ADIABATIC DECOMPOSITION OF THE ζ-DETERMINANT OF THE DIRAC LAPLACIAN I. THE CASE OF AN INVERTIBLE TANGENTIAL OPERATOR

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with an Appendix by Yoonweon Lee

Abstract. We discuss the decomposition of the ζ-determinant of the square of the Dirac operator into the contributions coming from the different parts of the manifold. The result was announced in [16]. The proof sketched in [16] was based on results of Brüning and Lesch (see [4]). In the meantime we have found another proof, more direct and elementary, and closer to the spirit of the original papers which initiated the study of the adiabatic decomposition of the spectral invariants (see [7] and [21]). We discuss this proof in detail. We study the general case (non-invertible tangential operator) in forthcoming work (see [17] and [18]). In the Appendix we present the computation of the cylinder contribution to the ζ-function of the Dirac Laplacian on a manifold with boundary, which we need in the main body of the paper. This computation is also used to show the vanishing result for the ζ-function on a manifold with boundary.

Results

Let $\mathcal{D} : C^\infty(M;S) \to C^\infty(M;S)$ be a compatible Dirac operator acting on sections of a bundle of Clifford modules S over a closed manifold M. Assume that we have a decomposition of M as $M_1 \cup M_2$, where M_1 and M_2 are compact manifolds with boundary such that

$$M = M_1 \cup M_2 \ \ , \ \ M_1 \cap M_2 = Y = \partial M_1 = \partial M_2 \,. \tag{0.1}$$

The ζ-determinant of the operator $\mathcal{D}$ is given by the formula

$$det_\zeta \mathcal{D} = e^{\frac{i\pi}{2}(\zeta_{\mathcal{D}^2}(0) - \eta_{\mathcal{D}}(0))} \cdot e^{-\frac{1}{2}\zeta'_{\mathcal{D}^2}(0)} \,, \tag{0.2}$$

(see [20], see also the Introduction of [19]). In this paper we study the decomposition of $det_\zeta \mathcal{D}$ on M into contributions coming from M_1 and M_2. This issue was already solved for the phase of the determinant

$$\frac{i\pi}{2}(\zeta_{\mathcal{D}^2}(0) - \eta_{\mathcal{D}}(0)) \,,$$

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and there remains only the modulus - the *square root* of the ζ-determinant of the Dirac Laplacian $\mathcal{D}^2$ - to study. We present here an *"adiabatic"* solution of the problem in the case of an *"invertible tangential operator"*. The general case will be presented in [18] (see also [17]). However, the discussion in this paper is an important part of the study of the general case.

We start with a brief discussion of the splitting of the phase of the ζ-determinant. The invariant $\zeta_{\mathcal{D}^2}(0)$ poses no problems. The value of the function $\zeta_{\mathcal{D}^2}(s)$ at $s = 0$ is a local invariant in the sense that it is given by a formula

$$\zeta_{\mathcal{D}^2}(0) = \int_M a(x)dx\,,$$

where $a(x)$ is a density determined at the point $x \in M$ by the coefficients of the operator $\mathcal{D}$ at this point (see for instance [8]). This is the reason why the index of an elliptic differential operator, which can be viewed as the difference of the values of two different ζ-functions determined by the operator $\mathcal{D}$, has a nice decomposition corresponding to the decomposition of the manifold.

The other contribution to the phase of $det_\zeta\mathcal{D}$ is the eta-invariant $\eta_{\mathcal{D}}(0)$ and this is not a local invariant (see [2]), hence at first sight it is difficult to expect a nice and clear splitting formula. It is therefore rather surprising that such a formula for $\eta_{\mathcal{D}}(0)$ actually exists.

In the following we concentrate on the odd-dimensional case

$$n = dim\ M = 2k + 1\,.$$

We further assume that M and the operator $\mathcal{D}$ have product structures in a neighborhood of the boundary Y. More precisely, we assume that there is a bicollar neighborhood $N = [-1, 1] \times Y$ of Y in M such that the Riemannian structure on M and the Hermitian structure on S are products when restricted to N. This implies that $\mathcal{D}$ has the following form when restricted to the submanifold N

$$\mathcal{D} = G(\partial_u + B)\,. \tag{0.3}$$

Here u denotes the normal variable, $G : S|_Y \to S|_Y$ is a bundle automorphism, and B is a corresponding Dirac operator on Y. Moreover, G and B do not depend on u and they satisfy

$$G^* = -G \quad , \quad G^2 = -Id\ , \quad B = B^* \quad \text{and} \quad GB = -BG\ . \tag{0.4}$$

The operator B has a discrete spectrum with infinitely many positive and infinitely many negative eigenvalues. In this work we consider only the case of an invertible tangential operator, i.e. we assume that $ker\ B = \{0\}$. The general case is more difficult to handle and we refer to [17] and [18] for the discussion of the *noninvertible case*. However, the present work plays an important part in the analysis of the general case.

Let $\Pi_>$ denote the spectral projection onto the subspace spanned by the eigensections of B corresponding to the positive eigenvalues. Then $\Pi_>$ is an elliptic boundary condition for $\mathcal{D}_2 = \mathcal{D}|_{M_2}$ (see [1]; see [3] for an exposition of the theory of elliptic boundary problems for Dirac operators). In fact, any orthogonal projection satisfying

$$(0.5) \qquad -GPG = Id - P \quad \text{and} \quad P - \Pi_> \ \textit{is a smoothing operator},$$

is a self-adjoint elliptic boundary condition for the operator $\mathcal{D}_2$. This means that the associated operator

$$(\mathcal{D}_2)_P : dom\ (\mathcal{D}_2)_P \to L^2(M_2; S|_{M_2})$$

with $dom\ (\mathcal{D}_2)_P = \{s \in H^1(M_2; S|_{M_2}) \mid P(s|_Y) = 0\}$ is a self-adjoint Fredholm operator with $ker((\mathcal{D}_2)_P) \subset C^\infty(M_2; S|_{M_2})$ and a discrete spectrum (see [25]).

The existence of the meromorphic extensions of the functions $\eta_{(\mathcal{D}_2)_P}(s)$, $\zeta_{(\mathcal{D}_2)_P^2}(s)$ to the whole complex plane and their nice behavior in a neighborhood of $s = 0$ was established in [25]. We denote by $\mathcal{G}r^*_\infty(\mathcal{D}_2)$ the space of P satisfying (0.5).

Let us observe that $Id - P \in \mathcal{G}r^*_\infty(\mathcal{D}_1)$, if P is an element of $\mathcal{G}r^*_\infty(\mathcal{D}_2)$. We denote by $\eta_{G(\partial_u + B)}(P_1, P_2)(s)$ the η-function of the operator $G(\partial_u + B)$ on $[0,1] \times Y$ subject to the boundary condition P_2 at $u = 0$ and $Id - P_1$ at $u = 1$. We have the following pasting formula proved in [25]

$$(0.6) \qquad \eta_{\mathcal{D}}(0) = \eta_{(\mathcal{D}_1)_{Id-P_1}}(0) + \eta_{(\mathcal{D}_2)_{P_2}}(0) + \eta_{G(\partial_u+B)}(P_1, P_2)(0) \ \ mod\ \mathbf{Z}\,.$$

A similar formula for finite-dimensional perturbations of $\Pi_>$ has been discussed by several authors (see [23, 24, 25] and references therein).

The proof of (0.6) offered by the second author goes as follows.
First, we replace the bicollar N by $N_R = [-R, R] \times Y$. Now $\eta_{\mathcal{D}}(0)$, which can be expressed using an appropriate heat-kernel formula, splits into contributions coming from each side, plus the cylinder contribution (vanishing in the case of $\mathcal{D}$) and error terms. The error terms disappear as $R \to \infty$.
Second, though $\eta_{\mathcal{D}}(0)$ is not local, its variation (for instance with respect to the parameter R) is local and therefore the value of the contributions does not vary with R. This is enough to make explicit calculations of the formula (0.6).

In this work we apply the strategy employed above to study

$$det_\zeta \mathcal{D}^2 = e^{-\frac{d}{ds}\zeta_{\mathcal{D}^2}(s)|_{s=0}}\,.$$

However, we have to take into account two additional difficulties, which arise in the case of the ζ-determinant of $\mathcal{D}^2$.

First of all, the invariant $-\frac{d}{ds}\zeta_{\mathcal{D}^2}(s)|_{s=0}$ is much more subtle than the η-invariant. Even the variation of $-\frac{d}{ds}\zeta_{\mathcal{D}^2}(s)|_{s=0}$ is not given by a local formula.

Second, the cylinder contribution is not trivial in this case.

We handled those difficulties in [16] using the technique developed in [4]. Here we choose a different path. The invariant $\frac{d}{ds}\zeta_{\mathcal{D}^2}(s)|_{s=0}$ is given by the formula

$$\frac{d}{ds}\zeta_{\mathcal{D}^2}(s)|_{s=0} = \int_0^\infty \frac{1}{t} Tr\ e^{-t\mathcal{D}^2} dt\,. \tag{0.7}$$

Let us explain how to interpret formula (0.7). The trace $Tr\ e^{-t\mathcal{D}^2}$ has an asymptotic expansion of the form

$$Tr\ e^{-t\mathcal{D}^2} = t^{-\frac{n}{2}}\sum_{k=0}^{N} a_k t^k + O(t^{N+\frac{1-n}{2}})\,,$$

where $a_k = \int_M \alpha_k(x)dx$, and the density $\alpha_k(x)$ at the point $x \in M$ is determined by coefficients of the operator $\mathcal{D}^2$ (see [8]). This shows that

$$\zeta_{\mathcal{D}^2}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}^2} dt$$

is a holomorphic function of s, for $Re(s) > \frac{n}{2}$, and that it has a meromorphic extension to the whole complex plane with (possible) simple poles at $s_k = \frac{n}{2} - k$. The Γ-function has the following form in a neighborhood of 0

$$\Gamma(s) = \frac{1}{s} + \gamma + s\,h(s)\,,$$

where γ is Euler's constant and $h(s)$ is a holomorphic function in a neighborhood of 0. This allows us to compute $\zeta_{\mathcal{D}^2}(0)$

$$\begin{aligned}\zeta_{\mathcal{D}^2}(0) &= \lim_{s\to 0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}^2} dt = \lim_{s\to 0} s\int_0^1 t^{s-1} Tr\ e^{-t\mathcal{D}^2} dt \\ &= \lim_{s\to 0} s\int_0^1 t^{s-1} t^{-\frac{n}{2}}(\sum_{k=0}^{N} t^k a_k)dt = \lim_{s\to 0} s\cdot\sum_{k=0}^{N}\frac{2a_k}{2s+2k-n} = a_{n/2}\,,\end{aligned}$$

where N denotes any sufficiently large natural number and we keep in mind that

$$a_{n/2} = 0 \quad \text{for } n \text{ odd.}$$

In particular, $\zeta_{\mathcal{D}^2}(0) = 0$ for n odd. Though $s = 0$ is a regular point, the ζ-function may have poles on the right side of 0, and the function

$$\kappa_{\mathcal{D}^2}(s) = \int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}^2} dt$$

has even more poles. In particular, following the computations presented above, we have

$$Res_{s=0}\kappa_{\mathcal{D}^2}(s) = a_{n/2}\,.$$

Now, the derivative of the ζ-function at $s=0$ is obtained as follows

$$\zeta'_{\mathcal{D}^2}(0) = \frac{d}{ds}(\frac{\kappa_{\mathcal{D}^2}(s)}{\Gamma(s)})|_{s=0} = \frac{d}{ds}(\frac{a_{n/2} + s(\kappa_{\mathcal{D}^2}(s) - \frac{a_{n/2}}{s})}{1 + s\gamma + s^2h(s)})|_{s=0} =$$

$$\frac{(\kappa_{\mathcal{D}^2}(s) - \frac{a_{n/2}}{s})(1 + s\gamma + s^2h(s)) - (a_{n/2} + s(\kappa_{\mathcal{D}^2}(s) - \frac{a_{n/2}}{s}))(\gamma + 2sh(s))}{(1 + s\gamma + s^2h(s))^2}|_{s=0} =$$

$$(\kappa_{\mathcal{D}^2}(s) - \frac{a_{n/2}}{s})|_{s=0} - \gamma a_{n/2} = (\kappa_{\mathcal{D}^2}(s) - \frac{a_{n/2}}{s})|_{s=0} - \gamma a_{n/2} \quad .$$

This discussion provides a justification for the a priori *"formal"* formula (0.7).

Remark 0.1. (a) For simplicity we presented here the ζ-function in the case $ker\ \mathcal{D} = \{0\}$. In general we define ζ-function as

$$\zeta_{\mathcal{D}^2}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(Tr\ e^{-t\mathcal{D}^2} - dim\ ker\ \mathcal{D})dt \quad ,$$

and

$$\zeta_{\mathcal{D}^2}(0) = a_{n/2} - dim\ ker\ \mathcal{D} \quad .$$

(b) The corresponding result for the boundary value problems is proved in the Appendix (see also [12]). It is shown that

$$\zeta_{\mathcal{D}^2_{i,P}}(0) = -dim\ ker\ \mathcal{D}_{i,P} \ \text{ for any } P \in Gr^*_\infty(\mathcal{D}_i)\,,$$

hence we can use formula (0.7) in the situation we discuss under the assumption $ker\ \mathcal{D} = \{0\}$.

We split $\zeta'_{\mathcal{D}^2_R}(0)$ into contributions coming from different submanifolds plus cylinder contributions and the error terms. Here $\mathcal{D}_R$ denotes the operator $\mathcal{D}$ on the manifold M_R equal to the manifold M with N replaced by N_R . We introduce a manifold with boundary

$$M_{1,R} = M_1 \cup [-R, 0] \times Y\,,$$

where we identify the "old" collar neighborhood of the boundary Y on M_1 with $[-R-1,-R]\times Y$. Similarly we introduce the manifold $M_{2,R}$. The bundle of Clifford modules S splits on Y into subbundles of spinors of positive and negative chirality

$$S|_Y = S^+ \oplus S^- \quad , \quad \text{with } S^\pm = Ran\ \frac{1}{2}(Id \mp i\Gamma)\,.$$

The operator $P_\pm = \frac{1}{2}(Id \mp i\Gamma)$ is the orthogonal projection of $S|_Y$ onto $S^\pm$ and provides $\mathcal{D}_i$ with a (local) *chiral* elliptic boundary condition. This again means that the operator $\mathcal{D}_{i,\pm} = \mathcal{D}_i$ with the domain

$$dom\ \mathcal{D}_{i,\pm} = \{s \in H^1(M_i;S) \mid P_\pm(s|_Y) = 0\}\,,$$

is Fredholm and that its kernel and cokernel consist of only smooth sections. We also have

$$\mathcal{D}^*_{i,+} = \mathcal{D}_{i,-}\,. \tag{0.8}$$

We study the ζ-determinants of the corresponding Laplacians

$$\Delta_{i,\pm} = \mathcal{D}_{i,\mp}\mathcal{D}_{i,\pm}\,. \tag{0.9}$$

We denote by $\Delta_{i,R,\pm}$ the corresponding operator on the manifold $M_{i,R}$.

In the present paper we avoid a discussion of the difficult issues related to the existence of the *"small"* eigenvalues of the operators involved. Therefore we assume that the tangential operator B is invertible, i.e. $ker\ B = \{0\}$. However, this condition alone does not make all small eigenvalues disappear. Careful analysis shows that we also need to assume that the operator $\mathcal{D}_{i,\infty}$, equal to the operator $\mathcal{D}_i$ extended in a natural way to the manifold $M_{i,\infty}$, has no L^2-solutions. The manifold $M_{i,\infty}$ is simply M_i with the infinite semicylinder $[0,\infty)\times Y$ (or $(-\infty,0]\times Y$) attached (see [6], see also [23]). The existence of L^2-solutions of $\mathcal{D}_{i,\infty}$ on $M_{i,\infty}$ is responsible for the existence of exponentially small eigenvalues of the operator $\mathcal{D}_R$. Therefore we assume $ker_{L^2}\mathcal{D}_{i,\infty} = \{0\}$. The conditions we posed make the small eigenvalues disappear. In particular, all the elliptic boundary problems we discuss in this paper are invertible. We refer to Proposition 1.1 and Remark 1.2 for more information.

Our first main result is the following theorem

Theorem 0.2. *Let us assume that*

$$ker_{L^2}\mathcal{D}_{1,\infty} = \{0\} = ker_{L^2}\mathcal{D}_{2,\infty} \quad \text{and} \quad ker\ B = \{0\}\,. \tag{0.10}$$

Then

$$\lim_{R\to\infty}\{ln\ det_\zeta\mathcal{D}^2_R - ln\ det_\zeta\Delta_{1,R,-} - ln\ det_\zeta\Delta_{1,R,+}\} = 0\,, \tag{0.11}$$

or equivalently

$$\lim_{R\to\infty} \frac{det_\zeta \mathcal{D}_R^2}{det_\zeta \Delta_{1,R,-} \cdot det_\zeta \Delta_{2,R,+}} = 1\,. \tag{0.12}$$

This Theorem is implicit in [11]. The focus of the authors was on the non-standard η-invariant introduced by Singer in [21] and on the analytic torsion. Therefore no statement was made about the ζ-determinant.

In Section 1 we use *Duhamel's Principle* to show that in order to study

$$\lim_{R\to\infty} \{ln\ det_\zeta \mathcal{D}_R^2 - ln\ det_\zeta \Delta_{1,R,+} - ln\ det_\zeta \Delta_{2,R,+}\} \tag{0.13}$$

it is enough to discuss the cylinder contributions.

In Section 2 we perform the computation on the cylinder and show that the limit (0.13) is indeed equal to 0 . Then we study the difference between the cylinder contribution for the chiral boundary condition and for the Atiyah-Patodi-Singer condition. Straightforward computations show that a new term appears which is equal to $-ln\ 2 \cdot \zeta_{B^2}(0)$. This gives the main result of the paper:

Theorem 0.3. *The following equality holds under the assumptions of our Theorem 0.2*

$$\lim_{R\to\infty} \frac{det_\zeta \mathcal{D}_R^2}{det_\zeta \mathcal{D}_{1,R,\Pi_<}^2 \cdot det_\zeta \mathcal{D}_{2,R,\Pi_>}^2} = 2^{-\zeta_{B^2}(0)}\,. \tag{0.14}$$

The Appendix by Yoonweon Lee contains a refined version of the computations of the cylinder contribution to the trace of the heat kernel of the Atiyah-Patodi-Singer problem performed by the second author in [25]. The more careful analysis by Lee proves *mod* $\mathbf{Z}$ vanishing of the function $P \mapsto \zeta_{\mathcal{D}_P^2}(0)$ on the Grassmannian $Gr_\infty^*(\mathcal{D}_i)$. Moreover, the formula (A.9) (see Appendix Proposition A.4) is used in the proof of Theorem 0.3.

Remark 0.4. This paper is related to many other works on the gluing formulas for the ζ-determinants. We refer to an excellent survey article [14] for the review of different approaches and the extensive bibliography. However, we want to mention that Theorem 0.3 is closely related to the results of [10]. In [10] only the operator $d + d^*$ is treated, but the gluing formula similar to (0.14) is obtained using the b-calculus technique, in the situation where the zero eigenvalues are allowed.

1. Duhamel's Principle. Reduction to the Cylinder

Our assumptions about the operator $\mathcal{D}_R$ (see (0.10)) allow us to apply the technique developed in [7] and to reduce the proof of Theorem 0.2 and Theorem 0.3 to the computations on the cylinder. The first important Corollary of (0.10) is the following Proposition

Proposition 1.1. *Let us assume that (0.10) holds. Then there exist positive constants c and R_0, such that*

$$\mu > c \tag{1.1}$$

for any eigenvalue μ of the operator $\mathcal{D}_R^2$, $\Delta_{i,R,\pm}$, $\mathcal{D}^2_{1,R,\Pi_<}$, $\mathcal{D}^2_{2,R,\Pi_>}$ and for any $R > R_0$.

Remark 1.2. The estimate (1.1) was observed by W. Müller. We refer to [7] (Theorem 6.1) for the proof in the case of the Atiyah-Patodi-Singer condition (operators $\mathcal{D}^2_{1,R,\Pi_<}$, $\mathcal{D}^2_{2,R,\Pi_>}$). A more general result was published in [15], Proposition 8.14. The proof for the "chiral" boundary conditions (operators $\Delta_{i,R,\pm}$) is even more simple. The case of the operator $\mathcal{D}_R$ was analyzed in [6] (see also [23]).

We need to recall the following result

Proposition 1.3. *Let $\mathcal{E}_R(t;x,y)$ denote the kernel of the heat operator for Δ_R, where Δ_R denotes one of the operators from Proposition 1.1. Assume that (0.10) holds. Then there exist positive constants c_1 , c_2 and c_3 such that*

$$\|\mathcal{E}_R(t;x,y)\| \le c_1 t^{-\frac{n}{2}} e^{c_2 t} e^{-c_3 \frac{d^2(x,y)}{t}}\,, \tag{1.2}$$

for any $t > 0$ and any $x, y \in M_R$ ($M_{1,R}$, or $M_{2,R}$ respectively) and for any $R > R_0$.

We refer to Sections 2 and 4 of [7] for the proof and related results. In particular Proposition 1.3 implies the following estimate

$$|Tr\ e^{-\Delta_R}| < c_4 \cdot R\,. \tag{1.3}$$

Now, we are ready to prove that we can neglect the *"large time contribution"* to the ζ-determinant of Δ_R.

Corollary 1.4. *Let us assume (0.10), then for any $\varepsilon > 0$ the following equality holds*

$$\lim_{R\to\infty}\int_{R^\varepsilon}^{\infty}\frac{1}{t}\cdot Tr\ e^{-t\Delta_R}dt = 0\,. \tag{1.4}$$

Proof. Assume that $R > R_0$ and let $\{\mu_k\}_{k=1}^{\infty}$ denote the set of eigenvalues of Δ_R. We have

$$\begin{aligned}\int_{R^\varepsilon}^{\infty}\frac{1}{t}\cdot Tr\ e^{-t\Delta_R}dt &= \int_{R^\varepsilon}^{\infty}\frac{1}{t}\cdot\sum_{k=1}^{\infty}e^{-t\mu_k}dt = \int_{R^\varepsilon}^{\infty}\frac{1}{t}\cdot\sum_{k=1}^{\infty}e^{-(t-1)\mu_k}e^{-\mu_k}dt\\ &< \int_{R^\varepsilon}^{\infty}\frac{1}{t}e^{-(t-1)c}\cdot Tr\ e^{-\Delta_R}dt \quad ,\end{aligned}$$

where c is the constant from Proposition 1.1. We use (1.3)

$$\int_{R^\varepsilon}^{\infty}\frac{1}{t}\cdot Tr\ e^{-t\Delta_R}dt < \int_{R^\varepsilon}^{\infty}\frac{1}{t}e^{-(t-1)c}\cdot Tr\ e^{-\Delta_R}dt < c_6R^{1-\varepsilon}\cdot e^{-c_7R^\varepsilon}$$

and the Corollary follows easily. $\square$

Now we follow [7]. Let $\tilde{\mathcal{E}}_R(t;x,y)$ denote the kernel of the operator $e^{-t\mathcal{D}_R^2}$ on the manifold M_R and let $\mathcal{E}_{cyl}(t;x,y)$ denote the kernel of the operator $e^{-t(-\partial_u^2+B^2)}$ on the infinite cylinder $(-\infty\,,+\infty)\times Y$, or the kernel of the APS - operator on $((-\infty,0]\cup[0,\infty))\times Y$. We introduce a smooth, increasing function $\rho(a,b):[0,\infty)\to[0,1]$ equal to 0 for $0\le u\le a$ and equal to 1 for $b\le u$. We use $\rho(a,b)(u)$ to define

$$\phi_1 = 1-\rho(\frac{5}{7}R\,,\frac{6}{7}R) \quad , \quad \psi_1 = 1-\psi_2$$

and

$$\phi_2 = \rho(\frac{1}{7}R\,,\frac{2}{7}R) \quad , \quad \psi_2 = \rho(\frac{3}{7}R\,,\frac{4}{7}R)\,.$$

We extend these functions to symmetric functions on the whole real line. All these functions are constant outside the interval $[-R\,,R]$ and we use them to define the corresponding functions on the manifold M_R. Now we define $Q_R(t;x,y)$ as a *"parametrix"* for the kernel $\tilde{\mathcal{E}}_R(t;x,y)$, actually using $\tilde{\mathcal{E}}_R(t;x,y)$, but the point here is that we will be able to separate the cylinder and the interior contribution

$$Q_R(t;x,y) = \phi_1(x)\mathcal{E}_{cyl}(t;x,y)\psi_1(y) + \phi_2(x)\tilde{\mathcal{E}}_R(t;x,y)\psi_2(y)\,. \tag{1.5}$$

Standard computations show that

$$\tilde{\mathcal{E}}_R(t;x,y) = Q_R(t;x,y) + (\tilde{\mathcal{E}}_R * \mathcal{C}_R)(t;x,y)\,, \tag{1.6}$$

where $\tilde{\mathcal{E}}_R * \mathcal{C}_R$ is a convolution given by

$$(\tilde{\mathcal{E}}_R * \mathcal{C}_R)(t;x,y) = \int_0^t ds \int_{M_R} dz \ \tilde{\mathcal{E}}_R(s;x,z)\mathcal{C}_R(t-s;z,y)\,,$$

and the correction term $\mathcal{C}_R(t;x,y)$ is given by the formula

$$\begin{aligned}\mathcal{C}_R(t;x,y) = &-\frac{\partial^2\phi_1}{\partial u^2}(x)\mathcal{E}_{cyl}(t;x,y)\psi_1(y) - \frac{\partial\phi_1}{\partial u}(x)\frac{\partial\mathcal{E}_{cyl}}{\partial u}(t;x,y)\psi_1(y)\\ &-\frac{\partial^2\phi_2}{\partial u^2}(x)\tilde{\mathcal{E}}_R(t;x,y)\psi_2(y) - \frac{\partial\phi_2}{\partial u}(x)\frac{\partial\tilde{\mathcal{E}}_R}{\partial u}(t;x,y)\psi_2(y)\,.\end{aligned} \tag{1.7}$$

The choice of the cut-off functions and the estimate (1.2) allow us to neglect the *"error" term contribution* to the logarithm of the determinant in the limit as $R \to \infty$.

Lemma 1.5. *The error term* $\mathcal{C}_R(t;x,y)$ *is equal to* 0 *outside of the cylinder* $[-\frac{6}{7}R\,,\frac{6}{7}R]\times Y$, *moreover it is equal to* 0 *if the distance between* x *and* y *is smaller than* $\frac{R}{7}$. *This fact combined with (1.2) proves the following estimate for certain positive constants*

$$\|(\tilde{\mathcal{E}}_R * \mathcal{C}_R)(t;x,y)\| \le c_1 e^{c_2 t} e^{-c_3\frac{R^2}{t}}\,. \tag{1.8}$$

The proof is elementary and follows the proof of the similar statement in [7] (see Proposition 5.2 of [7]).

Corollary 1.6. *Assume that* $0<\varepsilon<1$, *then*

$$\lim_{R\to 0}\int_0^{R^\varepsilon}\frac{dt}{t}\int_{\mathcal{M}_R} tr\ (\tilde{\mathcal{E}}_R * \mathcal{C}_R)(t;x,x)dx = 0\,, \tag{1.9}$$

where $\mathcal{M}_R$ *denotes any of the manifolds on which the operator* Δ_R *(of Proposition 1.3) is defined.*

Proof. This follows from the estimate on the kernel $(\tilde{\mathcal{E}}_R * \mathcal{C}_R)(t;x,x)$

$$\begin{aligned}|tr\ (\tilde{\mathcal{E}}_R * \mathcal{C}_R)(t;x,x)| &\le \|(\tilde{\mathcal{E}}_R * \mathcal{C}_R)(t;x,x)\|\\ &\le \int_0^t ds\int_{\mathcal{M}_R}\|\tilde{\mathcal{E}}_R(s;x,z)\mathcal{C}_R(t-s;z,x)\|dz\\ &\le c_1 e^{c_2 t}\cdot\int_0^t ds\int_{\mathcal{M}_R} e^{-c_3\frac{d^2(x,z)}{s}}e^{-c_3\frac{d^2(x,z)}{t-s}}dz\,.\end{aligned}$$

We use Lemma 1.5. It follows that the integral with respect to z is taken over the cylinder and moreover that the distance $d(x,z)$ is always larger than $\frac{R}{7}$, which gives

$$|tr\ (\tilde{\mathcal{E}}_R * \mathcal{C}_R)(t;x,x)| \leq c_1 e^{c_2 t} \cdot \int_0^t ds \int_{\mathcal{M}_R} e^{-c_3 \frac{d^2(x,z)}{s}} e^{-c_3 \frac{d^2(x,z)}{t-s}} dz$$
$$< c_1 e^{c_2 t} \int_0^t ds \int_{-R}^{R} e^{-c_4 \frac{tR^2}{s(t-s)}} dz < c_5 R e^{c_2 t} e^{-c_6 \frac{R^2}{t}} \int_0^t ds < c_5 R t e^{c_2 t} e^{-c_6 \frac{R^2}{t}} .$$

Now we have

$$\Big| \int_0^{R^\varepsilon} \frac{dt}{t} \int_{\mathcal{M}_R} tr\ (\tilde{\mathcal{E}}_R * \mathcal{C}_R)(t;x,x) dx \Big|$$
$$< \int_0^{R^\varepsilon} \frac{dt}{t} c_5 R t e^{c_2 t} e^{-c_6 \frac{R^2}{t}} < c_5 R^{1+\varepsilon} e^{-c_7 R^{1-\varepsilon}} ,$$

and (1.9) is proved. $\square$

The last result clearly explains that we have to analyze only the cylinder contribution to study the ratio of the determinants in the adiabatic limit. This is done in the next Section.

2. Computations on the Cylinder

Our study of the decomposition formula for the ζ-determinant involves the *"Laplacians"* $\Delta_{i,\pm} = \mathcal{D}_{i,\mp} \mathcal{D}_{i,\pm}$. It is well-known that $\Delta_{i,+}$ is the operator $\mathcal{D}_i^2$ subject to the Dirichlet boundary condition on S^+ and the Neumann boundary condition on S^- (see for instance Lemma 1.1. in [11]).

It was explained in the previous section that it is enough to analyze the cylinder contribution. Hence we have to write down the kernel of the *heat operator* defined by

$$-\partial_u^2 + B^2 : C^\infty([0\,,\infty) \times Y; S = S^+ \oplus S^-) \to C^\infty([0\,,\infty) \times Y; S = S^+ \oplus S^-)$$

subject to the Dirichlet condition on $S^+|_{\{0\}\times Y}$ and the Neumann condition on $S^-|_{\{0\}\times Y}$, in the case of the operator coming from the manifold $M_{2,R}$; and subject to the Dirichlet condition on $S^-|_{\{0\}\times Y}$ and the Neumann condition on $S^+|_{\{0\}\times Y}$, in the case of the operator coming from the manifold $M_{1,R}$.

Let $\mathcal{E}^{+}(t;x,y)$ denote the kernel of the first operator. The explicit formula is well-known (see [11] for instance)

(2.1)
$$\begin{aligned}\mathcal{E}^{+}(t;(u,x),(v,y)) =&\frac{1}{\sqrt{4\pi t}}\{e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}}\}e^{-tB^2}(t;x,y)P_{+}(y)\\ &+ \frac{1}{\sqrt{4\pi t}}\{e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}}\}e^{-tB^2}(t;x,y)P_{-}(y)\,,\end{aligned}$$

where $e^{-tB^2}(t;x,y)$ denotes the kernel of the operator e^{-tB^2}. This formula determines the cylinder contribution coming from the manifold $M_{2,R}$. The inward normal coordinate on M_1 is equal to $-u$. As a consequence the chirality of spinors, which is determined by the Clifford multiplication by the normal vector, is switched as G is replaced by $-G$. The corresponding cylinder contribution for the manifold $M_{1,R}$ is determined by the kernel

(2.2)
$$\begin{aligned}\mathcal{E}^{-}(t;(u,x),(v,y)) =&\frac{1}{\sqrt{4\pi t}}\{e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}}\}e^{-tB^2}(t;x,y)P_{+}(y)\\ &+ \frac{1}{\sqrt{4\pi t}}\{e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}}\}e^{-tB^2}(t;x,y)P_{-}(y)\,.\end{aligned}$$

Now we sum up the formulas (2.1) and (2.2) and put $u=v$, $x=y$. This gives

(2.3)
$$\begin{aligned}&\int_0^R du\frac{1}{\sqrt{4\pi t}}\{1-e^{-\frac{u^2}{t}}\}Tr_Y e^{-tB^2}P_{+}+\int_0^R du\frac{1}{\sqrt{4\pi t}}\{1+e^{-\frac{u^2}{t}}\}Tr_Y e^{-tB^2}P_{-}+\\ &\int_0^R du\frac{1}{\sqrt{4\pi t}}\{1+e^{-\frac{u^2}{t}}\}Tr_Y e^{-tB^2}P_{+}+\int_0^R du\frac{1}{\sqrt{4\pi t}}\{1-e^{-\frac{u^2}{t}}\}Tr_Y e^{-tB^2}P_{-}\,.\end{aligned}$$

In the formula (2.3) we neglect the presence of the cut-off functions introduced in Section 1. We also denote by $Tr_Y e^{-tB^2}$ the trace of the operator e^{-tB^2} on the manifold Y. Therefore modulo a term exponentially decaying with R, the boundary contribution to the sum $Tr\ e^{-t\Delta_{R,1}} + Tr\ e^{-t\Delta_{R,2}}$ is equal to

(2.4)
$$\frac{2}{\sqrt{4\pi t}}\cdot\int_0^R du\,(Tr_Y e^{-tB^2}P_{+} + Tr_Y e^{-tB^2}P_{-}) = \frac{1}{\sqrt{4\pi t}}\cdot\int_{-R}^R du\cdot Tr_Y e^{-tB^2}\,.$$

The right side of (2.4) is exactly equal to the trace of the heat kernel of the operator $-\partial_u^2 + B^2$ on the cylinder $(-\infty, +\infty) \times Y$, which is the cylinder contribution of the operator $\mathcal{D}_R^2$ modulo terms which disappear as $R \to \infty$. This ends the proof of Theorem 0.2.

Now, we have to analyze the difference between the trace of $\mathcal{E}^+(t;x,y)$ and the trace $\mathcal{E}^>(t;x,y)$, where $\mathcal{E}^>(t;x,y)$ denotes the kernel of the *heat operator* defined by the operator $G(\partial_u + B)$ subject to the Atiyah-Patodi-Singer boundary condition. We introduce $\phi(u)$ a smooth cut-off function, equal to 1 for $0 \leq u \leq R$ and vanishing for $2R \leq u$, with derivatives bounded by $\frac{c}{R}$, and we study the following function

$$\tag{2.5} \mathcal{T}(s) = \int_0^\infty t^{s-1} dt \int_{[0,\infty)\times Y} \phi(u) \cdot tr(\mathcal{E}^>(t;(u,y),(u,y)) \\ - \mathcal{E}^+(t;(u,y),(u,y))) dy du\,.$$

Long, but elementary computations give us the following formula for the contribution made by the Atiyah-Patodi-Singer part (see Appendix, Proposition A.4.)

$$\tag{2.6} \begin{aligned} &\int_0^\infty t^{s-1} dt \int_{[0,\infty)\times Y} \phi(u) \cdot tr(\mathcal{E}^>(t;(u,y),(u,y)) dy du \\ &= \frac{1}{\sqrt{4\pi}} \int_0^\infty \phi(u) du \int_0^\infty t^{s-\frac{3}{2}} Tr_Y e^{-tB^2} dt \\ &\quad + \frac{1}{2} \cdot \int_0^\infty t^{s-1} dt \int_0^\infty \phi'(u) du \sum_{n=1}^\infty e^{2u\lambda_n} \operatorname{erfc}\Big(\frac{u}{\sqrt{t}} + \lambda_n \sqrt{t}\Big) \\ &\quad + \frac{\Gamma(s+\frac{1}{2})}{4s\sqrt{\pi}} \zeta_{B^2}(s) - \frac{1}{2\sqrt{\pi}} \cdot \int_0^\infty t^{s-\frac{3}{2}}\, Tr_Y e^{-tB^2} \Big(\int_0^\infty \phi(u) e^{-\frac{u^2}{t}} du \Big)\, dt\,. \end{aligned}$$

We have three terms on the right side of (2.6), which we denote by $\mathcal{T}_1(s)$, $\mathcal{T}_2(s)$ and $\mathcal{T}_3(s)$. The sum in $\mathcal{T}_2(s)$ is taken over all positive eigenvalues λ_n of the tangential operator B and the function $\operatorname{erfc}(u)$ is given by the formula

$$\tag{2.7} \operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-s^2} ds\,.$$

The first contribution $\mathcal{T}_1(s)$ corresponds to the contribution to (2.5) given by the kernel $\mathcal{E}^+(t;(u,y),(u,y))$ and they cancel each other when we take the difference. We can also easily deal with the second contribution:

Proposition 2.1. *The function*

$$\mathcal{T}_2(s) = \frac{1}{2} \cdot \int_0^\infty t^{s-1} dt \int_0^\infty \phi'(u) du \sum_{n=1}^\infty e^{2u\lambda_n} \mathrm{erfc}(\frac{u}{\sqrt{t}} + \lambda_n \sqrt{t})$$

is a holomorphic function of s vanishing as $R \to \infty$.

Proof. We estimate using $\int_u^\infty e^{-s^2} ds \leq e^{-u^2}$

$$\int_0^\infty t^{s-1} dt \int_0^\infty \phi'(u) du \sum_{n=1}^\infty e^{2u\lambda_n} \mathrm{erfc}(\frac{u}{\sqrt{t}} + \lambda_n \sqrt{t}) \leq$$

$$\frac{2}{\sqrt{\pi}} \int_0^\infty t^{s-1} dt \int_R^{2R} \phi'(u) du \sum_{n=1}^\infty e^{-\frac{u^2}{t}} e^{-t\lambda_n^2} \leq$$

$$\frac{c}{R} \frac{2}{\sqrt{\pi}} \int_0^\infty t^{s-1} (\sum_{n=1}^\infty e^{-t\lambda_n^2}) dt \int_R^{2R} e^{-\frac{u^2}{t}} du \leq$$

$$\frac{c_1}{R} \int_0^\infty t^{s-1} Tr\ e^{-tB^2} dt \int_R^\infty e^{-\frac{u^2}{t}} du \leq$$

$$\frac{c_1}{R} \int_0^\infty t^{s-\frac{1}{2}} Tr\ e^{-tB^2} dt \int_R^\infty e^{-\frac{u^2}{t}} \frac{du}{\sqrt{t}} \leq \frac{c_1}{R} \int_0^\infty t^{s-\frac{1}{2}} e^{-\frac{R^2}{t}} Tr\ e^{-tB^2} dt \,,$$

and the Proposition follows. □

Now we see that $\mathcal{T}_3(s)$ is the only source of an additional contribution. It is not difficult to see that, modulo a function holomorphic on the whole complex plane, $\mathcal{T}_3(s)$ is equal to

$$\begin{aligned} \mathcal{S}(s) &= \frac{\Gamma(s+\frac{1}{2})}{4s\sqrt{\pi}} \zeta_{B^2}(s) - \frac{1}{4} \cdot \int_0^\infty t^{s-1} Tr_Y e^{-tB^2} dt \\ &= \frac{\Gamma(s+\frac{1}{2})}{4s\sqrt{\pi}} \zeta_{B^2}(s) - \frac{\Gamma(s)}{4} \cdot \zeta_{B^2}(s) \,. \end{aligned}$$

Indeed, the difference

$$g_R(s) = \mathcal{T}_3(s) - \mathcal{S}(s) = \frac{1}{2\sqrt{\pi}} \int_0^\infty t^{s-1} Tr\ e^{-tB^2} (\int_0^\infty (1-\phi(u)) e^{-\frac{u^2}{t}} \frac{du}{\sqrt{t}}) dt$$

is a holomorphic function on the complex plane, which depends on the parameter R. We use the following elementary result:

Lemma 2.2. *The following equality holds for any complex s*

$$\lim_{R\to\infty} g_R(s) = \lim_{R\to\infty} g'_R(s) = 0\,. \tag{2.8}$$

Proof. We have to estimate

$$\Big|\frac{1}{2\sqrt{\pi}}\int_0^\infty t^{s-1} Tr\ e^{-tB^2}\int_0^\infty \big(1-\phi(u)\big)e^{-\frac{u^2}{t}}\frac{du}{\sqrt{t}}dt\Big|\,. \tag{2.9}$$

We use the following elementary inequality

$$\Big|\int_0^\infty (1-\phi(u))e^{-\frac{u^2}{t}}\frac{du}{\sqrt{t}}\Big| \le \Big|\int_R^\infty e^{-\frac{u^2}{t}}\frac{du}{\sqrt{t}}\Big| = \Big|\int_{\frac{R}{\sqrt{t}}}^\infty (-\frac{1}{2s})(-2s)e^{-s^2}ds\Big|$$
$$\le \Big|-\frac{\sqrt{t}}{2R}\int_{\frac{R}{\sqrt{t}}}^\infty \frac{d}{ds}(e^{-s^2})ds\Big| = \frac{\sqrt{t}}{2R}e^{-\frac{R^2}{t}}\,.$$

This allows us to estimate (2.9)

$$\Big|\frac{1}{2\sqrt{\pi}}\int_0^\infty t^{s-1} Tr\ e^{-tB^2}\int_0^\infty (1-\phi(u))e^{-\frac{u^2}{t}}\frac{du}{\sqrt{t}}dt\Big|$$
$$< \frac{1}{4\sqrt{\pi}R}\int_0^\infty t^{s-\frac12}e^{-\frac{R^2}{t}}Tr_Y e^{-tB^2}dt\,.$$

The last expression goes to 0 as $R\to\infty$. The estimates on the derivatives with respect to s go exactly in the same way. □

The function $\mathcal{S}(s)$ was given by the formula

$$\mathcal{S}(s) = \Big(\frac{\Gamma(s+\frac12)}{4s\sqrt{\pi}} - \frac{\Gamma(s)}{4}\Big)\cdot\zeta_{B^2}(s)\,.$$

We see that $\mathcal{S}(s)$ is a holomorphic function for $Re(s) > \frac{n}{2}$ and that it has a meromorphic extension to the whole complex plane with simple poles on the real line, provided by both factors. Hence the poles at the positive integers come from $\zeta_{B^2}(s)$ and the ζ-function is regular in the neighborhood of 0. The first factor

$$\frac{\Gamma(s+\frac12)}{4s\sqrt{\pi}} - \frac{\Gamma(s)}{4}$$

is holomorphic for $Re(s) > 0$ and it is not very difficult to show that in fact it is holomorphic in a neighborhood of $s = 0$. We have

$$\frac{\Gamma(s+\frac12)}{4s\sqrt{\pi}} - \frac{\Gamma(s)}{4} = \frac{1}{4\sqrt{\pi}}\cdot\frac{\Gamma(s+\frac12) - s\Gamma(s)\sqrt{\pi}}{s}$$
$$= \frac{1}{4\sqrt{\pi}}\cdot\Big(\frac{\Gamma(s+1/2)-\Gamma(1/2)}{s} + \Gamma(1/2)\frac{1-\Gamma(s+1)}{s}\Big)\,,$$

and we see that

$$\lim_{s\to 0}\frac{\Gamma(s+1/2)}{4s\sqrt{\pi}}-\frac{\Gamma(s)}{4}=\frac{1}{4\sqrt{\pi}}\cdot\Big(\Gamma'(\frac{1}{2})-\sqrt{\pi}\cdot\Gamma'(1)\Big)\,.$$

It is well-known that $\Gamma'(1)=\gamma$ (once again, γ denotes Euler's constant), and it is not difficult to compute $\Gamma'(1/2)$ using, for instance, the formula

$$\Gamma(z+\frac{1}{2})=\frac{\sqrt{\pi}\cdot\Gamma(2z)}{2^{2z-1}\cdot\Gamma(z)}\,,$$

(see for instance [22], formula (A22) on page 265).

We have

$$\begin{aligned}&\lim_{s\to 0}\frac{\Gamma(s+\frac{1}{2})-\Gamma(\frac{1}{2})}{s}=\sqrt{\pi}\cdot\lim_{s\to 0}\frac{\Gamma(2s)/2^{2s-1}\cdot\Gamma(s)-1}{s}\\ =&\sqrt{\pi}\cdot\lim_{s\to 0}\frac{2^{1-2s}\Gamma(2s)-\Gamma(s)}{s\Gamma(s)}=\sqrt{\pi}\cdot\lim_{s\to 0}\frac{1}{\Gamma(1+s)}\cdot\lim_{s\to 0}(2^{1-2s}\Gamma(2s)-\Gamma(s))\\ =&\sqrt{\pi}\cdot\lim_{s\to 0}(2^{1-2s}(\frac{1}{2s}+\gamma+2sh(2s))-(\frac{1}{s}+\gamma+sh(s)))\,,\end{aligned}$$

where $h(s)$ is a holomorphic function in the neighborhood of $s=0$. Hence we finally obtain

$$\begin{aligned}&\lim_{s\to 0}\frac{\Gamma(s+\frac{1}{2})-\Gamma(\frac{1}{2})}{s}\\ =&\sqrt{\pi}\cdot\lim_{s\to 0}\Big(\frac{2^{1-2s}-2}{2s}+2^{1-2s}\gamma-\gamma+2^{1-2s}2s\,h(2s)-s\,h(s)\Big)\\ =&-2\sqrt{\pi}\cdot ln\ 2+\sqrt{\pi}\gamma\,,\end{aligned}$$

and

$$\lim_{s\to 0}\frac{\Gamma(s+\frac{1}{2})}{4s\sqrt{\pi}}-\frac{\Gamma(s)}{4}=-\frac{1}{4\sqrt{\pi}}\cdot 2\sqrt{\pi}\cdot ln\ 2=-\frac{1}{2}ln\ 2\,. \tag{2.10}$$

This gives us the following result

Proposition 2.3. *The adiabatic limit of the difference between the logarithm of the ζ-determinant of the operator $\mathcal{D}^2_{2,R,\Pi_>}$ and the logarithm of the ζ-determinant of the operator $\Delta_{2,R,+}$ is given by*

$$\lim_{R\to\infty}(ln\ det_\zeta\mathcal{D}^2_{2,R,\Pi_>}-ln\ det_\zeta\Delta_{2,R,+})=\frac{ln\ 2}{2}\cdot\zeta_{B^2}(0)\,. \tag{2.11}$$

We have obtained *"half"* of the correction term which appears in Theorem 0.3 (see (0.14)). The other *"half"* is equal to the contribution of the manifold $M_{1,R}$. Now Theorem 0.3 is proved.

Appendix A. The value of the ζ-function at $s=0$ on the smooth, self-adjoint Grassmannian

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In this Appendix we write M instead of M_2 and $\mathcal{D}$ instead of $\mathcal{D}_2$. The goal is to prove the following result

Proposition A.1. *For any $P \in Gr_\infty^*(\mathcal{D})$, the value of $\zeta_{\mathcal{D}_P^2}(s)$ at $s=0$ is equal to $-dim\ ker\mathcal{D}_P$.*

Remark A.2. (1) The proof depends only on the assumption that the perturbation of $\Pi_>$ is an operator of the trace class. Therefore the result holds for any orthogonal projection $P = Id + GPG$, such that $P - \Pi_>$ is a pseudodifferential operator of order $-dim\ Y - 1$.

(2) One of the formulas we obtain for the cylinder contribution to the invariant $\zeta_{\mathcal{D}^2_{\Pi_>}}(0)$ (see Proposition A.4.) is used in the proof of the decomposition formula for the ζ-determinant discussed in the main body of the paper.

We start with the proof of Proposition A.1 in the most simple case. We assume

$$\text{(A.1)} \qquad dim\ ker\ B = 0 \ \ and \ \ dim\ ker\ \mathcal{D}_{\Pi_>} = 0 \ .$$

It was explained earlier that the first condition in (A.1) implies that $\Pi_> \in Gr_\infty^*(\mathcal{D})$, hence $\mathcal{D}_{\Pi_>}$ is a self-adjoint operator. The second condition implies the invertibility of $\mathcal{D}_{\Pi_>}$. We have to show that $\zeta_{\mathcal{D}^2_{\Pi_>}}(0) = 0$.

We start with selecting a smooth cut-off function $\rho : M \to [0,1]$ equal to 1 on $[0\,,1/3] \times Y$ and equal to 0 on $M \setminus ([0\,,2/3] \times Y)$. We also choose $\rho_1\,,\rho_2 : M \to [0,1]$ such that

$$\rho_1|_{supp\ \rho} \equiv 1 \ \ and \ \ \rho_1 \equiv 0 \ on \ M \setminus N \ \ and$$
$$\rho_2|_{supp\ (1-\rho)} \equiv 1 \ \ and \ \ \rho_2 \equiv 0 \ on \ [0,1/4] \times Y\,.$$

Let $\mathcal{E}_{cyl}(t;x,y)$ denote the heat kernel of the Atiyah-Patodi-Singer problem on the cylinder $[0\,,\infty) \times Y$ and $\tilde{\mathcal{E}}(t;x,y)$ denote the kernel of the operator $e^{-t\tilde{\mathcal{D}}^2}$, where $\tilde{\mathcal{D}}$ is the double of the operator $\mathcal{D}$, living on $\tilde{M}$ the double of M (see [3] for details of the construction). Finally let $\mathcal{E}_>(t;x,y)$ denote the kernel of the *heat operator* of $\mathcal{D}^2_{\Pi_>}$ on M. A standard application of *Duhamel's Principle* shows that there exists a positive constant c, such that

(A.2)
$$\mathcal{E}_>(t;x,y) = \rho_1(x)\mathcal{E}_{cyl}(t;x,y)\rho(y) + \rho_2(x)\tilde{\mathcal{E}}(t;x,y)(1-\rho(y)) + O(e^{-c/t}),$$

for $0 < t \leq 1$. Now the ζ-function is given by the formula

$$\zeta_{\mathcal{D}^2_{\Pi_>}}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}^2_{\Pi_>}} dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}dt \int_M tr\ \mathcal{E}_>(t;x,x)dx\,.$$

Equation (A.2) implies that there exist positive constants c_1 and c_2, such that

$$|tr\ \mathcal{E}_>(t;x,x) - \rho(x)\cdot tr\ \mathcal{E}_{cyl}(t;x,x) - (1-\rho(x))\cdot tr\ \tilde{\mathcal{E}}(t;x,x)| < c_1 e^{-c_2/t} \tag{A.3}$$

for $0 < t \leq 1$, which implies that

$$\int_0^\infty t^{s-1}dt \int_M (tr\ \mathcal{E}_>(t;x,x) - \rho(x)\cdot tr\ \mathcal{E}_{cyl}(t;x,x) - (1-\rho(x))\cdot tr\ \tilde{\mathcal{E}}(t;x,x))dx$$

is a well-defined holomorphic function of s on the whole complex plane. In particular, we have obtained the following result

Lemma A.3.

$$\zeta_{\mathcal{D}^2_{\Pi_>}}(0) = \lim_{s\to 0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}dt \int_M \rho(x)\cdot tr\ \mathcal{E}_{cyl}(t;x,x)dx\,. \tag{A.4}$$

Proof. Equation (A.3) implies the following equality

$$\zeta_{\mathcal{D}^2_{\Pi_>}}(0) = \lim_{s\to 0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}dt \int_M \Big(\rho(x)\cdot tr\ \mathcal{E}_{cyl}(t;x,x) + (1-\rho(x))\cdot tr\ \tilde{\mathcal{E}}(t;x,x)\Big)dx\,.$$

It is well-known that in the case of the Dirac Laplacian on a closed, odd-dimensional manifold, the *"local"* ζ-function disappears (see for instance [8]), hence

$$\lim_{s\to 0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\cdot(1-\rho(x))\cdot tr\ \tilde{\mathcal{E}}(t;x,x)dt = 0\,, \tag{A.5}$$

which gives the result. ∎

Now let us recall that B has a symmetric spectrum and its spectral decomposition has the form

$$\{\lambda_n\,,\phi_n\ ;\ -\lambda_n\,,G\phi_n\}_{n=1}^{\infty}\,.$$

The explicit representation of the kernel $\mathcal{E}_{cyl}(t;x,y)$ with respect to this decomposition is as follows

(A.6)

$$\begin{aligned}\mathcal{E}_{cyl}(t;(u,x),(v,y)) &= \sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2 t}}{\sqrt{4\pi t}}\{e^{-(u-v)^2/4t}-e^{-(u+v)^2/4t}\}\phi_n(x)\otimes\overline{\phi_n(y)}\\ &+\sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2 t}}{\sqrt{4\pi t}}\{e^{-(u-v)^2/4t}+e^{-(u+v)^2/4t}\}G(x)\phi_n(x)\otimes\overline{G(y)\phi_n(y)}\\ &-\sum_{n=1}^{\infty}\lambda_n e^{\lambda_n(u+v)}\mathrm{erfc}\Big((u+v)/2\sqrt{t}+\lambda_n\sqrt{t}\Big)G(x)\phi_n(x)\otimes\overline{G(y)\phi_n(y)}\,,\end{aligned}$$

where $\mathrm{erfc}(r)$ is defined as in (2.7):

$$\mathrm{erfc}(r)=\frac{2}{\sqrt{\pi}}\int_r^{\infty}e^{-\xi^2}d\xi\,.$$

We now have an explicit representation of the integral in (A.4)

$$\begin{aligned}\text{(A.7)}\qquad &\int_0^{\infty}t^{s-1}dt\int_M\rho(x)\cdot tr\ \mathcal{E}_{cyl}(t;x,x)dx\\ &=\int_0^{\infty}\rho(u)du\int_0^{\infty}t^{s-1}2\cdot\Big\{\sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2 t}}{\sqrt{4\pi t}}\Big\}dt\\ &\quad-\int_0^{\infty}\int_0^{\infty}t^{s-1}\rho(u)\Big\{\sum_{n=1}^{\infty}\lambda_n e^{2\lambda_n u}\mathrm{erfc}\Big(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t}\Big)\Big\}dudt\\ &=\frac{1}{2\sqrt{\pi}}\int_0^{\infty}\rho(u)du\int_0^{\infty}t^{s-3/2}Tr_Y e^{-tB^2}dt\\ &\quad-\int_0^{\infty}t^{s-1}dt\int_0^{\infty}\rho(u)\Big\{\sum_{n=1}^{\infty}\lambda_n e^{2\lambda_n u}\mathrm{erfc}\Big(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t}\Big)\Big\}du\,.\end{aligned}$$

We start with the second integral on the right side of (A.7)

$$\begin{aligned}&\int_0^{\infty}t^{s-1}dt\int_0^{\infty}\rho(u)\{\sum_{n=1}^{\infty}\lambda_n e^{2\lambda_n u}\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\}du\\ =&\frac{1}{2}\int_0^{\infty}t^{s-1}dt\int_0^{\infty}\rho(u)\{\sum_{n=1}^{\infty}(\frac{d}{du}e^{2u\lambda_n})\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\}du.\end{aligned}$$

Integration by parts leads to

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty t^{s-1}\{[\sum_{n=1}^\infty \rho(u)e^{2u\lambda_n}\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})]|_0^\infty\}dt \\
&-\frac{1}{2}\int_0^\infty t^{s-1}dt\int_0^\infty \rho'(u)\{\sum_{n=1}^\infty e^{2\lambda_n u}\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\}du \\
&-\frac{1}{2}\int_0^\infty t^{s-1}dt\int_0^\infty \rho(u)\{\sum_{n=1}^\infty e^{2\lambda_n u}\mathrm{erfc}'(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\frac{1}{\sqrt{t}}\}du \\
=&-\frac{1}{2}\int_0^\infty t^{s-1}\{\sum_{n=1}^\infty \mathrm{erfc}(\lambda_n\sqrt{t})\}dt \\
&-\frac{1}{2}\int_0^\infty t^{s-1}dt\int_0^\infty \rho'(u)\{\sum_{n=1}^\infty e^{2\lambda_n u}\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\}du \\
&+\frac{1}{\sqrt{\pi}}\int_0^\infty t^{s-1}dt\int_0^\infty \rho(u)\{\sum_{n=1}^\infty e^{2\lambda_n u}e^{-(u^2/t+2u\lambda_n+\lambda_n^2 t)}\}\frac{du}{\sqrt{t}}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{2}\int_0^\infty t^{s-1}\{\sum_{n=1}^\infty e^{-\lambda_n^2 t}\}dt\frac{2}{\sqrt{\pi}}\int_0^\infty \rho(u)e^{-u^2/t}\frac{du}{\sqrt{t}} \\
&-\frac{1}{2}\int_0^\infty t^{s-1}\{\sum_{n=1}^\infty \mathrm{erfc}(\lambda_n\sqrt{t})\}dt \\
&-\frac{1}{2}\int_0^\infty t^{s-1}dt\int_0^\infty \rho'(u)\{\sum_{n=1}^\infty e^{2\lambda_n u}\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\}du\,.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\int_0^\infty t^{s-1}dt\int_0^\infty \rho(u)\{\sum_{n=1}^\infty \lambda_n e^{2\lambda_n u}\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\}du \\
=&\int_0^\infty t^{s-3/2}Tr\ e^{-tB^2}dt\cdot\frac{1}{2\sqrt{\pi}}\int_0^\infty \rho(u)e^{-u^2/t}du \\
&-\frac{1}{2}\int_0^\infty t^{s-1}\{\sum_{n=1}^\infty \mathrm{erfc}(\lambda_n\sqrt{t})\}dt \\
&-\frac{1}{2}\int_0^\infty t^{s-1}dt\int_0^\infty \rho'(u)\{\sum_{n=1}^\infty e^{2\lambda_n u}\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\}du\,.
\end{aligned}
\tag{A.8}
$$

First, we analyze the middle term on the right side. The following calculations hold for a single eigenvalue:

$$\begin{aligned}\int_0^\infty t^{s-1}\mathrm{erfc}(\lambda_n\sqrt{t})dt &= \frac{1}{s}\int_0^\infty d/dt(t^s)\mathrm{erfc}(\lambda_n\sqrt{t})dt \\ &= \frac{t^s}{s}\mathrm{erfc}(\lambda_n\sqrt{t})|_0^\infty - \frac{1}{s}\int_0^\infty t^s\mathrm{erfc}'(\lambda_n\sqrt{t})\frac{\lambda_n}{2\sqrt{t}}dt \\ &= -\frac{\lambda_n}{2s}\int_0^\infty t^{s-1/2}(-\frac{2}{\sqrt{\pi}}e^{-\lambda_n^2 t})dt \\ &= \frac{\lambda_n}{s\sqrt{\pi}}\int_0^\infty t^{s-1/2}e^{-\lambda_n^2 t}dt = \frac{\Gamma(s+1/2)}{s\sqrt{\pi}}\lambda_n^{-2s}\,.\end{aligned}$$

It follows that for $Re(s)$ large, the middle term on the right side of (A.8) is equal to

$$-\frac{1}{2}\int_0^\infty t^{s-1}\{\sum_{n=1}^\infty \mathrm{erfc}(\lambda_n\sqrt{t})\}dt = -\frac{1}{2}\frac{\Gamma(s+1/2)}{2s\sqrt{\pi}}\zeta_{B^2}(s)\,.$$

This has a nice meromorphic extension, with simple poles, to the whole complex plane. We rewrite (A.8) as

$$\begin{aligned}&\int_0^\infty t^{s-1}dt\int_0^\infty \rho(u)\{\sum_{n=1}^\infty \lambda_n e^{2\lambda_n u}\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\}du \\ =&\int_0^\infty t^{s-3/2}Tr_Y e^{-tB^2}dt\cdot\frac{1}{2\sqrt{\pi}}\int_0^\infty \rho(u)e^{-u^2/t}du \\ &-\frac{\Gamma(s+1/2)}{4s\sqrt{\pi}}\zeta_{B^2}(s) - \frac{1}{2}\int_0^\infty t^{s-1}dt\int_0^\infty \rho'(u)\{\sum_{n=1}^\infty e^{2\lambda_n u}\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\}du\ ,\end{aligned}$$

and we substitute this into (A.7).

We put the final result of the computation as an independent statement.

Proposition A.4. *The following equality describes the cylinder contribution to the ζ-function of the operator $\mathcal{D}^2_{\Pi_>}$*

(A.9)
$$
\begin{aligned}
&\int_0^\infty t^{s-1}dt \int_M \rho(x)\cdot tr\ \mathcal{E}_{cyl}(t;x,x)dx \\
=&\frac{1}{2\sqrt{\pi}}\int_0^\infty \rho(u)du \int_0^\infty t^{s-3/2}Tr_Y e^{-tB^2}dt \\
&-\{\int_0^\infty t^{s-3/2}Tr_Y e^{-tB^2}dt\cdot \frac{1}{2\sqrt{\pi}}\int_0^\infty \rho(u)e^{-u^2/t}du \ - \ \frac{\Gamma(s+1/2)}{4s\sqrt{\pi}}\zeta_{B^2}(s) \\
&\quad -\frac{1}{2}\int_0^\infty t^{s-1}dt\int_0^\infty \rho'(u)\{\sum_{n=1}^\infty e^{2\lambda_n u}\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\}du\}\,.
\end{aligned}
$$

The formula (A.9) is used in the study of the *adiabatic decomposition* of the ζ-determinant presented in Section 2. We have to analyze (A.9) further in order to get information about the value of the ζ-function at $s=0$.

Lemma A.5. *The function*

$$
\mathcal{F}_1(s) = \int_0^\infty t^{s-1}dt\int_0^\infty \rho'(u)\{\sum_{n=1}^\infty e^{2\lambda_n u}\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\}du \tag{A.10}
$$

is a holomorphic function on the whole complex plane.

Proof. We use the fact that *supp* $\rho' \subset [1/3\,,2/3]\times Y$, which guarantees a nice behavior of the integral with respect to the u-variable since the sum over the eigenvalues is absolutely convergent. We just have to show that $|\mathcal{F}_1(s)|$ behaves nicely with respect to s. We use the fact that $\mathrm{erfc}(r)\le e^{-r^2}$ and estimate

$$
\begin{aligned}
|\mathcal{F}_1(s)| &\le \int_0^\infty t^{s-1}dt\int_0^\infty |\rho'(u)|\{\sum_{n=1}^\infty e^{2\lambda_n u}\mathrm{erfc}(\frac{u}{\sqrt{t}}+\lambda_n\sqrt{t})\}du \\
&\le \int_0^\infty t^{s-1}dt\int_0^\infty |\rho'(u)|\{\sum_{n=1}^\infty e^{-u^2/t-t\lambda_n^2}\}du \\
&= \frac{1}{2}\cdot\int_0^\infty t^{s-1}Tr_Y e^{-tB^2}dt\int_{1/3}^{2/3}|\rho'(u)|e^{-u^2/t}du \\
&\le c_1\int_0^\infty t^{s-1}e^{-c_2/t}Tr_Y e^{-tB^2}dt
\end{aligned}
$$

for some positive constants c_1, c_2 and now the Lemma follows from the well-known asymptotics of $Tr_Y e^{-tB^2}$ as $t\to 0$ and $t\to\infty$. $\square$

Now, we consider the term

$$\mathcal{F}_2(s) = \int_0^\infty t^{s-3/2} Tr_Y e^{-tB^2} dt \cdot \frac{1}{2\sqrt{\pi}} \int_0^\infty \rho(u) e^{-u^2/t} du \,. \tag{A.11}$$

The function $\rho(u)$ is equal to 1 for $0 \le u \le 1/3$ and we split the integral accordingly

$$\begin{aligned}\mathcal{F}_2(s) = &\int_0^\infty t^{s-3/2} Tr_Y e^{-tB^2} dt \cdot \frac{1}{2\sqrt{\pi}} \int_0^{1/3} e^{-u^2/t} du \\ &+ \int_0^\infty t^{s-3/2} Tr_Y e^{-tB^2} dt \cdot \frac{1}{2\sqrt{\pi}} \int_{1/3}^{2/3} \rho(u) e^{-u^2/t} du \,.\end{aligned}$$

Let us observe that

$$\begin{aligned}\int_0^{1/3} e^{-u^2/t} du = \sqrt{t} \cdot \int_0^{1/3\sqrt{t}} e^{-y^2} dy = \sqrt{t} \cdot \int_0^\infty e^{-y^2} dy - \sqrt{t} \cdot \int_{1/3\sqrt{t}}^\infty e^{-y^2} dy \\ = \frac{\sqrt{\pi}}{2}\sqrt{t} - \frac{\sqrt{\pi}}{2}\sqrt{t} \cdot \mathrm{erfc}(1/3\sqrt{t}) \ ,\end{aligned}$$

which allows us to represent $\mathcal{F}_2(s)$ in the following form

$$\begin{aligned}\mathcal{F}_2(s) = &\frac{1}{4} \int_0^\infty t^{s-1} Tr_Y e^{-tB^2} dt - \frac{1}{4} \int_0^\infty t^{s-1} Tr_Y e^{-tB^2} \cdot \mathrm{erfc}(1/3\sqrt{t}) dt \\ &+ \int_0^\infty t^{s-3/2} Tr_Y e^{-tB^2} dt \cdot \frac{1}{2\sqrt{\pi}} \int_{1/3}^{2/3} \rho(u) e^{-u^2/t} du \ .\end{aligned} \tag{A.12}$$

The middle term on the right side of the above equality is again holomorphic on the whole complex plane due to the inequality

$$\Big| \int_0^\infty t^{s-1} Tr_Y e^{-tB^2} \cdot \mathrm{erfc}\big(\frac{1}{3\sqrt{t}}\big) dt \Big| \le c \int_0^\infty t^{s-1} Tr_Y e^{-tB^2} \cdot e^{-1/9t} dt \ .$$

We estimate the last term on the right side of (A.12) in the same way to show that it is a holomorphic function of s as well. Finally, we evaluate the

ζ-function at $s=0$, using Lemma A.3:

$$\begin{aligned}
\zeta_{\mathcal{D}^2_{\Pi_>}}(0) &= \lim_{s\to 0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}dt\int_M \rho(x)\cdot tr\ \mathcal{E}_{cyl}(t;x,x)dx \\
&= \lim_{s\to 0}\frac{1}{\Gamma(s)}\{\frac{1}{2\sqrt{\pi}}\int_0^\infty \rho(u)du\int_0^\infty t^{s-3/2}Tr_Y e^{-tB^2}dt \\
&\qquad -\mathcal{F}_2(s)+\frac{\Gamma(s+1/2)}{4s\sqrt{\pi}}\zeta_{B^2}(s)+\frac{1}{2}\mathcal{F}_1(s)\} \\
&= \lim_{s\to 0} s\cdot\{\frac{1}{2\sqrt{\pi}}\int_0^\infty \rho(u)du\int_0^\infty t^{s-3/2}Tr_Y e^{-tB^2}dt \\
&\qquad -\mathcal{F}_2(s)+\frac{\Gamma(s+1/2)}{4s\sqrt{\pi}}\zeta_{B^2}(s)+\frac{1}{2}\mathcal{F}_1(s)\} \\
&= \lim_{s\to 0} s\cdot\{\frac{1}{2\sqrt{\pi}}\int_0^\infty \rho(u)du\int_0^\infty t^{s-3/2}Tr_Y e^{-tB^2}dt \\
&\qquad -\frac{1}{4}\int_0^\infty t^{s-1}Tr_Y e^{-tB^2}dt+\frac{\Gamma(s+1/2)}{4s\sqrt{\pi}}\zeta_{B^2}(s)\} \\
&= \lim_{s\to 0} s\cdot\frac{1}{2\sqrt{\pi}}\int_0^\infty \rho(u)du\int_0^\infty t^{s-3/2}Tr_Y e^{-tB^2}dt \\
&\quad +\lim_{s\to 0}\{s\frac{\Gamma(s+1/2)}{4s\sqrt{\pi}}\zeta_{B^2}(s)-s\frac{1}{4}\int_0^\infty t^{s-1}Tr_Y e^{-tB^2}dt\} \\
&= 0+\{\frac{1}{4}\zeta_{B^2}(0)-\frac{1}{4}\zeta_{B^2}(0)\}=0\,.
\end{aligned}$$

The situation is not different in the case of non-invertible $\mathcal{D}_{\Pi_>}$. We have

$$\zeta_{\mathcal{D}^2_{\Pi_>}}(0)=\lim_{s\to 0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(Tr\ e^{-t\mathcal{D}^2_{\Pi_>}}-dim\ ker\mathcal{D}^2_{\Pi_>})dt \quad ,$$

where the dimension of the kernel is present in order to make the integral $\int_1^\infty$ convergent. Now we have

$$\begin{aligned}
\zeta_{\mathcal{D}^2_{\Pi_>}}(0) &= \lim_{s\to 0}\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}(Tr\ e^{-t\mathcal{D}^2_{\Pi_>}}-dim\ ker\ \mathcal{D}^2_{\Pi_>})dt \\
&= \lim_{s\to 0}(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}Tr\ e^{-t\mathcal{D}^2_{\Pi_>}}dt)-dim\ ker\ \mathcal{D}^2_{\Pi_>} \\
&= \lim_{s\to 0}(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}dt\int_M \rho(x)\cdot tr\ \mathcal{E}_{cyl}(t;x,x)dx)-dim\ ker\ \mathcal{D}^2_{\Pi_>} \\
&= -\,dim\ ker\ \mathcal{D}^2_{\Pi_>} \quad .
\end{aligned}$$

We also do not have problem with the case $ker\ B \neq \{0\}$. The *Cobordism Theorem* for the Dirac operators (see for instance [3]) implies the existence of the involution

$$\sigma : ker\ B \to ker\ B \quad ,$$

such that $G\sigma = -\sigma G$. Let $\pi_\sigma : ker\ B \to ker\ B$ denote orthogonal projection onto $+1$-eigenspace of σ . The orthogonal projection $\Pi_\sigma = \Pi_> + \pi_\sigma$ is an element of $Gr^*_\infty(\mathcal{D})$ and we can repeat our computations to obtain

$$\zeta_{\mathcal{D}^2_{\Pi_\sigma}}(0) = -dim\ ker\ \mathcal{D}^2_{\Pi_\sigma} \quad .$$

Finally the result for arbitrary element $P \in Gr^*_\infty(\mathcal{D})$ follows from the existence of a positive constant $c > 0$, such that for any $0 < t < 1$

$$|Tr\ e^{-t\mathcal{D}^2_P} - Tr\ e^{-t\mathcal{D}^2_{\Pi_\sigma}}| < c\sqrt{t} \quad .$$

This result is stated as Theorem 3.2 in [25]. The proof consists of a straightforward computation and the details are presented in [25]. The idea is easy to understand. It was explained in Section 1 of [25], that $\mathcal{D}^2_P$ is unitarily equivalent to the operator of the form $\mathcal{D}^2_{\Pi_\sigma} + \mathcal{K}$, where $\mathcal{K} : L^2(M;S) \to L^2(M;S)$ is a bounded operator, with kernel $\mathcal{K}(x,y)$ supported in $N = [0,1] \times Y$. Moreover, $\mathcal{K}(x,y)$ is smoothing in Y-direction. By the Duhamel's Principle we have

$$Tr\ e^{-t\mathcal{D}^2_P} - Tr\ e^{-t\mathcal{D}^2_{\Pi_\sigma}} = -Tr \int_0^t e^{-s\mathcal{D}^2_P}\mathcal{K}e^{-(t-s)\mathcal{D}^2_{\Pi_\sigma}} ds \quad .$$

The expression on the right side can be written as the series, where each next term has the better behavior with respect to t , than the previous one. The first term is

$$-Tr \int_0^t e^{-s\mathcal{D}^2_{\Pi_\sigma}}\mathcal{K}e^{-(t-s)\mathcal{D}^2_{\Pi_\sigma}} ds = -\int_0^t Tr\ \mathcal{K}e^{-t\mathcal{D}^2_{\Pi_\sigma}} = -t \cdot Tr\ \mathcal{K}e^{-t\mathcal{D}^2_{\Pi_\sigma}} \quad .$$

Now the kernel of the operator $\mathcal{K}$ is smoothing in the Y-direction, hence the only singularity left is in the normal direction and we obtain

$$|Tr\ e^{-t\mathcal{D}^2_P} - Tr\ e^{-t\mathcal{D}^2_{\Pi_\sigma}}| \sim_{t\to 0} t|Tr\ \mathcal{K}e^{-t\mathcal{D}^2_{\Pi_\sigma}}| \leq t \cdot c/\sqrt{t} \leq c \cdot \sqrt{t}$$

(we refer to [25] for the detailed presentation). It follows that

$$|\lim_{s\to 0} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(Tr\ e^{-t\mathcal{D}^2_P} - Tr\ e^{-t\mathcal{D}^2_{\Pi_\sigma}})dt| \leq$$

$$\lim_{s\to 0}\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}|Tr\ e^{-t\mathcal{D}_P^2}-Tr\ e^{-t\mathcal{D}_{\Pi_\sigma}^2}|dt\leq$$

$$c\cdot\lim_{s\to 0}s\cdot\int_0^1 t^{s-1/2}ds=0\quad,$$

and as a result we have

$$\zeta_{\mathcal{D}_P^2}(0)-\zeta_{\mathcal{D}_{\Pi_\sigma}^2}(0)=dim\ ker\ \mathcal{D}_{\Pi_\sigma}-dim\ ker\mathcal{D}_P\quad.$$

This ends the proof of the Proposition A.1.

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