sigma)\to C^\infty(X;{}^0\Sigma)$ and for all ψ, $E(\lambda)\psi$ is a non L^2-solution of $(D^2+\lambda^2)\sigma=0$; more precisely it is defined using the following

Lemma 4.1. *Let $\psi\in C^\infty(\partial\bar{X};{}^0\Sigma)$, and $\lambda\in\mathbb{C}\setminus(-\mathbb{N}/2)$ not a pole of $R(\lambda)$, then there exists $\sigma\in C^\infty(X;{}^0\Sigma)$ solution of $(D^2+\lambda^2)\sigma=0$, unique when $\Re(\lambda)\geq 0$, and such that there exist $\sigma^\pm\in C^\infty(\bar{X};{}^0\Sigma)$ with $\sigma^-|_{\partial\bar{X}}=\psi$ and $\sigma=x^{\frac{d}{2}-\lambda}\sigma^-+x^{\frac{d}{2}+\lambda}\sigma^+$. Moreover $\sigma^\pm$ are meromorphic in $\lambda\in\mathbb{C}\setminus(-\mathbb{N}/2)$.*

Proof. This is essentially the same construction as for the Laplacian on functions in [11]: using the indicial equation (3.6) and Borel lemma, it is possible to construct a spinor $\sigma_\infty\in x^{-\lambda+\frac{d}{2}}C^\infty(\bar{X};{}^0\Sigma)$, holomorphic in $\mathbb{C}\setminus(\mathbb{Z}/2)$ such that $(D^2+\lambda^2)\sigma_\infty=O(x^\infty)$ and $(x^{\lambda-\frac{d}{2}}\sigma_\infty)|_{\partial\bar{X}}=\psi$. Note that this spinor is meromorphic in $\lambda\in\mathbb{C}$ with only simple poles at $\mathbb{N}/2$ coming from the roots of the indicial equation. Then we can set

$$\sigma:=\sigma_\infty-R(\lambda)(D^2+\lambda^2)\sigma_\infty.$$

If λ is not a pole of $R(\lambda)$, this solves the problem and defines $\sigma^\pm$ by using the mapping property of $R(\lambda)$ stated in Corollary 3.5. The meromorphy of $\sigma^\pm$ is also a consequence of the construction and of the meromorphy of $R(\lambda)$. The uniqueness of the solution is due to the fact that for two solutions σ_1,σ_2 of the problem, the indicial equation implies that $\sigma_1-\sigma_2\in x^{\lambda+\frac{d}{2}}C^\infty(\bar{X};{}^0\Sigma)$, and then for $\Re(\lambda)>0$ this would be L^2 and λ would be a pole of the resolvent in the physical plane. For $\Re(\lambda)=0$, this can be proved using an application of Green formula like in [11]: if $\widetilde{\sigma}_1,\widetilde{\sigma}_2$ are two solutions of the problem, then the difference $\widetilde{\sigma}_1-\widetilde{\sigma}_2$ is in $x^{\frac{d}{2}+\lambda}C^\infty(\bar{X};{}^0\Sigma)$ by the indicial equation and it is also in the kernel of $D^2+\lambda^2$, so we may apply the Lemma 3.3 with $\sigma_1=\widetilde{\sigma}_1-\widetilde{\sigma}_2$ and $\sigma_2:=R(-\lambda)\varphi$ where $\varphi\in\dot{C}^\infty(\bar{X};{}^0\Sigma)$ is chosen arbitrarily. This clearly implies that $\int_X\langle\widetilde{\sigma}_1-\widetilde{\sigma}_2,\varphi\rangle\mathrm{dv}_g=0$ and thus $\widetilde{\sigma}_1=\widetilde{\sigma}_2$. □

Remark. By uniqueness of the solution, σ and $\sigma^\pm|_{\partial\bar{X}}$ depend linearly on ψ.

Definition 4.2. The Eisenstein series is the operator $E(\lambda):C^\infty(\partial\bar{X};{}^0\Sigma)\to C^\infty(X;{}^0\Sigma)$ defined by $E(\lambda)\psi:=\sigma$ where σ is the smooth spinor in Lemma 4.1.

Definition 4.3. The scattering operator $S(\lambda):C^\infty(\partial\bar{X};{}^0\Sigma)\to C^\infty(\partial\bar{X};{}^0\Sigma)$ is defined by $S(\lambda)\psi:=\sigma^+|_{\partial\bar{X}}$ where σ^+ is the smooth spinor in Lemma 4.1.

It is rather easy to prove that the scattering operator is off-diagonal with respect to the splitting ${}^0\Sigma(\partial\bar{X})={}^0\Sigma_+(\partial\bar{X})\oplus{}^0\Sigma_-(\partial\bar{X})$. To that end, we give an alternative construction of the Eisenstein series $E(\lambda)\psi$ when $\psi\in{}^0\Sigma_+$ or $\psi\in{}^0\Sigma_-$. Let us first define a useful meromorphic function on $\mathbb{C}$

$$C(\lambda):=2^{-2\lambda}\frac{\Gamma(\frac{1}{2}-\lambda)}{\Gamma(\frac{1}{2}+\lambda)}. \tag{4.1}$$

which satisfies $C(\lambda)C(-\lambda)=1$.

Lemma 4.4. *Let $\psi\in C^\infty(\partial\bar{X};{}^0\Sigma_\pm)$, and $\lambda\in\mathbb{C}\setminus(-\mathbb{N}/2)$ be not a pole of $R_\pm(\lambda)$, then there exists a unique $\sigma\in C^\infty(X;{}^0\Sigma)$ solution of $(D\pm i\lambda)\sigma=0$ and such that there exists $\sigma^\pm\in C^\infty(\bar{X};{}^0\Sigma)$ with $\sigma^-|_{\partial\bar{X}}=\psi$ and $\sigma=x^{\frac{d}{2}-\lambda}\sigma^-+x^{\frac{d}{2}+\lambda}\sigma^+$. Moreover one has $\sigma^+|_{\partial\bar{X}}\in C^\infty(\partial\bar{X};{}^0\Sigma_\mp)$ and $\sigma^\pm$ are meromorphic in $\lambda\in\mathbb{C}\setminus(-\mathbb{N}/2)$. If in addition the metric g is even, then $\sigma^\pm/C(\lambda)$ are holomorphic in $\{\Re(\lambda)\geq 0\}$ where $C(\lambda)$ is the function in* (4.1)*.*

Proof. Recall the indicial equation for $(D\pm i\lambda)$: let $j\in\mathbb{N}$ and $\psi_\pm\in C^\infty(\partial\bar{X};{}^0\Sigma_\pm)$ then there exist some smooth spinor $F_{\lambda,j}$ near $\partial\bar{X}$ such that

$$x^{\lambda-\frac{d}{2}}(D\pm i\lambda)x^{-\lambda+\frac{d}{2}+j}(\psi_++\psi_-)=ix^j\Big((j-\lambda\pm\lambda)\psi_++(\lambda-j\pm\lambda)\psi_-\Big)+x^{j+1}F_{\lambda,j}. \tag{4.2}$$

Using this indicial equation inductively, we can construct for all $\psi\in C^\infty(\partial\bar{X};{}^0\Sigma_\pm)$ a formal Taylor series, and thus a true spinor $\sigma_{\infty,\pm}\in x^{-\lambda+\frac{d}{2}}C^\infty(\bar{X};{}^0\Sigma)$ by Borel lemma, such that $(D\pm i\lambda)\sigma_{\infty,\pm}=$

$O(x^\infty)$ and $(x^{\lambda-\frac{d}{2}}\sigma_{\infty,\pm})|_{\partial\bar{X}} = \psi$. This can be done holomorphically in λ as long as λ is not a root of the indicial equation (4.2). The λ such that the indicial numbers $j-\lambda\pm\lambda$ and $\lambda-j\pm\lambda$ in (4.2) vanish are $\mathbb{N}/2$ and they vanish only on the $\Sigma_\mp$ part of the bundle, therefore since $C(\lambda)$ has first order poles at $1/2+\mathbb{N}_0$, we see that $\sigma_\infty/C(\lambda)$ can be chosen holomorphically in $\lambda\in\mathbb{C}\setminus\mathbb{N}$ and that it has at most poles of order 1 at each $\lambda = k$ with $k\in\mathbb{N}$; since it does not involve new arguments we refer the reader who needs more details to the paper of Graham-Zworski [11] where it was studied in the case of the Laplacian on functions. Now consider the case of a metric g even. Since $\mathcal{A}_+,\mathcal{A}_-$ defined in (3.4) are preserved by $x^{-\frac{d}{2}}Dx^{\frac{d}{2}}$ if g is even, then clearly $x^{\lambda-\frac{d}{2}}(D\pm i\lambda)x^{-\lambda+\frac{d}{2}} = \lambda(-\mathrm{cl}(x\partial_x)\pm i) + x^{-\frac{d}{2}}Dx^{\frac{d}{2}}$ also preserves both $\mathcal{A}_+,\mathcal{A}_-$. In particular, if $\psi_-=0$ in (4.2), then $x^{2j+1}F_{\lambda,2j}\in\mathcal{A}_+$ and $x^{2j+2}F_{\lambda,2j+1}\in\mathcal{A}_-$, while the converse is true if $\psi_+=0$. This implies that the spinor $\sigma_{\infty,\pm}$ can be taken so that $x^{\lambda-\frac{d}{2}}\sigma_{\infty,\pm}\in\mathcal{A}_\pm$ and the λ which are actually solution of the indicial equation (4.2) for $D\pm i\lambda$ are only at $1/2+\mathbb{N}_0$. The spinor $\sigma_\infty/C(\lambda)$ can be taken holomorphic also at $\lambda\in\mathbb{N}$. It remains to set

$$\sigma := \sigma_{\infty,\pm} - R_\pm(\lambda)(D\pm i\lambda)\sigma_{\infty,\pm} \tag{4.3}$$

which solves our problem, using the mapping property of $R_\pm(\lambda)$ stated in Proposition 3.2. □

By uniqueness, the solution in Lemma 4.4 is clearly the same as the one of Lemma 4.1 when the initial data ψ is either in $\sigma_+|_{\partial\bar{X}}$ or $\sigma_-|_{\partial\bar{X}}$, which implies

Corollary 4.5. *The scattering operator $S(\lambda)$ maps $C^\infty(\partial\bar{X};{}^0\Sigma_\pm)$ to $C^\infty(\partial\bar{X};{}^0\Sigma_\mp)$.*

Let us define the natural projection and inclusion

$$P_\pm : C^\infty(\partial\bar{X};{}^0\Sigma)\to C^\infty(\partial\bar{X};{}^0\Sigma_\pm);\quad I_\pm : C^\infty(\partial\bar{X};{}^0\Sigma_\pm)\to C^\infty(\partial\bar{X};{}^0\Sigma)$$

and also the maps corresponding to the two off-diagonal components of $S(\lambda)$

$$S_\pm(\lambda) := P_\mp S(\lambda)I_\pm : C^\infty(\partial\bar{X};{}^0\Sigma_\pm)\to C^\infty(\partial\bar{X};{}^0\Sigma_\mp)$$
$$E_\pm(\lambda) := E(\lambda)I_\pm P_\pm : C^\infty(\partial\bar{X};{}^0\Sigma)\to C^\infty(X;{}^0\Sigma).$$

4.2. **Some relations between resolvent, scattering operator and Eisenstein series.** Like for the Laplacian on functions, the Schwartz kernels of $R(\lambda)$, $E(\lambda)$ and $S(\lambda)$ are related by the following

Proposition 4.6. *Let $\lambda\in\mathbb{C}$ be such that $\lambda\notin-\mathbb{N}/2$ and λ not a pole of $R(\lambda)$, then the Schwartz kernel $E(\lambda;m,y')$ and $E_\pm(\lambda;m,y')$ in $C^{-\infty}(\bar{X}\times\partial\bar{X};\mathcal{E})$ of respectively $E(\lambda)$ and $E_\pm(\lambda)$ can be expressed by*

$$\begin{aligned} E(\lambda;m,y') &= 2\lambda[x'^{-\frac{d}{2}-\lambda}R(\lambda;m,x',y')]|_{x'=0},\\ E_\pm(\lambda;m,y') &= [x'^{-\frac{d}{2}-\lambda}R_\pm(\lambda;m,x',y')]|_{x'=0}\mathrm{cl}(\nu)\end{aligned} \tag{4.4}$$

where we use the product decomposition $(x',y')\in[0,\epsilon)\times\partial\bar{X}$ near the boundary in the right variable of $\bar{X}\times\bar{X}$. If in addition $\Re(\lambda)<-\frac{d}{2}$, the Schwartz kernel $S(\lambda;y,y')$ of $S(\lambda)$ is in $C^0(\partial\bar{X}\times\partial\bar{X};\mathcal{E})$ and can be expressed by

$$S(\lambda;y,y') = [x^{-\frac{d}{2}-\lambda}E(\lambda;x,y,y')]_{x=0},\quad S_\pm(\lambda;y,y') = [x^{-\frac{d}{2}-\lambda}E_\pm(\lambda;x,y,y')]|_{x=0}. \tag{4.5}$$

Proof. Let σ_∞ as in Lemma 4.1, then $E(\lambda)\psi = \sigma_\infty - R(\lambda)(D^2+\lambda^2)\sigma_\infty$. The first statement of Proposition is simply obtained by integration by part in (x',y') of

$$\int_{x'\geq\epsilon}\langle R(\lambda;m,x',y'),(D^2+\lambda^2)\sigma_\infty(x',y')\rangle\mathrm{dv}_g(x',y')$$

and letting $\epsilon\to0$, this gives the term $\sigma_\infty(m)$ plus a term

$$\int_{\partial\bar{X}}\left[x'^{-d}\Big(\langle R(\lambda;m,x',y'),\nabla_{x'\partial_{x'}}\sigma_\infty(x',y')\rangle - \langle\nabla_{x'\partial'_x}R(\lambda;m,x',y'),\sigma_\infty(x',y')\rangle\Big)\right]_{x'=0}\mathrm{dv}_{h_0}.$$

But from the analysis of the resolvent $R(\lambda)$, we have for $m \in X$ and as $x' \to 0$

$$R(\lambda; m, x', y') = x'^{\frac{d}{2}+\lambda}(L(\lambda; m, y') + O(x')),$$

$$\nabla_{x'\partial_{x'}} R(\lambda; m, x', y') = (\frac{d}{2} + \lambda)x'^{\frac{d}{2}+\lambda}(L(\lambda; m, y') + O(x'))$$

$$\sigma_\infty(x', y') = x'^{\frac{d}{2}-\lambda}(\psi(y') + O(x')), \quad \nabla_{x'\partial_{x'}} \sigma_\infty(x', y') = (\frac{d}{2} - \lambda)x'^{\frac{d}{2}-\lambda}(\psi(y') + O(x'))$$

for some $L \in C^\infty(X \times \partial\bar{X}; \mathcal{E})$ and where $\psi \in C^\infty(\partial\bar{X}, \Sigma)$ is arbitrarily chosen. We can then deduce that $E(\lambda; m, y') = 2\lambda L(\lambda; m, y')$ as distributions in $C^\infty(X \times \partial\bar{X}; \mathcal{E})$. Using the structure of $R(\lambda)$ in Proposition 3.2, we observe that the kernel of $E(\lambda)$ is also a distribution in $C^{-\infty}(\bar{X} \times \partial\bar{X}; \mathcal{E})$ since its lift to $\bar{X} \times_0 \partial\bar{X}$ is a conormal distribution on $\bar{X} \times_0 \partial\bar{X}$, more precisely it is an element in $\rho_{\rm lb}^{\lambda+\frac{d}{2}} \rho_{\rm ff}^{-\lambda-\frac{d}{2}} C^\infty(\bar{X} \times_0 \partial\bar{X}; \mathcal{E})$. This is exactly the same argument for the $E_\pm(\lambda)$ formula in (4.4) by using the representation (4.3) and integration by part.

Now for the scattering operator, we take $\Re(\lambda) < -\frac{d}{2}$ and use the definition of $S(\lambda)\psi$ to deduce that

$$S(\lambda)\psi = (x^{-\lambda-\frac{d}{2}} E(\lambda)\psi)|_{x=0}.$$

From the fact that the lift of the kernel $x^{-\lambda-\frac{d}{2}} E(\lambda)$ to $\bar{X} \times_0 \partial\bar{X}$ is in $\rho_{\rm ff}^{-2\lambda-d} C^\infty(\bar{X} \times_0 \partial\bar{X}; \mathcal{E})$, thus in $C^0(\bar{X} \times \partial\bar{X})$, we see that

$$\int_{\partial\bar{X}} x^{-\lambda-\frac{d}{2}} E(\lambda; x, y, y')\psi(y'){\rm dv}_{h_0}(y')$$

is $C^0(\bar{X}; {}^0\Sigma)$ and its restriction to $\partial\bar{X}$ is given by the pairing of $[x^{-\lambda-\frac{d}{2}} E(\lambda)]|_{x=0}$ with ψ, which ends the proof for $S(\lambda)$. The argument is the same for $S_\pm(\lambda)$. □

In the same way as $E(\lambda)$, we define the operator $E^\sharp(\lambda), E^\sharp_\pm(\lambda)$ so that the Schwartz kernel of $E^\sharp(\lambda), E^\sharp_\pm(\lambda)$ are given for $\lambda \in \mathbb{C} \setminus (-\mathbb{N}/2)$ not a pole of $R(\lambda)$ by

$$\begin{aligned} E^\sharp(\lambda; y, m') &:= 2\lambda[x^{-\lambda-\frac{d}{2}} R(\lambda; x, y, m')]|_{x=0}, \\ E^\sharp_\pm(\lambda; y, m') &:= -{\rm cl}(\nu)[x^{-\lambda-\frac{d}{2}} R_\pm(\lambda; x, y, m')]|_{x=0} \end{aligned} \tag{4.6}$$

using the product decomposition $[0, \epsilon)_x \times \partial\bar{X}$ near $\partial\bar{X}$. Like for the analysis of the kernel of $E(\lambda)$ above, the structure of the kernel $E^\sharp(\lambda)$ on the blow-up $\partial\bar{X} \times_0 \bar{X}$ is clear from the analysis of $R(\lambda)$. Note also that $E^\sharp_\pm(\lambda)$ maps $\dot{C}^\infty(\bar{X}; {}^0\Sigma)$ to $C^\infty(\partial\bar{X}; {}^0\Sigma_\pm)$ by using Corollary 3.5 and

$$(E^\sharp_\pm(\lambda)f)(y) = -{\rm cl}(\nu) \lim_{x\to 0} \int_X x^{-\lambda-\frac{d}{2}} R_\pm(\lambda; x, y, m')\sigma(m'){\rm dv}_g(m').$$

We see also from the remark following Corollary 3.5 that

$$E^\sharp(\bar{\lambda}; y, m') = E(\lambda; m', y)^*, \qquad E^\sharp_\pm(\bar{\lambda}; y, m') = E_\mp(\lambda; m', y)^* \tag{4.7}$$

when these are considered as linear maps from ${}^0\Sigma_{m'}$ to ${}^0\Sigma_y$.

Lemma 4.7. *Let $m, m' \in X$, then for $\lambda \notin \mathbb{Z}/2$ neither a pole of $R(\lambda)$ nor of $R(-\lambda)$, we have*

$$\begin{aligned} R(\lambda; m, m') - R(-\lambda; m, m') &= (2\lambda)^{-1} \int_{\partial\bar{X}} E(\lambda; m, y)E^\sharp(-\lambda; y, m'){\rm dv}_{h_0}(y), \\ R_\pm(\lambda; m, m') - R_\mp(-\lambda; m, m') &= -\int_{\partial\bar{X}} E_\pm(\lambda; m, y){\rm cl}(\nu)E^\sharp_\mp(-\lambda; y, m'){\rm dv}_{h_0}(y) \end{aligned} \tag{4.8}$$

or in terms of operators

$$R(\lambda) - R(-\lambda) = (2\lambda)^{-1} E(\lambda)E^\sharp(-\lambda), \qquad R_\pm(\lambda) - R_\mp(-\lambda) = -E_\pm(\lambda){\rm cl}(\nu)E^\sharp_\mp(-\lambda).$$

Proof. This is a straightforward application of Green formula and does not involve anything more than in the proof given by Guillopé [16] for the Laplacian on functions on a surface. It is based on the fact that $(D^2 + \lambda^2)R(\lambda; m, m') = (D^2 + \lambda^2)R(-\lambda; m, m') = \delta(m - m')$ and $(D \pm i\lambda)R_\pm(\lambda; m, m') = (D \pm i\lambda)R_\mp(-\lambda; m, m') = \delta(m - m')$, where $\delta(m - m')$ denotes the Dirac mass on the diagonal. □

A corollary of this is some functional equations relating $E(\lambda), E^\sharp(\lambda)$ and $S(\lambda)$.

Corollary 4.8. *The following meromorphic identities hold*

$$E(\lambda) = E(-\lambda)S(\lambda), \qquad E^\sharp(\lambda) = S(\lambda)E^\sharp(-\lambda),$$
$$E_\mp(\lambda) = E_\pm(-\lambda)S_\mp(\lambda), \qquad E^\sharp_\pm(\lambda) = S_\pm(\lambda)E^\sharp_\mp(-\lambda).$$

Proof. Let us consider the second identity: assume $\Re(\lambda) < -\frac{d}{2}$, it suffices to multiply the first line of (4.8) by $x(m)^{-\lambda-\frac{d}{2}}$ and take the limit as $x(m) \to 0$ when $m' \in X$ is fixed, the limit makes sense in view of our analysis of the Schwartz kernels of $R(\lambda), E(\lambda), E^\sharp(\lambda)$ and $S(\lambda)$. Then we use Proposition 4.6 and the definition of $E^\sharp(\lambda)$ and this gives the proof of the second identity of the Corollary, at least for $\Re(\lambda) < -\frac{d}{2}$, but this extends meromorphically to $\lambda \in \mathbb{C}$. The proofs of other identities are similar. □

4.3. **Properties of $S(\lambda)$.**

Proposition 4.9. *For λ such that $\Re(\lambda) < -\frac{d}{2}$, $\lambda \notin -\mathbb{N}/2$ and λ not a pole of $R(\lambda)$, the operator $S(\lambda)$ is a classical pseudo-differential operator on $\partial\bar{X}$ of order 2λ, with principal symbol*

$$\sigma_{\mathrm{pr}}(S(\lambda))(\xi) = C(\lambda)\mathrm{cl}(\nu)|\xi|^{2\lambda-1}_{h_0} i\mathrm{cl}(\xi), \qquad \text{with } C(\lambda) := 2^{-2\lambda}\frac{\Gamma(-\lambda+1/2)}{\Gamma(\lambda+1/2)}. \tag{4.9}$$

Moreover $S(\lambda)$ can be meromorphically extended to $\mathbb{C} \setminus (-\mathbb{N}/2)$ as a family of pseudo-differential operators in $\Psi^{2\lambda}(\partial\bar{X}; {}^0\Sigma)$.

Proof. Let $\beta : \bar{X} \times_0 \bar{X} \to \bar{X} \times \bar{X}$ be the blow-down map, $\partial\bar{X} \times_0 \partial\bar{X} := [\partial\bar{X}, \partial\bar{X}, \Delta_\partial]$ be the blow-up of $\partial\bar{X} \times \partial\bar{X}$ around the diagonal Δ_∂ and $\beta_\partial : \partial\bar{X} \times_0 \partial\bar{X}$ the associated blow-down map. Then the expression (4.5) can also be written for $\Re(\lambda) < -\frac{d}{2}$ ($S(\lambda)$ and $R(\lambda)$ denote also the Schwartz kernel)

$$S(\lambda) = 2\lambda\beta_{\partial *}\Big(\beta^*((xx')^{-\lambda-\frac{d}{2}}R(\lambda))|_{\mathrm{lb}\cap\mathrm{rb}}\Big) \tag{4.10}$$

where $\mathrm{lb} \cap \mathrm{rb}$ is naturally identified with $\partial\bar{X} \times_0 \partial\bar{X}$. For more details, we refer to the article of Joshi-Sa Barreto [19] which deals with the Laplacian on functions. Now using the fact that $R(\lambda) \in \Psi_0^{-2,\lambda+\frac{d}{2},\lambda+\frac{d}{2}}(\bar{X}; {}^0\Sigma)$, we deduce that

$$((xx')^{-\lambda-\frac{d}{2}}R(\lambda))|_{\mathrm{lb}\cap\mathrm{rb}} \in \rho_{\mathrm{ff},\partial}^{-2\lambda-d}C^\infty(\partial\bar{X} \times_0 \partial\bar{X}; \mathcal{E})$$

where $\rho_{\mathrm{ff},\partial} := \rho_{\mathrm{ff}}|_{\mathrm{lb}\cap\mathrm{rb}}$ is a boundary defining function of the boundary (i.e., the face obtained by blowing-up) of $\partial\bar{X} \times_0 \partial\bar{X}$. This shows that the kernel $S(\lambda)$ is classically (or polyhomogeneous) conormal to the diagonal and the leading singularity at $y = y' = p$ given in polar coordinates in the conormal bundle is given by

$$S(\lambda; y, y') \sim c(\lambda)|y - y'|^{-2\lambda-d}U_p(p'), \qquad p' = \frac{y - y'}{|y - y'|} \in S^{d-1}$$

for some $c(\lambda) \in \mathbb{C}$ and where $U_p(p') \in \mathrm{End}({}^0\Sigma_p(\bar{X}))$ denote the limit of the parallel transport $U(e_p, m)$ in the fiber ff_p when $m \to p' \in \mathrm{lb}\cap\mathrm{rb}\cap\mathrm{ff}_p \simeq S^{d-1}$ (here e_p is the center of ff_p defined by the intersection of the interior diagonal with ff_p and identified with the center of hyperbolic space). Thus we obtain $S(\lambda) \in \Psi^{2\lambda}(\partial\bar{X}; {}^0\Sigma)$, moreover the expression (4.10) can be meromorphically extended to $\mathbb{C} \setminus (-\mathbb{N}/2)$ as a distribution classically conormal to the diagonal, thus as a family $S(\lambda) \in \Psi^{2\lambda}(\partial\bar{X}; {}^0\Sigma)$. As for the principal symbol, we use the expression of $U_p(p')$ for $\mathbb{H}^{d+1}$ given in Corollary A.4 and Fourier transform to obtain (4.9). Notice that there might be first order poles of infinite multiplicity at $\mathbb{N}/2$ coming from the meromorphic extension of the distribution $|y - y'|^{2\lambda+j}$ to $\lambda \in \mathbb{C}$. This phenomenon is described in [11] for the case of the Laplacian on functions. □

Like for functions, the scattering operator is a unitary operator on the continuous spectrum and satisfies a functional equation.

Lemma 4.10. *The operator $S(\lambda)$ is unitary on $\{\Re(\lambda)=0\}$, it satisfies $S(\lambda)S(-\lambda)=\mathrm{Id}$ for λ such that $S(\pm\lambda)$ is defined, and it is conformally covariant in the sense that for another choice $\hat{x}=e^{\omega}x$ of geodesic boundary defining function, the corresponding scattering operator is $\hat{S}(\lambda)=e^{-(\frac{d}{2}+\lambda)\omega_0}S(\lambda)e^{(\frac{d}{2}-\lambda)\omega_0}$, where $\omega_0=\omega|_{\partial\bar{X}}$.*

Proof. The functional equation is a straightforward consequence of the uniqueness in Lemma 4.1 or the first equality of Corollary 4.8. The unitarity follows easily from Lemma 3.3 by taking the solutions σ_1,σ_2 of Lemma 4.1 for two initial data $\psi_1,\psi_2\in C^\infty(\partial\bar{X};{}^0\Sigma)$. The conformal covariance of $S(\lambda)$ is straightforward by using the uniqueness of the solution in Lemma 4.1. □

Corollary 4.11. *If the metric g is even in the sense of* (3.2)*, the operator $\widetilde{S}(\lambda):=S(\lambda)/C(\lambda)$ is finite meromorphic in $\mathbb{C}$, and it is holomorphic in $\{\Re(\lambda)\geq 0\}$.*

Proof. The analyticity in the right half-plane is a consequence of the last statement in Lemma 4.4 and the fact that $S(\lambda)\psi=\sigma^+|_{\partial\bar{X}}$ with the notation of this Lemma. We already know the meromorphic extension outside $-\mathbb{N}/2$ so we can write, using Proposition 4.9,

$$S(\lambda)/C(\lambda)=\mathrm{cl}(\nu)(D_{h_0}+i)(|D_{h_0}|+1)^{2\lambda-1}(\mathrm{Id}+K(\lambda))$$

for some $K(\lambda)$ compact on $L^2(\partial\bar{X};{}^0\Sigma)$ and analytic in $\{\Re(\lambda)\geq 0\}$. We know from Lemma 4.10 that $\mathrm{Id}+K(\lambda)$ is invertible for almost all $\lambda\in\mathbb{C}$, so we may use Fredholm analytic theorem to show that $(S(\lambda)/C(\lambda))^{-1}$ is a meromorphic family of operators with poles of finite multiplicity at most in $\Re(\lambda)>0$, so by the functional equation in Lemma 4.10, we deduce that $S(\lambda)/C(\lambda)$ is meromorphic in $\Re(\lambda)<0$ with poles of finite multiplicity. □

We give another corollary of the properties of $\widetilde{S}(\lambda)$.

Corollary 4.12. *For an AH manifold with a metric g even in the sense of* (3.2)*, the resolvent $R_\pm(\lambda)$ is finite meromorphic.*

Proof. According to Proposition 3.2, the only problem for the meromorphy of $R_\pm(\lambda)$ can be at $-\mathbb{N}/2$, so consider the half plane $\{\Re(\lambda)<0\}$. Since $R(-\lambda),E(-\lambda),E^\sharp(-\lambda)$ are holomorphic in $\{\Re(\lambda)<0\}$, the result is a straightforward consequence of Corollary 4.11 together with the formula

$$R(\lambda)=R(-\lambda)+(2\lambda)^{-1}C(\lambda)E(-\lambda)\frac{S(\lambda)}{C(\lambda)}E^\sharp(-\lambda),$$

itself a consequence of Lemma 4.7 and Corollary 4.8. □

4.4. **Representation of $E(\lambda)$ and $S(\lambda)$ in the case X_Γ.** Using Proposition 4.6 and (3.10), we obtain directly an explicit representation modulo a smoothing term. We use the functions η_j,ϕ_j^i,χ_j^i of Section 3.6. We denote by $S_{\mathbb{H}^{d+1}}(\lambda)$ and $E_{\mathbb{H}^{d+1}}(\lambda)$ the scattering operator and Eisenstein series on $\mathbb{H}^{d+1}$, defined using a defining function of $\partial\overline{\mathbb{H}^{d+1}}$ which is equal to x_0 on the half ball $\bar{B}$ in the model $\mathbb{H}^{d+1}=\{(x_0,y_0)\in(0,\infty)\times\mathbb{R}^{d-1}\}$, i.e., in terms of distribution kernel on $\bar{B}\times\partial\bar{B}$ and on $\partial\bar{B}\times\partial\bar{B}$

$$\begin{aligned}E_{\mathbb{H}^{d+1}}(\lambda;x_0,y_0,y_0')&:=2\lambda[{x_0'}^{-\lambda-\frac{d}{2}}R_{\mathbb{H}^{d+1}}(\lambda;x_0,y_0,x_0',y_0')]_{x_0'=0},\\ S_{\mathbb{H}^{d+1}}(\lambda;y_0,y_0')&:=[x_0^{-\lambda-\frac{d}{2}}E_{\mathbb{H}^{d+1}}(\lambda;x_0,y_0,y_0')]|_{x_0=0}.\end{aligned}$$

Lemma 4.13. *If $\lambda\in\mathbb{C}\setminus(-\mathbb{N}_0/2)$ is not a pole of $R(\lambda)$, then the Eisenstein series $E(\lambda)$ for D^2 on a convex co-compact quotient $X:=\Gamma\backslash\mathbb{H}^{d+1}$ has the kernel $E(\lambda)=E_0(\lambda)+E_\infty(\lambda)$ where*

$$\begin{aligned}E_0(\lambda)&:=\sum_j \iota_j^*\chi_j^2E_{\mathbb{H}^{d+1}}(\lambda)\phi_j^1\eta_j^{-\lambda-\frac{d}{2}}\iota_{j*},\\ E_\infty(\lambda)&=2\lambda[{x'}^{-\lambda-\frac{d}{2}}(R(\lambda)-R_0(\lambda))]|_{x'=0}\in x^{\lambda+\frac{d}{2}}C^\infty(\bar{X}\times\partial\bar{X};\mathcal{E}).\end{aligned}$$

Similarly, the scattering operator $S(\lambda)$ for D^2 on X has the kernel $S(\lambda) = S_0(\lambda) + S_\infty(\lambda)$ where

$$S_0(\lambda) := \sum_j \iota_j^* \eta_j^{-\lambda-\frac{d}{2}} \phi_j^2 S_{\mathbb{H}^{d+1}}(\lambda) \phi_j^1 \eta_j^{-\lambda-\frac{d}{2}} \iota_{j*},$$

$$S_\infty(\lambda) = 2\lambda[(xx')^{-\lambda-\frac{d}{2}}(R(\lambda) - R_0(\lambda))]|_{x=x'=0} \in C^\infty(\partial\bar{X} \times \partial\bar{X}; \mathcal{E}).$$

5. Selberg zeta function of odd type

In this section, we will assume that the dimension $d+1 = 2n+1$ is odd except in Lemma 5.2.

5.1. **Odd heat kernel of Dirac operator on $\mathbb{H}^{d+1}$.**

By the identification (2.4), the kernel of the odd heat operator $D_{\mathbb{H}^{d+1}} e^{-tD^2_{\mathbb{H}^{d+1}}}$ on $L^2(\mathbb{H}^{d+1}; \Sigma(\mathbb{H}^{d+1}))$ can be considered as a τ_d-radial function P_t over G. Hence there exists a function P_t from G to $\mathrm{End}(V_{\tau_d})$ satisfying the K-equivariance condition

$$P_t(k_1 g k_2) = \tau_d(k_2)^{-1} P_t(g) \tau_d(k_1)^{-1} \qquad \text{for } g \in G,\ k_1, k_2 \in K \tag{5.1}$$

such that

$$D_{\mathbb{H}^{d+1}} e^{-tD^2_{\mathbb{H}^{d+1}}}(gK, hK) = P_t(h^{-1}g) \qquad \text{for } g, h \in G. \tag{5.2}$$

Let us remark that $P_t(h^{-1}g)$ and $P_t(k_1^{-1}h^{-1}gk_2)$ for $k_1, k_2 \in K$ give the same map by the condition (5.1), so that the right hand side of (5.2) does not depend on the choice of the representatives of the K-cosets.

Recalling the Cartan decomposition $G = KA^+K$ with $A^+ := \{a_r = \exp(rH) \,|\, r > 0\}$, any element $g \in G$ can be written as $g = h a_r k$ where $a_r = \exp(rH)$ and r is the same as the hyperbolic distance $d_{\mathbb{H}^{d+1}}(eK, gK)$ between two points eK and gK in $\mathbb{H}^{d+1} \cong G/K$. Here e denotes the identity element in G. Now let us recall that the spin representation τ_d decomposes into two half spin representations σ_+, σ_- when restricting to $M = \mathrm{Spin}(d)$,

$$\tau_d|_M = \sigma_+ \oplus \sigma_-,$$

hence the representation space V_{τ_d} also decomposes into $V_{\sigma^+} \oplus V_{\sigma^-}$ as M-representation spaces. By Schur's lemma there exists a function $p_t^\pm : \mathbb{R} \to \mathbb{C}$ such that

$$P_t(a_r)|_{V_{\sigma_\pm}} = p_t^\pm(r)\,\mathrm{Id}_{V_{\sigma_\pm}}$$

where $a_r \in A^+$. As in the proof of Theorem 8.5 of [6] using Theorem 8.3 of [6], one can easily derive

Proposition 5.1. *The scalar components of $D_{\mathbb{H}^{d+1}} e^{-tD^2_{\mathbb{H}^{d+1}}}$ are given by*

$$p_t^\pm(r) = \pm \frac{\sinh(r/2)}{i 2^{3n+3/2} \Gamma(n+3/2) t^{3/2}} \left(-\frac{d}{d(\cosh r)} \right)^n r \sinh^{-1}(r/2)\, e^{-\frac{r^2}{4t}}. \tag{5.3}$$

Let us observe that the equalities (5.1) and (5.3) determine the odd heat kernel $D_{\mathbb{H}^{d+1}} e^{-tD^2_{\mathbb{H}^{d+1}}}$ by the Cartan decomposition $G = KA^+K$.

5.2. **Odd heat kernels over convex co-compact hyperbolic manifolds.**

By abuse of notation, g, h will also denote the points in $\mathbb{H}^{d+1}$ corresponding to the cosets gK, hK in G/K. By a usual construction, the kernel of the odd heat operator De^{-tD^2} over X_Γ is given (as an automorphic kernel) by

$$De^{-tD^2}(g, h) = \sum_{\gamma \in \Gamma} D_{\mathbb{H}^{d+1}} e^{-tD^2_{\mathbb{H}^{d+1}}}(g, \gamma h) \tag{5.4}$$

where g, h denote points in $X_\Gamma = \Gamma \backslash \mathbb{H}^{d+1}$ which we view as a fundamental domain in $\mathbb{H}^{d+1}$ with sides identified through Γ. Using $d_{\mathbb{H}^{d+1}}(g, h) = d_{\mathbb{H}^{d+1}}(e, g^{-1}h)$ where e denotes the origin in the unit disc model of $\mathbb{H}^{d+1}$, then by (5.1), (5.3) and some elementary calculations, we have

$$||D_{\mathbb{H}^{d+1}} e^{-tD^2_{\mathbb{H}^{d+1}}}(g, \gamma h)|| \le C\, t^{-\frac{3}{2}}\, e^{-\frac{d}{2} r_\gamma(g,h) - \frac{r_\gamma(g,h)^2}{4t}} \sum_{\substack{0 \le j \le n+1 \\ 0 \le k \le n}} r_\gamma(g, h)^j t^{-k}. \tag{5.5}$$

Here C is a positive constant independent on t and $r_\gamma(g,h) := d_{\mathbb{H}^{d+1}}(g,\gamma h)$. In particular the number $r_\gamma(g,g)$ is independent of g in the axis of γ and is called the translation length of γ, denoted by ℓ_γ.

Lemma 5.2. *Let $\mathcal{F}$ be a fundamental domain of Γ and $\widetilde{x}$ be a boundary defining function of X_Γ which we view as well as a function on $\mathcal{F}$. There are positive constants C_1, C_2 such that for all $\gamma \in \Gamma$ with translation length $\ell_\gamma > C_1$ and all $g,h \in \mathcal{F}$,*

$$e^{-r_\gamma(g,h)} \leq C_2\, e^{-\ell_\gamma}\widetilde{x}(g)\widetilde{x}(h).$$

Proof. By conjugating by an isometry of $\mathbb{H}^{d+1}$, we can assume that, in the half space model $\mathbb{H}^{d+1} = \mathbb{R}^+_x \times \mathbb{R}^d_y$, the point at ∞ is not in the limit set $\Lambda(\Gamma)$ of the group Γ. Then, since the group is convex co-compact, there exists a fundamental domain $\mathcal{F}$ which satisfies the following: there exists $C > 0$ and $\epsilon > 0$ such that

$$\mathcal{F} \cup \Lambda(\Gamma) \subset B := \{z \in [0,\infty) \times \mathbb{R}^d; |z| \leq C, d_{\rm eucl}(z;\Lambda(\Gamma)) \geq \epsilon\}$$

where $d_{\rm eucl}$ denotes the Euclidean distance in $\mathbb{R}^{d+1}$. Notice that the function $\widetilde{x}$ is comparable to the function x on $\mathcal{F}$ in the sense that $1/A < \widetilde{x}/x < A$ for some constant $A > 0$. Let now $\gamma \in \Gamma$ be an isometry, whose fixed points p^1_γ, p^2_γ must belong to $\mathbb{R}^d_y \cap B$. Composing a translation $z \mapsto z - (p^1_\gamma + p^2_\gamma)/2$ with a rotation in the $\mathbb{R}^d_y$ variable, we define an isometry q_γ which maps p^1_γ to $p_\gamma := (0, |p^1_\gamma - p^2_\gamma|/2, 0\ldots,0)$ and p^2_γ to $-p_\gamma$. Notice that, since q_γ is also an Euclidean isometry, then

$$q_\gamma(\mathcal{F}) \subset B' := \{z \in [0,\infty) \times \mathbb{R}^d; |z| \leq 2C, d_{\rm eucl}(z;\pm p_\gamma) \geq \epsilon\}.$$

We identify the (x,y_1) half-plane inside $\mathbb{H}^{d+1}$ with $\mathbb{H}^2$ by setting $z_0 := y_1 + ix$, in particular $\pm p_\gamma$ belong to the boundary of $\mathbb{H}^2$. We consider the isometry s_γ of $\mathbb{H}^2$ defined by

$$s_\gamma : z_0 \mapsto \frac{z_0 + p_\gamma}{-z_0 + p_\gamma}. \tag{5.6}$$

This isometry maps p_γ to ∞ and $-p_\gamma$ to 0 in $\mathbb{H}^{d+1}$. We extend s_γ recursively to $\mathbb{H}^{d+1}$ as follows: the isometry s_γ of $\mathbb{H}^k$ is extended to an isometry of $\mathbb{H}^{k+1}$ by identifying $\mathbb{H}^{k+1}$ with $(0,\pi)_\theta \times \mathbb{H}^k$ via the map

$$\iota_k : (\theta, x, y_1, \ldots, y_{k-1}) \mapsto (x\sin\theta, y_1, \ldots, y_{k-1}, x\cos\theta)$$

and defining the extension, still denoted s_γ, by $s_\gamma(\iota_k(\theta,w)) := \iota_k(\theta, s_\gamma(w))$. Note that $|\iota_k(\theta,w)| = |w|$ for each k, and thus $|s_\gamma(z_0,y_2,\ldots,y_d)| = |s_\gamma(z_0)|$. Using this fact and the explicit formula (5.6) in $\mathbb{H}^2$, it is easy to see that for all $z \in B'$, we have

$$\frac{\epsilon}{4C} \leq |s_\gamma(z)| \leq \frac{4C}{\epsilon}.$$

We conclude that $t_\gamma := s_\gamma \circ q_\gamma$ maps $\mathcal{F}$ into $\{z \in \mathbb{R}^+ \times \mathbb{R}^d, \varepsilon \leq |z| \leq \varepsilon^{-1}\}$ for some $\varepsilon > 0$ which does not depend on γ. But $t_\gamma \circ \gamma \circ t_\gamma^{-1}$ is an isometry fixing the line $\{y = 0\}$ and thus it can be written under the form

$$t_\gamma \circ \gamma \circ t_\gamma^{-1} : (x,y) \mapsto e^{\ell_\gamma}(x, \mathcal{O}_\gamma(y))$$

for some $\mathcal{O}_\gamma \in {\rm SO}(d)$ where ℓ_γ is the translation length of γ. Then we have for $m = (x,y), m' = (x',y')$ in the half-space model

$$\cosh^2(d_{\mathbb{H}^{d+1}}(m,m')/2) = \frac{|y-y'|^2 + |x+x'|^2}{4xx'}$$

and by writing $t_\gamma g = (x,y)$ and $t_\gamma h = (x',y')$ in the half-space model,

$$\cosh^2(r_\gamma(g,h)/2) = e^{\ell_\gamma}\frac{|e^{-\ell_\gamma}y - \mathcal{O}_\gamma(y')|^2 + |e^{-\ell_\gamma}x + x'|^2}{4xx'}. \tag{5.7}$$

But since $t_\gamma g, t_\gamma h \in t_\gamma(\mathcal{F})$, one has $\varepsilon \leq (x^2 + |y|^2)^{\frac{1}{2}} \leq \varepsilon^{-1}$ and the same for (x',y'), which from (5.7) implies that

$$\varepsilon^2 \leq \cosh^2(r_\gamma(g,h)/2)e^{-\ell_\gamma}xx' \leq \varepsilon^{-2} \tag{5.8}$$

for ℓ_γ large enough (depending only on ε). Observe now that, using the embeddings $\mathbb{H}^2 \subset \cdots \subset \mathbb{H}^{d+1}$ as above using the maps ι_k, we can view the point p_γ as an element of (the boundary of) each $\mathbb{H}^k$ for $k = 2, \dots, d+1$. Moreover, as Euclidean norms, one clearly has $|\iota_k(\theta,\omega) - p_\gamma| = |\omega - p_\gamma|$ for all $\omega \in \mathbb{H}^k$, and thus $|z - p_\gamma| = |z_0 - p_\gamma|$ where $z \in \mathbb{H}^{d+1}$ corresponds to a unique $(\theta_1, \dots, \theta_{d-1}, z_0) \in (0,\pi)^{d-1} \times \mathbb{H}^2$ by the maps ι_k described above. Let us denote $\mathrm{Im}(z) := x$ when $z = (x, y_1, \dots, y_d) \in \mathbb{H}^{d+1}$, then each $z \in \mathbb{H}^{d+1}$ is associated to a unique $(\theta_1, \dots, \theta_{d-1}, z_0) \in (0,\pi)^{d-1} \times \mathbb{H}^2$ by the maps ι_k (here z_0 is a complex coordinate on $\mathbb{H}^2$ viewed as the half-space $\mathrm{Im}(z_0) > 0$) and we have $\mathrm{Im}(z) = \mathrm{Im}(z_0)\sin(\theta_1)\dots\sin(\theta_{d-1})$ and $\mathrm{Im}(s_\gamma(z)) = \mathrm{Im}(s_\gamma(z_0))\sin(\theta_1)\dots\sin(\theta_{d-1})$ by construction of s_γ acting on $\mathbb{H}^{n+1}$. But a short computation gives

$$\mathrm{Im}(s_\gamma(z_0)) = 2p_\gamma \frac{\mathrm{Im}(z_0)}{|z_0 - p_\gamma|^2} = 2p_\gamma \frac{\mathrm{Im}(z_0)}{|z - p_\gamma|^2}$$

where $p_\gamma \in \mathbb{H}^2$ is viewed as a positive real number in $\mathbb{C}$, which therefore implies $\mathrm{Im}(s_\gamma(z))/\mathrm{Im}(z) = 2p_\gamma/|z - p_\gamma|^2$ for all $z \in \mathbb{H}^{d+1}$. We have also shown that $\inf_{z \in q_\gamma(\mathcal{F})} |z - p_\gamma| > \epsilon$ and $0 < p_\gamma < 1/\epsilon$ for some $\epsilon > 0$ uniform in $\gamma \in \Gamma$ by convex co-compactness of Γ, thus we can combine this with (5.8) and the fact that $x \circ q_\gamma = x$ where x is comparable to $\widetilde{x}$ on $\mathcal{F}$ to deduce that there exists a constant $C > 0$ uniform in γ so that $(x \circ t_\gamma)/\widetilde{x} \le C$ on $\mathcal{F}$. This ends the proof. □

By (1.2), (5.5) and Lemma 5.2, we get that the right hand side of (5.4) converges uniformly in g, h in a fundamental domain. We denote by tr the local trace over ${}^0\Sigma_m(X_\Gamma) \cong V_{\tau_d}$ for m in a fundamental domain of Γ.

Proposition 5.3. *For m in a fundamental domain of Γ, we have*

$$\mathrm{tr}(De^{-tD^2})(m) = \sum_{\gamma \in \Gamma \setminus \{e\}} \mathrm{tr}(D_{\mathbb{H}^{d+1}} e^{-tD^2_{\mathbb{H}^{d+1}}})(m, \gamma m). \tag{5.9}$$

Proof. It is enough to show that $\mathrm{tr}(D_{\mathbb{H}^{d+1}} e^{-tD^2_{\mathbb{H}^{d+1}}})(m,m) = 0$, which is a consequence of

$$\mathrm{tr}(D_{\mathbb{H}^{d+1}} e^{-tD^2_{\mathbb{H}^{d+1}}})(m,m) = \mathrm{tr}(P_t)(e) = d(\sigma_\pm)(p_t^+(0) + p_t^-(0)) = 0$$

where $d(\sigma_\pm)$ denotes the dimension of $V_{\sigma_\pm}$ and the last equality follows from (5.3). □

By equations (5.5), (5.7) and Proposition 5.3, we deduce that there is $\epsilon > 0$ such that

$$\left|\mathrm{tr}(De^{-tD^2})(m)\right| \le C_\epsilon(t)\, x(m)^{2(\frac{d}{2}+\epsilon)} \sum_{\gamma \in \Gamma \setminus \{e\}} e^{-(d+\epsilon) r_\gamma}$$

where $C_\epsilon(t)$ is a constant depending only on ϵ, t and x a boundary defining function. Hence the local trace function $\mathrm{tr}(De^{-tD^2})(m,m)$ is integrable over X_Γ. Now we can define

$$\mathrm{Tr}(De^{-tD^2}) := \int_{X_\Gamma} \mathrm{tr}(De^{-tD^2})(m)\, \mathrm{dv}(m) \tag{5.10}$$

where $\mathrm{dv}(m)$ denotes the metric over X_Γ induced from the hyperbolic metric $\mathrm{dv}_{\mathbb{H}^{d+1}}$.

By our assumption on Γ, $\Gamma \setminus \{e\}$ consists of hyperbolic elements and decomposes into Γ-conjugacy classes of hyperbolic elements. We denote by Γ_{hyp} the set of Γ-conjugacy classes of hyperbolic elements. Each element $[\gamma]$ in the set Γ_{hyp} corresponds to a closed geodesic C_γ in X_Γ. We denote by $l(C_\gamma)$ the length of C_γ and by $j(\gamma)$ the positive integer such that $\gamma = \gamma_0^{j(\gamma)}$ with a primitive γ_0. A primitive hyperbolic element γ means that it can not be given by a power of any other elements in Γ, so that Γ-conjugacy class of a primitive γ corresponds to a prime geodesic C_γ in X_Γ. The trace of the monodromy in $\Sigma(X_\Gamma) \cong \Gamma\backslash\big(G \times_{\tau_d} V_{\tau_d}\big)$ along a closed geodesic C_γ is given by $\chi_{\sigma_+}(m_\gamma) + \chi_{\sigma_-}(m_\gamma)$ since any hyperbolic element γ can be conjugated to $m_\gamma a_\gamma \in MA^+$. A closed geodesic C_γ corresponds to a fixed point of the geodesic flow on the unit sphere bundle over X_Γ. The Poincaré map $P(C_\gamma)$ is the differential of the geodesic flow at C_γ, which is given by $P(C_\gamma) = \mathrm{Ad}(m_\gamma a_\gamma)$ if $\gamma = m_\gamma a_\gamma$. The unit sphere bundle SX_Γ of X_Γ is given by $\Gamma\backslash G/M$, and its tangent bundle TSX_Γ is given by

$$TSX_\Gamma = \Gamma\backslash G \times_M (\bar{\mathfrak{n}} \oplus \mathfrak{a} \oplus \mathfrak{n})$$

where $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$ and M acts on $\overline{\mathfrak{n}} \oplus \mathfrak{a} \oplus \mathfrak{n}$ by the adjoint action Ad. Hence $P(C_\gamma)$ preserves the decomposition $\overline{\mathfrak{n}} \oplus \mathfrak{a} \oplus \mathfrak{n}$. We denote by $P(C_\gamma)_{|\mathfrak{n}}$, $P(C_\gamma)_{|\overline{\mathfrak{n}}}$ its restriction to $\mathfrak{n}$, $\overline{\mathfrak{n}}$ part respectively. Now we put

(5.11)
$$D(\gamma) := |\det(P(C_\gamma)_{|\mathfrak{n}\oplus\bar{\mathfrak{n}}} - \mathrm{Id})|^{1/2} = e^{-nl(C_\gamma)}|\det(P(C_\gamma)_{|\mathfrak{n}} - \mathrm{Id})| = e^{nl(C_\gamma)}\det(\mathrm{Id} - P(C_\gamma)_{|\bar{\mathfrak{n}}}).$$

Proposition 5.4. *The following identity holds*

$$\mathrm{Tr}(De^{-tD^2}) = \frac{2\pi i}{(4\pi t)^{\frac{3}{2}}} \sum_{[\gamma]\in\Gamma_{\mathrm{hyp}}} \frac{l(C_\gamma)^2}{j(\gamma)D(\gamma)}(\chi_{\sigma_+}(m_\gamma) - \chi_{\sigma_-}(m_\gamma))e^{-\frac{l(C_\gamma)^2}{4t}}. \tag{5.12}$$

Proof. By equalities (5.2), (5.9) and (5.10),

$$\mathrm{Tr}(De^{-tD^2}) = \sum_{\gamma\in\Gamma\setminus\{e\}} \int_{\Gamma\backslash G} \mathrm{tr}\, P_t(g^{-1}\gamma g)\, d(\Gamma g). \tag{5.13}$$

By Theorem 2.2 in [27], the scalar function $p_t(g) := \mathrm{tr}\, P_t(g)$ is in the Harish-Chandra L^1-space. (Note that $p_t(g)$ should not be confused with $p_t^{\pm}(r)$ in the subsection 5.1.) Hence we can follow the well known path of Selberg on p. 63–66 of his famous paper [32] to obtain

$$\sum_{\gamma\in\Gamma\setminus\{e\}} \int_{\Gamma\backslash G} p_t(g^{-1}\gamma g)\, d(\Gamma g) = \sum_{[\gamma]\in\Gamma_{\mathrm{hyp}}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} p_t(g^{-1}\gamma g)\, d(G_\gamma g) \tag{5.14}$$

where Γ_γ, G_γ denote the centralizer of γ in Γ and G respectively. Now we show the following equality,

$$\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} p_t(g^{-1}\gamma g)\, d(G_\gamma g) = \frac{2\pi i}{(4\pi t)^{\frac{3}{2}}} \frac{l(C_\gamma)^2}{j(\gamma)D(\gamma)}(\chi_{\sigma_+}(m_\gamma) - \chi_{\sigma_-}(m_\gamma))e^{-\frac{l(C_\gamma)^2}{4t}}. \tag{5.15}$$

We may assume that a hyperbolic element $\gamma \in \Gamma$ has the form $m_\gamma a_\gamma \in MA^+$. If $\gamma \in MA^+$,

$$\int_{G_\gamma\backslash G} p_t(g^{-1}\gamma g)\, d(G_\gamma g) = \mathrm{vol}(G_\gamma/A)^{-1} \int_{G/A} p_t(g\gamma g^{-1})\, d(gA). \tag{5.16}$$

We also have

$$\int_{G/A} p_t(g\gamma g^{-1})\, d(gA) = D(\gamma)^{-1} F_{p_t}(m_\gamma a_\gamma) \tag{5.17}$$

where the Abel transform of p_t is given by

$$F_{p_t}(m_\gamma a_\gamma) = a_\gamma^\rho \int_N \int_K p_t(k\, m_\gamma a_\gamma n k^{-1})\, dk\, dn$$

with $a_\gamma^\rho = \exp(nl(C_\gamma))$. By Theorem 6.2 in [35], we have

$$\Theta_{\sigma,\lambda}(p_t) = \int_M \int_{-\infty}^{\infty} F_{p_t}(m\exp(rH))\, \mathrm{tr}\, \sigma(m)\, e^{i\lambda r}\, dr\, dm. \tag{5.18}$$

Here $\Theta_{\sigma,\lambda}(p_t)$ is defined by

$$\Theta_{\sigma,\lambda}(p_t) := \mathrm{Tr}\ \pi_{\sigma,\lambda}(p_t) = \mathrm{Tr} \int_G p_t(g)\pi_{\sigma,\lambda}(g)\, dg$$

and for $(\sigma, H_\sigma) \in \widehat{M}$ (where $\widehat{M}$ denotes the set of equivalence classes of irreducible unitary representations of M) and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ the principal representation $\pi_{\sigma,\lambda} := \mathrm{Ind}_{MAN}^G(\sigma \otimes e^{i\lambda} \otimes \mathrm{Id})$ of G acts on the space

$$\mathcal{H}_{\sigma,\lambda} := \{\, f : G \to H_\sigma \mid f(xman) = a^{-(i\lambda+\rho)}\sigma(m)^{-1}f(x),\ f|_K \in L^2(K) \,\}$$

by the left translation $\pi_{\sigma,\lambda}(g)f(x) = f(g^{-1}x)$. Applying the Fourier inversion theorem and the Peter-Weyl theorem to the equality (5.18), we get

$$F_{p_t}(m_\gamma a_\gamma) = \sum_{\sigma\in\widehat{M}} \overline{\mathrm{tr}\, \sigma(m_\gamma)}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \Theta_{\sigma,\lambda}(p_t) e^{-il(C_\gamma)\lambda}\, d\lambda. \tag{5.19}$$

Now let observe that $\Theta_{\sigma,\lambda}(p_t)$ vanishes unless $\sigma = \sigma_\pm$ since $\tau_d|_M = \sigma_+ \oplus \sigma_-$. Moreover, we have

$$\Theta_{\sigma_\pm,\lambda}(p_t) = \pm\lambda e^{-t\lambda^2} \tag{5.20}$$

as in Proposition 3.1 in [29] by (4.5) in [28]. Note that the analysis for this does not depend on Γ, but is performed over G. Combining (5.16), (5.17), (5.19), (5.20) and observing that $\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)/\mathrm{vol}(G_\gamma/A) = l(C_\gamma)/j(\gamma)$, we conclude

$$\begin{aligned}\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} p_t(g^{-1}\gamma g)\,d(G_\gamma g) &= \sum_{\sigma\in\widehat{M}} \frac{l(C_\gamma)\overline{\mathrm{tr}\,\sigma(m_\gamma)}}{j(\gamma)D(\gamma)}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\Theta_{\sigma,\lambda}(h)e^{-il(C_\gamma)\lambda}\,d\lambda \\ &= \frac{l(C_\gamma)\big(\chi_{\sigma_+}(m_\gamma)-\chi_{\sigma_-}(m_\gamma)\big)}{j(\gamma)D(\gamma)}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\lambda e^{-t\lambda^2}e^{il(C_\gamma)\lambda}\,d\lambda \\ &= \frac{2\pi i}{(4\pi t)^{\frac{3}{2}}}\,\frac{l(C_\gamma)^2\big(\chi_{\sigma_+}(m_\gamma)-\chi_{\sigma_-}(m_\gamma)\big)}{j(\gamma)D(\gamma)}\; e^{-\frac{l(C_\gamma)^2}{4t}}.\end{aligned}$$

Taking the sum over $[\gamma]\in\Gamma_{\mathrm{hyp}}$ of this equality and by (5.13) and (5.14), we obtain (5.12). □

From Proposition 5.4, putting $c := \min_{[\gamma]\in\Gamma_{\mathrm{hyp}}} l(C_\gamma) > 0$ we obtain the

Corollary 5.5. *The following estimate holds*

$$\mathrm{Tr}(De^{-tD^2}) = O(e^{-c^2/t}) \qquad as \quad t\to 0.$$

5.3. **Selberg zeta function of odd type.** We define the Selberg zeta functions attached to half spinor representations $\sigma_\pm$ by

$$Z_\Gamma(\sigma_\pm,\lambda) := \exp\Big(-\sum_{[\gamma]\in\Gamma_{\mathrm{hyp}}}\frac{\chi_{\sigma_\pm}(m_\gamma)}{j(\gamma)D(\gamma)}e^{-\lambda\, l(C_\gamma)}\Big) \tag{5.21}$$

for $\Re(\lambda) > \delta_\Gamma - n$. It is easy to see that $Z_\Gamma(\sigma_\pm,\lambda)$ absolutely converges for $\Re(\lambda) > \delta_\Gamma - n$ by (1.2).

Proposition 5.6. *For* $\Re(\lambda) > \delta_\Gamma - n$,

$$Z_\Gamma(\sigma_\pm,\lambda) = \prod_{[\gamma]\in\mathrm{P}\Gamma_{\mathrm{hyp}}}\prod_{k=0}^{\infty}\det\Big(\mathrm{Id}-\sigma_\pm(m_\gamma)\otimes S^k(P(C_\gamma)_{|\bar{\mathfrak{n}}})e^{-(\lambda+n)l(C_\gamma)}\Big) \tag{5.22}$$

where $\mathrm{P}\Gamma_{\mathrm{hyp}}$ *is the set of* Γ*-conjugacy classes of primitive hyperbolic elements, and for an endomorphism* $L : V\to V$, $S^k(L)$ *denotes the action of* L *on the symmetric tensor product* $V_{\mathrm{sym}}^{\otimes k}$.

Proof. It is easy to see that log of the right hand side (5.22) is the same as

$$\begin{aligned}&\sum_{[\gamma]\in\mathrm{P}\Gamma_{\mathrm{hyp}}}\sum_{k=0}^{\infty}\mathrm{tr}\,\log\Big(\mathrm{Id}-\sigma_\pm(m_\gamma)\otimes S^k(P(C_\gamma)_{|\bar{\mathfrak{n}}})e^{-(\lambda+n)l(C_\gamma)}\Big) \\ &= -\sum_{[\gamma]\in\mathrm{P}\Gamma_{\mathrm{hyp}}}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty} j^{-1}\mathrm{tr}\Big(\sigma_\pm(m_\gamma)\otimes S^k(P(C_\gamma)_{|\bar{\mathfrak{n}}})e^{-(\lambda+n)l(C_\gamma)}\Big)^j \\ &= -\sum_{[\gamma]\in\Gamma_{\mathrm{hyp}}}\sum_{k=0}^{\infty} j(\gamma)^{-1}\mathrm{tr}(\sigma_\pm(m_\gamma))\,\mathrm{tr}(S^k(P(C_\gamma)_{|\bar{\mathfrak{n}}}))e^{-(\lambda+n)l(C_\gamma)} \\ &= -\sum_{[\gamma]\in\Gamma_{\mathrm{hyp}}} j(\gamma)^{-1}\det(\mathrm{Id}-P(C_\gamma)_{|\bar{\mathfrak{n}}})^{-1}\mathrm{tr}(\sigma_\pm(m_\gamma))\,e^{-(\lambda+n)l(C_\gamma)}.\end{aligned}$$

Now these equalities complete the proof if we use (5.11). □

The Selberg zeta function of odd type is defined by

$$Z^o_{\Gamma,\Sigma}(\lambda) = \frac{Z_\Gamma(\sigma_+,\lambda)}{Z_\Gamma(\sigma_-,\lambda)} \qquad \text{for} \quad \Re(\lambda) > \delta_\Gamma - n. \tag{5.23}$$

Note that the definition in (5.23) is shifted by $-n$ from the one in [27], [29]. From this definition, the following equality follows easily

$$\partial_\lambda \log Z^o_{\Gamma,\Sigma}(\lambda) = \sum_{[\gamma]\in\Gamma_{\rm hyp}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} (\chi_{\sigma_+}(m_\gamma) - \chi_{\sigma_-}(m_\gamma)) e^{-\lambda l(C_\gamma)}. \tag{5.24}$$

By Proposition 5.4, and from the identity

$$\int_0^\infty e^{-t\lambda^2}(4\pi t)^{-\frac{3}{2}} e^{-\frac{r^2}{4t}}\, dt = \frac{e^{-\lambda r}}{4\pi r} \qquad \text{for} \quad \Re(\lambda^2) > 0,$$

we have

Corollary 5.7. *For λ such that $\Re(\lambda) > \delta_\Gamma - n$ and $\Re(\lambda^2) > 0$,*

$$\int_0^\infty e^{-t\lambda^2}\, \mathrm{Tr}(De^{-tD^2})\, dt = \frac{i}{2} \partial_\lambda \log Z^o_{\Gamma,\Sigma}(\lambda). \tag{5.25}$$

Let us define the function ω_λ on the fundamental domain X_Γ by

$$\omega_\lambda(m) := \mathrm{tr}[DR(\lambda; m, m') - DR_{\mathbb{H}^{d+1}}(\lambda; m, m')]_{m=m'} \tag{5.26}$$

where $R_{\mathbb{H}^{d+1}}(\lambda)$ is the meromorphic extension of the resolvent of $D^2_{\mathbb{H}^{d+1}}$. The kernel of $D(R(\lambda) - R_{\mathbb{H}^{d+1}}(\lambda))$ is smooth in $X_\Gamma \times X_\Gamma$ and, since the function ω_λ is automorphic on $\mathbb{H}^{d+1}$ with respect to Γ, it induces a smooth function on $X \times X$. From the analysis of the resolvent above, we see that ω_λ is in $x^{2\lambda+d} C^\infty(\bar{X})$ and thus integrable when $\Re(\lambda) > 0$; moreover it is meromorphic in $\mathbb{C}$. By reversing the order of integration and trace in (5.25), we can write for $\Re(\lambda) > \max(\delta_\Gamma - n, 0)$

$$\partial_\lambda \log\, Z^o_{\Gamma,\Sigma}(\lambda) = \frac{\partial_\lambda Z^o_{\Gamma,\Sigma}(\lambda)}{Z^o_{\Gamma,\Sigma}(\lambda)} = -2i \int_{X_\Gamma} \omega_\lambda(m)\, \mathrm{dv}(m) \tag{5.27}$$

and the integral of $\omega_\lambda(m)$ can be decomposed under the form $\int_{x>\epsilon_0}$ and $\int_{x<\epsilon_0}$ for some boundary defining function x, so that it can be decomposed under the sum of a meromorphic function of λ and of

$$\lim_{\epsilon\to 0} \int_\epsilon^{\epsilon_0} x^{2\lambda-1} \omega'_\lambda(x,y)\, dx\, \mathrm{dv}_{h_0}(y)$$

for some ω'_λ smooth and meromorphic in λ. As $\epsilon \to 0$, this has an expansion of the form $A(\lambda) + \sum_{j=0}^\infty \epsilon^{2\lambda+j} C_j(\lambda)$ for some meromorphic $C_j(\lambda)$, $A(\lambda)$, and for $\Re(\lambda) > \max(\delta_\Gamma - n, 0)$ we have (5.27) which is equal to $A(\lambda) + \int_{x(m)>\epsilon_0} \omega_\lambda(m)\, \mathrm{dv}(m)$. This shows

Lemma 5.8. *The function $\partial_\lambda Z^o_{\Gamma,\Sigma}(\lambda)/Z^o_{\Gamma,\Sigma}(\lambda)$ has a meromorphic extension to $\mathbb{C}$ given by the value*

$$\frac{\partial_\lambda Z^o_{\Gamma,\Sigma}(\lambda)}{Z^o_{\Gamma,\Sigma}(\lambda)} = -2i\, \mathrm{FP}_{\epsilon\to 0} \int_{x(m)>\epsilon} \omega_\lambda(m)\, \mathrm{dv}(m)$$

where $\mathrm{FP}_{\epsilon\to 0}$ means the finite part as $\epsilon \to 0$, that is the constant coefficient in the expansion in powers of ϵ, and ω_λ is given in (5.26).

In addition, using that $\mathrm{tr}[DR_{\mathbb{H}^{d+1}}(\lambda; m, m') - DR_{\mathbb{H}^{d+1}}(-\lambda; m, m')]_{m=m'} = 0$ for all $\lambda \in \mathbb{C}$, we obtain

$$\frac{\partial_\lambda Z^o_{\Gamma,\Sigma}(\lambda)}{Z^o_{\Gamma,\Sigma}(\lambda)} - \frac{\partial_\lambda Z^o_{\Gamma,\Sigma}(-\lambda)}{Z^o_{\Gamma,\Sigma}(-\lambda)} = -2i\, \mathrm{FP}_{\epsilon\to 0} \int_{x(m)>\epsilon} \mathrm{tr}[D\Pi(\lambda; m, m')]_{m=m'}\, \mathrm{dv}(m) \tag{5.28}$$

where $\Pi(\lambda; m, m') := R(\lambda; m, m') - R(-\lambda; m, m')$.

6. Spectral side of trace formula, Maass-Selberg relation

With the only exception of Theorem 6.9, the dimension $d+1$ in this Section can be either odd or even, and X can be any asymptotically hyperbolic manifold with constant curvature near infinity. By convention, if $J(\lambda)$ is an operator depending on λ, we shall use the following notation throughout the section

$$\widetilde{J}(\lambda) := J(\lambda)/C(\lambda), \quad \text{with} \quad C(\lambda) = 2^{-2\lambda}\frac{\Gamma(-\lambda+1/2)}{\Gamma(\lambda+1/2)}$$

where the function $C(\lambda)$, already introduced in (4.9), satisfies $C(\lambda)C(-\lambda)=1$.

6.1. **The Maass-Selberg relation.** We now describe the Maass-Selberg relation in order to study the singularities of the odd Selberg zeta function in terms of scattering data.

A corollary of the Lemma 4.7 is that the kernel of $\Pi(\lambda) := R(\lambda) - R(-\lambda)$ is smooth on $X \times X$. Actually, in the Mazzeo-Melrose construction described before, one can choose the same term $Q_0(\lambda)$ for the parametrix of $R(\lambda)$ and $R(-\lambda)$, proving directly that $\Pi(\lambda)$ is the sum of a term whose lift under β is smooth on $\bar{X} \times_0 \bar{X} \setminus (\mathrm{lb} \cup \mathrm{rb})$ with a term in $(xx')^{\lambda+\frac{d}{2}} C^\infty(\bar{X} \times \bar{X}; \mathcal{E}) + (xx')^{-\lambda+\frac{d}{2}} C^\infty(\bar{X} \times \bar{X}; \mathcal{E})$. The local trace of $\Pi(\lambda)$, i.e., the trace of the endomorphism $\Pi(\lambda; m, m)$, satisfies

$$\mathrm{tr}(\Pi(\lambda; m, m)) \in C^\infty(\bar{X}) + x^{2\lambda+d} C^\infty(\bar{X}) + x^{-2\lambda+d} C^\infty(\bar{X}).$$

From the composition properties of $\Psi_0^{*,*,*}(\bar{X}; {}^0\Sigma)$ in Mazzeo [24], the operator[1] $D\Pi(\lambda)$ has a kernel which has the exact same properties as $\Pi(\lambda)$ and thus its local trace satisfies

$$\mathrm{tr}((D\Pi)(\lambda; m, m)) \in C^\infty(\bar{X}) + x^{2\lambda+d} C^\infty(\bar{X}) + x^{-2\lambda+d} C^\infty(\bar{X}). \tag{6.1}$$

Lemma 6.1. *Let $\epsilon > 0$ and $\lambda \in \mathbb{C}$ neither a pole of $R(\lambda)$ nor of $R(-\lambda)$, then*

$$\begin{aligned} &\int_{x(m)>\epsilon} \mathrm{tr}(D\Pi)(\lambda; m, m) \mathrm{dv}_g(m) \\ &= -\frac{\epsilon^{-d}}{2} \int_{x(m)=\epsilon} \int_{\partial\bar{X}} \mathrm{tr}\Big(\mathrm{cl}(\nu)\partial_\lambda \widetilde{E}(\lambda; x, y, y') \widetilde{E}^\sharp(-\lambda; y', x, y)\Big) \mathrm{dv}_{h_0}(y') \mathrm{dv}_{h_\epsilon}(y). \end{aligned}$$

Proof. First observe that

$$D\widetilde{E}(\lambda) = -\lambda \widetilde{E}(\lambda)\mathrm{cl}(\nu), \quad \widetilde{E}^\sharp(-\lambda) D = -\lambda \mathrm{cl}(\nu) \widetilde{E}^\sharp(-\lambda)$$

which is a consequence of Lemma 4.4 and the remark that follows. Then we get for small $t \in \mathbb{C}$

$$\frac{1}{2t}\Big(D\widetilde{E}(\lambda+t)\widetilde{E}^\sharp(-\lambda) - \widetilde{E}(\lambda+t)(\widetilde{E}^\sharp(-\lambda)D)\Big) = -\frac{1}{2}\widetilde{E}(\lambda+t)\mathrm{cl}(\nu)\widetilde{E}^\sharp(-\lambda).$$

From Lemma 4.7, the limit as $t \to 0$ on the right hand side is $-\frac{1}{2}\widetilde{E}(\lambda)\mathrm{cl}(\nu)\widetilde{E}^\sharp(-\lambda) = D\Pi(\lambda)$, which by taking the local trace and using the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ gives

$$\mathrm{tr}(D\Pi(\lambda; m, m)) = \lim_{t\to 0} \frac{1}{2t} \mathrm{tr}\Big(\widetilde{E}^\sharp(-\lambda) D\widetilde{E}(\lambda+t)(m,m) - (\widetilde{E}^\sharp(-\lambda)D)\widetilde{E}(\lambda+t)(m,m)\Big).$$

In particular, remark that the local trace on the right hand side has to vanish at $t=0$, which will be used in the last equality below. We use this expression and Green's formula on $\{x(m) > \epsilon\}$ in

[1] The notation $D\Pi$ should not be confused with the usual notation $d\Pi$ for the spectral measure!

the $\mathrm{dv}_g(m)$ integral to get

$$\begin{aligned}
&\int_{x(m)>\epsilon} \mathrm{tr}(D\Pi(\lambda;m,m))\mathrm{dv}_g(m)\\
&=\lim_{t\to 0}\frac{1}{2t}\int_{\partial\bar X}\int_{x(m)>\epsilon}\mathrm{tr}\Big[\widetilde{E}^\sharp(-\lambda;y',m)D\widetilde{E}(\lambda+t;m,y')\\
&\quad -(\widetilde{E}^\sharp(-\lambda)D)(y',m)\widetilde{E}(\lambda+t;m,y')\Big]\,\mathrm{dv}_g(m)\mathrm{dv}_{h_0}(y')\\
&=-\,\epsilon^{-d}\lim_{t\to 0}\frac{1}{2t}\int_{x(m)=\epsilon}\int_{\partial\bar X}\mathrm{tr}\left(\widetilde{E}^\sharp(-\lambda;y',m)\mathrm{cl}(\nu)\widetilde{E}(\lambda+t;m,y')\right)\,\mathrm{dv}_{h_0}(y')\,\mathrm{dv}_{h_\epsilon}(y)\\
&=-\,\frac{\epsilon^{-d}}{2}\int_{x(m)=\epsilon}\int_{\partial\bar X}\mathrm{tr}\left(\widetilde{E}^\sharp(-\lambda;y',m)\mathrm{cl}(\nu)\partial_\lambda\widetilde{E}(\lambda;m,y')\right)\,\mathrm{dv}_{h_0}(y')\,\mathrm{dv}_{h_\epsilon}(y).
\end{aligned}$$

The Lemma is proved using the cyclicity of the local trace once again. □

We now consider the limit of the expression in Lemma 6.1 as $\epsilon\to 0$. For that we introduce the representation of $\widetilde{E}(\lambda)$ given in Lemma 4.13 and a similar one for $\widetilde{E}^\sharp(\lambda)$: $\widetilde{E}^\sharp(\lambda)=\widetilde{E}_0^\sharp(\lambda)+\widetilde{E}_\infty^\sharp(\lambda)$ obtained by restricting (3.12) times $x(m)^{-\lambda-\frac{d}{2}}$ at $m\in\partial\bar X$ and using (3.11), that is

$$\begin{aligned}
&\widetilde{E}_0^\sharp(\lambda):=\sum_j \iota_j^*\phi_j^1\eta_j^{-\lambda-\frac{d}{2}}\widetilde{E}^\sharp_{\mathbb{H}^{d+1}}(\lambda)\chi_j^2{\iota_j}_*,\\
&\widetilde{E}^\sharp_\infty(\lambda):=2\lambda[x^{-\lambda-\frac{d}{2}}(\widetilde{R}(\lambda)-\widetilde{R}_0^\sharp(\lambda))]_{x=0}\in {x'}^{\lambda+\frac{d}{2}}C^\infty(\partial\bar X\times\bar X;\mathcal{E})
\end{aligned}\tag{6.2}$$

where again $\widetilde{E}^\sharp_{\mathbb{H}^{d+1}}(\lambda)$ is the corresponding operator on $\mathbb{H}^{d+1}$ like in Lemma 4.13 and $R_0^\sharp(\lambda):=R_0(\bar\lambda)^*$. Notice that, using the same arguments as in (4.7), we obtain $\widetilde{E}_0^\sharp(\lambda;y',m)=\widetilde{E}_0(\bar\lambda;m,y')^*$. Similarly we have $\widetilde{S}(\lambda)=\widetilde{S}_0^\sharp(\lambda)+\widetilde{S}_\infty^\sharp(\lambda)$ with

$$\begin{aligned}
&\widetilde{S}_0^\sharp(\lambda):=\sum_j \iota_j^*\phi_j^1\eta_j^{-\lambda-\frac{d}{2}}\widetilde{S}_{\mathbb{H}^{d+1}}(\lambda)\eta_j^{-\lambda-\frac{d}{2}}\phi_j^2{\iota_j}_*,\\
&\widetilde{S}^\sharp_\infty(\lambda):=2\lambda[x{x'}^{-\lambda-\frac{d}{2}}(\widetilde{R}(\lambda)-\widetilde{R}_0^\sharp(\lambda))]_{x=x'=0}\in C^\infty(\partial\bar X\times\partial\bar X;\mathcal{E})
\end{aligned}\tag{6.3}$$

and $\widetilde{S}_{\mathbb{H}^{d+1}}(\lambda)$ is the operator on $\mathbb{H}^{d+1}$ like in Lemma 4.13. Then we can prove

Proposition 6.2. *The meromorphic identity holds in $\lambda\in\mathbb{C}$,*

$$\begin{aligned}
&\int_{x(m)>\epsilon}\mathrm{tr}(D\Pi(\lambda;m,m))\mathrm{dv}_g(m)\\
&=-\,\frac{\epsilon^{-d}}{2}\int_{x(m)=\epsilon}\int_{\partial\bar X}\Big[\mathrm{tr}\left(\mathrm{cl}(\nu)\partial_\lambda\widetilde{E}(\lambda;x,y,y')\widetilde{E}^\sharp_\infty(-\lambda;y',x,y)\right)\\
&\quad+\mathrm{tr}\left(\mathrm{cl}(\nu)\partial_\lambda\widetilde{E}_\infty(\lambda;x,y,y')\widetilde{E}^\sharp_0(-\lambda;y',x,y)\right)\Big]\,\mathrm{dv}_{h_0}(y')\mathrm{dv}_{h_\epsilon}(y)
\end{aligned}\tag{6.4}$$

where tr *means the local trace on* $\mathrm{End}({}^0\Sigma)$.

Proof. The point is to prove the vanishing of

$$\mathrm{tr}\Big(\mathrm{cl}(\nu)\partial_\lambda\widetilde{E}_0(\lambda;x,y,y')\widetilde{E}^\sharp_0(-\lambda;y',x,y)\Big)$$

so we use the explicit formula for $\widetilde{E}_0(\lambda)$ and $\widetilde{E}_0^\sharp(\lambda)$ given in Lemma 4.13 and (6.2). We have to deal with terms of the form

$$\mathrm{tr}\Big(\iota_j^*\mathrm{cl}(X_j)\partial_\lambda(\chi_j^2\widetilde{E}_{\mathbb{H}^{d+1}}(\lambda)\phi_j^1\eta_j^{-\lambda-\frac{d}{2}})\gamma_{jk}^*\phi_k^1\eta_k^{\lambda-\frac{d}{2}}\widetilde{E}^\sharp_{\mathbb{H}^{d+1}}(-\lambda)\chi_k^2\iota_{k*}(m,m)\Big)\tag{6.5}$$

where γ_{jk} is the unique isometry of $\mathbb{H}^{d+1}$ extending $\iota_k\circ\iota_j^{-1}:\iota_j(U_k\cap U_j)\to\iota_k(U_k\cap U_j)$ (and which acts smoothly up to the boundary) and X_j is the vector field $X_j:={\iota_j}_*(\nu)$. We use the fact that

$\gamma_{jk}^* R_{\mathbb{H}^{d+1}}(\lambda) = R_{\mathbb{H}^{d+1}}(\lambda)\gamma_{jk}^*$ since γ_{jk} is an isometry so if $\alpha_{jk} := [\gamma_{jk}^*(x_0)/x_0]|_{x_0=0} \in C^\infty(\mathbb{R}^d)$, then one deduces that $\gamma_{jk}^* E^\sharp_{\mathbb{H}^{d+1}}(-\lambda) = \alpha_{jk}^{\lambda-\frac{d}{2}} E^\sharp_{\mathbb{H}^{d+1}}(-\lambda)\gamma_{jk}^*$. Let us consider (6.5), it can be written as

$$A_{jk}(\lambda; x_0, y_0, y_0') := \chi_j^2(x_0, y_0)\chi_k^2(\gamma_{jk}(x_0, y_0)) \\ \mathrm{tr}\left(\mathrm{cl}(X_j)(\partial_\lambda[\widetilde{E}_{\mathbb{H}^{d+1}}(\lambda; x_0, y_0, y_0')\eta_j(y_0')^{-\lambda-\frac{d}{2}}]\beta_{jk}(\lambda; y_0')\widetilde{E}^\sharp_{\mathbb{H}^{d+1}}(-\lambda; y_0', x_0, y_0)\right)$$

where $\iota_j(m) = (x_0, y_0)$ and $\beta_{jk}(\lambda; y') := \eta_j(y')^{\lambda-\frac{d}{2}}\phi_j^1(y')\phi_k^1(\gamma_{jk}(y'))$. We shall show that A_{jk} vanishes for algebraic reasons. First we recall from (2.5) that

$$\widetilde{E}_{\mathbb{H}^{d+1}}(\lambda; x_0, y_0, y_0') = f(\lambda)x_0^{\lambda+\frac{d}{2}}(x_0^2 + |y_0 - y_0'|^2)^{-\lambda-\frac{d}{2}}U((x_0, y_0), y_0')$$

and a similar expression for $\widetilde{E}^\sharp_{\mathbb{H}^{d+1}}$, here U is the parallel transport on $\mathbb{H}^{d+1}$ extended to the boundary (see Appendix A) and $f(\lambda)$ some explicit meromorphic function. Thus using the fact that $U(m_0, y_0')U(y_0', m_0) = \mathrm{Id}$, A_{jk} can be written under the form

$$A_{jk}(\lambda; m_0, y_0') = b_{jk}(\lambda; m_0, y_0')\mathrm{tr}(\mathrm{cl}(X_j)U(m_0, y_0')U(y_0', m_0)) = b_{jk}(\lambda; m_0, y_0')\mathrm{tr}(\mathrm{cl}(X_j))$$

for some b_{jk} where $m_0 = (x_0, y_0)$. But since the dimension $d > 1$, the trace vanishes. □

We deduce from this formula

Proposition 6.3. *For $\lambda \in \mathbb{C}$ not a pole of $S(\lambda)$ and $S(-\lambda)$, the right-hand side term in Proposition 6.2 has a limit as $\epsilon \to 0$, given by*

$$\lim_{\epsilon\to 0}\int_{x(m)>\epsilon} \mathrm{tr}(D\Pi(\lambda; m, m))\mathrm{dv}_g(m) = -\frac{1}{2}\,\mathrm{Tr}\left(\mathrm{cl}(\nu)[\partial_\lambda \widetilde{S}(\lambda)\widetilde{S}^\sharp_\infty(-\lambda) + \partial_\lambda \widetilde{S}_\infty(\lambda)\widetilde{S}^\sharp_0(-\lambda)]\right), \tag{6.6}$$

where Tr *denotes the trace for trace-class operators.*

Proof. First when $d+1$ is odd, we know from the discussion before Lemma 5.8 that the term (6.4) has an expansion as $\epsilon \to 0$ of the form $A(\lambda) + \sum_{j=0}^\infty \epsilon^{-2\lambda+j}C_j^-(\lambda) + \sum_{j=0}^\infty \epsilon^{2\lambda+j}C_j^+(\lambda)$ for some meromorphic functions $A(\lambda), C_j^\pm(\lambda)$. But actually the same result holds for the general AH manifolds where the metric has constant curvature near ∞ and $d+1$ odd or even: indeed, using the parametrix (3.10) and the fact that the local trace $\mathrm{tr}(D(R_{\mathbb{H}^{d+1}}(\lambda) - R_{\mathbb{H}^{d+1}}(-\lambda)))$ vanishes as explained in the Remark following Proposition 2.1, it is clear that $\mathrm{tr}(D\Pi(\lambda; m, m))$ is a function in the class $x^{2\lambda}C^\infty(\bar{X}) + x^{-2\lambda}C^\infty(\bar{X})$. Let us then take the limit as $\epsilon \to 0$ in (6.4). For instance consider

$$x^{-d}\mathrm{tr}\Big(\mathrm{cl}(\nu)\partial_\lambda \widetilde{E}(\lambda; x, y, y'))\widetilde{E}^\sharp_\infty(-\lambda; y', x, y)\Big),$$

we can use the arguments used in the proof of Theorem 3.10 of [14] (in the present case this is even simpler since they correspond only to the mixed terms there, which comes from the regularity $\widetilde{E}^\sharp_\infty(-\lambda) \in x^{-\lambda+\frac{d}{2}}C^\infty(\partial\bar{X} \times \bar{X}; \mathcal{E})$) and we obtain

$$\begin{aligned} &x^{-d}\mathrm{tr}\left(\mathrm{cl}(\nu)\partial_\lambda \widetilde{E}(\lambda; x, y, y')\widetilde{E}^\sharp_\infty(-\lambda; y', x, y)\right) \\ &= \log(x)\mathrm{tr}\left(\mathrm{cl}(\nu)\widetilde{S}(\lambda; y, y')\widetilde{S}^\sharp_\infty(-\lambda; y', y)\right) + \mathrm{tr}\left(\mathrm{cl}(\nu)\partial_\lambda \widetilde{S}(\lambda; y, y')\widetilde{S}^\sharp_\infty(-\lambda; y', y)\right) \\ &\quad + O(x\log(x)). \end{aligned} \tag{6.7}$$

Similarly we have

$$\begin{aligned} &x^{-d}\mathrm{tr}\Big(\mathrm{cl}(\nu)\partial_\lambda \widetilde{E}_\infty(\lambda; x, y, y')\widetilde{E}^\sharp_0(-\lambda; y', x, y)\Big) \\ &= \log(x)\mathrm{tr}\Big(\mathrm{cl}(\nu)\widetilde{S}_\infty(\lambda; y, y')\widetilde{S}^\sharp_0(-\lambda; y', y)\Big) + \mathrm{tr}\Big(\mathrm{cl}(\nu)\partial_\lambda \widetilde{S}_\infty(\lambda; y, y')\widetilde{S}^\sharp_0(-\lambda; y', y)\Big) \\ &\quad + O(x\log(x)). \end{aligned} \tag{6.8}$$

Thus the sum of (6.7) and (6.8) integrates in y, y' to a function of x of the form $\alpha(\lambda)\log(x) + \beta(\lambda) + O(x\log(x))$ for some meromorphic function $\alpha(\lambda), \beta(\lambda)$ which we can express in terms of the scattering operators. But from the discussion before, we also know that this trace has no $\log(x)$ coefficients and so $\alpha(\lambda) = 0$, which ends the proof by letting $x \to 0$ and writing $\beta(\lambda)$ in terms of $\widetilde{S}(\lambda), \widetilde{S}_\infty(\lambda), \widetilde{S}^\sharp_\infty(-\lambda)$ and $\widetilde{S}^\sharp_0(-\lambda)$ from (6.7), (6.8). □

Remark 6.4. By holomorphy on the continuous spectrum, the terms $O(x\log x)$ in (6.7) and (6.8) are continuous functions of $\lambda\in i\mathbb{R}$, and thus $G(\epsilon,\lambda):=\int_{x(m)>\epsilon}\mathrm{tr}(D\Pi(\lambda;m,m))\mathrm{dv}_g(m)$ is a continuous function on $[0,\epsilon_0]\times i\mathbb{R}$ for some small ϵ_0.

Let us define the super trace of a trace class operator A on $L^2(\partial\bar{X},\Sigma)$ by

$$\text{s-Tr}(A):=\frac{1}{i}\mathrm{tr}(\mathrm{cl}(\nu)A). \tag{6.9}$$

Corollary 6.5. *Let $\lambda\in\mathbb{C}$ be such that $S(z)$ and $S(-z)$ are analytic at $z=\lambda$, then the super trace* $\text{s-Tr}(\partial_\lambda\widetilde{S}(\lambda)\widetilde{S}(-\mu))$ *extends meromorphically in μ from $\Re(\lambda-\mu)<-d$ to $\mu\in\mathbb{C}$, it is analytic in $\mu=\lambda$, and the following identity holds*

$$\lim_{\epsilon\to 0}\int_{x(m)>\epsilon}\mathrm{tr}(D\Pi(\lambda;m,m))\mathrm{dv}_g(m)=-\frac{i}{2}\,\text{s-Tr}(\partial_\lambda\widetilde{S}(\lambda)\widetilde{S}(-\mu))|_{\mu=\lambda}.$$

Proof. Since $\partial_\lambda\widetilde{S}(\lambda)\widetilde{S}^\sharp_\infty(-\mu)$ and $\partial_\lambda\widetilde{S}_\infty(\lambda)\widetilde{S}^\sharp_0(-\mu)$ have smooth kernels, it is clear that their supertrace extends meromorphically to $\mathbb{C}$ and is analytic at $\mu=\lambda$ by assumption on λ. Now if we show

$$\text{s-Tr}(\partial_\lambda\widetilde{S}_0(\lambda)\widetilde{S}^\sharp_0(-\mu))=0 \tag{6.10}$$

then we have proved the corollary in view of Proposition 6.3. We have to study terms of the form

$$\mathrm{tr}\Big(\iota_j^*\mathrm{cl}(\nu)\partial_\lambda[\eta_j^{-\lambda-\frac{d}{2}}\phi_j^2\widetilde{S}_{\mathbb{H}^{d+1}}(\lambda)\phi_j^1\eta_j^{-\lambda-\frac{d}{2}}]\gamma_{jk}^*\eta_k^{\mu-\frac{d}{2}}\phi_k^1\widetilde{S}_{\mathbb{H}^{d+1}}(-\mu)\phi_k^2\eta_k^{\mu-\frac{d}{2}}\iota_{k*}\Big) \tag{6.11}$$

where γ_{jk} is the unique isometry of $\mathbb{H}^{d+1}$ extending $\iota_k\circ\iota_j^{-1}:\iota_j(U_k\cap U_j)\to\iota_k(U_k\cap U_j)$, which acts also as a conformal transformation on $\partial\bar{B}\subset\mathbb{R}^d$. As above we use the fact that $\gamma_{jk}^*R_{\mathbb{H}^{d+1}}(-\mu)=R_{\mathbb{H}^{d+1}}(-\mu)\gamma_{jk}^*$ since γ_{jk} is an isometry so if $\alpha_{jk}:=[\gamma_{jk}^*(x_0)/x_0]|_{x=0}\in C^\infty(\mathbb{R}^d)$, then one deduces that $\gamma^*_{jk}S_{\mathbb{H}^{d+1}}(-\mu)=\alpha_{jk}^{\mu-\frac{d}{2}}S_{\mathbb{H}^{d+1}}(-\mu)\alpha_{jk}^{\mu-\frac{d}{2}}\gamma_{jk}^*$. So the term (6.11) is equal to

$$\begin{aligned}\int_{\partial\bar{B}\times\partial\bar{B}}\mathrm{tr}\,\Big(\mathrm{cl}(\nu)\partial_\lambda\Big[\eta_j(y)^{-\lambda-\frac{d}{2}}\phi_j^2(y)\widetilde{S}_{\mathbb{H}^{d+1}}(\lambda;y,y')\phi_j^1(y')\eta_j(y')^{-\lambda-\frac{d}{2}}\Big]\\ \eta_j(y')^{\mu-\frac{d}{2}}\phi_k^1(\gamma_{jk}(y'))\widetilde{S}_{\mathbb{H}^{d+1}}(-\mu;y',y)\phi_k^2(\gamma_{jk}(y))\eta_j(y)^{\mu-\frac{d}{2}}\Big)\,(\eta_j(y)\eta_j(y'))^d dydy'.\end{aligned} \tag{6.12}$$

Using the explicit formula of $S_{\mathbb{H}^{d+1}}(\lambda;y,y')$, we see that the local trace of the operator above, i.e., the integrand in (6.12), can be written under the form

$$f_{jk}(\lambda,\mu;y,y')\mathrm{tr}(\mathrm{cl}(\nu)U(y,y')U(y',y))$$

for some function f_{jk} and where $U(y,y')$ is the parallel transport map on spinors on $\mathbb{H}^{d+1}$ studied in Appendix A and extended down to the boundary $\mathbb{R}^d$. Thus since $U(y,y')U(y',y)=\mathrm{Id}$ and $\mathrm{tr}(\mathrm{cl}(\nu))=0$, we obtain that (6.12) vanishes, which finishes the proof. □

6.2. **Analysis of residues of** $\text{s-Tr}(\partial_\lambda\widetilde{S}(\lambda)\widetilde{S}(-\lambda))$**.** Let us define $F(\lambda)$ for the value at $\mu=\lambda$ of the meromorphic extension in μ of $\text{s-Tr}(\partial_\lambda\widetilde{S}(\lambda)\widetilde{S}(-\mu))$

$$F(\lambda):=\text{s-Tr}(\partial_\lambda\widetilde{S}(\lambda)\widetilde{S}(-\mu))|_{\mu=\lambda}. \tag{6.13}$$

It is clear from Corollary 6.5 that $F(\lambda)$ is meromorphic in $\lambda\in\mathbb{C}$, but we want to prove that it has only first order poles, the residues of which are integers. Since $\widetilde{S}(\lambda)$ is unitary on $\{\Re(\lambda)=0\}$ it is analytic at $\lambda=0$, so one can define

$$\mathcal{S}_\pm(\lambda):=\widetilde{S}_\pm(\lambda)\widetilde{S}_\mp(0):C^\infty(\partial\bar{X};{}^0\Sigma_\mp)\to C^\infty(\partial\bar{X};{}^0\Sigma_\mp) \tag{6.14}$$

which are the two diagonal components of $\mathcal{S}(\lambda):=\widetilde{S}(\lambda)\widetilde{S}(0)$ in the splitting ${}^0\Sigma_+\oplus{}^0\Sigma_-$. These two operators are elliptic pseudo-differential operators of complex order 2λ by Proposition 4.9

$$\mathcal{S}_\pm(\lambda)\in\Psi^{2\lambda}(\partial\bar{X};{}^0\Sigma_\mp\boxtimes{}^0\Sigma_\mp^*),$$

and their principal symbol is $|\xi|^{2\lambda}$. Let D be the Dirac operator on $(\partial\bar{X},h_0)$ and let $D_\mp=P_\pm DI_\mp:C^\infty(\partial\bar{X};{}^0\Sigma_\mp)\to C^\infty(\partial\bar{X};{}^0\Sigma_\pm)$ be the off diagonal components of D. If $|D|_\mp:=(D_\pm D_\mp)^{\frac{1}{2}}$, it is possible to factorize $\mathcal{S}_\pm(\lambda)$ by $(\mathrm{Id}+|D|_\mp)^{-\lambda}\mathcal{S}_\pm(\lambda)(\mathrm{Id}+|D|_\mp)^{-\lambda}$ and this operator is of the form $\mathrm{Id}+K(\lambda)$ for some meromorphic family of compact operators on $L^2(\partial\bar{X};{}^0\Sigma_\mp)$, it is thus

Fredholm on this space. Then we can use the theory of Gohberg-Sigal [10] like in [18] or Section 2 of [13] for these operators. In particular, one can define the null multiplicities $N_{\lambda_0}(\mathcal{S}_\pm(\lambda))$ of $\mathcal{S}_\pm(\lambda)$ at a point λ_0 as follows: by the theory of [10], for a meromorphic family of operators $L(\lambda) = \mathrm{Id} + K(\lambda)$ acting on a Hilbert space $\mathcal{H}$ with $K(\lambda)$ compact, with $L(\lambda)$ invertible for some λ, there exist holomorphically invertible operators $U_1(\lambda), U_2(\lambda)$ near λ_0, some $(k_l)_{l=0,\dots,m} \in \mathbb{Z}\backslash\{0\}$ (with $m \in \mathbb{N}$) and some orthogonal projectors P_l on $L^2(\partial\bar{X}; {}^0\Sigma_\pm)$ such that $\mathrm{rank}(P_l) = 1$ if $l > 0$, $P_iP_j = \delta_{ij}$ and

$$L(\lambda) = U_1(\lambda)\Big(P_0 + \sum_{l=1}^{m}(\lambda-\lambda_0)^{k_l}P_l\Big)U_2(\lambda), \tag{6.15}$$

then we define the null multiplicity at λ_0 by

$$N_{\lambda_0}(L(\lambda)) := \sum_{k_l>0} k_l. \tag{6.16}$$

Note that, by [10], this is an integer depending only on $\mathcal{S}_\pm(\lambda)$ and not on the factorization (6.15) and that $N_{\lambda_0}(L(\lambda)) = 0$ if and only if $L(\lambda)^{-1}$ is holomorphic at $\lambda = \lambda_0$.

Proposition 6.6. *The function $F(\lambda)$ of (6.13) is meromorphic in $\lambda \in \mathbb{C}$, one has*

$$F(\lambda) = \mathrm{FP}_{\mu=\lambda}\mathrm{TR}(\partial_\lambda\mathcal{S}_-(\lambda)\mathcal{S}_-^{-1}(\mu)) - \mathrm{FP}_{\mu=\lambda}\mathrm{TR}(\partial_\lambda\mathcal{S}_+(\lambda)\mathcal{S}_+^{-1}(\mu)) \tag{6.17}$$

where TR is the Kontsevich-Vishik trace of [20] and $\mathrm{FP}_{\mu=\lambda}$ means the finite part (or regular value) of the meromorphic function of μ at $\mu = \lambda$. The poles of $F(\lambda)$ are first order poles, with residue at a pole λ_0 given by

$$\begin{aligned}\mathrm{Res}_{\lambda=\lambda_0}F(\lambda) =& \ \mathrm{Tr}(\mathrm{Res}_{\lambda=\lambda_0}(\partial_\lambda\mathcal{S}_-(\lambda)\mathcal{S}_-(\lambda)^{-1})) - \mathrm{Tr}(\mathrm{Res}_{\lambda=\lambda_0}(\partial_\lambda\mathcal{S}_+(\lambda)\mathcal{S}_+(\lambda)^{-1}))\\ =& \ (N_{\lambda_0}(\mathcal{S}_-(\lambda)) - N_{\lambda_0}(\mathcal{S}_-(\lambda)^{-1})) - (N_{\lambda_0}(\mathcal{S}_+(\lambda)) - N_{\lambda_0}(\mathcal{S}_+(\lambda)^{-1}))\end{aligned}$$

where N_{λ_0} is the null multiplicity defined in (6.16).

Proof. The first statement is straightforward since

$$\text{s-Tr}(\partial_\lambda\widetilde{S}(\lambda)\widetilde{S}(-\mu)) = \mathrm{TR}(\partial_\lambda\mathcal{S}_-(\lambda)\mathcal{S}_-^{-1}(\mu)) - \mathrm{TR}(\partial_\lambda\mathcal{S}_+(\lambda)\mathcal{S}_+^{-1}(\mu))$$

and we know from the work of Lesch [21] that the Kontsevich-Vishik trace of an analytic family of log-polyhomogeneous operators $A(\mu)$ extend meromorphically to $\mu \in \mathbb{C}$, so it suffices to use the fact that s-Tr$(\partial_\lambda\widetilde{S}(\lambda)\widetilde{S}(-\mu))$ analytically continues to $\mu \in \mathbb{C}$ and is analytic at $\mu = \lambda$ to prove (6.17).

As shown in Proposition 6.3, $F(\lambda)$ can be written as a trace of a meromorphic family of trace class operators, more precisely, using the fact that $\mathrm{cl}(\nu)$ anti-commutes with $\widetilde{S}(\lambda)$ and $\widetilde{S}_0(\lambda)$ for all λ where they are defined,

$$F(\lambda) = \frac{1}{i}\mathrm{tr}\Big(\mathrm{cl}(\nu)(\partial_\lambda\widetilde{S}(\lambda)\widetilde{S}(\lambda)^{-1} - \partial_\lambda\widetilde{S}_0(\lambda)\widetilde{S}_0^\sharp(-\lambda))\Big).$$

Consequently, the polar part of $F(\lambda)$ at a pole λ_0 is given by the trace of the polar part (which is finite rank) of $\mathrm{cl}(\nu)(\partial_\lambda\widetilde{S}(\lambda)\widetilde{S}(\lambda)^{-1} - \partial_\lambda\widetilde{S}_0(\lambda)\widetilde{S}_0^\sharp(-\lambda))$. But clearly from the explicit formula of $S_{\mathbb{H}^{d+1}}(\lambda)$, we see that $\partial_\lambda\widetilde{S}_0(\lambda)\widetilde{S}_0^\sharp(-\lambda)$ is holomorphic in $\lambda \in \mathbb{C}$. Now use that $\widetilde{S}(0)^2 = \mathrm{Id}$ to write

$$\begin{aligned}\mathrm{cl}(\nu)\partial_\lambda\widetilde{S}(\lambda)\widetilde{S}(\lambda)^{-1} =& + iI_+\partial_\lambda(\widetilde{S}_-(\lambda)\widetilde{S}_+(0))(\widetilde{S}_-(\lambda)\widetilde{S}_+(0))^{-1}P_+\\ &- iI_-\partial_\lambda(\widetilde{S}_+(\lambda)\widetilde{S}_-(0))(\widetilde{S}_+(\lambda)\widetilde{S}_-(0))^{-1}P_-\end{aligned} \tag{6.18}$$

and write a factorization of the form (6.15) for $\mathcal{S}_\pm(\lambda)$, from which it is clear that $\partial_\lambda\mathcal{S}_\pm(\lambda)\mathcal{S}_\pm(\lambda)^{-1}$ has only first order poles except possibly

$$\begin{aligned}&U_1(\lambda)(P_0 + \sum_{l=1}^{m}(\lambda-\lambda_0)^{k_l}P_l)\partial_\lambda U_2(\lambda)U_2(\lambda)^{-1}(\sum_{l=1}^{m}(\lambda-\lambda_0)^{-k_l}P_l)U_1(\lambda)^{-1}\\ &+U_1(\lambda)(\sum_{l=1}^{m}(\lambda-\lambda_0)^{k_l}P_l)\partial_\lambda U_2(\lambda)U_2(\lambda)^{-1}(P_0 + \sum_{l=1}^{m}(\lambda-\lambda_0)^{-k_l}P_l)U_1(\lambda)^{-1}.\end{aligned}$$

But this is a finite rank operator and so by the cyclicity of the trace, we deduce that the trace of this term is holomorphic in λ. To finish the proof, it suffices to apply the main result of [10]:

$$\mathrm{tr}(\mathrm{Res}_{\lambda=\lambda_0}(\partial_\lambda \mathcal{S}_\pm(\lambda)\mathcal{S}_\pm(\lambda)^{-1})) = N_{\lambda_0}(\mathcal{S}_\pm(\lambda)) - N_{\lambda_0}(\mathcal{S}_\pm(\lambda)^{-1}).$$

□

Let us define the multiplicity of resonances as follows

$$m_\pm(\lambda_0) := \mathrm{rank}(\mathrm{Res}_{\lambda=\lambda_0} R_\pm(\lambda)) \tag{6.19}$$

We want to identify scattering poles and resonances.

Proposition 6.7. *Let $\lambda_0 \in \mathbb{C}$, then the following identity holds*

$$N_{\lambda_0}(\mathcal{S}_\mp(-\lambda)) = m_\pm(\lambda_0) + 1\!\!1_{-1/2-\mathbb{N}_0}(\lambda_0) \dim\ker \mathcal{S}_\mp(-\lambda_0). \tag{6.20}$$

Proof. We just sketch the proof since it is very similar to that of Theorem 1.1. of [13], and we strongly encourage the reader to look at [13]. The first thing to notice is that $N_{\lambda_0}(\mathcal{S}_\pm(-\lambda)) = N_{-\lambda_0}(\mathcal{S}_\pm(\lambda))$ and that $N_{\lambda_0}(\mathcal{S}_\pm(\lambda)^{-1}) = N_{\lambda_0}(\mathcal{S}_\mp(-\lambda))$ since $\mathcal{S}_\pm(\lambda)^{-1} = \widetilde{S}_\pm(0)\mathcal{S}_\mp(-\lambda)\widetilde{S}_\pm(0)^{-1}$. Remark that $R_\pm(\lambda)$ and $\mathcal{S}_\pm(\lambda)$ are analytic in $\{\Re(\lambda) \geq 0\}$ and so the identity (6.20) is trivial (all terms are 0) for $\Re(\lambda_0) \geq 0$.

Now suppose that $\Re(\lambda_0) < 0$. First we prove that

$$N_{\lambda_0}(\mathcal{S}_\mp(-\lambda)) - 1\!\!1_{-1/2-\mathbb{N}_0}(\lambda_0) \dim\ker \mathcal{S}_\mp(-\lambda_0) \leq m_\pm(\lambda_0). \tag{6.21}$$

By (4.4) and (4.5), $S_\pm(\lambda)$ can be represented for $\Re(\lambda) < -\frac{d}{2}$ by

$$S_\pm(\lambda; y, y') = \pm i[(xx')^{-\lambda-\frac{d}{2}} R_\pm(\lambda; x, y, x', y')]|_{x=x'=0}$$

and the expression can be extended to $\lambda \in \mathbb{C}$ meromorphically as a singular integral kernel using the blow-down maps like in (4.10). Then we can apply mutatis mutandis Lemma 3.2 of [13], where $S(\lambda)$ there is replaced by $S_\pm(\lambda)$ here, the function $z(\lambda)$ there is λ here, and we have to multiply the factorization (3.11) of [13] by $\widetilde{S}_\mp(0)$ on the right, which is harmless since it does not depend on λ. We want to apply the factorization of $\mathcal{S}(\lambda)$ obtained from this Lemma 3.2 of [13] to prove (6.21), in a way similar to Corollary 3.3 of [13]. First Corollary 3.3 in [13] can also be rewritten (using the notations of [13]) under the form

$$N_{\lambda_0}(\widetilde{S}(n-\lambda)) - 1\!\!1_{-n/2-\mathbb{N}_0}(\lambda_0) \dim\ker \widetilde{S}(n-\lambda_0) \leq m_{\lambda_0}(z'(\lambda)R(\lambda))$$

by using equation (3.19) in [13] if $\lambda_0 \in n/2 - \mathbb{N}$ and the fact that $c(n-\lambda)$ is holomorphic at all $\lambda_0 \notin n/2 - \mathbb{N}$. Then the proof of this Corollary 3.3 in [13] can be copied word by word by replacing $\widetilde{S}(n-\lambda)$ and $c(n-\lambda)$ there by $\mathcal{S}_\mp(-\lambda)$ and $C(-\lambda)$ here, and $m_{\lambda_0}(z'(\lambda)R(\lambda))$ by $m_\pm(\lambda_0)$. This finally proves (6.21).

Then we need to prove the converse inequality of (6.21). From Lemma 4.7, Corollary 4.8 and the fact that $\widetilde{S}_\mp(0)\widetilde{S}_\pm(0) = \mathrm{Id}|_{{}^0\Sigma_\mp}$, we deduce

$$R_\pm(\lambda) - R_\mp(-\lambda) = -E_\mp(-\lambda)C(\lambda)\mathcal{S}_\pm(\lambda)\widetilde{S}_\pm(0)\mathrm{cl}(\nu)E^\sharp_\mp(-\lambda) \tag{6.22}$$

which is the equivalent in our setting to the identity (3.15) of [13]. Since Lemma 3.4 of [13] is only based on the identity (3.15) in [13], the structure of the resolvent kernel at the boundary and the unique continuation principle of Mazzeo [23], the same proof applies and is actually easier in our case since there is no pure point spectrum thus no resonance in the physical sheet $\{\Re(\lambda) \geq 0\}$. This implies

$$N_{\lambda_0}(\mathcal{S}_\mp(-\lambda)) - 1\!\!1_{-1/2-\mathbb{N}_0}(\lambda_0) \dim\ker \mathcal{S}_\mp(-\lambda_0) \geq m_\pm(\lambda_0).$$

The idea of the proof of Lemma 3.4 of [13] is to use (6.22) to write the residue of $R_\pm(\lambda)$ at λ_0 with $\Re(\lambda_0) < 0$ in terms of the singular part of the Laurent expansion of $\mathcal{S}_\pm(\lambda)$, itself obtained from a factorization of the form (6.15), then use the fact that $R_\mp(-\lambda)$, $E_\mp(-\lambda)$ and $E^\sharp_\mp(-\lambda)$ are holomorphic in $\{\Re(\lambda) < 0\}$ and finally count the rank of the residue in terms of the k_l of the factorization (6.15). □

Theorem 6.8. *The function $F(\lambda)$ is meromorphic with first order poles and integer residues given by*

$$\begin{aligned}\mathrm{Res}_{\lambda=\lambda_0} F(\lambda) &= m_+(\lambda_0) - m_-(\lambda_0) + 1\!\!1_{-1/2-\mathbb{N}_0}(\lambda_0)\mathrm{Ind}(\mathcal{S}_-(-\lambda_0)) \\ &= m_+(\lambda_0) - m_-(\lambda_0).\end{aligned}$$

for $\Re(\lambda_0) \leq 0$, where $m_\pm(\lambda_0)$ is defined in (6.19).

Proof. Apply Proposition 6.7 with Proposition 6.6. To see the index of $\mathcal{S}_-(-\lambda_0)$ appearing, we also use that $\mathcal{S}_+(-\lambda_0)^* = \mathcal{S}_-(-\lambda_0)$ for $\lambda_0 \in \mathbb{R}$, which comes from the self-adjointness of $\widetilde{S}(-\lambda_0)$. The fact that the index of $\mathcal{S}_-(-\lambda_0)$ vanishes comes from the invariance of the index by continuous deformation and the invertibility of $\widetilde{S}(\lambda)$ except on a discrete set of $\lambda \in \mathbb{C}$. □

We deduce directly our main theorem from this Theorem, Corollary 6.5 and the identity (5.28):

Theorem 6.9. *The odd Selberg zeta function $Z^o_{\Gamma,\Sigma}(\lambda)$ on a spin convex co-compact hyperbolic manifold X_Γ of dimension $2n+1$ has a meromorphic extension to $\mathbb{C}$, is analytic in a neighborhood of the right half plane $\{\Re(\lambda) \geq 0\}$, and λ_0 is a zero or pole if and only if the meromorphic extension $R_+(\lambda)$ or $R_-(\lambda)$ of $(D \pm i\lambda)^{-1}$ from $\{\Re(\lambda) > 0\}$ to $\mathbb{C}$ have a pole at λ_0, in which case the order of λ_0 as a zero or pole of $Z^o_{\Gamma,\Sigma}(\lambda)$ (with the positive sign convention for zeros) is given by*

$$\mathrm{rank}\,\mathrm{Res}_{\lambda_0} R_-(\lambda) - \mathrm{rank}\,\mathrm{Res}_{\lambda_0} R_+(\lambda).$$

7. Eta invariant of Dirac operator

In this section, we will assume that the dimension of X_Γ is odd, that is, $d+1 = 2n+1$. First, we prove

Proposition 7.1. *Using notation* (5.10)*, the following estimate holds*

$$\mathrm{Tr}(De^{-tD^2}) = O(t^{-1}) \qquad as \quad t \to \infty.$$

Proof. Since the claim easily follows from (5.5) when $\delta_\Gamma < n$ (and actually in that case one gets directly $O(t^{-3/2})$ instead of just $O(t^{-1})$), we assume that $\delta_\Gamma \geq n$ in the following proof.

Let us write the operator De^{-tD^2} as a contour integral

$$De^{-tD^2} = \frac{1}{2\pi i}\int_\Lambda e^{-t\lambda^2} D(D^2-\lambda^2)^{-1} 2\lambda d\lambda \tag{7.1}$$

where $\Lambda = \{re^{i(\pi+\pi/8)}; r > 0\} \cup \{re^{-i\pi/8}; r > 0\}$ oriented from $+\infty e^{-i\pi/8}$ towards $+\infty e^{i(\pi+\pi/8)}$. Let us check the identity (7.1): by Corollary 3.5, we have that $D(D^2-\lambda^2)^{-1} = DR(i\lambda)$ is holomorphic in $\Im(\lambda) \leq 0$ as an operator bounded from $x^\epsilon L^2$ to $x^{-\epsilon}L^2$ for all $\epsilon > 0$, moreover $||DR(i\lambda)||_{L^2\to L^2} = O(1/|\Im(\lambda)|)$ when $|\lambda| \to \infty$ thus the integral converges in the operator norm of $\mathcal{L}(x^\epsilon L^2, x^{-\epsilon}L^2)$. Moreover, applying the integral (7.1) to a $C_0^\infty(X)$ function f defines a function $u(t)$ and since $-\lambda^2(D^2-\lambda^2)^{-1} = \mathrm{Id} - D^2(D^2-\lambda^2)^{-1}$, we see that the integral converges in $C^\infty(X)$ uniformly in $t \in [0,\infty)$ and solves $\partial_t u = -D^2 u$ with $u(0) = Df$. It is easy to prove that the C^k norms of the integral kernels in (7.1) also converge by applying powers of D^2 on the right and the left and using Sobolev embeddings. The same is true for the integral kernel of the operator $D_{\mathbb{H}^{d+1}}e^{-tD^2_{\mathbb{H}^{d+1}}}$ in terms of the resolvent kernel $D_{\mathbb{H}^{d+1}}(D^2_{\mathbb{H}^{d+1}}-\lambda^2)^{-1}$ and by Proposition 5.3 we deduce that

$$\mathrm{tr}(De^{-tD^2})(m) = \frac{1}{2\pi i}\int_\Lambda e^{-t\lambda^2}\omega_{i\lambda}(m)2\lambda d\lambda$$

with $\omega_\lambda(m) = \mathrm{tr}(DR(\lambda) - DR_{\mathbb{H}^{d+1}}(\lambda))(m)$. Since $\omega_{i\lambda} \in x^{2i\lambda+d}C^\infty(\bar{X})$ is holomorphic in $\Im(\lambda) \leq 0$, it is in $L^1(X)$ if $\Im(\lambda) < 0$, and for $\Im(\lambda) \leq 0$ one has the asymptotic

$$K(\lambda,\epsilon) := 2\lambda\int_{x(m)>\epsilon}\omega_{i\lambda}(m)\mathrm{dv}(m) = \epsilon^{2i\lambda}F(\lambda) + \lambda G(\lambda,\epsilon)$$

where $F(\lambda)$ is a holomorphic function of λ and $G(\lambda,\epsilon)$ is continuous in (λ,ϵ) down to $\epsilon = 0$ and holomorphic in λ. In particular for $|\lambda| < 1$, we have $|K(\lambda,\epsilon)| \leq C$ for some constant $C > 0$

independent of (λ, ϵ) and $\lim_{\epsilon\to 0} K(\lambda,\epsilon) = \lambda G(\lambda, 0)$ as long as $\Im(\lambda) < 0$. We can thus use Lebesgue's dominated convergence theorem to deduce that

$$\lim_{\epsilon\to 0}\int_{x(m)>\epsilon}\Big(\int_{\Lambda\cap\{|\lambda|<1\}} e^{-t\lambda^2}\omega_{i\lambda}(m)2\lambda d\lambda\Big)\,\mathrm{dv}(m) = \int_{\Lambda\cap\{|\lambda|<1\}} e^{-t\lambda^2}\lambda G(\lambda,0)d\lambda.$$

Now if we can show that , for $\lambda\in\Lambda\cap\{|\lambda|>1\}$, the estimate $||\omega_{i\lambda}||_{L^1(X)} = O(e^{\alpha|\lambda|})$ holds for some $\alpha>0$ we can use Lebesgue's dominated convergence theorem again for the integral corresponding to $\lambda\in\Lambda\cap\{|\lambda|>1\}$, and then changing variable $u=\sqrt{t}\lambda$ in the whole Λ integral gives directly

$$\lim_{\epsilon\to 0}\int_{x(m)>\epsilon}\mathrm{tr}(De^{-tD^2})(m)\mathrm{dv}(m) = O(t^{-1}) \quad \text{ as } t\to\infty.$$

We thus use that for $\Re(\lambda)>d$, one has

$$\omega_\lambda(m) = \mathrm{tr}\Big([DR(\lambda;m,m') - DR_{\mathbb{H}^{d+1}}(\lambda;m,m')]_{m=m'}) = \sum_{\gamma\in\Gamma\backslash\mathrm{Id}} \mathrm{tr}(DR_{\mathbb{H}^{d+1}}(\lambda;m,\gamma m)\gamma^*_m)$$

where $\gamma^*_m : {}^0\Sigma_m \to {}^0\Sigma_{\gamma m}$ is the action induced by γ on the spinor bundle. The map γ^* is given by $\gamma^*[g,v]=[\gamma g, v]$ for $[g,v]\in{}^0\Sigma_m$, $[\gamma g,v]\in{}^0\Sigma_{\gamma m}$ under the identification in (2.4). Using (2.5), the Euler integral formula for the hypergeometric function

$$F\left(\frac{d+1}{2}+\lambda,\lambda+1,2\lambda+1;z\right) = \frac{\Gamma(2\lambda+1)}{\Gamma(\lambda+1)\Gamma(\lambda)}\int_0^1 \frac{t^\lambda(1-t)^{\lambda-1}}{(1-tz)^{\frac{d+1}{2}+\lambda}}dt$$

the expression of the Beta function $B(\lambda+1,\lambda)$ in terms of Gamma functions, and the obvious bound $1-tz > 1-z = \tanh^2(d_{\mathbb{H}^{d+1}}(m,\gamma m)/2)$ when $z=\cosh^{-2}(d_{\mathbb{H}^{d+1}}(m,\gamma m)/2)$, we obtain

$$|\omega_\lambda(m)| \leq C\left|\tfrac{\Gamma(\frac{d+1}{2}+\lambda)\Gamma(\lambda+1)}{\Gamma(2\lambda+1)}\right| \sum_{\gamma\in\Gamma\backslash\{\mathrm{Id}\}} \sinh(d_{\mathbb{H}^{d+1}}(m,\gamma m)/2)^{-d-2\Re(\lambda)}.$$

Using the Legendre duplication formula, the term containing Gamma functions is uniformly bounded for $\Re(\lambda)>0$. Since the injectivity radius of $X=\Gamma\backslash\mathbb{H}^{d+1}$ is strictly positive, i.e.,

$$\inf_{m\in\mathbb{H}^{d+1},\gamma\in\Gamma\backslash\{\mathrm{Id}\}}\{\sinh(d_{\mathbb{H}^{d+1}}(m,\gamma m)/2)\} > \epsilon$$

for some $\epsilon>0$, we deduce the estimate

$$|\omega_\lambda(m)| \leq e^{C\Re(\lambda)}\sum_{\gamma\in\Gamma\backslash\{\mathrm{Id}\}} e^{-(d/2+\Re(\lambda))d_{\mathbb{H}^{d+1}}(m,\gamma m)}. \tag{7.2}$$

Now for those (finitely many) $\gamma\in\Gamma\setminus\{\mathrm{Id}\}$ for which Lemma 5.2 possibly does not hold we still have the weaker inequality

$$e^{-d_{\mathbb{H}^{d+1}}(m,\gamma m)} \leq Cx(m)^2$$

for some $C>0$, where x is a boundary defining function. In particular $e^{-\lambda d_{\mathbb{H}^{d+1}}(m,\gamma m)}$ descends to an L^1 function on X if $\Re(\lambda)>d/2$. Combining finally with Lemma 5.2 and (7.2), the convergence of Poincaré series in $\{\Re(\lambda)>d\}$ implies the bound in the same half-space

$$||\omega_\lambda||_{L^1(X)} \leq e^{C\Re(\lambda)}$$

and this ends the proof. □

We define the eta invariant of D by

$$\eta(D) = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-\frac12}\,\mathrm{Tr}(De^{-tD^2})\,dt \tag{7.3}$$

where the trace Tr means the integral of the local trace like in (5.10). Note that the integral on the right hand side of (7.3) is finite by Corollary 5.5 and Proposition 7.1. Theorem 6.9 about meromorphic extension of $Z^o_{\Gamma,\Sigma}(\lambda)$ and its analyticity on $[0,\infty)$ implies directly the following result:

Theorem 7.2. *The eta invariant of the Dirac operator over a convex co-compact hyperbolic manifold X_Γ satisfies*

$$\exp(\pi i\eta(D)) = Z^o_{\Gamma,\Sigma}(0). \tag{7.4}$$

Proof. We start by writing

$$t^{-1/2} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\lambda^2 t} d\lambda,$$

then we have by (5.25)

$$\eta(D) = \frac{2}{\pi} \int_0^\infty \int_0^\infty e^{-\lambda^2 t} \mathrm{Tr}(De^{-tD^2})\, d\lambda\, dt = \frac{i}{\pi} \int_0^\infty \frac{\partial_\lambda Z^o_{\Gamma,\Sigma}(\lambda)}{Z^o_{\Gamma,\Sigma}(\lambda)} d\lambda$$

and this concludes the proof by Theorem 6.9, in particular, the meromorphic extension of $Z^o_{\Gamma,\Sigma}(\lambda)$ over $\mathbb{C}$ with $\lambda = 0$ as a regular value. □

8. Eta invariant of odd signature operator and its structure on Schottky space

For a $(4m-1)$-dimensional convex co-compact hyperbolic manifold

$$X_\Gamma = \Gamma \backslash \mathrm{SO}_0(4m-1,1)/\mathrm{SO}(4m-1),$$

we consider the odd signature operator A on odd forms $\Lambda^{\mathrm{odd}} = \oplus_{p=0}^{2m} \Lambda^{2p-1}$ acting by $(-1)^{m+p}(\star d + d\star)$ over Λ^{2p-1} as in Millson's paper [27]. Recall that $A^2 = \Delta$ and A is self adjoint. We want to make a sense of

$$\eta(A) := \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \int_{X_\Gamma} \mathrm{Tr}_\Lambda(Ae^{-t\Delta})(m)\, \mathrm{dv}(m)\, dt \tag{8.1}$$

where Tr_Λ is the local trace on the bundle Λ^{odd}. First it is easy to see that $\mathrm{Tr}_\Lambda(Ae^{-t\Delta})$ is the same as $\mathrm{Tr}_{\Lambda^{2m-1}}(\star d\, e^{-t\Delta})$ since the other parts are off diagonal if we write $Ae^{-t\Delta}$ as a matrix with respect to the natural basis of Λ^{odd}. First we show that the local trace $\mathrm{tr}_{\Lambda^{2m-1}}(\star d\, e^{-t\Delta})$ is integrable on X_Γ. As in the spinor bundle Σ, the bundle of $(2m-1)$-forms can be understood as a homogeneous vector bundle given by the representation $\Lambda^{2m-1}\phi$ with the standard representation ϕ of $\mathrm{SO}(4m-1)$, which decomposes into

$$\Lambda^{2m-1}\phi\,|_{\mathrm{SO}(4m-2)} = \Lambda^{2m-1}_+ \bar{\phi} \oplus \Lambda^{2m-1}_- \bar{\phi} \oplus \Lambda^{2m-2}\bar{\phi} \tag{8.2}$$

where $\bar{\phi}$ denotes the standard representation of $\mathrm{SO}(4m-2)$. As in the subsection 5.1, there is the $\Lambda^{2m-1}\phi$-radial function P_t associated to $\star d\, e^{-t\Delta}$, and we have the corresponding scalar functions $p^\pm_t(r)$, $p^{2m-2}_t(r)$ of P_t restricting to the representation spaces on the right hand side of (8.2). Now, as in Proposition 5.1, we have

Proposition 8.1. *The scalar components $p^\pm_t(r)$, $p^{2m-2}_t(r)$ are given by*

$$p^\pm_t(r) = \pm \frac{(4m-1)\sinh(r)}{i 2^{2m-1/2} \pi^{2m+1/2} t^{3/2}} \left(-\frac{d}{d(\cosh r)} \right)^{2m-1} r \sinh^{-1}(r)\, e^{-\frac{r^2}{4t}}, \qquad p^{2m-2}_t(r) \equiv 0.$$

Proof. The equalities follow from Lemma 7.4 and Theorem 7.6 in [31] and Theorem 1.1 in [27]. □

Using this proposition and repeating the same argument as in Section 5, one can easily show that $\mathrm{tr}_{\Lambda^{2m-1}}(\star d\, e^{-t\Delta})$ is integrable over X_Γ. By the same argument as in Proposition 5.4 and Corollary 5.5, one can also obtain the corresponding results, which implies that the small time part of the integral $\int_0^\infty \cdot\, dt$ in (8.1) converges. The convergence of the large time part also follows from the corresponding computations to (5.5) and Lemma 5.2 under the condition $\delta_\Gamma < 2m-1$. Hence the eta invariant $\eta(A)$ given in (8.1) is well defined if $\delta_\Gamma < 2m-1$. For $\Re(\lambda) > \delta_\Gamma - (2m-1)$, we also have the Selberg zeta function of odd type $Z^o_{\Gamma,\Lambda}(\lambda)$ just putting $\sigma_\pm = \Lambda^{2m-1}_\pm \bar{\phi}$ in (5.21) and (5.23), which coincides with the one in (1.1) introduced by Millson [27]. We first have a result similar to the case of spinor bundle dealt with above:

Theorem 8.2. *If $X_\Gamma := \Gamma \backslash \mathbb{H}^{4m-1}$ is a convex co-compact hyperbolic manifold with the Poincaré exponent $\delta_\Gamma < 2m-1$, then the local trace $\mathrm{Tr}_{\Lambda^{2m-1}}(\star d\, e^{-t\Delta})$ is integrable on X for all $t > 0$, so that the integral* (8.1) *converges and defines the eta invariant $\eta(A)$. Moreover, we also have*

$$e^{\pi i \eta(A)} = Z^o_{\Gamma,\Lambda}(0). \tag{8.3}$$

Proof. We already showed the first claim. The equality (8.3) also easily follows from the corresponding results to Proposition 5.4 and the assumption $\delta_\Gamma < 2m-1$, from which we do not need to show the meromorphic extension of $Z^o_{\Gamma,\Lambda}(\lambda)$ at $\lambda = 0$. □

It turns out that this eta invariant $\eta(A)$ of the odd signature operator A has an intimate relationship with the deformation space of the hyperbolic structures when X_Γ is 3-dimensional. To explain this, first we review the work of Zograf [36].

8.1. **Zograf factorization formula.** A *marked Schottky group* is a discrete subgroup Γ of the linear fractional transformations $\mathrm{PSL}(2,\mathbb{C})$, with distinguished free generators $\gamma_1, \gamma_2, \dots, \gamma_g$ satisfying the following condition: there exist $2g$ smooth Jordan curves C_r, $r = \pm 1, \dots, \pm g$, which form the oriented boundary of a domain Ω_0 in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ such that $\gamma_r C_r = -C_{-r}$ for $r = 1, \dots, g$. If Ω is the union of images of Ω_0 under Γ, then $Y_\Gamma = \Gamma\backslash\Omega$ is a compact Riemann surface of genus g. The action of Γ on $\mathbb{C}$ naturally extends to the action on $\mathbb{H}^3$ where $\partial\mathbb{H}^3 = \mathbb{C}$ and the quotient space $X_\Gamma = \Gamma\backslash\mathbb{H}^3$ is a *Schottky hyperbolic* 3*-manifold* whose boundary is the Riemann surface Y_Γ. Here let us remark that δ_Γ is the Hausdorff dimension of the limit set Λ in $\partial\mathbb{H}^3$ of Γ and δ_Γ is also the smallest number such that $\prod_{\{\gamma\}}(1-q_\gamma^s)$ absolutely converges whenever $\Re(s) > \delta_\Gamma$. The function $\prod_{\{\gamma\}}(1-q_\gamma^s)$ was briefly described in [2] where it was asserted without the proof that with the values of q_γ^s chosen appropriately, the infinite product is defined for $\Re(s) > \delta_\Gamma$ and has an analytic continuation to $\mathbb{C}$.

Each nontrivial element $\gamma \in \Gamma$ is loxodromic: there exists a unique number $q_\gamma \in \mathbb{C}$ (the multiplier) such that $0 < |q_\gamma| < 1$ and γ is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to $z \mapsto q_\gamma z$, that is,

$$\frac{\gamma z - a_\gamma}{\gamma z - b_\gamma} = q_\gamma \frac{z - a_\gamma}{z - b_\gamma}$$

for some $a_\gamma, b_\gamma \in \hat{\mathbb{C}}$ (the attracting and repelling fixed points respectively). A marked Schottky group with an ordered set of free generators $\gamma_1, \dots, \gamma_g$ is *normalized* if $a_{\gamma_1} = 0$, $b_{\gamma_1} = \infty$, $a_{\gamma_2} = 1$. The *Schottky space* $\mathfrak{S}_g$ is the space of marked normalized Schottky groups with g generators. It is a complex manifold of dimension $3g-3$, covering the Riemann moduli space $\mathfrak{M}_g$ and with universal cover the Teichmüller space $\mathfrak{T}_g$.

Like the Teichmüller space $\mathfrak{T}_g$, the Schottky space $\mathfrak{S}_g$ has a natural Kähler metric, the Weil-Petersson metric. In [33], Takhtajan-Zograf constructed a Kähler potential S called *classical Liouville action* of the Weil-Petersson metric on $\mathfrak{S}_g$, that is,

$$\partial\overline{\partial} S = 2i\,\omega_{WP} \tag{8.4}$$

where ∂ and $\overline{\partial}$ are the $(1,0)$ and $(0,1)$ components of the de Rham differential d on $\mathfrak{S}_g$ respectively, and ω_{WP} is the symplectic form of the Weil-Petersson metric. On the other hand, from the local index theorem for families of $\overline{\partial}$-operators in Takhtajan-Zograf [34], the following equality also follows

$$\partial\overline{\partial} \log \frac{\mathrm{Det}\Delta}{\det \mathrm{Im}\,\tau} = -\frac{i}{6\pi}\omega_{WP} \tag{8.5}$$

where $\mathrm{Det}\Delta$ and τ denote the ζ-regularized determinant of the Laplacian Δ of hyperbolic metric and the period matrix respectively over the Riemann surface corresponding to an inverse image in $\mathfrak{T}_g$ of a point in $\mathfrak{S}_g$. Let us remark that $\mathrm{Det}\Delta$ and $\det \mathrm{Im}\,\tau$ descend to well-defined functions on $\mathfrak{S}_g$. Comparing (8.4) with (8.5), one can expect a nontrivial relationship between S and $\log \frac{\mathrm{Det}\Delta}{\det \mathrm{Im}\,\tau}$. Indeed, in [36], [37] Zograf proved

Theorem 8.3. [Zograf] *There exists a holomorphic function $F(\Gamma) : \mathfrak{S}_g \to \mathbb{C}$ such that*

$$\frac{\mathrm{Det}\Delta}{\det \mathrm{Im}\,\tau} = c_g \exp\left(-\frac{1}{12\pi} S\right) |F(\Gamma)|^2 \tag{8.6}$$

where c_g is a constant depending only on g. For points in $\mathfrak{S}_g$ corresponding to Schottky groups Γ with $\delta_\Gamma < 1$, the function $F(\Gamma)$ is given by the following absolutely convergent product:

$$F(\Gamma) = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - q_\gamma^{1+m}) \tag{8.7}$$

where q_γ is the multiplier of $\gamma \in \Gamma$, and $\{\gamma\}$ runs over all distinct primitive conjugacy classes in Γ excluding the identity.

Combining the equalities (8.6) and (8.7), these are called *Zograf factorization formula.* This result was extended by McIntyre-Takhtajan to the Schottky groups without the condition for δ_Γ in [26]. Here they used the ζ-regularized determinant of Δ_n acting on the space of n-differentials so that the corresponding holomorphic function is $F_n(\Gamma) = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - q_\gamma^{n+m})$ which absolutely converges for any Schottky group Γ if $n > 1$.

8.2. **Eta invariant as a functional over the Schottky space.** By the construction of X_Γ and its boundary Y_Γ, the eta invariant $\eta(A)$ can be understood as a functional over the Schottky space $\mathfrak{S}_g$. Now a natural question is to describe the eta invariant $\eta(A)$ as a functional over $\mathfrak{S}_g$. For this, we have

Theorem 8.4. *Let $\mathfrak{S}_g^0$ be a subset of $\mathfrak{S}_g$ consisting of normalized Schottky groups Γ's with the property $\delta_\Gamma < 1$. Then we have*

$$F(\Gamma) = |F(\Gamma)| \exp\left(-\frac{\pi i}{2}\eta(A)\right) \qquad \text{over } \ \mathfrak{S}_g^0,$$

in particular, $\eta(A)$ is a pluriharmonic function over $\mathfrak{S}_g^0$.

Proof. The proof is a simple application of the equality (8.3). For this, as in Proposition 5.6, we rewrite $Z^o_{\Gamma,\Lambda}(\lambda)$ with respect to the group $\mathrm{PSL}(2,\mathbb{C})$ as follows:

$$Z^o_{\Gamma,\Lambda}(\lambda) = \prod_{[\gamma]\in \mathrm{P}\Gamma_{\mathrm{lox}}} \prod_{k,\ell=0}^{\infty} \frac{\left(1 - e^{i\theta_\gamma} (\mu_\gamma)^{-2k} (\bar{\mu}_\gamma)^{-2\ell} |\mu_\gamma|^{-2(\lambda+1)}\right)}{\left(1 - e^{-i\theta_\gamma} (\mu_\gamma)^{-2k} (\bar{\mu}_\gamma)^{-2\ell} |\mu_\gamma|^{-2(\lambda+1)}\right)}. \tag{8.8}$$

Here γ runs over the set of Γ-conjugacy classes of the loxodromic elements in Γ and a loxodromic element γ can be conjugated to a diagonal matrix with the diagonal elements $\mu_\gamma = \exp(\frac{1}{2}(l_\gamma + i\theta_\gamma))$, $\mu_\gamma^{-1} = \exp(-\frac{1}{2}(l_\gamma + i\theta_\gamma))$ in $\mathrm{PSL}(2,\mathbb{C})$ ($|\mu_\gamma| > 1$). Let us remark that the infinite product on the right hand side of (8.8) absolutely converges for $\Re(\lambda) > \delta_\Gamma - 1$, in particular, at $\lambda = 0$ since $\delta_\Gamma < 1$.

Now comparing the definition of q_γ and μ_γ, one can see that $q_\gamma = \mu_\gamma^{-2}$, that is, $q_\gamma^{1/2} = \mu_\gamma^{-1}$. Hence the odd Selberg zeta function $Z^o_{\Gamma,\Lambda}(0)$ has the following expression in terms of q_γ,

$$\begin{aligned} Z^o_{\Gamma,\Lambda}(0) &= \prod_{[\gamma]\in \mathrm{P}\Gamma_{\mathrm{lox}}} \prod_{k,\ell=0}^{\infty} \frac{\left(1 - (\bar{q}_\gamma q_\gamma^{-1})^{\frac{1}{2}} q_\gamma^k \bar{q}_\gamma^\ell (q_\gamma \bar{q}_\gamma)^{\frac{1}{2}}\right)}{\left(1 - (q_\gamma \bar{q}_\gamma^{-1})^{\frac{1}{2}} q_\gamma^k \bar{q}_\gamma^\ell (q_\gamma \bar{q}_\gamma)^{\frac{1}{2}}\right)} \\ &= \prod_{[\gamma]\in \mathrm{P}\Gamma_{\mathrm{lox}}} \prod_{k,\ell=0}^{\infty} \frac{\left(1 - q_\gamma^k \bar{q}_\gamma^{\ell+1}\right)}{\left(1 - q_\gamma^{k+1} \bar{q}_\gamma^{\ell}\right)} = \prod_{[\gamma]\in \mathrm{P}\Gamma_{\mathrm{lox}}} \prod_{m=0}^{\infty} \frac{\left(1 - \bar{q}_\gamma^{1+m}\right)}{\left(1 - q_\gamma^{1+m}\right)}. \end{aligned}$$

Combining this and (8.3) completes the proof.

□

Remark. In the proof of Theorem 8.4, we assume the condition $\delta_\Gamma < 1$ which simplifies the proof in several steps. But, one can expect that a similar result still holds over the whole Schottky space $\mathfrak{S}_g$. This extension to $\mathfrak{S}_g$ is also related to the proof of the assertion of Bowen in [2] about the meromorphic extension of $\prod_{\{\gamma\}}(1 - q_\gamma^s)$ over $\mathbb{C}$. These problems will be discussed elsewhere.

Appendix A. Computation of parallel transport in the spinor bundle

Let $\tau_m^{m'}$ denote parallel transport in the tangent bundle of the upper half-space model $\mathbb{H}^{d+1}$ of hyperbolic space, between points $m = (x, y), m' = (x', y')$, along the unique geodesic linking them. We identify $T_m\mathbb{H}^{d+1}$ with $\mathbb{R}^{d+1}$ using the orthonormal basis at m given by $\{x\partial_x, x\partial_{y_1}, \ldots, x\partial_{y_d}\}$. We denote by $\tau(m, m')$ the matrix of the transformation $\tau_m^{m'}$ written in these bases.

Proposition A.1. *Let* $r := |y - y'|$, $\rho_{\rm ff} := \sqrt{(x + x')^2 + r^2}$. *The special orthogonal matrix* $\tau(m, m')$ *has the following coefficients:*

$$\begin{aligned}
\tau_{00} &= 1 - 2r^2/\rho_{\rm ff}^2 \\
\tau_{0j} &= -2(x + x')(y_j - y'_j)/\rho_{\rm ff}^2 && \text{for } j = 1, \dots, d \\
\tau_{j0} &= 2(x + x')(y_j - y'_j)/\rho_{\rm ff}^2 && \text{for } j = 1, \dots, d \\
\tau_{jl} &= \delta_j^l - 2(y_j - y'_j)(y_l - y'_l)/\rho_{\rm ff}^2 && \text{for } j, l \in \{1, \dots, d\}.
\end{aligned}$$

which are smooth on the stretched product $\mathbb{H}^{d+1} \times_0 \mathbb{H}^{d+1}$ *defined in subsection 3.2.*

Proof. Let A be the translation by $(0, y')$ in $\mathbb{R}^{d+1}$, composed to the left by the homothety of factor $1/x'$. This isometry of $\mathbb{H}^{d+1}$ maps m' to $(1, 0)$. In the above trivialization of the tangent bundle, A_* acts as the identity. Moreover, since it is an isometry, A transforms the geodesic from m to m' into the geodesic from $A(m)$ to $(1, 0)$ and preserves parallelism. Thus (as matrices)

$$\tau(m, m') = \tau(A(m), (1, 0)), \qquad \text{where } A(m) = \left(\frac{x}{x'}, \frac{y - y'}{x'}\right). \tag{A.1}$$

We now concentrate on $\tau(m, (1, 0))$.

If $y = 0$ it is clear that $\tau((x, 0), (1, 0))$ is just the identity matrix. Suppose $y \neq 0$. Let $\partial_r := \frac{1}{r} \sum_{j=1}^d y_j \partial_{y_j}$ denote the radial vector field, defined outside the vertical line through the origin. Define $R := x\partial_r$, $X := x\partial_x$. For each $j \geq 1$ set $e_j := x\partial_{y_j}$, and let $T_j := e_j - \langle e_j, R\rangle R$ denote the component of e_j which is tangent to the sphere S^{d-1}. The geodesic from m to $(1, 0)$ lives in the totally geodesic plane Π_m passing through $(1, 0)$ and m, which is a copy of the hyperbolic 2-space. Along this plane the vector fields T_j extend smoothly at the vertical line through the origin. It is clear that the vector fields T_j are parallel along Π_m.

Lemma A.2. *In the plane* Π_m, *parallel transport between* m *and* $(1, 0)$ *is given by the complex number*

$$\frac{-r + i(1 + x)}{r + i(1 + x)}$$

Proof. We use as (real) basis for $T\Pi_m$ the orthonormal vector fields X and R. The complex structure rotates R to X. The formula is deduced from the similar formula in $\mathbb{H}^2$. □

Equivalently, in the basis $\{X, R\}$, parallel transport is given by the 2×2 orthogonal matrix

$$\rho_{\rm ff}^{-2} \begin{bmatrix} (x + 1)^2 - r^2 & 2r(x + 1) \\ -2r(x + 1) & (x + 1)^2 - r^2 \end{bmatrix}.$$

We decompose a vector $V = a_0 X + \sum_{j=1}^d a_j e_j$ into its tangent, respectively orthogonal components to Π_m as follows:

$$V = a_0 X + \left(\sum_{j=1}^d a_j \langle e_j, R\rangle\right) R + \sum_{j=1}^d a_j T_j.$$

Since T_j are parallel, the orthogonal component is constant during parallel transport. Using (A.1) and Lemma A.2, we write

$$\begin{aligned}
\tau(m, m')(a_0 X + \sum_{j=1}^d a_j e_j) =& a_0 \tau(X) + \sum_{j=1}^d a_j \langle e_j, R\rangle \tau(R) + \sum_{j=1}^d a_j T_j \\
=& a_0 \frac{(x + x')^2 - r^2}{\rho_{\rm ff}^2} X + 2a_0 \frac{r(x + x')}{\rho_{\rm ff}^2} R + \sum_{j=1}^d a_j (e_j - \langle e_j, R\rangle R) \\
&+ \sum_{j=1}^d a_j \langle e_j, R\rangle \left(-\frac{2r(x + x')}{\rho_{\rm ff}^2} X + \frac{(x + x')^2 - r^2}{\rho_{\rm ff}^2} R\right)
\end{aligned}$$

from which the proposition follows, since $(\rho_{\rm ff}, (y - y')/\rho_{\rm ff}, y', x/\rho_{\rm ff}, x'/\rho_{\rm ff})$ are smooth coordinates on the blow-up $\mathbb{H}^{d+1} \times_0 \mathbb{H}^{d+1}$. □

The above oriented basis $\{X, e_1, \ldots, e_d\}$ of $T\mathbb{H}^{d+1}$ extends smoothly to the boundary $\{0\} \times \mathbb{R}^d$ of $\overline{\mathbb{H}}^{d+1}$ as an orthonormal basis of the zero tangent bundle with respect to the hyperbolic metric. Therefore, by the Proposition above, $\tau(m, m')$ is a smooth section on $\overline{\mathbb{H}}^{d+1} \times_0 \overline{\mathbb{H}}^{d+1}$ in the pull-back vector bundle

$$\beta^*(\pi_1^{*0}T\overline{\mathbb{H}}^{d+1} \boxtimes \pi_2^{*0}T^*\overline{\mathbb{H}}^{d+1}).$$

The orthonormal frame bundle P_{SO} of the 0-tangent bundle is trivialized over $\overline{\mathbb{H}}^{d+1}$ by the frame $p = \{X, e_1, \ldots, e_d\}$, therefore the (unique) spin structure P_{Spin} is identified with $\overline{\mathbb{H}}^{d+1} \times \mathrm{Spin}(d+1)$. Denote by $\tilde{p}$ one of the lifts of p to P_{Spin}. By definition of the lifted connection, parallel transport in P_{Spin} of the section $\tilde{p}$ along the geodesic from m to m' is $\tilde{p}\,U(m, m')$, where $U(m, m')$ the unique lift of the $\mathrm{SO}(d+1)$-valued function $\tau(m, m')$ to the $\mathrm{Spin}(d+1)$ group, starting at the identity for $m = m'$. Thus, parallel transport of a constant section σ (with respect to the trivialization $\tilde{p}$) in the spinor bundle is simply

$$\tau_m^{m'}[\tilde{p}, \sigma] = [\tilde{p}\,U(m, m'), \sigma] = [\tilde{p}, U(m, m')\sigma]$$

where multiplication in the last term is the spinor representation. By abuse of notation we write $U(m, m')$ for $\tau_m^{m'}$.

Proposition A.3. *Let $m = (x, y), m' = (x', y') \in \overline{\mathbb{H}}^{d+1}$. In the above trivialization of the spinor bundle, parallel transport takes the form*

$$U(m, m') = \frac{x + x'}{\rho} - \frac{r}{\rho}\mathrm{cl(X)cl(R)}.$$

Proof. We view the $\mathrm{Spin}(d+1)$ group inside the Clifford algebra as the group generated by even Clifford products of unit vectors. The projection $\pi : \mathrm{Spin}(d+1) \to \mathrm{SO}(d+1)$ is given by the adjoint action in the Clifford algebra on vectors:

$$\pi(c)(V) := cVc^{-1},$$

the kernel being precisely $\{\pm 1\}$. We must therefore examine the adjoint action of $A(m, m') := \frac{x+x'}{\rho_{\mathrm{ff}}} - \frac{r}{\rho_{\mathrm{ff}}}\mathrm{cl(X)cl(R)}$ on ${}^0T\overline{\mathbb{H}}^{d+1}$. Note that any Clifford element of the form $\alpha + \beta\mathrm{cl(X)cl(R)}$ with $\alpha^2 + \beta^2 = 1$ belongs to the Spin group. Next, $A^{-1}(m, m') = \frac{x+x'}{\rho_{\mathrm{ff}}} + \frac{r}{\rho_{\mathrm{ff}}}\mathrm{cl(X)cl(R)}$ so

$$\pi(A(m, m'))X = \left(\frac{(x + x')^2}{\rho_{\mathrm{ff}}^2} - \frac{r^2}{\rho_{\mathrm{ff}}^2}\right)X - 2\frac{(x + x')r}{\rho_{\mathrm{ff}}^2}R$$

which coincides with the action of $\tau(m, m')$ on X from Proposition A.1. Similarly, for the vector fields T_j from the proof of Proposition A.1 we have $\pi(A(m, m'))T_j = T_j = \tau(m, m')T_j$. Thus $\pi(A(m, m')) = \tau(m, m')$. The proof is finished by noting that $A(m, m')$ was normalized so that $A(m, m) = 1$. □

Corollary A.4. *Let $m' = (1, 0)$, $m = (0, r\omega)$. In the limit $r \to \infty$, the parallel transport $U(m, m')$ tends to $-\mathrm{cl(X)cl(R)}$.*

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