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Tau function and Chern–Simons invariant

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ABSTRACT

We define a Chern–Simons invariant for Schottky hyperbolic 3-manifolds of infinite volume. We then prove an expression relating the Bergman tau function on a fiber space over the Teichmüller space to the lifting of the function F defined by Zograf on Teichmüller space, and a holomorphic function on this space which we introduce. If the point in this space corresponds to a marked Riemann surface X, then this function is constructed from the renormalized volume and our Chern–Simons invariant for the bounding 3-manifold of X given by Schottky uniformization, together with a regularized Polyakov integral. We also obtain a relation between the Chern–Simons invariant and the eta invariant of the bounding 3-manifold, with defect given by the phase of the Bergman tau function of X.

1. Introduction

For a closed Riemannian 3-manifold M, the Chern–Simons invariant $CS(M)$ and the eta invariant $\eta(M)$ are two of the most important invariants which play central roles in

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the study of the geometric structures of M. Although they are introduced from totally different origins, they satisfy the following surprising equality

$$2CS(M) = 3\eta(M) \mod \mathbb{Z}. \tag{1.1}$$

The Chern–Simons invariant was introduced by Chern and Simons as a secondary characteristic class in [5]. Because of a choice of framing needed to pull back the Chern–Simons 3-form to the base manifold, this is well-defined only up to integer. The eta invariant was discovered by Atiyah, Patodi, and Singer in [2] as the boundary defect term for the signature theorem for manifolds with boundary. Since the eta invariant is well-defined without any integer ambiguity, the eta invariant can be understood as the lifting of the Chern–Simons invariant from the circle to the real line. We refer to [3] for the proof of (1.1) and other related materials.

The Chern–Simons invariant is particularly important in the study of hyperbolic 3-manifolds since this invariant and the hyperbolic volume together define a complex valued invariant naturally for hyperbolic 3-manifolds. For finite volume hyperbolic 3-manifolds with cusps, they define a local holomorphic function over an open neighborhood of the origin in the deformation space of the incomplete hyperbolic structures. This had been conjectured by Neumann and Zagier in [19] and was proved by Yoshida in [21].

There is another class of hyperbolic 3-manifolds whose hyperbolic volumes are infinite, that is, convex co-compact hyperbolic manifolds. Although one cannot obtain a nontrivial meaningful invariant from the honest hyperbolic volume for these hyperbolic 3-manifolds, a renormalization process to define a nontrivial invariant was introduced by Krasnov in [14] for Schottky hyperbolic 3-manifolds. This has been generalized to a class of convex co-compact hyperbolic 3-manifolds, which includes quasi-Fuchsian hyperbolic 3-manifolds, in [20,15]. This invariant is now referred to as the renormalized volume.

One of the motivations of this paper is to introduce a Chern–Simons invariant for convex co-compact hyperbolic 3-manifolds, in particular, for Schottky hyperbolic 3-manifolds, and to study its properties under the change of geometric structures. This is a natural problem since the aforementioned renormalized volume for these hyperbolic 3-manifolds encodes many interesting geometric meanings [14,20,15]. In particular, it turns out that the function defined by renormalized volumes is a Kähler potential of the Weil–Petersson Kähler 2-form over the deformation space of the hyperbolic structures for Schottky hyperbolic 3-manifolds.

Although our approach can be easily generalized to the same class of convex co-compact hyperbolic 3-manifolds as in [20,15], in this paper we restrict to the case of the Schottky hyperbolic 3-manifolds for simplicity, and for its application to the relation with other invariants. In the study of the Chern–Simons invariant of Schottky hyperbolic 3-manifolds, one of the new features is that the Schottky hyperbolic 3-manifold has a boundary Riemann surface after a proper compactification, which is referred to as a conformal boundary. In general, the Chern–Simons invariant can be defined for manifolds with boundary, but the resulting Chern–Simons invariant depends on the choice of the framing over the boundary. This phenomenon also occurs for the case of finite volume hyperbolic 3-manifolds with cusps and Yoshida used a special framing over flat torus, which can be understood as the boundary of the cusp, to define the Chern–Simons invariant in [21]. In our case, the Schottky hyperbolic 3-manifold has a Riemann surface boundary of higher genus in general, so there is no natural special framing as in the case of hyperbolic manifolds with cusps. Hence it is natural to regard a choice of framing over the boundary Riemann surface as another parameter when we consider a function defined by the resulting Chern–Simons invariant for Schottky hyperbolic 3-manifolds.

To explain in more detail, let us introduce some basic notions related to Schottky hyperbolic 3-manifolds. Each marked compact Riemann surface X of genus $g \geq 1$ has a Schottky uniformization given by a unique marked normalized Schottky group Γ; by its action on hyperbolic 3-space, the group also naturally defines an infinite volume hyperbolic 3-manifold M_X whose conformal boundary is X. Hence, the space of the marked normalized Schottky groups can be considered as the deformation space of the marked hyperbolic structures for Schottky hyperbolic 3-manifolds. This space is referred to as the Schottky space, denoted $\mathfrak{S}_g$, and it is known that its universal covering space is the Teichmüller space $\mathfrak{T}_g$. A holomorphic 1-form Φ on X determines a singular framing on X, and there exists a singular framing s_Φ on M_X which extends the framing on X in a sense we prescribe. In Section 4 we define an invariant $\mathbb{CS}(M,s)$ for a certain class of 3-manifolds M and singular framings s on M. The value of $\mathbb{CS}$ at a point corresponding to (X,Φ) is then defined to equal $\mathbb{CS}(M_X,s_\Phi)$. Our definition of $\mathbb{CS}(M,s)$ is motivated by the work of Meyerhoff [18] and Yoshida [21] for finite volume hyperbolic 3-manifolds with cusps. In Section 4.5 we show

$$\mathbb{CS}(M,s) = \frac{1}{\pi^2}W(M) + 2iCS(M,s), \tag{1.2}$$

where $W(M)$ is the aforementioned renormalized volume of M (see [14,20,15]; we use the definition of Section 8 of [15]), and $CS(M,s)$ is the integral over M of the pull back of the Chern–Simons 3-form by the framing s, together with a correction term corresponding to the singularities of the framing. Let us remark that $CS(M,s)$ is finite by our construction without any renormalization process and is well-defined only up to $\frac{1}{2}\mathbb{Z}$.

Our construction of $\mathbb{CS}(M,s)$ follows the one in [18,21] in principle, and this can be also understood in terms of $\mathfrak{sl}_2(\mathbb{C})$-flat connections. For a closed Riemannian 3-manifold M, a representation $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ defines a $PSL_2(\mathbb{C})$-principal bundle P_ρ over M with an $\mathfrak{sl}_2(\mathbb{C})$-flat connection. Using this connection one can define the Chern–Simons form over P_ρ, and the corresponding Chern–Simons invariant $CS_{PSL_2(\mathbb{C})}(M,\rho)$ is complex valued. In particular, if M is a closed hyperbolic manifold and ρ is the holonomy representation induced by the hyperbolic structure on M, the pull-back of the Chern–Simons form over P_ρ to the $SO(3)$-frame bundle of M can be given in terms of the Chern–Simons form and the volume form (with an exact form) defined by the hyperbolic metric. Then we have

$$CS_{PSL_2(\mathbb{C})}(M,\rho) = 2CS(M) - \frac{i}{\pi^2}\operatorname{Vol}(M) \tag{1.3}$$

where $CS(M)$ and $\operatorname{Vol}(M)$ denote the Chern–Simons invariant and the volume of M for the hyperbolic metric respectively. For more detail about this, we refer to [8,4]. (Note that the term $2CS(M)$ on the right hand side of (1.3) means the notation $CS(M)$ in [8,4].) Comparing (1.2) with (1.3), our invariant $\mathbb{CS}(M,s)$ may be understood as a generalization of $iCS_{PSL_2(\mathbb{C})}(M,\rho)$ for a class of noncompact 3-manifolds of infinite volume.

From the above construction, we can see that the invariant $\mathbb{CS}(M_X, s_\Phi)$ defines a function $\mathbb{CS}$ over the space consisting of pairs (X,Φ) where X is a marked Riemann surface of genus $g \geq 1$ and Φ is a holomorphic 1-form on X. For simplicity, we assume that Φ has only simple zeroes throughout this paper. The resulting space consisting of such pairs (X,Φ) is denoted by $\tilde{\mathcal{H}}_g$. Forgetting the holomorphic 1-form Φ data, one obtains a natural projection from $\tilde{\mathcal{H}}_g$ to $\mathfrak{T}_g$. Now one might expect that the resulting complex valued function $\mathbb{CS}$ would behave as in the case of finite volume hyperbolic 3-manifolds with cusps, for instance, it might be a holomorphic function over $\tilde{\mathcal{H}}_g$. But this cannot be true since its real part, the renormalized volume, does not define a harmonic function over $\tilde{\mathcal{H}}_g$. Hence, we need to modify the real part to make the resulting function become a holomorphic function. For this we introduce a real valued invariant $I(X,\Phi)$, which can be considered as a regularized Polyakov integral, interpolating the hyperbolic metric on X and the singular flat metric determined by Φ. We refer to (6.1) for the precise definition of $I(X,\Phi)$ and to Corollary 6.2 for the Polyakov type formula. Then we work with the following complex valued invariant

$$4\pi\mathbb{CS}(M_X,s_\Phi) + \frac{1}{\pi}I(X,\Phi) = \frac{4}{\pi}W(M_X) + \frac{1}{\pi}I(X,\Phi) + 8\pi i CS(M_X,s_\Phi). \tag{1.4}$$

The main result of this paper is the following theorem.

Theorem 1.1. *The complex valued invariant* $\exp(4\pi\mathbb{CS}(M_X,s_\Phi) + \frac{1}{\pi}I(X,\Phi))$ *determines a holomorphic function over* $\tilde{\mathcal{H}}_g$, $g \geq 1$, *and the following equality holds:*

$$\tau_B^{24} = c\exp\left(4\pi\mathbb{CS} + \frac{1}{\pi}I\right)F^{24}. \tag{1.5}$$

Here τ_B is the Bergman tau function defined by Kokotov and Korotkin in [12], with the property that τ_B^{24} is a globally well-defined holomorphic function on $\tilde{\mathcal{H}}_g$. The c represents a constant, depending on g and a topological choice that will be explained in Remark 4.11. The function F is the lifting to $\tilde{\mathcal{H}}_g$ of the holomorphic function over $\mathfrak{S}_g$ defined by Zograf in [22]. We refer to the Sections 2.1 and 2.2 respectively for the precise definitions of τ_B and F and remarks about the related works.

Remark 1.2. The construction of $CS(M_X, s_\Phi)$ involves a topological choice of singular framing over M_X, and this depends on the Teichmüller marking on X rather than the Schottky marking of X. Hence the function $\exp(4\pi\mathbb{CS} + \frac{1}{\pi}I)$ cannot descend to a fiber space over $\mathfrak{S}_g$ whose fiber is the space of the holomorphic 1-forms with simple zeroes. For more details, see the paragraph before Theorem 4.10, and Remark 4.11.

The equality (1.5) allows us to interpret the Bergman tau function as a higher genus generalization of the Dedekind eta function. When $g = 1$, it is known that $\tau_B = \eta(\tau)^2$ and $F = \prod_{m=1}^{\infty}(1 - q^m)^2$ on $\tilde{\mathcal{H}}_1 \simeq \mathfrak{T}_1 \times \mathbb{C}^*$ where $q = e^{2\pi i\tau}$, $\tau \in H^2 \simeq \mathfrak{T}_1$. By some computations given in Section 4.6 we have $\mathbb{CS} = i\tau$, and it is easy to see that $I = 0$ from the definition. Consequently in this case, Theorem 1.1 reduces to the 48-th power of the defining equation of the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m).$$

In [11,12], it was shown that τ_B^{24} satisfies a modular property with respect to the mapping class group, which reduces to the modular property of η^{48} in genus 1.

The relation between objects on the 2-manifold X and the bounding infinite volume 3-manifold M_X given by Theorem 1.1 fits well with principle of "holography"—for example, see [16] and [20]. In this context, the Schottky uniformization provides a natural choice of bounding 3-manifold M_X.

To state a corollary of Theorem 1.1 which is a generalization of (1.1) for Schottky hyperbolic 3-manifolds, we need a result about the phase of F. In [7], it is shown that the eta invariant $\eta(M_X)$ of the odd signature operator over M_X is well-defined, without any additional renormalization, and it is proved that the phase of F at X is $\exp(-\frac{\pi i}{2}\eta(M_X))$, whenever the marked Schottky group Γ has exponent of convergence $\delta < 1$. We refer to [7] for more details. Combining this with (1.5), we have

Corollary 1.3. *The following equality holds*

$$\exp(8\pi i CS - 12\pi i\eta) = c\left(\frac{\tau_B}{|\tau_B|}\right)^{24}$$

over the subset of $\tilde{\mathcal{H}}_g$, $g \geq 1$, for which the corresponding marked Schottky group Γ has exponent of convergence $\delta < 1$.

Comparing the equality (1.1), which holds for closed 3-manifolds, with Corollary 1.3, we can see that this is a generalization of (1.1) for Schottky hyperbolic 3-manifolds, where the boundary Riemann surface X produces a defect term given by the phase of τ_B.

2. Preliminary background

2.1. Bergman tau functions

We define the space $\mathcal{H}_g$ to be the moduli space of pairs (X,Φ) where X is a compact Riemann surface of genus $g \geq 1$ and Φ is a holomorphic 1-form with only simple zeroes over X. For more details about this space, we refer to [13]. (Note that $\mathcal{H}_g$ denotes the moduli space of pairs (X,Φ) without the simple zero condition of Φ in [13].)

Following [12], we introduce a covering $\hat{\mathcal{H}}_g$ of $\mathcal{H}_g$ consisting of triples $(X,\Phi,\{a_i,b_i \mid 1 \leq i \leq g\})$ where $\{a_i,b_i \mid 1 \leq i \leq g\}$ denotes a Torelli marking on X, that is, a canonical basis of $H_1(X,\mathbb{Z})$. Cutting the Riemann surface along the cycles given by a Torelli marking $\{a_i,b_i \mid 1 \leq i \leq g\}$, we get the fundamental polygon $\hat{X}$. Inside of $\hat{X}$ we choose $(2g-3)$-paths l_j which connect the zero p_1 with the other zeros p_j for $j = 2,\dots,2g-2$. The set of paths a_i, b_i, l_j gives a basis in the relative homology group $H_1(X,(\Phi),\mathbb{Z})$ where $(\Phi) = \sum_{j=1}^{2g-2} p_j$ denotes the divisor of Φ. As in [12], we introduce local coordinates on $\hat{\mathcal{H}}_g$ as follows:

$$A_i := \int_{a_i} \Phi, \qquad B_i := \int_{b_i} \Phi, \qquad Z_j := \int_{l_{j+1}} \Phi, \tag{2.1}$$

where $i = 1,\dots,g$ and $j = 1,\dots,2g-3$. For simplicity, we also use another notation ζ_i for the coordinates defined by

$$\zeta_i := A_i, \qquad \zeta_{g+i} := B_i, \qquad \zeta_{2g+j} := Z_j. \tag{2.2}$$

Define cycles s_i for $i = 1,\dots,4g-3$ by $s_i = -b_i$, $s_{g+i} = a_i$ for $i = 1,\dots,g$ and define the cycle s_{2g+i} to be a small circle with positive orientation around p_{i+1}. Note that the above construction is valid even for the case of $g = 1$ where there are only A_1, B_1 coordinates.

The Bergman kernel on a Riemann surface X with a Torelli marking is defined by $B(p,q) := d_p d_q \log E(p,q)$ for $p,q \in X$ where $E(p,q)$ is the prime form on X. Near the diagonal $p = q$, the Bergman kernel $B(p,q)$ has the expression

$$B\big(z(p),z(q)\big) = \Big(\big(z(p)-z(q)\big)^{-2} + H\big(z(p),z(q)\big)\Big) dz(p)dz(q)$$

where $z(p), z(q)$ are local coordinates of points p,q in X, and the Bergman projective connection R_B is defined in a local coordinate by

$$R_B\big(z(p)\big) = 6 \lim_{q\to p} H\big(z(p),z(q)\big). \tag{2.3}$$

In [12], Kokotov and Korotkin define the Bergman tau function τ_B over $\hat{\mathcal{H}}_g$ to be a holomorphic solution of the following compatible system of equations:

$$\frac{\partial \log \tau_B}{\partial \zeta_i} = \frac{\sqrt{-1}}{12\pi} \int\limits_{s_i} \frac{R_B - R_\Phi}{h}\, dz \quad \text{for } i = 1, \dots, 4g-3, \tag{2.4}$$

where $\Phi(z) = h(z)\, dz$ for a local coordinate z. Here R_Φ is the projective connection given by the Schwarzian derivative $\mathcal{S}(\int^z \Phi)$ with respect to a local coordinate z. For a holomorphic function f, the Schwarzian derivative $\mathcal{S}(f)$ is defined by

$$\mathcal{S}(f) = \left(\frac{f_{zz}}{f_z}\right)_z - \frac{1}{2}\left(\frac{f_{zz}}{f_z}\right)^2.$$

It is shown in [12] that τ_B does not depend on the choice of the l_j, and that τ_B^{24} is a globally well-defined function on $\hat{\mathcal{H}}_g$.

Originally the Bergman tau function was introduced over the space $\hat{H}_{g,n}$ in [9,11]. The space $\hat{H}_{g,n}$ is the space of equivalence classes of $[\lambda : X \to \mathbb{CP}^1]$ where X is a compact Riemann surface with a Torelli marking and λ is an n-fold branched covering with only simple ramification points. In this case, $d\lambda$ plays the role of Φ in (2.4). Over $\hat{H}_{g,n}$, the Bergman tau function τ_B satisfies the equality $\tau_B^{-2} = \tau_I$ where τ_I denotes the isomonodromic tau function of Dubrovin (see [10]). We also refer to (6.14) for the holomorphic factorization formula of the regularized determinant of the flat Laplacian defined by Φ, which is given in terms of $|\tau_B|^2$ over $\tilde{\mathcal{H}}_g$.

Finally we introduce covering space $\tilde{\mathcal{H}}_g$ of $\hat{\mathcal{H}}_g$ by marking an ordered set of generators $\{a_i, b_i \mid 1 \le i \le g\}$ of $\pi_1(X)$ rather than of $H_1(X, \mathbb{Z})$. There is a canonical map from $\tilde{\mathcal{H}}_g$ to the Teichmüller space $\mathfrak{T}_g$ of marked Riemann surfaces of genus g. Note that τ_B^{24} can be lifted to $\tilde{\mathcal{H}}_g$.

2.2. Schottky groups, Schottky spaces, and Zograf F-functions

Given a compact Riemann surface X of genus $g \ge 1$, there exists a Schottky uniformization of X, described as follows. A subgroup Γ of $PSL_2(\mathbb{C})$ is called a *Schottky group* if it is generated by $L_1, \dots, L_g$ satisfying the following condition: there exist $2g$ smooth Jordan curves C_r, $r = \pm 1, \dots, \pm g$, which form the oriented boundary of a domain $D \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ such that $L_r C_r = -C_{-r}$, $r = 1, \dots, g$ where $PSL_2(\mathbb{C})$ acts on $\hat{\mathbb{C}}$ in the usual way and the negative signs indicate opposite orientation. Any Schottky group gives a compact Riemann surface $X = \Gamma \backslash \Omega$ where $\Omega = \bigcup_{\gamma \in \Gamma} \gamma D$ is the set of discontinuity of the action of Γ on $\hat{\mathbb{C}}$, and every compact Riemann surface arises in this way. A Schottky group is *marked* if it is equipped with a particular choice of ordered set of free generators $L_1, \dots, L_g$. If the Riemann surface X is marked, then requiring the $b_1, \dots, b_g \in \pi_1(X)$ to map to $L_1, \dots, L_g$ fixes the marked Schottky group up to overall conjugation in $PSL_2(\mathbb{C})$.

We define a *Schottky 3-manifold* to be a smooth 3-manifold with boundary that is topologically a closed solid 3-dimensional handlebody $\overline{M} := M \cup X$, where M is the corresponding open handlebody, and the boundary X is a compact smooth 2-dimensional surface. We call a Schottky 3-manifold *hyperbolic* if it is equipped with a complete hyperbolic metric g_M on M, and we call it *marked* if it is equipped with an ordered choice of generators of $\pi_1(M)$.

Any compact Riemann surface X with a uniformization by a marked Schottky group Γ gives a marked Schottky hyperbolic 3-manifold $M \cup X$ in the following way: The action of Γ on $\hat{\mathbb{C}} = \partial \overline{H^3}$ canonically extends to the action on H^3 where $\overline{H^3} := H^3 \cup \hat{\mathbb{C}}$. Then $M = \Gamma \backslash H^3$ with $X = \Gamma \backslash \Omega$ has the inherited topology on from $\overline{H^3}$. The choice of the ordered set of generators $L_1, \ldots, L_g$ gives the marking on $\pi_1(M)$, by identifying elements of Γ with deck transformations of the universal cover of $\overline{M}$. Conversely, by means of the developing map, every marked Schottky hyperbolic 3-manifold M arises from a marked Schottky group in this way, and the group is unique up to an overall conjugation in $PSL_2(\mathbb{C})$. When a marked Schottky group Γ and a marked Schottky hyperbolic 3-manifold $M \cup X$ correspond in this way, we will say that the group Γ *uniformizes* the manifold $\overline{M} = M \cup X$.

In summary, given a compact marked Riemann surface X, we obtain a unique marked Schottky hyperbolic 3-manifold $M \cup X$ whose conformal boundary is X. We will sometimes write $M = M_X$ if we want to emphasize that the manifold M is determined by the marked surface X.

For a fixed g, the *Schottky space of genus* g, denoted by $\mathfrak{S}_g$, is the set of all marked Schottky groups with g generators, modulo overall conjugation in $PSL_2(\mathbb{C})$. It is known that $\mathfrak{S}_g$ has a canonical complex manifold structure of dimension $3g-3$, and its universal cover is the Teichmüller space $\mathfrak{T}_g$, with the covering map being holomorphic. The generators L_i, $i = 1, \ldots, g$, are holomorphic maps from $\mathfrak{S}_g$ to $PSL_2(\mathbb{C})$. In view of the uniformization discussed above, we implicitly identify $\mathfrak{S}_g$ with the deformation space of marked Schottky hyperbolic 3-manifolds.

In [22], Zograf defines a holomorphic function F on $\mathfrak{S}_g$ which can be characterized by the following equality

$$\partial F = \frac{1}{6\pi}(R_B - R_S). \tag{2.5}$$

Here ∂ denotes the $(1,0)$ component of the deRham differential d on $\mathfrak{S}_g$ and R_S is a projective connection defined by $\mathcal{S}(\pi_S^{-1})$ with the Schottky uniformization $\pi_S : \Omega \to X$. The function F was discovered as one of the ingredients for the holomorphic factorization formula of the regularized determinant of the hyperbolic Laplacian which also involves the classical Liouville action. We refer to (6.13) for a brief explanation of this equality and [22,17] for more details.

The function F was shown in [23] (see also [17]) to have an infinite product expansion on a subset of $\mathfrak{S}_g$:

$$F = \prod_{\{\gamma\}} \prod_{m=1}^{\infty} (1 - q_\gamma^m). \tag{2.6}$$

Here the first product runs over all primitive closed geodesics γ in M_X, and the complex number q_γ has modulus $e^{-\mathrm{length}(\gamma)}$ and argument given by the holonomy around γ in an orthogonal plane. Eq. (2.6) is valid whenever the exponent of convergence δ of Γ is strictly less than 1. Recalling the chain of coverings $\tilde{\mathcal{H}}_g \to \mathfrak{T}_g \to \mathfrak{S}_g$, note that F can be lifted to the space $\tilde{\mathcal{H}}_g$.

3. Framings over Schottky hyperbolic 3-manifolds

In this section, $\overline{M} = M \cup X$ will denote a marked Schottky hyperbolic 3-manifold with conformal boundary X. We define what we mean by a "singular framing" over M or over X, and we define a class of "admissible" singular framings which we will use to define the Chern–Simons invariant. We then describe how to assign, to each holomorphic 1-form Φ on X with only simple zeroes, an admissible singular framing on X. Finally, we prove that an admissible singular framing on X "extends" (in a sense to be defined below) to an admissible singular framing on M.

3.1. Admissible singular framings

Let $F(M)$ denote the $SO(3)$ frame bundle with the projection map $p : F(M) \to M$. For a subset $U \subset M$, by a *framing over* U we mean a section of $F(M)$ over U.

Let $\mathcal{L}$ denote a union of disjoint simple curves in M. A framing over $\mathcal{L}$ in M, written as $(e_1(y), e_2(y), e_3(y)) \in T_yM \oplus T_yM \oplus T_yM$ for each $y \in \mathcal{L}$, is called a *reference framing on* $\mathcal{L}$, if $e_1(y)$ is tangent to $\mathcal{L}$ at each $y \in \mathcal{L}$.

Let $\mathcal{N}^\epsilon(\mathcal{L})$ be an ϵ-neighborhood of $\mathcal{L}$ in the metric g_M. A choice of reference framing κ over $\mathcal{L}$ allows us to construct the *deleted ϵ-tube around* $\mathcal{L}$, which by definition we take to be a map

$$\alpha : (0, \epsilon) \times \mathcal{L} \times S^1 \to \big(\mathcal{N}^\epsilon(\mathcal{L})\big) \subset M,$$

constructed as follows: for each $(\rho, y, v) \in (0, \epsilon) \times \mathcal{L} \times S^1$, we take the unique geodesic starting at y with initial vector $\cos(v)e_2(y) + \sin(v)e_3(y)$, and travel a distance ρ from y to the point $\alpha(\rho, y, v)$.

Given a reference framing κ on $\mathcal{L}$, we define the corresponding *reference framing of the deleted ϵ-tube around* $\mathcal{L}$ by parallel translating the reference framing κ along the unique geodesic connecting y and $\alpha(\rho, y, v)$. This gives a lifting

$$\tilde{\alpha} : (0, \epsilon) \times \mathcal{L} \times S^1 \to p^{-1}\big(\mathcal{N}^\epsilon(\mathcal{L})\big) \subset F(M)$$

of the map α. The *standard cylinder over* $\mathcal{L}$ is the map

$$\psi : \mathcal{L} \times S^1 \to p^{-1}(\mathcal{L}) \subset F(M)$$

which takes the point $(y, v) \in \mathcal{L} \times S^1$ to the framing

$$\psi(y,v) := \big(e_1(y), \cos(v)e_2(y) + \sin(v)e_3(y), -\sin(v)e_2(y) + \cos(v)e_3(y)\big)$$

at the point y.

A matrix function

$$A : (0,\epsilon) \times \mathcal{L} \times S^1 \to SO(3)$$

acts on a framing $\tilde{\alpha}$ of the deleted ϵ-tube around $\mathcal{L}$ by fiberwise right multiplication:

$$(e_1, e_2, e_3) \cdot A(\rho, y, v) = \left(\sum_{i=1}^{3} e_i a_{i1}, \sum_{i=1}^{3} e_i a_{i2}, \sum_{i=1}^{3} e_i a_{i3} \right),$$

over a point $\alpha(\rho, y, v)$ where a_{ij} denotes (i,j)-entry of $A(\rho, y, v)$. We denote the resulting framing by $\tilde{\alpha} \cdot A$. A matrix function $A : \mathcal{L} \times S^1 \to SO(3)$ acts on the standard cylinder ψ to give $\psi \cdot A$ in the same fashion.

For a connected simple curve $\ell \subset M$, the *special singularity of index n at ℓ* is the framing $\tilde{\alpha} \cdot A_n$ over the deleted ϵ-tube around ℓ, where $\tilde{\alpha}$ is the reference framing on the deleted ϵ-tube around ℓ, and A_n is the matrix function on $(0,\epsilon) \times \ell \times S^1$ defined by

$$A_n(\rho, y, v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(nv) & -\sin(nv) \\ 0 & \sin(nv) & \cos(nv) \end{pmatrix}.$$

For fixed $y \in \ell$ and $v \in S^1$, the limit of $\tilde{\alpha} \cdot A_n$ as $\rho \to 0$ exists, and equals the framing $(e_1(y), \cos(nv)e_2(y) + \sin(nv)e_3(y), -\sin(nv)e_2(y) + \cos(nv)e_3(y))$ over y. Hence the map consisting of these limits as $\rho \to 0$ for all $y \in \ell$ and $v \in S^1$ is given by n-copies of the standard cylinder over ℓ. Here a negative integer n indicates opposite orientation. For $\mathcal{L}$ a disjoint union of simple curves, we say that a framing $\mathcal{F}$ over $M \setminus \mathcal{L}$ has a *special singularity at $\mathcal{L}$* if $\mathcal{F} \circ \alpha$ has the special singularity of index n for an integer n on each connected component of $(0,\epsilon) \times \mathcal{L} \times S^1$. Let us remark that n could be different over each component of $\mathcal{L}$. Our definition of special singularity coincides with Meyerhoff's [18] when $n = 1$.

For a connected simple curve $\ell \subset M$, the *admissible singularity of index n at ℓ* is the special singularity framing of index n at ℓ, acted on by a matrix function A:

$$\tilde{\alpha} \cdot A_n \cdot A : (0,\epsilon) \times \ell \times S^1 \to p^{-1}\big(\mathcal{N}^\epsilon(\ell)\big) \subset F(M), \tag{3.1}$$

where $A : (0,\epsilon) \times \ell \times S^1 \to SO(3)$ satisfies the condition that $\lim_{\rho \to 0} A(\rho, y, v)$ exists and is independent of v, for all $y \in \ell$ and $v \in S^1$. We say that a framing $\mathcal{F}$ over $M \setminus \mathcal{L}$ has an *admissible singularity at $\mathcal{L}$* if the limit of $\mathcal{F} \circ \alpha$ as $\rho \to 0$ exists for all $y \in \mathcal{L}$ and $v \in S^1$ and the map given by this limit is the same as the map given by the limit of $\tilde{\alpha} \cdot A_n \cdot A$ as $\rho \to 0$, that is, n-copies of the standard cylinder acted by A over each connected component of $\mathcal{L}$.

3.2. Standard admissible framings

Every Schottky hyperbolic 3-manifold is conformally compact: in some neighborhood $N \subset \overline{M}$ of X, there exists a smooth boundary defining function $r : N \to \mathbb{R}_{\geq 0}$ such that

i) $r > 0$ on $N \cap M$, $r = 0$ on X, and $dr = 0$ restricted to X,
ii) the rescaled metric $\bar{g} := r^2 g_M$ extends smoothly to $N \cap \overline{M}$,
iii) $|dr|^2_{\bar{g}} = 1$ in N.

We also write $\bar{g}$ for the extension of the metric $\bar{g}$ to $N \cap \overline{M}$. The conformal class of the metric $\bar{g}|_{TX}$ is independent of the choice of boundary defining function; hence the choice of a metric g_M induces a unique conformal class of metrics on the conformal boundary X. For genus $g > 1$, in each conformal class of metrics on X, there is a unique hyperbolic metric g_X of constant curvature -1. For genus $g = 1$, in each conformal class of metrics on X there is a unique flat metric g_X in which $\mathrm{Area}(X) = 1$. We will need a parametrization of a neighborhood $N \subset \overline{M}$ of the conformal boundary X. If we demand that $\bar{g}|_{TX}$ is equal to the metric g_X, then the boundary defining function satisfying the conditions above is unique. For a sufficiently small $a > 0$, this defining function r determines an identification of $X \times [0, a)$ with a subneighborhood $N_{[0,a)} \subset N$, by letting $(p, t) \in X \times [0, a)$ correspond to the point obtained by following the integral curve ϕ_t of $\nabla_{\bar{g}} r$ emanating from p for t units of time. Throughout the rest of the paper, we will fix such an a. For this defining function r, the t-coordinate is just r and $\nabla_{\bar{g}} r$ is orthogonal to the slices $X \times \{t\}$. Hence identifying t with r on $X \times [0, a)$, the hyperbolic metric g_M over M has the form

$$g_M = r^{-2}\bigl(g_r + dr^2\bigr) \tag{3.2}$$

over $N_{[0,a)}$, where g_r denotes a Riemannian metric over $X^r := X \times \{r\}$. See [6] for more details.

An *admissible singular framing* $(\mathcal{F}, \kappa, \mathcal{L})$ over M consists of a union of disjoint simple curves $\mathcal{L}$ in M, a reference framing κ over $\mathcal{L}$, and a framing $\mathcal{F}$ over $M \setminus \mathcal{L}$, satisfying

i) the closure $\overline{\mathcal{L}}$ is smooth in $\overline{M}$, and $\overline{\mathcal{L}}$ is orthogonal to X in $\bar{g}$ at the intersection,
ii) the framings $r^{-1}\mathcal{F}$ and $r^{-1}\kappa$ extend smoothly to $\overline{M} \setminus \overline{\mathcal{L}}$ and $\overline{\mathcal{L}}$ respectively,
iii) the first vector e_1 of $\mathcal{F}$ is tangent to the gradient flow curves of r over $N_{(0,\epsilon)} \setminus \mathcal{L}$ for $0 < \epsilon < a$, and
iv) the framing $\mathcal{F}$ has an admissible singularity at $\mathcal{L}$.

Let $\ell_1, \ldots, \ell_g$ be closed curves in M representing the marked generators of $\pi_1(M)$, with the property that there exist discs $D_1, \ldots, D_{g-1}$ such that $M \setminus \bigcup D_i$ is the disjoint union of g solid tori $\ell_i \times D$, where D is the unit disc. Given an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$, define $\mathcal{L}^1$ to be the disjoint union of connected components of $\mathcal{L}$ that are closed, and define $\mathcal{L}^2 := \mathcal{L} \setminus \mathcal{L}^1$. Then $(\mathcal{F}, \kappa, \mathcal{L})$ will be called *standard* if

i) $\mathcal{F}$ has a special singularity of index 1 at each curve in $\mathcal{L}^1$ where the set $\mathcal{L}^1$ is a subset of $\bigsqcup_{i=1}^g \ell_i$ and
ii) the index of the admissible singularity of $\mathcal{F}$ at each curve in $\mathcal{L}^2$ is -1.

3.3. Admissible singular framings associated to holomorphic 1-forms

We define an admissible singular framing on a surface X with a metric g_X. Let Z consist of finitely many points in X. A reference framing on Z is a choice of a frame (e_2, e_3) at each point $z \in Z$, orthonormal with respect to the metric g_X. A reference framing on Z defines a geodesic polar coordinate $\alpha : (0, \epsilon) \times Z \times S^1 \to \mathcal{N}^\epsilon(Z) \setminus Z$ which takes (ρ, z, v) to the point at distance ρ from $z \in Z$ along the geodesic with initial vector $\cos(v)e_2(y) + \sin(v)e_3(y)$. Parallel translation gives a corresponding reference framing $\tilde{\alpha}$ over $(0, \epsilon) \times Z \times S^1$. The special singularity of index n at $z \in Z$ is the framing $\tilde{\alpha} \cdot A_n$ on $(0, \epsilon) \times \{z\} \times S^1$ where $\tilde{\alpha}$ denotes the reference framing and A_n is the matrix function given by

$$A_n(\rho, v) = \begin{pmatrix} \cos(nv) & -\sin(nv) \\ \sin(nv) & \cos(nv) \end{pmatrix}.$$

An admissible singularity of index n at z is the special singularity, right-multiplied by a matrix function $A(\rho, z, v)$ with the property that $\lim_{\rho \to 0} A(\rho, z, v)$ exists and is independent of v. An admissible singular framing $(\mathcal{F}, \kappa, Z)$ on X consists of a finite set Z in X, a reference framing on Z, and a framing $\mathcal{F}$ of $X \setminus Z$ such that the limit of $\mathcal{F}$ as $\rho \to 0$ exists for all $v \in S^1$ and the map given by this limit is the same as the map given by the limit of an admissible singularity at each point of Z.

Suppose that X is a Riemann surface, with a metric g_X compatible with its complex structure. We now describe how to assign, to a holomorphic 1-form Φ with only simple zeroes, an admissible singular framing with index -1 singular points at the zeroes of Φ.

The metric g_X is a collection $\{e^{\phi_\alpha}|dz_\alpha|^2\}_{\alpha \in A}$ on an atlas $\{(U_\alpha, z_\alpha)\}_{\alpha \in A}$ of X for which the functions $\phi_\alpha \in C^\infty(U_\alpha, \mathbb{R})$ satisfy

$$\phi_\alpha + \log\left|f'_{\alpha\beta}(z_\beta)\right|^2 = \phi_\beta \quad \text{on } U_\alpha \cap U_\beta, \tag{3.3}$$

where $f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$ are the holomorphic transition functions. A holomorphic 1-form Φ on X is a collection $\{h_\alpha dz_\alpha\}$ for the atlas $\{(U_\alpha, z_\alpha)\}$ for which h_α is a holomorphic function on U_α satisfying

$$h_\alpha f'_{\alpha\beta}(z_\beta) = h_\beta \quad \text{on } U_\alpha \cap U_\beta. \tag{3.4}$$

The phase function $e^{i\theta_\alpha} := h_\alpha/|h_\alpha|$ is well-defined over $X \setminus Z$ where Z denotes the zero set of Φ. The transformation law (3.4) implies

$$i\theta_\alpha + \log \frac{f'_{\alpha\beta}(z_\beta)}{|f'_{\alpha\beta}(z_\beta)|} = i\theta_\beta \quad \text{on } U_\alpha \cap U_\beta. \tag{3.5}$$

Note that θ_α is defined only up to an integer multiple of 2π. By (3.3), (3.5), it follows that $e^{\phi_\alpha/2+i\theta_\alpha}dz_\alpha$ defines an orthonormal co-framing ω_2, ω_3 given by

$$\omega_{2\alpha} = e^{\phi_\alpha/2}(\cos\theta_\alpha dx_\alpha - \sin\theta_\alpha dy_\alpha), \qquad \omega_{3\alpha} = e^{\phi_\alpha/2}(\sin\theta_\alpha dx_\alpha + \cos\theta_\alpha dy_\alpha)$$

on $U_\alpha \setminus Z$ where $z_\alpha = x_\alpha + iy_\alpha$. Now we obtain an orthonormal framing

$$\mathcal{F}_\Phi = (f_2, f_3) \quad \text{where } f_2 = \omega_2^*, f_3 = \omega_3^*$$

over $X \setminus Z$, which has admissible singularities at Z of index -1.

For the singular part Z, let $z_{i\alpha}$ denote the co-ordinate of a zero of Φ in a patch U_α. Then h_α has an expression $h_\alpha = (z_\alpha - z_{i\alpha})\tilde{h}_{i\alpha}$, where $\tilde{h}_{i\alpha}$ is non-vanishing at the zero. Now we put $e^{i\tilde{\theta}_{i,\alpha}} := \tilde{h}_{i,\alpha}/|\tilde{h}_{i,\alpha}|$. Since $\tilde{h}_{i\alpha}$ is non-vanishing at the zero, $\tilde{\theta}_{i\alpha}$ is well-defined at the zero up to an integer multiple of 2π. By (3.3), (3.4), it follows that $e^{\frac{1}{2}(\phi_\alpha+i\tilde{\theta}_{i,\alpha})}dz_\alpha$ defines the following orthonormal co-framing at the zero,

$$\begin{aligned} \tilde{\omega}_{2\alpha} &= e^{\phi_\alpha/2}\big(\cos(\tilde{\theta}_\alpha/2)dx_\alpha - \sin(\tilde{\theta}_\alpha/2)dy_\alpha\big), \\ \tilde{\omega}_{3\alpha} &= e^{\phi_\alpha/2}\big(\sin(\tilde{\theta}_\alpha/2)dx_\alpha + \cos(\tilde{\theta}_\alpha/2)dy_\alpha\big), \end{aligned} \tag{3.6}$$

and the corresponding orthonormal framing $(\tilde{f}_2, \tilde{f}_3)$ at the zero. By the transformation law for $\tilde{h}$, this orthonormal framing transforms correctly under change of coordinate. Note however that this co-frame and frame are well-defined only up to sign.

We select $g-1$ of the points in Z to have the framing $(\tilde{f}_2, \tilde{f}_3)$, and let the other $g-1$ points in Z have the framing $(\tilde{f}_2, -\tilde{f}_3)$; we denote the resulting framing at Z by κ_Φ. When we extend the framing $\mathcal{F}_\Phi$ to M, these will correspond to "outgoing" and "incoming" endpoints of curves in M respectively.

3.4. Existence of admissible extensions

On a subset of X, we can identify any $SO(2)$ framing with respect to g_X with an $SO(3)$ framing with respect to $\bar{g}$, by taking each framing (f_2, f_3) to the framing (f_1, f_2, f_3), where f_1 is the inward unit normal vector to X with respect to $\bar{g}$. We say that an admissible singular framing $(\mathcal{F}_X, \kappa_X, Z)$ has an *admissible extension* to M if there exists an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$ over M such that $\partial\overline{\mathcal{L}} = Z$, and such that the extension of $r^{-1}\mathcal{F}$ and $r^{-1}\kappa$ equals the given framing $\mathcal{F}_X$ and κ_X, respectively, under the identification above.

Now, our goal is to show that, for a holomorphic 1-form Φ with only simple zeroes on X, the associated admissible singular framing $(\mathcal{F}_\Phi, \kappa_\Phi, Z)$ on X extends to an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$ on M. (A similar proof shows that any admissible singular framing on X extends to M.)

Before proving the existence of such an admissible extension, we establish two lemmas.

Lemma 3.1. *Suppose $\overline{W} = W \cup \partial W$ is a marked smooth 3-dimensional closed handlebody of genus p with metric $g_{\overline{W}}$, and suppose that $\mathcal{F}_{\partial W}$ is a smooth (non-singular) $SO(3)$ framing of ∂W. Then there exists an admissible extension of $\mathcal{F}_{\partial W}$ to W which has a special singularity of index 1 at $\mathcal{L}^1$. Its set of singular curves $\mathcal{L}^1$ may be taken to consist of at most p closed curves, each representing a distinct marked generator of $\pi_1(W)$.*

Proof. There exists a smooth embedding of W into $\mathbb{R}^3$, which gives a global framing $\mathcal{F}_0$ on W, by which we can identify any other framing on W with a map to $SO(3)$. Let $\mathcal{L}^0$ be the union of p closed simple curves representing the marked generators of $\pi_1(W)$. Given a connected curve ℓ in $\mathcal{L}^0$, there exists a disc D in W such that $W \setminus D$ is the disjoint union of a handlebody of genus $p-1$ and a solid torus T satisfying $T \cap \mathcal{L}^0 = \ell$ and $\partial T \simeq \ell \times S^1$. Since ∂D is homologically trivial in ∂W, it is a commutator in $\pi_1(\partial W)$ and so its image in $SO(3)$ under the framing $\mathcal{F}_{\partial W}$ is homotopically trivial. Hence $\mathcal{F}_{\partial W}$ can be smoothly extended to $D \subset \partial(W \setminus D)$. In this way the problem reduces to finding a framing on each solid torus T. If $\pi_1(T)$ is represented by ℓ, identify ∂T with $\ell \times S^1$. The image of this S^1 in $SO(3)$ given by $\mathcal{F}_{\partial W}$ is either homotopically trivial, in which case the framing extends smoothly to all of T, or it is homotopically nontrivial, in which case the framing has the same homotopy type as a special singularity framing of index 1 around ℓ and can thus be extended to a framing on $T \setminus \ell$ with this singularity. □

We will apply Lemma 3.1 to a handlebody of genus $p = 2g-1$ made by removing some parts of M in the proof of Theorem 3.3. From now on, we put $a_1 = \frac{a}{4}$ for simplicity, where a neighborhood of the conformal boundary X in $\overline{M}$ can be identified with $X \times [0, a)$ (see Section 3.2).

Lemma 3.2. *Let $\overline{M} = M \cup X$ be a marked Schottky hyperbolic 3-manifold, and let $a > 0$ be such that the neighborhood $N_{[0,a]} \subset \overline{M}$ of X exists. Let Φ be a holomorphic 1-form with only simple zeroes on X and $(\mathcal{F}_\Phi, \kappa_\Phi, Z)$ be the associated admissible singular framing as defined above. Then $(\mathcal{F}_\Phi, \kappa_\Phi, Z)$ has an admissible extension to $N_{(0,a_1]}$.*

Proof. If Z is the singular set of the framing $\mathcal{F}_\Phi$ on X, then we can take the set of singular curves to be the g_M geodesics given by $\mathcal{L} = \{\phi_r(x) : x \in Z, r \in (0, a_1]\}$. Given an admissible singular framing $\mathcal{F}_\Phi = (f_1, f_2, f_3)$ over $X \setminus Z$ with respect to $\bar{g} = r^2 g_M$, one can find an admissible singular framing $\mathcal{F} = (e_1, e_2, e_3)$ with respect to g_M that is parallel near infinity and extends $\mathcal{F}_\Phi$, by rewriting the parallel transport equation for e_i with respect to g_M in terms of b_i, where $e_i(r) = rb_i(r) = r(b_i^1(r)\frac{\partial}{\partial t} + b_i^2(r)\frac{\partial}{\partial x} + b_i^3(r)\frac{\partial}{\partial y})$. The parallel transport equation along the gradient flow curve ϕ_r becomes

$$r\dot{b_i^m}(r) + b_i^m(r) + r\sum_{j,k} \Gamma_{j,k}^m(\phi_r)\dot{\phi_r^j} b_i^k(r) = 0,$$

and we use the solution, with initial conditions $b_i(0) = f_i$, to define e_i. We extend the reference framing on $\mathcal{L}$ in the same manner, using the reference framing on Z as the initial condition. □

Theorem 3.3. *If $\overline{M} = M \cup X$ is a marked Schottky hyperbolic 3-manifold and Φ is a holomorphic 1-form with only simple zeroes on X, then the associated admissible singular framing $(\mathcal{F}_\Phi, \kappa_\Phi, Z)$ on X extends to an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$ on M. The framing $(\mathcal{F}, \kappa, \mathcal{L})$ can be taken to be standard.*

Proof. We begin by defining the $\mathcal{L}^2$ part of the singular curve of $\mathcal{F}$. In Lemma 3.2, the $\mathcal{L}^2$ part in $N_{(0,\frac{a}{4}]}$ is defined to be the gradient flow curves. Now we extend them by taking pairs of two ends in X^{a_1} of those curves and making curves to connect them smoothly within $N_{(0,a)}$. We may assume that each connected curve ℓ_i, $i = 1, \ldots, g-1$ in $\mathcal{L}^2$ meets level surface X^ϵ at two points for $a_1 \leq \epsilon < \frac{a}{2}$ and at one point for $\epsilon = \frac{a}{2}$. By construction, the end points of $\mathcal{L}^2$ are given by the zero set $Z = \{p_1, \ldots, p_{2g-2}\}$ of Φ. As we mentioned in the end of Section 3.3, we may assume that if the reference framing is taken to be $(\tilde{f}_2, \tilde{f}_3)$ on one end of ℓ_i, then the reference framing is taken to be $(\tilde{f}_2, -\tilde{f}_3)$ on the other end of ℓ_i.

Let us choose a reference framing κ^2 on $\mathcal{L}^2$ which extends $(\tilde{f}_2, \tilde{f}_3)$ and $(\tilde{f}_2, -\tilde{f}_3)$ at each end point respectively, and which satisfies the parallel condition over $\mathcal{L}^2 \cap N_{(0,a_1]}$. We also let $\mathcal{F}$ be the admissible extension of $\mathcal{F}_\Phi$ on the set $N_{(0,\frac{a}{4}]}$ guaranteed to exist by Lemma 3.2. Note that $\mathcal{F}$ has an admissible singularity of index -1 at $\mathcal{L}^2 \cap N_{(0,a_1]}$ by definition.

Now we define $\mathcal{F}$ over $\mathcal{N}^\epsilon(\mathcal{L}^2) \cap N_{[a_1,a)}$ so that $\mathcal{F}$ has an admissible singularity of index -1 at $\mathcal{L}^2 \cap N_{[a_1,a)}$. Let β_i be a diffeomorphism from $\bar{\ell}_i \subset \overline{M}$ to $[-1,1]$ which maps the end with the reference framing $(\tilde{f}_2, \tilde{f}_3)$ to -1 and the end with the reference framing $(\tilde{f}_2, -\tilde{f}_3)$ to 1, and maps $\ell_i \cap N_{[a_1,a)}$ to $[-\frac{1}{2}, \frac{1}{2}]$. Let ξ be a smooth increasing function on the interval $[-1,1]$ whose derivative is supported in $(-\frac{1}{3}, \frac{1}{3})$ whose values are 0 on $[-1, -\frac{1}{3}]$ and π on $[\frac{1}{3}, 1]$. We define $\chi : \overline{\mathcal{L}^2} \to [0,\pi]$ by the composition of ξ and β_i over ℓ_i and let

$$A(\rho, v, y) = \begin{pmatrix} \cos\chi(y) & 0 & -\sin\chi(y) \\ 0 & 1 & 0 \\ \sin\chi(y) & 0 & \cos\chi(y) \end{pmatrix} \quad \text{on } (0,\epsilon) \times \left(\mathcal{L}^2 \cap N_{[\frac{a}{3},a)}\right) \times S^1 \tag{3.7}$$

and A over $(0,\epsilon) \times (\mathcal{L}^2 \cap N_{[a_1,\frac{a}{3}]}) \times S^1$ is defined to connect the above matrix in (3.7) and the matrix A determining the admissible framing $\mathcal{F}$ over $\mathcal{N}^\epsilon(\mathcal{L}^2) \cap X^{a_1}$. We may assume that $\lim_{\rho\to 0} A(\rho, v, y)$ exists and is independent of v, for all $y \in \mathcal{L}^2$ and $v \in S^1$. Then, for the reference framing $\tilde{\alpha}$ of the deleted ϵ-tube around $\mathcal{L}$ obtained from κ^2, we define $\mathcal{F}$ by the equality $\mathcal{F} \circ \alpha = \tilde{\alpha} \cdot A_{-1} \cdot A$ over $\mathcal{N}^\epsilon(\mathcal{L}^2) \cap N_{[a_1,a)}$, which extends the previously constructed framing $\mathcal{F}$ over $N_{(0,a_1]}$. Note that this extension of $\mathcal{F}$ is independent of the choice of a reference framing κ^2 on $\mathcal{L}^2$ satisfying the conditions above. In particular, the extension of $\mathcal{F}$ does not depend on the choice of signs in κ_Φ. By definition, this framing $\mathcal{F}$ has an admissible singularity of index -1 at $\mathcal{L}^2 \cap N_{[a_1,a)}$.

So far an admissible framing $\mathcal{F}$ has been constructed over $N_{(0,a_1]} \cup \mathcal{N}^\epsilon(\mathcal{L}^2)$. Now we extend it over $M \setminus (\mathcal{L}^1 \cup \mathcal{L}^2)$ by appropriately choosing $\mathcal{L}^1$. First let W_0 denote the closure of $M^{a_1} \setminus \mathcal{N}^\epsilon(\mathcal{L}^2)$ where $M^{a_1} = M \setminus N_{(0,a_1)}$. Then there is a homotopy which deforms W_0 to a closed handlebody W_1 of genus $2g-1$. Given a set of generators of $\pi_1(M) \simeq \pi_1(M^{a_1})$, there exist $(g-1)$-closed discs $D_i \subset W_1$, $i = 1, \dots, g-1$ such that these decompose W_1 into one handlebody of genus g and solid tori T_i, $i = 1, \dots, g-1$ satisfying the following conditions: the decomposed handlebody of genus g contains the homotopic images of loops realizing the given generators of $\pi_1(M^{\frac{a}{4}})$. For a generator $\tilde{\gamma}_i$ of $\pi_1(T_i)$, there is a closed curve γ_i in W_0 given by the (inverse) homotopic image of the loop realizing $\tilde{\gamma}_i$. By this construction, the set G of generators of $\pi_1(W_0)$ is given by the union of the chosen generators of $\pi_1(M^{a_1})$ by marking and the set of $\gamma_1, \dots, \gamma_{g-1}$.

Applying Lemma 3.1 for the framing defined as above over the boundary of the closure of W_0, we obtain an admissible extension of $(\mathcal{F}_\Phi, \kappa_\Phi, Z)$. To show that we can take it to be standard, we have to modify the construction so that $\mathcal{L}^1$ consists of representatives of the marked generators of $\pi_1(M)$. Suppose that $\mathcal{L}^1$ contains a representative of a generator γ_i. Then we may replace the reference framing $\tilde{\alpha}$ with another framing with an additional rotation 2π along the corresponding part of $\mathcal{L}^2$. This will change the homotopy type of the admissible singular framing $\mathcal{F}$ along it since $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$. Hence it can be extended over the subset of W_0 corresponding to T_i without removing a curve representing γ_i. This means $\mathcal{L}^1$ can be taken to represent a subset of the given generators of $\pi_1(M)$. Then this completes the proof. $\square$

4. Definition of the invariant $\mathbb{CS}$

4.1. The form C on $PSL_2(\mathbb{C})$

If H^3 is the hyperbolic space of dimension 3, the frame bundle $F(H^3)$ can be identified with $PSL_2(\mathbb{C})$ canonically. Let

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\{h, e, f\}$ form a base of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of $PSL_2(\mathbb{C})$. Let $\{h^*_{\mathbb{C}}, e^*_{\mathbb{C}}, f^*_{\mathbb{C}}\}$ be its dual base of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{sl}_2(\mathbb{C}), \mathbb{C})$. In Section 3 in [21], Yoshida defines the form C as the left-invariant differential form on $PSL_2(\mathbb{C})$ whose value at the identity is given by $\frac{i}{\pi^2} h^*_{\mathbb{C}} \wedge e^*_{\mathbb{C}} \wedge f^*_{\mathbb{C}}$, and proves the following:

Proposition 4.1. *The form C on $PSL_2(\mathbb{C})$ is complex analytic, closed, and bi-invariant, and has the following expression*

$$C = \frac{1}{4\pi^2}\big(4\theta_1 \wedge \theta_2 \wedge \theta_3 - d(\theta_1 \wedge \theta_{23} + \theta_2 \wedge \theta_{31} + \theta_3 \wedge \theta_{12})\big)$$
$$+ \frac{i}{4\pi^2}(\theta_{12} \wedge \theta_{13} \wedge \theta_{23} - \theta_{12} \wedge \theta_1 \wedge \theta_2 - \theta_{13} \wedge \theta_1 \wedge \theta_3 - \theta_{23} \wedge \theta_2 \wedge \theta_3).$$

Here θ_i and θ_{ij} denote the fundamental form and the connection form respectively on $PSL_2(\mathbb{C})$ of the Riemannian connection of H^3.

Since H^3 has constant sectional curvature -1, $\Omega_{ij} = -\theta_i \wedge \theta_j$ for $i, j = 1, 2, 3$ where Ω_{ij} denotes the curvature form of the connection form θ_{ij}. Thus C is a complex analytic form on $PSL_2(\mathbb{C})$ whose real part, up to scalar multiplication, is the volume form plus an exact form, and whose imaginary part, up to scalar multiplication, is the Chern–Simons form defined in [5]. Using the equalities $d\theta_i = -\sum_j \theta_{ij} \wedge \theta_j$, $d\theta_{ij} = -\sum_k \theta_{ik} \wedge \theta_{kj} + \Omega_{ij}$, one can obtain

Proposition 4.2. *The form C on $PSL_2(\mathbb{C})$ has the following expressions*

$$C = -\frac{i}{4\pi^2}\eta \wedge d\eta$$
$$= -\frac{1}{4\pi^2}(d\theta_{23} \wedge \theta_1 + d\theta_1 \wedge \theta_{23}) + \frac{i}{4\pi^2}(d\theta_{23} \wedge \theta_{23} - d\theta_1 \wedge \theta_1),$$

where $\eta = \theta_1 - i\theta_{23}$.

For an oriented smooth hyperbolic manifold $M = \Gamma\backslash H^3$ of dimension 3, let $\tilde{M}$ be the universal cover of M and $d : \tilde{M} \to H^3$ be a developing map. Taking the differential of d, we obtain the $SO(3)$-bundle map $\tilde{d} : F(\tilde{M}) \to PSL_2(\mathbb{C})$. Since the form C is left invariant, $\tilde{d}^*C$ projects to a closed form on $F(M) = \Gamma\backslash F(\tilde{M})$ which by abuse of notation we denote also by C.

Now, for the rest of this section, suppose that M is a marked Schottky hyperbolic 3-manifold. For an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$ over M, we introduce a map

$$s : (M \setminus \mathcal{L}) \cup \mathcal{L} \to F(M) \tag{4.1}$$

defined by the admissible singular framing $\mathcal{F}$ over $M \setminus \mathcal{L}$ and the reference framing κ on $\mathcal{L}$. For $0 < \epsilon < a_1$, we now define

$$\mathbb{CS}^\epsilon(M, s) = \int\limits_{s(M^\epsilon \setminus \mathcal{L})} C - \sum_j \frac{n(j)}{2\pi} \int\limits_{s(\ell_j^\epsilon)} (\theta_1 - i\theta_{23}) \tag{4.2}$$

where $M^\epsilon := M \setminus N_{(0,\epsilon)}$, ℓ_j denotes a connected component of $\mathcal{L}$, and $\ell_j^\epsilon := \ell_j \cap M^\epsilon$. Here the sum is over the connected components ℓ_j of $\mathcal{L}$ and $n(j)$ is the index of the admissible singularity of $\mathcal{F}$ at ℓ_j. The complex valued invariant we define will be a suitably regularized value of $\mathbb{CS}^\epsilon(M, s)$ as $\epsilon \to 0$.

For a standard admissible framing $(\mathcal{F}, \kappa, \mathcal{L})$ over M, the singular curve $\mathcal{L}$ consists of two parts: $\mathcal{L}^1$ is a union of simple closed curves and $\mathcal{L}^2$ is a union of curves connecting two end points in $X = \partial\overline{M}$. Then the quantity defined in (4.2) is given by

$$\mathbb{CS}^\epsilon(M, s) = \int_{s(M^\epsilon \setminus \mathcal{L})} C - \frac{1}{2\pi} \int_{s(\mathcal{L}^1)} (\theta_1 - i\theta_{23}) + \frac{1}{2\pi} \int_{s(\mathcal{L}^{2,\epsilon})} (\theta_1 - i\theta_{23}) \tag{4.3}$$

where $\mathcal{L}^{2,\epsilon} := \mathcal{L}^2 \cap M^\epsilon$.

4.2. Boundaries of $\overline{s(M^\epsilon \setminus \mathcal{L})}$

For a standard admissible framing $(\mathcal{F}, \kappa, \mathcal{L})$ over M, we investigate the structure of the boundaries of $\overline{s(M^\epsilon \setminus \mathcal{L})}$ where the closure is taken in $F(M)$. The boundary $\partial(\overline{s(M^\epsilon \setminus \mathcal{L})})$ consists of three parts which we are going to describe below.

One part of the boundary $\partial(\overline{s(M^\epsilon \setminus \mathcal{L})})$ is given by the closure of $s(X^\epsilon \setminus \mathcal{L}^2)$ in $F(M)$, which we denote by $B^{0,\epsilon}$. Note that the boundary of $B^{0,\epsilon}$ consists of a disjoint union of circles.

The second part of the boundary $\partial(\overline{s(M^\epsilon \setminus \mathcal{L})})$ is given by $\bigcup_{y \in \mathcal{L}^1} \lim_{\delta \to 0} s(S_\delta(y))$, where $S_\delta(y)$ denotes the circle consisting of points in the orthogonal disc to $\mathcal{L}^1$ of distance δ from $y \in \mathcal{L}^1$. For $y \in \mathcal{L}^1$, the limit of $s(S_\delta(y))$ as $\delta \to 0$ exists since the framing $\mathcal{F}$ has a special singularity of index 1 at $\mathcal{L}^1$. We denote this part of boundary, which does not depend on ϵ, by B^1. Actually B^1 is given by the standard cylinder over $\mathcal{L}^1$: there is a map

$$\psi : \mathcal{L}^1 \times S^1 \to p^{-1}\big(\mathcal{L}^1\big) \subset F(M)$$

which takes the point $(y, v) \in \mathcal{L}^1 \times S^1$ to the framing

$$\psi(y, v) := \big(e_1(y), \cos(v)e_2(y) + \sin(v)e_3(y), -\sin(v)e_2(y) + \cos(v)e_3(y)\big) \tag{4.4}$$

at the point $y \in \mathcal{L}^1$. Here (e_1, e_2, e_3) is the reference framing κ^1 on $\mathcal{L}^1$. The boundary orientation of B^1 is induced from $\mathcal{F}$ and is given by $(\psi_* \frac{\partial}{\partial y}, \psi_* \frac{\partial}{\partial v})$ so that ψ is orientation-preserving.

The remaining part of boundary $\partial(\overline{s(M^\epsilon \setminus \mathcal{L})})$ is given by $\bigcup_{y \in \mathcal{L}^{2,\epsilon}} \lim_{\delta \to 0} s(S_\delta(y))$. For $y \in \mathcal{L}^2$, the limit of $s(S_\delta(y))$ as $\delta \to 0$ exists since the framing $\mathcal{F}$ has an admissible singularity of index -1 at $\mathcal{L}^2$. We denote this part by $B^{2,\epsilon}$. Note that $B^{2,\epsilon}$ has circle boundaries which are the boundaries of $B^{0,\epsilon}$ with the opposite orientation. As the case of B^1, $B^2 = \lim_{\epsilon \to 0} B^{2,\epsilon}$ can be described in terms of the standard cylinder over $\mathcal{L}^2$ with some modification. There is a map

$$\psi : \mathcal{L}^2 \times S^1 \to p^{-1}\big(\mathcal{L}^2\big) \subset F(M)$$

which takes the point $(y, v) \in \mathcal{L}^2 \times S^1$ to the framing given by

$$\psi(y,v) = \big(e_1(y), \cos(v)e_2(y) + \sin(v)e_3(y), -\sin(v)e_2(y) + \cos(v)e_3(y)\big), \tag{4.5}$$

where (e_1, e_2, e_3) is the reference framing κ^2 on $\mathcal{L}^2$. We denote by $\tilde{B}^2$ the image of ψ. We take the orientation of $\tilde{B}^2$ to be given by $(\psi_* \frac{\partial}{\partial y}, -\psi_* \frac{\partial}{\partial v})$, so that ψ is orientation-reversing by definition. The $\tilde{B}^2$ and B^2 do not coincide completely, but we can describe their difference explicitly:

Lemma 4.3. *The fiberwise right multiplication of A appearing in Eq. (3.1) induces an orientation preserving diffeomorphism $\mathcal{A}$ of $p^{-1}(\overline{\mathcal{N}^\epsilon(\mathcal{L}^2)}) \subset F(M)$ mapping $\tilde{B}^2$ to B^2 over $\mathcal{L}^2$.*

Proof. The claim follows directly from the definition of admissible singularity. □

4.3. Real part of $\mathbb{CS}^\epsilon(M,s)$

We start with

Lemma 4.4. *For s corresponding to an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$, the following equalities hold over $N_{(0,a_1)} \setminus \mathcal{L}^2$,*

$$\omega_{12} = II(e_2,e_2)\omega_2 + II(e_3,e_2)\omega_3, \qquad \omega_{13} = II(e_2,e_3)\omega_2 + II(e_3,e_3)\omega_3,$$

where $\omega_i = s^*\theta_i$, $\omega_{ij} = s^*\theta_{ij}$ denote the fundamental forms and connection forms pulled back by s respectively, and $II(*,*)$ denotes the second fundamental form.

Proof. By definition of $\mathcal{F} = (e_1, e_2, e_3)$, e_1 is tangent to a geodesic which is also trajectory of the gradient flow of the defining function r and e_2, e_3 are tangent to the level surface X^ϵ with $r = \epsilon$. We use the equality $\omega_{ij}(e_k) = -g_M(\nabla_{e_k} e_i, e_j)$ to obtain $\omega_{1j}(e_1) = 0$ and $\omega_{1j}(e_k) = -g_M(\nabla_{e_k} e_1, e_j) = II(e_k, e_j)$ for $j = 2,3$, $k = 2,3$. This completes the proof. □

The mean curvature H is defined to be the trace of II. (Note that H is defined to the half of the trace of II in some of the literature.) In [15], W-volume of M^ϵ is defined by

$$W\big(M^\epsilon\big) := \mathrm{Vol}\big(M^\epsilon\big) - \frac{1}{4} \int\limits_{X^\epsilon} H \,\mathrm{dvol}$$

where $\mathrm{Vol}(M^\epsilon)$ denotes the volume of M^ϵ and dvol denotes the area form over X^ϵ induced by g_M. One nice property of W-volume proved in Lemma 4.5 in [15] is the following equality: for $0 < \epsilon < a_1$,

$$W\big(M^\epsilon\big) = 2\pi(1-g)\log\epsilon + W_{\mathrm{f.p.}}\big(M^\epsilon\big), \tag{4.6}$$

where $W(M) := \lim_{\epsilon\to 0} W_{\text{f.p.}}(M^\epsilon)$ exists and defines the renormalized volume $W(M)$ of M as in Section 8 of [15].

Proposition 4.5. *For s defined by a standard admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$,*

$$\operatorname{Re} \mathbb{CS}^\epsilon(M, s) = \frac{1}{\pi^2} W\big(M^\epsilon\big) \quad \text{for } 0 < \epsilon < a_1.$$

Proof. By the definition, we have

$$\begin{aligned}\int_{\overline{s(M^\epsilon\setminus\mathcal{L})}} \operatorname{Re} C &= \frac{1}{4\pi^2} \int_{\overline{s(M^\epsilon\setminus\mathcal{L})}} \big(4\theta_1\wedge\theta_2\wedge\theta_3 - d(\theta_1\wedge\theta_{23} + \theta_2\wedge\theta_{31} + \theta_3\wedge\theta_{12})\big) \\ &= \frac{1}{\pi^2}\operatorname{Vol}(M_\epsilon) - \frac{1}{4\pi^2} \int_{\partial(\overline{s(M^\epsilon\setminus\mathcal{L})})} \theta_1\wedge\theta_{23} + \theta_2\wedge\theta_{31} + \theta_3\wedge\theta_{12}. \end{aligned} \tag{4.7}$$

For the second equality in (4.7), we apply Stokes' theorem. Now we consider the integrals over the boundary $\partial(\overline{s(M^\epsilon \setminus \mathcal{L})}) = B^{0,\epsilon} \cup B^1 \cup B^{2,\epsilon}$. For the boundary integral over $B^{0,\epsilon}$, we have

$$\begin{aligned} &-\frac{1}{4\pi^2}\int_{B^{0,\epsilon}} \theta_1\wedge\theta_{23} + \theta_2\wedge\theta_{31} + \theta_3\wedge\theta_{12} \\ &\quad= \frac{1}{4\pi^2}\int_{X^\epsilon} \omega_1\wedge\omega_{23} + \omega_2\wedge\omega_{31} + \omega_3\wedge\omega_{12} \\ &\quad= -\frac{1}{4\pi^2}\int_{X^\epsilon} \operatorname{tr} II\, \omega_2\wedge\omega_3 = -\frac{1}{4\pi^2}\int_{X^\epsilon} H \operatorname{dvol}, \end{aligned} \tag{4.8}$$

where X^ϵ is oriented by $\omega_2 \wedge \omega_3$ and the second equality follows from Lemma 4.4.

For the boundary integral over B^1, recall that the boundary B^1 is diffeomorphic to $\mathcal{L}^1 \times S^1$ by ψ in (4.4), and that $\psi_* \frac{\partial}{\partial v}$ is a vertical vector field and $\psi^*\theta_{1j}(\frac{\partial}{\partial v}) = 0$ for $j = 2, 3$ by definition of B^1, hence $\psi^*(\theta_2 \wedge \theta_{31})(\frac{\partial}{\partial v}, *) = 0$, $\psi^*(\theta_3 \wedge \theta_{12})(\frac{\partial}{\partial v}, *) = 0$. Moreover, by definition, $\psi^*\theta_{23}(\frac{\partial}{\partial v}) = -1$. This implies

$$\begin{aligned} &-\frac{1}{4\pi^2}\int_{B^1} \theta_1\wedge\theta_{23} + \theta_2\wedge\theta_{31} + \theta_3\wedge\theta_{12} \\ &\quad= -\frac{1}{4\pi^2}\int_{\mathcal{L}^1\times S^1} \psi^*(\theta_1\wedge\theta_{23}) \\ &\quad= \frac{1}{2\pi}\int_{\mathcal{L}^1} \psi^*\theta_1 = \frac{1}{2\pi}\int_{\mathcal{L}^1} s^*\theta_1 = \frac{1}{2\pi}\int_{s(\mathcal{L}^1)} \theta_1. \end{aligned} \tag{4.9}$$

Hence the boundary integral over B^1 cancels the real part of the second integral in (4.3).

For the boundary integral $B^{2,\epsilon}$,

$$-\frac{1}{4\pi^2}\int\limits_{B^{2,\epsilon}} \theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12}$$
$$=-\frac{1}{4\pi^2}\int\limits_{\mathcal{L}^{2,\epsilon}\times S^1} \psi^*\mathcal{A}^*(\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12}),$$

where ψ is given by (4.5). Using $\mathcal{A}^*\theta=A^{-1}\cdot\theta$ and $\mathcal{A}^*\Theta=A^{-1}\cdot dA+A^{-1}\cdot\Theta\cdot A$ with $\theta=(\theta_1,\theta_2,\theta_3)^t$, $\Theta=(\theta_{ij})$,

$$\begin{aligned}&\mathcal{A}^*(\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12})\\&\quad=\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12}\\&\qquad+\sum_{j=1}^{3}\theta_j\wedge(a_{j1}A_2\cdot dA_3+a_{j2}A_3\cdot dA_1+a_{j3}A_1\cdot dA_2)\end{aligned}\tag{4.10}$$

where a_{jk} denotes the entry in A and A_j denotes the column vector of A, and $A_j\cdot dA_k$ denotes the inner product of two vectors. By Lemma 4.3 and (4.10), and repeating the computation of the integral over B^1,

$$\begin{aligned}&-\frac{1}{4\pi^2}\int\limits_{\mathcal{L}^{2,\epsilon}\times S^1}\psi^*\mathcal{A}^*(\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12})\\&=-\frac{1}{2\pi}\int\limits_{s(\mathcal{L}^{2,\epsilon})}\theta_1\\&\quad-\frac{1}{4\pi^2}\int\limits_{\mathcal{L}^{2,\epsilon}\times S^1}\psi^*\left(\sum_{j=1}^{3}\theta_j\wedge(a_{j1}A_2\cdot dA_3+a_{j2}A_3\cdot dA_1+a_{j3}A_1\cdot dA_2)\right)\\&=-\frac{1}{2\pi}\int\limits_{s(\mathcal{L}^{2,\epsilon})}\theta_1.\end{aligned}$$

Here we use that $\psi:\mathcal{L}^2\times S^1\to\tilde{B}^2$ in (4.5) is orientation reversing, and that the form involving A vanishes on the vertical vector field $\psi_*\frac{\partial}{\partial v}$. Hence the boundary integral over $B^{2,\epsilon}$ cancels the real part of the third integral in (4.3). This completes the proof. □

4.4. Imaginary part of $\mathbb{CS}^\epsilon(M,s)$

Now we prove

Proposition 4.6. *For s corresponding to an admissible singular framing $(\mathcal{F},\kappa,\mathcal{L})$, the imaginary part of $\mathbb{CS}^\epsilon(M,s)$ converges to a finite value as $\epsilon\to 0$.*

Proof. Over $N_{(0,a_1)} \setminus \mathcal{L}^2$, the pull back of the imaginary part of C by s is given by

$$\frac{1}{4\pi^2}(\omega_{12} \wedge \omega_{13} \wedge \omega_{23} - \omega_{12} \wedge \omega_1 \wedge \omega_2 - \omega_{13} \wedge \omega_1 \wedge \omega_3 - \omega_{23} \wedge \omega_2 \wedge \omega_3). \tag{4.11}$$

The first and the last terms in (4.11) vanish respectively since they are sum of triple wedge products of ω_2, ω_3 by Lemma 4.4. The second and the third terms in (4.11) cancel each other by Lemma 4.4 and the fact $II(e_2, e_3) = II(e_3, e_2)$. Hence the imaginary part of the first integral in (4.2) is finite and independent of $0 < \epsilon < a_1$. For the imaginary part of the line integral over $\mathcal{L}$, note that for $\ell_j \in \mathcal{L}^2$, the integral $\int_{\ell_j \cap N_{[\epsilon,a_1]}} \omega_{23}$ measures the total rotation of κ with respect to parallel translation on $\ell_j \cap N_{[\epsilon,a_1]}$. Since $r^{-1}\kappa$ extends smoothly to $\overline{M}$ by definition, the limit of the line integral as $\epsilon \to 0$ has a finite value. This completes the proof. $\square$

Proposition 4.7. *For a given marked Schottky hyperbolic 3-manifold M, if s_0, s_1 are defined by standard admissible framings $(\mathcal{F}_0, \kappa_0, \mathcal{L}_0)$ and $(\mathcal{F}_1, \kappa_1, \mathcal{L}_1)$ on M which are related by a homotopy of standard admissible framings which are fixed outside of M^{a_1}, then*

$$\operatorname{Im} \mathbb{CS}^\epsilon(M, s_0) = \operatorname{Im} \mathbb{CS}^\epsilon(M, s_1).$$

Proof. Let $(\mathcal{F}_u, \kappa_u, \mathcal{L}_u)$, with $u \in [0,1]$ be the homotopy connecting $(\mathcal{F}_0, \kappa_0, \mathcal{L}_0)$ and $(\mathcal{F}_1, \kappa_1, \mathcal{L}_1)$. The framing $\mathcal{F}_u$ defines a section $s : W_\epsilon \to F(M)$ over $W_\epsilon := [0,1] \times M^\epsilon \setminus \{(u, y_u) \mid y_u \in \mathcal{L}_u, u \in [0,1]\}$. Denoting by Q the imaginary part of C, we have

$$0 = \int\limits_{s(W_\epsilon)} dQ = \int\limits_{s_1(M^\epsilon \setminus \mathcal{L}_1)} Q - \int\limits_{s_0(M^\epsilon \setminus \mathcal{L}_0)} Q + \int\limits_{B_W} Q. \tag{4.12}$$

The boundary B_W consists of three parts $\hat{B}^0$, $\hat{B}^1$, and $\hat{B}^2$, consisting of the trajectories under the homotopy $\mathcal{F}_u$ of $B^{0,\epsilon}$, B^1, and $B^{2,\epsilon}$ respectively. For the integral over the part $\hat{B}^0$, $\theta_i(s_* \frac{\partial}{\partial u}) = 0$ and $\theta_{ij}(s_* \frac{\partial}{\partial u}) = 0$ over $\hat{B}^0 = B^{0,\epsilon} \subset F(M)$. Therefore

$$\int\limits_{\hat{B}^0} Q = 0. \tag{4.13}$$

The boundary $\hat{B}^1$ is diffeomorphic to $[0,1] \times \mathcal{L}^1 \times S^1$ by

$$\psi(u, y, v) = \{u\} \times \big(e_1(y), \cos(v)e_2(y) + \sin(v)e_3(y), -\sin(v)e_2(y) + \cos(v)e_3(y)\big)$$

where $(e_1(y), e_2(y), e_3(y))$ denotes the reference framing $\kappa_u(y)$ for $y \in \mathcal{L}^1$. Here and below, we identify $\mathcal{L}^1_u$ with $\mathcal{L}^1 = \mathcal{L}^1_0$ implicitly. The orientation $(\frac{\partial}{\partial u}, \frac{\partial}{\partial y}, -\frac{\partial}{\partial v})$ on $[0,1] \times \mathcal{L}^1 \times S^1$ makes ψ orientation preserving. As before, $\psi^* \Omega_{ij}(\frac{\partial}{\partial v}, *) = 0$ for $1 \le i, j \le 3$, $(\psi^*\theta_{12})(\frac{\partial}{\partial v}) = 0$, $(\psi^*\theta_{13})(\frac{\partial}{\partial v}) = 0$, and $(\psi^*\theta_{23})(\frac{\partial}{\partial v}) = -1$. From above facts, we have

$$\psi^*Q = \frac{1}{4\pi^2}\psi^*(\theta_{12}\wedge\theta_{13}\wedge\theta_{23}+\theta_{23}\wedge\Omega_{23}) = \frac{1}{4\pi^2}\psi^*(\theta_{23}\wedge d\theta_{23}),$$

and

$$\psi^*\theta_{23} = -dv + q^*s^*\theta_{23},$$

where $q : [0,1]\times\mathcal{L}^1\times S^1 \to [0,1]\times\mathcal{L}^1$ is the natural projection, $s_u : \mathcal{L}^1 \to F(M)$ is the section defined by κ_u, and $s : [0,1]\times\mathcal{L}^1 \to F(M)$ is the corresponding family given by $s(u,\cdot) = s_u$. It follows that $\psi^*Q = -\frac{1}{4\pi^2}dv\wedge d(q^*s^*\theta_{23})$. With the above orientation convention, by Stokes' theorem, we have

$$\begin{aligned}\int\limits_{\hat{B}^1} Q &= \int\limits_{[0,1]\times\mathcal{L}^1\times S^1}\psi^*Q = -\frac{1}{4\pi^2}\int\limits_{[0,1]\times\mathcal{L}^1\times S^1} dv\wedge d\big(q^*s^*\theta_{23}\big)\\ &= \frac{1}{2\pi}\int\limits_{[0,1]\times\mathcal{L}^1} d\big(s^*\theta_{23}\big) = \frac{1}{2\pi}\Bigg(\int\limits_{\mathcal{L}^1} s_1^*\theta_{23} - \int\limits_{\mathcal{L}^1} s_0^*\theta_{23}\Bigg).\end{aligned} \tag{4.14}$$

The right hand side of (4.14) is the same as the difference of the imaginary parts of the second integrals for $u=1$ and $u=0$ in the definition of $\mathbb{CS}^\epsilon(M,s)$ in (4.3).

For the boundary integral over $\hat{B}^{2,\epsilon}$, as in the proof of Proposition 4.5 we have

$$\int\limits_{\hat{B}^{2,\epsilon}} Q = \int\limits_{[0,1]\times\mathcal{L}^{2,\epsilon}\times S^1} \psi^*\mathcal{A}^*Q$$

where ψ is the orientation reversing diffeomorphism defined in (4.5). We also have

$$\begin{aligned}\mathcal{A}^*Q = Q &+ \frac{1}{24\pi^2}\operatorname{Tr}\big(\big(A^{-1}dA\big)^3\big)\\ &+ \frac{1}{4\pi^2}d(\theta_{12}\wedge d\hat{A}_1\cdot\hat{A}_2 + \theta_{13}\wedge d\hat{A}_1\cdot\hat{A}_3 + \theta_{23}\wedge d\hat{A}_2\cdot\hat{A}_3)\end{aligned} \tag{4.15}$$

where $\hat{A}_j$ denotes the row vector of A. Hence, in a similar way as (4.14),

$$\begin{aligned}\int\limits_{\hat{B}^{2,\epsilon}} Q &= -\frac{1}{2\pi}\Bigg(\int\limits_{\mathcal{L}^{2,\epsilon}} s_1^*\theta_{23} - \int\limits_{\mathcal{L}^{2,\epsilon}} s_0^*\theta_{23}\Bigg)\\ &\qquad - \frac{1}{2\pi}\Bigg(\int\limits_{\mathcal{L}^{2,\epsilon}} \psi_1^* d\hat{A}_2\cdot\hat{A}_3 - \int\limits_{\mathcal{L}^{2,\epsilon}} \psi_0^* d\hat{A}_2\cdot\hat{A}_3\Bigg)\\ &= -\frac{1}{2\pi}\Bigg(\int\limits_{\mathcal{L}^{2,\epsilon}} s_1^*\theta_{23} - \int\limits_{\mathcal{L}^{2,\epsilon}} s_0^*\theta_{23}\Bigg)\end{aligned} \tag{4.16}$$

where ψ_1 and ψ_0 represent ψ taken at $u=1$ and $u=0$. Here the first equality follows from that $\psi^*\theta_{12}$, $\psi^*\theta_{13}$, and the form involving A vanish on the vertical vector field

$\psi_* \frac{\partial}{\partial v}$. The expression $(1/2\pi) \int_{\mathcal{L}^{2,\epsilon}} \psi_u^* d\hat{A}_2 \cdot \hat{A}_3$ can be shown to be the total rotation angle of $\hat{A}_2$ about the axis $\hat{A}_1$ along $\mathcal{L}^{2,\epsilon}$. Since A is fixed at the endpoints of $\mathcal{L}^{2,\epsilon}$ through the homotopy, this total rotation angle does not change, so the second equality follows. The right hand side of (4.16) is the same as the difference of the imaginary parts of the third integrals for $u = 1$ and $u = 0$ in the definition of $\mathbb{CS}^\epsilon(M, s)$ in (4.3). Combining (4.12), (4.13), (4.14) and (4.16) completes the proof. □

4.5. Definition of the invariant $\mathbb{CS}(M, s)$ and the function $\mathbb{CS}$

For $s : M \to F(M)$ corresponding to an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$ as explained after Eq. (4.1), we define the Chern–Simons invariant of (M, s) to be

$$CS(M, s) := \frac{1}{2} \lim_{\epsilon \to 0} \operatorname{Im} \mathbb{CS}^\epsilon(M, s),$$

where the limit exists by Proposition 4.6. By Proposition 4.7, $CS(M, s)$ is independent of a homotopic change of an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$ inside of M^{a_1}. We can now define the invariant $\mathbb{CS}(M, s)$.

Definition 4.8. For $s : M \to F(M)$ corresponding to an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$,

$$\mathbb{CS}(M, s) := \lim_{\epsilon \to 0} \left(\mathbb{CS}^\epsilon(M, s) + \frac{2}{\pi}(g - 1) \log \epsilon \right).$$

By (4.6) and Proposition 4.5, as we stated in (1.2), we have

$$\mathbb{CS}(M, s) = \frac{1}{\pi^2} W(M) + 2i\, CS(M, s).$$

Now, suppose we are given a compact marked Riemann surface X and a holomorphic 1-form Φ on X, with a corresponding admissible singular framing $(\mathcal{F}_\Phi, \kappa_\Phi, Z)$ over X. Then we have associated to this data a unique marked Schottky hyperbolic 3-manifold M_X and $s_\Phi : M_X \to F(M_X)$ corresponding to a standard admissible singular extension $(\mathcal{F}, \kappa, \mathcal{L})$ over M_X constructed in Theorem 3.3. We now consider to what extent the invariant $\mathbb{CS}(M_X, s_\Phi)$ depends on our choice of admissible extension s_Φ. We have already shown in Proposition 4.7 that it is independent of a homotopic change of $(\mathcal{F}, \kappa, \mathcal{L})$ in $M_X^{a_1}$. Now we show

Proposition 4.9. *The quantity* $\exp(4\pi \mathbb{CS}(M_X, s_\Phi))$ *is independent of the choice of signs in* κ_Φ *and of the choice of* κ.

Proof. Note that the modulus of $\exp(4\pi \mathbb{CS}(M_X, s_\Phi))$ depends only on M_X by Proposition 4.5. For the argument of $\exp(4\pi \mathbb{CS}(M_X, s_\Phi))$, there is a choice of a reference framing κ which can rotate along $\mathcal{L}$, but a change of a rotation number results in only an integer difference in the imaginary part of $\mathbb{CS}(M_X, s_\Phi)$ through the second and third integrals in (4.3). There are sign ambiguities in the definition of the reference framing at zeroes of Φ mentioned just after (3.6). Hence the imaginary part of $\mathbb{CS}(M_X, s_\Phi)$ is well-defined only up to addition of a half-integer, but this ambiguity will disappear for $\exp(4\pi\mathbb{CS}(M_X, s_\Phi))$. □

To state the main result of this section, we need to introduce an auxiliary space. For each point (X, Φ) in $\tilde{\mathcal{H}}_g$, we attach the data of a choice of isotopy class of $g-1$ simple, pairwise disjoint curves in $M_X \setminus \bigsqcup_{i=1}^g \ell_i$ whose endpoints are the zeroes of Φ where the set $\{\ell_1, \ldots, \ell_g\}$ represents the marked generators of $\pi_1(M_X)$ (see the last paragraph of Section 3.2). Here we distinguish two choices of isotopy classes of $g-1$ simple pairwise disjoint curves if they are not isotopic to each other in $M_X \setminus \bigsqcup_{i=1}^g \ell_i$. The resulting space $\tilde{\mathcal{H}}_g^*$ is locally isomorphic to $\tilde{\mathcal{H}}_g$, and there is a natural projection map to $\tilde{\mathcal{H}}_g$ corresponding to forgetting the added data. Note that each connected component of $\tilde{\mathcal{H}}_g^*$ covers $\tilde{\mathcal{H}}_g$ by this projection map.

Theorem 4.10. *The expression* $\exp(4\pi\mathbb{CS}(M_X, s_\Phi))$ *determines a globally well-defined function* $\exp(4\pi\mathbb{CS}) : \tilde{\mathcal{H}}_g^* \to \mathbb{C}$.

Proof. Given a point in $\tilde{\mathcal{H}}_g^*$, we use Theorem 3.3 to construct a standard singular admissible framing on M_X, whose $\mathcal{L}^2$ curves are isotopic to the given $g-1$ curves. It is clear from the construction that any two such framings are related by a homotopy, which is an isotopy of the corresponding set of curves $\mathcal{L}$. It then follows from Propositions 4.7 and 4.9 that the value of $\exp(4\pi\mathbb{CS}(M_X, s_\Phi))$ is uniquely determined by this data. □

Remark 4.11. The proof of Theorem 1.1 in Section 7 will show that, in fact, the function $\exp(4\pi\mathbb{CS})$, restricted to any connected component of $\tilde{\mathcal{H}}_g^*$, descends to a well-defined function on $\tilde{\mathcal{H}}_g$. But, restricting to a different connected component of $\tilde{\mathcal{H}}_g^*$ will give a different function on $\tilde{\mathcal{H}}_g$ differing by a multiplicative constant. This results in a different constant c in (1.5) of Theorem 1.1. On the other hand, the function $\exp(4\pi\mathbb{CS})$ on $\tilde{\mathcal{H}}_g$ cannot descend to the space of triples $(X, \Phi, \{b_1, \ldots, b_g\})$ where $\{b_1, \ldots, b_g\}$ denotes a Schottky marking of $\pi_1(M_X)$ given by the set of curves $\{\ell_1, \ldots, \ell_g\}$. This is because $\mathcal{L}^2$ curves are distinguished if two choices of them are not isotopic to each other in $M_X \setminus \bigsqcup_{i=1}^g \ell_i$, that is, the isotopy type of $\mathcal{L}^2$ curves makes sense in $\pi_1(M_X \setminus \bigsqcup_{i=1}^g \ell_i)$, not in $\pi_1(M_X)$.

4.6. The case of genus 1

For a complex number τ with $\operatorname{Re}\tau > 0$, an elliptic curve X_τ is defined by the action $q_\tau \cdot w = q_\tau w$ for $q_\tau = \exp(2\pi i\tau)$, $w \in \mathbb{C} \setminus \{0\}$. That is, $X_\tau = (\mathbb{C} \setminus \{0\})/\langle q_\tau \rangle$. Then the Schottky hyperbolic manifold M_τ with $\partial\overline{M}_\tau = X_\tau$ can be realized by the quotient of

$$H^3 = \{(t, w) \in \mathbb{R}^+ \times \mathbb{C}\} \quad \text{under the action } q_\tau \cdot (t, w) = (q_\tau t, q_\tau w). \tag{4.17}$$

In this case, the complex valued invariant $\mathbb{CS}(M_\tau, s_\Phi)$ can be computed explicitly where $\Phi = c\frac{dw}{w}$ ($c = \frac{1}{2\pi\sqrt{\operatorname{Im}\tau}}$) is the holomorphic 1-form on X_τ.

The following flat metric

$$|\Phi|^2 = c^2 \left|\frac{dw}{w}\right|^2 = c^2 \left(\frac{d\ell^2}{\ell^2} + d\varphi^2\right) \quad \text{for } w = \ell e^{i\varphi} \in \mathbb{C} \setminus \{0\} \tag{4.18}$$

descends to X_τ since it is invariant under the action of q_τ on $\mathbb{C}\setminus\{0\}$. Then the co-framing on X_τ associated to $\Phi = c\frac{dw}{w}$ is given by

$$\bar{\omega}_2 = c\frac{d\ell}{\ell}, \qquad \bar{\omega}_3 = c\, d\varphi.$$

To consider the standard admissible co-framing on M_τ which is the extension of $(\bar{\omega}_2, \bar{\omega}_3)$, let us recall that the hyperbolic metric g_{H^3} on H^3 is given by

$$g_{H^3} = \frac{1}{\ell^2 \sin^2\vartheta}\left(\ell^2 d\vartheta^2 + d\ell^2 + \ell^2\cos^2\vartheta\, d\varphi^2\right)$$

where $t = \ell\sin\vartheta$, $w = \ell\cos\vartheta e^{i\varphi}$. It is easy to see that g_{H^3} is invariant under the action in (4.17), hence it descends to the hyperbolic metric g_{M_τ} on M_τ. Taking the defining function $r = \frac{2c(1-\cos\vartheta)}{\sin\vartheta}$, the metric g_M has the following form, which is claimed in (3.2),

$$g_{M_\tau} = \frac{dr^2}{r^2} + \frac{1}{r^2}\left(\alpha(r)\frac{d\ell^2}{\ell^2} + \beta(r)d\varphi^2\right) \tag{4.19}$$

where $\alpha(r) = 4c^2\frac{(1-\cos\vartheta(r))^2}{\sin^4\vartheta(r)}$, $\beta(r) = 4c^2\frac{(1-\cos\vartheta(r))^2\cos^2\vartheta(r)}{\sin^4\vartheta(r)}$. The metric $\alpha(r)\frac{d\ell^2}{\ell^2} + \beta(r)d\varphi^2$ extends to the flat metric $|\Phi|^2$ in (4.18) as $r \to 0$. Now the orthonormal co-framing with respect to (4.19), which is the standard admissible extension of $(\bar{\omega}_2, \bar{\omega}_3)$, is given by

$$\omega_1 = \frac{d\vartheta}{\sin\vartheta}, \qquad \omega_2 = \frac{1}{\sin\vartheta}\frac{d\ell}{\ell}, \qquad \omega_3 = \frac{\cos\vartheta}{\sin\vartheta}d\varphi. \tag{4.20}$$

Note that this co-framing is not defined along curve $\mathcal{L}^1$ which is the quotient of the line $w = 0$ under the action in (4.17). From this and using $d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j$, we obtain

$$\omega_{12} = \frac{\cos\vartheta}{\sin\vartheta}\frac{d\ell}{\ell}, \qquad \omega_{13} = \frac{1}{\sin\vartheta}d\varphi, \qquad \omega_{23} = 0. \tag{4.21}$$

Now let us compute $\mathbb{CS}^\epsilon(M_\tau, \Phi)$ for a small $\epsilon > 0$ using all these preliminary formulae. Recalling the definition in (4.3), the real part of the first integral is given by

$$\begin{aligned}
&\int\limits_{s_\Phi(M_\tau^\epsilon\setminus\mathcal{L}^1)} \operatorname{Re} C \\
&= \frac{1}{\pi^2}\int\limits_{M_\tau^\epsilon} \omega_1\wedge\omega_2\wedge\omega_3 - \frac{1}{4\pi^2}\int\limits_{s_\Phi(M_\tau^\epsilon\setminus\mathcal{L}^1)} d(\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12}) \\
&= \frac{1}{\pi^2}\int\limits_{M_\tau^\epsilon} \omega_1\wedge\omega_2\wedge\omega_3 - \frac{1}{4\pi^2}\int\limits_{X^\epsilon} H\,\mathrm{dvol} + \frac{1}{2\pi}\int\limits_{s(\mathcal{L}^1)} \theta_1 \\
&= \frac{1}{\pi}(-\log|q_\tau|)\left(\frac{1}{\sin^2\vartheta_\epsilon}-1\right) - \frac{1}{\pi}(-\log|q_\tau|)\left(\frac{1}{\sin^2\vartheta_\epsilon}-\frac{1}{2}\right) + \frac{1}{2\pi}(-\log|q_\tau|) \\
&= 0 \qquad (4.22)
\end{aligned}$$

where ϑ_ϵ is defined by $\epsilon = \frac{2c(1-\cos\vartheta_\epsilon)}{\sin\vartheta_\epsilon}$. For the second equality in (4.22) we used equalities (4.8) and (4.9) in the proof of Proposition 4.5. By (4.21), we can easily derive that $\operatorname{Im} C = 0$. Hence, for the imaginary part of the first integral of $\mathbb{CS}^\epsilon(M_\tau,\Phi)$ we have

$$\int\limits_{s_\Phi(M_\tau^\epsilon\setminus\mathcal{L}^1)} \operatorname{Im} C = 0. \qquad (4.23)$$

For the second line integral in (4.3), we need a reference framing κ defined over a tubular neighborhood of $\mathcal{L}^1$, for which we take the following co-framing

$$\omega_1 = \frac{dt}{t}, \qquad \omega_2 = \frac{1}{t}(\cos\beta dx + \sin\beta dy), \qquad \omega_3 = \frac{1}{t}(-\sin\beta dx + \cos\beta dy)$$

where $\beta = 2\pi\mathrm{Re}\tau\frac{\log t}{\log|q_\tau|}$ and $dw = dx + idy$. With respect to this co-framing, the corresponding connection 1-form ω_{23} is given by

$$\omega_{23} = -\frac{2\pi\mathrm{Re}\tau}{\log|q_\tau|}\frac{1}{\ell^2}(tdt + xdx + ydy).$$

Then we have

$$-\frac{1}{2\pi}\int\limits_{s_\kappa(\mathcal{L}^1)} (\theta_1 - i\theta_{23}) = -\frac{1}{2\pi}\int\limits_{|q_\tau|}^{1} \omega_1 - i\omega_{23} = i\tau. \qquad (4.24)$$

Combining (4.22), (4.23), and (4.24), we conclude that

$$\mathbb{CS}(M_\tau,\Phi) = \lim_{\epsilon\to 0}\mathbb{CS}^\epsilon(M_\tau,\Phi) = i\tau. \qquad (4.25)$$

Note that, in genus 1, $\mathbb{CS}^\epsilon(M_\tau,\Phi)$ is in fact independent of ϵ, for sufficiently small $\epsilon > 0$.

5. Variation of the invariant $\mathbb{CS}$

Suppose we are given a contractible open set U in $\tilde{\mathcal{H}}_g^*$. By the results of the previous section, the invariant $\mathbb{CS}(M_X, s_\Phi)$ determines a function $\mathbb{CS} : U \to \mathbb{C}$, which is well-defined up to addition of $\frac{1}{2}ni$ for $n \in \mathbb{Z}$. In this section we find expressions for the derivatives $\partial\mathbb{CS}$ and $\bar{\partial}\mathbb{CS}$ of this function.

Before beginning the proof of this variation formula for $\mathbb{CS}$, it may be helpful to explain the underlying idea and the structure of the proof, since the computation is somewhat lengthy and technical. The invariant $\mathbb{CS}$ is a regularized limit of $\mathbb{CS}^\epsilon$, as stated in Definition 4.8; and, as explained in Section 4.1, the key term in $\mathbb{CS}^\epsilon$ is an integral of the constant 3-form C, over a domain in $PSL_2(\mathbb{C})$. The domain is obtained by taking a fundamental domain in the hyperbolic 3-space H^3 for the group Γ defining our 3-manifold M_X, deleting both a neighborhood of infinity and the link $\mathcal{L}$, and lifting to $PSL_2(\mathbb{C})$ (i.e. the frame bundle) by our choice of framing s. We choose a basepoint in $\tilde{\mathcal{H}}_g^*$ and a one-complex-parameter deformation, and we pull the form C back to the domain in $PSL_2(\mathbb{C})$ associated to the basepoint by the corresponding deformation map of H^3. We then apply Stokes' theorem to the compactification of this domain. This is the same strategy used by a number of previous authors, in particular [18,21], and [8]. The main differences in our case are the presence of additional boundary components, the limit near infinity, and the translation to the deformation theory of the Riemann surface X, each of which complicates the analysis.

In Section 5.1, we introduce some (admittedly involved) notation to explain the domain precisely, which is needed for the application of Stokes' theorem. Because of the singular curves $\mathcal{L}$ and the conformal boundary X of $\overline{M}_X$, our domain is rather complicated. Since this domain is varying depending on the variation parameter, we need to introduce the pull back map H in order to apply Stokes' theorem to a fixed domain with boundary. In Section 5.2, we start with the equality (5.2) which is a modified version of Stokes' theorem. In this step, we take our domain to be the subset corresponding to M_X^ϵ rather than the whole domain. We deal with the boundary contributions in (5.2) arising from singular curves $\mathcal{L}$ in Lemmas 5.1, 5.2. In Lemma 5.3, we deal with another boundary contribution corresponding to ∂M_X^ϵ. Then in Section 5.3, we analyze the limit of this boundary contribution as $\epsilon \to 0$. Since the hyperbolic metric g_M blows up as we approach to X, a careful analysis is needed to deal with this limit and the exact result for this is given in Proposition 5.7. The final step is to translate the formula in Proposition 5.7 in terms of the (meromorphic) quadratic differential and the Beltrami differential on X. This is done in Section 5.4 and the main results of this section are given in Theorem 5.14 and Corollary 5.15. The main tool for this part is the holomorphic variation technique developed in [24,17].

In summary, combining the techniques developed mainly in [21,24], we obtain a variation formula of $\mathbb{CS}$, which is given in terms of a certain integral over the boundary Riemann surface X with some additional data at the zeroes of Φ. This variation formula will be also given in terms of the coordinates (2.1) in Proposition 7.1.

5.1. Basic notations for variation

Each point $u \in U$ determines a compact marked Riemann surface X_u together with a holomorphic 1-form $\varPhi_u$ on X_u. We fix a basepoint $u_0 \in U$, and for simplicity we write $X = X_{u_0}$ and similarly below. We will always assume that X_u is uniformized by a marked Schottky group, $X_u = \varGamma_u \backslash \varOmega_u$, where $\varGamma_u$ is a marked Schottky group with marked normalized generators $\{L_1(u), \dots, L_g(u)\}$ and ordinary set $\varOmega_u$. The group $\varGamma_u$ simultaneously defines a marked Schottky hyperbolic 3-manifold $M_u := M_{X_u} = \varGamma_u \backslash H^3$. For each $u \in U$, we have a quasi-conformal mapping $f_u : \varOmega \to \varOmega_u$. Define $P_u^\epsilon : \varOmega_u \to H^3$ to be the map translating points along the integral curve ϕ_t of $\nabla_{\bar{g}_u} r_u$ emanating from $z \in \varOmega_u$ for ϵ units of time, where $\bar{g}$ and r are defined as in Section 2.2. Then we define a map $\mathbf{f}_u : \bigcup_{0<\epsilon<a_1} P^\epsilon(\varOmega) \to H^3$ by

$$\mathbf{f}_u|_{P^\epsilon(\varOmega)} = P_u^\epsilon \circ f_u \circ \left(P^\epsilon\right)^{-1}. \tag{5.1}$$

(Here $a_1 = \frac{a}{4}$, where a is defined as in Section 2.2.) This map extends to a diffeomorphism $\mathbf{f}_u : H^3 \to H^3$, satisfying $\mathbf{f}_u \circ \gamma = \gamma_u \circ \mathbf{f}_u$ for all $\gamma_u \in \varGamma_u$.

Corresponding to the family $\varPhi_u$ and the given homotopy class of $g-1$ curves in M_u connecting the zeroes of $\varPhi_u$, we take a smooth family of sections $s_u := s_{\varPhi_u} : (M_u \setminus \mathcal{L}_u) \cup \mathcal{L}_u \to F(M_u)$, constructed as in Theorem 4.10. Here $\mathcal{L}^2$ is taken to be isotopic to the given $g-1$ curves, and $\mathcal{L}_u = \mathbf{f}_u(\mathcal{L})$. We also denote by $\mathcal{L}_u$ and s_u the corresponding liftings $\mathcal{L}_u \subset H^3$ and $s_u : (H^3 \setminus \mathcal{L}_u) \cup \mathcal{L}_u \to F(H^3) \cong PSL_2(\mathbb{C})$. The family defines a map $s : U \times H^3 = \{(u,x) \mid u \in U, x \in (H^3 \setminus \mathcal{L}_u) \cup \mathcal{L}_u\} \to PSL_2(\mathbb{C})$. We let K be the unique map $K : U \times s_0((H^3 \setminus \mathcal{L}) \cup \mathcal{L}) \to PSL_2(\mathbb{C})$ satisfying

$$K \circ (\mathrm{id}, s_0) = s \circ (\mathrm{id}, \mathbf{f}),$$

where $s \circ (\mathrm{id}, \mathbf{f})(u,x) := s_u \circ \mathbf{f}_u(x)$ for $(u,x) \in U \times H^3$. As observed in Section 4.2, the closure $\overline{s_0(H^3 \setminus \mathcal{L})}$ of $s_0(H^3 \setminus \mathcal{L})$ in $PSL_2(\mathbb{C})$ provides a natural compactification of $s_0(H^3 \setminus \mathcal{L})$, and K extends smoothly to $U \times (\overline{s_0(H^3 \setminus \mathcal{L})} \cup s_0(\mathcal{L}))$ (we also denote the extension by K). Note that the generators $L_r(u)$ of $\varGamma_u$, $r = 1, \dots, g$, can be considered as giving holomorphic functions

$$L_r : U \to PSL_2(\mathbb{C}).$$

We let $\mathbf{D}$ be a fundamental domain for the action of $\varGamma$ on H^3, such that $\partial\mathbf{D} \subset H^3$ consists of $2g$ smooth surfaces D_r, $-L_r(0)D_r$, for $r = 1, \dots, g$ (the negative sign indicates opposite orientation). Define $\mathbf{D}_u := \mathbf{f}_u(\mathbf{D})$. Considering H^3 as $\{(t,x,y) \in \mathbb{R}^3 | t > 0\}$, define D and C_r to be the intersection of the closure of $\mathbf{D}$ and D_r respectively with the set $t = 0$. Then D is a fundamental domain of the action of $\varGamma$ on $\varOmega$, ∂D consists of smooth curves C_r, $-L_r(0)C_r$, and we define $D_u := f_u(D)$. We denote $\mathbf{D}' := \mathbf{D} \setminus \mathcal{L}$, and define $\Delta := s_0(\mathbf{D}')$. As above, the closure $\overline{\Delta}$ of Δ in $PSL_2(\mathbb{C})$ provides a natural compactification of Δ. Let $T_r := \overline{s_0(D_r)}$ for $r = 1, \dots, g$. The boundary components of $\overline{\Delta}$ consist of $B^0 \cup B^1 \cup B^2$ as defined in Section 4.2, and $\bigcup_{r=1}^{g} (T_r - L_r(0)T_r)$. We denote by $\mathbf{D}^\epsilon$ and $\overline{\Delta}^\epsilon$ the subsets of $\mathbf{D}$ and $\overline{\Delta}$ respectively corresponding to M^ϵ. Define $D^\epsilon := \mathbf{D} \cap P^\epsilon(D)$. The boundary components of $\overline{\Delta}^\epsilon$ consist of $B^{0,\epsilon} \cup B^1 \cup B^{2,\epsilon}$ and $\bigcup_{r=1}^{g} (T_r^\epsilon - L_r(0)T_r^\epsilon)$ where $B^{0,\epsilon}$ is diffeomorphic to B^0, and $B^{2,\epsilon}$ and $\bigcup_{r=1}^{g} (T_r^\epsilon - L_r(0)T_r^\epsilon)$ are subsets of B^2 and $\bigcup_{r=1}^{g} (T_r - L_r(0)T_r)$ respectively. The notations $\mathbf{D}'_u$, Δ_u, etc., denote the corresponding constructions for $\mathbf{D}_u$.

Since we will always be working in a fixed fundamental domain $\mathbf{D}_u$, from now on, we will write $\mathcal{L}_u = \mathcal{L}_u^1 \cup \mathcal{L}_u^2$ to mean the intersection $\mathcal{L}_u \cap \mathbf{D}_u$. The boundary points of $\mathcal{L}_u^1 \cup \mathcal{L}_u^2$ consist of finitely many matched pairs $y_j(u)$ and $L_{r(j)}(u)y_j(u)$, $j \in \mathcal{J}$, together with $2g-2$ points which are the zeros of the holomorphic 1-form Φ_u. We may assume that every curve in $\mathcal{L}_u^1$ has exactly two points $y_j(u)$, $L_{r(j)}(u)y_j(u)$ in its boundary, and we assume the orientation of $\mathcal{L}_u^1$ given by the reference framing κ_u^1 is such that the component connecting $L_{r(j)}(u)y_j(u)$ to $y_j(u)$ is oriented towards $y_j(u)$.

Under the canonical map from $\tilde{\mathcal{H}}_g^*$ to $\mathfrak{S}_g$, a holomorphic tangent vector ϖ in $T^{1,0}U$ at u_0 maps to a holomorphic tangent vector in $T^{1,0}\mathfrak{S}_g$, which corresponds to a harmonic Beltrami differential $\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma)$. Then μ defines a quasi-conformal mapping $f_{w\mu} : X \to X_w$ for all w in some neighborhood W of the origin in $\mathbb{C}$. There exists a holomorphic family $\{\Phi(w)\}$, where $\Phi(w)$ is a holomorphic 1-form on X_w, such that the derivative at u_0 of the complex curve in U given by the family $\{(X_w, \Phi(w))\}$ is ϖ. (Here we are using the local isomorphism of $\tilde{\mathcal{H}}_g^*$ and $\tilde{\mathcal{H}}_g$.) In this way we obtain a complex curve $u : W \to U$, such that $\frac{\partial u}{\partial w} = \varpi$ and $\frac{\partial u}{\partial \bar{w}} = 0$ (with w a local coordinate in W).

For the curve $u : W \to U$ we define $f : W \times \Omega \to \mathbb{C}$ by $f(w,z) = f_{u(w)}(z) = f_{w\mu}(z)$ and $\mathbf{f} := W \times H^3 \to H^3$ by $\mathbf{f}(w,x) = \mathbf{f}_{u(w)}(x)$. We also define

$$H = K \circ (u, \mathrm{id}) : W \times \left(\overline{s_0\left(H^3 \setminus \mathcal{L}\right)} \cup s_0(\mathcal{L})\right) \to PSL_2(\mathbb{C}),$$

and

$$\sigma = s \circ (u, \mathbf{f}) : W \times H^3 \to PSL_2(\mathbb{C}).$$

5.2. Contributions of boundaries

For technical reasons we consider the holomorphic variation of $\overline{\mathbb{CS}}$ rather than $\mathbb{CS}$. To derive a variation formula for $\overline{\mathbb{CS}}$, we start with the following equality:

$$\begin{aligned} 0 &= \int'_{\overline{\Delta}^\epsilon} H^* d\overline{C} = \int'_{\overline{\Delta}^\epsilon} (d_W + d_\Delta) H^* \overline{C} = d_W \int'_{\overline{\Delta}^\epsilon} H^* \overline{C} - \int'_{\partial \overline{\Delta}^\epsilon} H^* \overline{C} \\ &= d_W \int'_{\overline{\Delta}^\epsilon} H^* \overline{C} - \int'_{B^{0,\epsilon} \cup B^1 \cup B^{2,\epsilon}} H^* \overline{C} - \sum_{r=1}^{g} \int'_{T_r^\epsilon - L_r T_r^\epsilon} H^* \overline{C}. \end{aligned} \tag{5.2}$$

Here the notation $\int'_{\overline{\Delta}^\epsilon}$ denotes the *partial integral*: we consider the integrand as a form on $\overline{\Delta}$ taking values in forms on W, and integrate over $\{w\} \times \overline{\Delta}^\epsilon$, obtaining a 1-form on W. The notation $d = d_W + d_\Delta$ denotes the splitting of d on $W \times \Delta$ in the obvious way. Note that we use the orientation from $W \times \Delta$; for this reason, we have $d_W \int'_{\overline{\Delta}^\epsilon} H^*\overline{C} = \int'_{\overline{\Delta}^\epsilon} d_W H^*\overline{C}$, but when we apply Stokes' theorem, we have $\int'_{\overline{\Delta}^\epsilon} d_\Delta H^*\overline{C} = -\int'_{\partial\overline{\Delta}^\epsilon} H^*\overline{C}$. We use a similar convention for partial integrals throughout this section.

The next three lemmas deal with the partial integrals $\int'_{B^1} H^*\overline{C}$, $\int'_{B^{2,\epsilon}} H^*\overline{C}$, and $\int'_{B^{0,\epsilon}} H^*\overline{C}$ respectively.

Lemma 5.1. *Let $u : W \to U$ be a complex curve as defined above, with $w \in W$. Then we have the following equality of 1-forms over W:*

$$\int'_{B^1} H^*\overline{C} - \frac{1}{2\pi} d_W \int'_{\mathcal{L}^1} \sigma^*(\theta_1 + i\theta_{23}) = -\frac{1}{2\pi} \sum_{y \in \partial\mathcal{L}^1} \sigma^*(\theta_1 + i\theta_{23})|_y.$$

Proof. Recall that the integral over B^1 is independent of ϵ for small $\epsilon > 0$. As in the proof of Proposition 4.6, we have the diffeomorphism

$$\psi : W \times \mathcal{L}^1 \times S^1 \longrightarrow W \times B^1$$

defined by

$$\begin{aligned} \psi(w, y, v) = {} & \{w\} \times H(w, \cdot)^{-1} \\ & \times \big(e_1(y), \cos(v)e_2(y) + \sin(v)e_3(y), -\sin(v)e_2(y) + \cos(v)e_3(y)\big) \end{aligned}$$

for $w \in W$, $y \in \mathcal{L}^1$ and $v \in S^1$. The notation $H(w, \cdot)^{-1}$ denotes the inverse of $H(w, \cdot)$ restricted to its image. The orientation of $\mathcal{L}^1 \times S^1$ is given by $(\frac{\partial}{\partial y}, \frac{\partial}{\partial v})$. As in the proof of Propositions 4.5 and 4.6, we have $(\psi^* H^* \theta_i)(\frac{\partial}{\partial v}) = (\psi^* H^* \theta_{1i})(\frac{\partial}{\partial v}) = 0$ $(i = 1, 2, 3)$ and $(\psi^* H^* \theta_{23})(\frac{\partial}{\partial v}) = -1$. It follows that

$$\psi^* H^* \overline{C} = -\frac{1}{4\pi^2} \psi^* H^* \big(d(\theta_1 \wedge \theta_{23})\big) - \frac{i}{4\pi^2} \psi^* H^* (\theta_{23} \wedge d\theta_{23}).$$

Let $q : W \times \mathcal{L}^1 \times S^1 \to W \times \mathcal{L}^1$ be the natural projection. Then

$$\psi^* H^* \theta_{23} = -dv + q^* \sigma^* \theta_{23}.$$

It follows that $\psi^* H^* (\theta_{23} \wedge d\theta_{23}) = -dv \wedge d(q^* \sigma^* \theta_{23})$. From the above orientation convention, by Stokes' theorem, we have

$$\begin{aligned}
&\frac{1}{4\pi^2}\int'_{\mathcal{L}^1\times S^1}\psi^*H^*\big(d(\theta_1\wedge\theta_{23})\big)\\
&\quad=\frac{1}{4\pi^2}d_W\int'_{\mathcal{L}^1\times S^1}\psi^*H^*(\theta_1\wedge\theta_{23})-\frac{1}{4\pi^2}\int'_{\partial\mathcal{L}^1\times S^1}\psi^*H^*(\theta_1\wedge\theta_{23})\\
&\quad=-\frac{1}{2\pi}d_W\int'_{\mathcal{L}^1}\sigma^*\theta_1+\frac{1}{2\pi}\sum_{y\in\partial\mathcal{L}^1}\sigma^*\theta_1|_y
\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}
&\frac{1}{4\pi^2}\int'_{\mathcal{L}^1\times S^1}\psi^*H^*(\theta_{23}\wedge d\theta_{23})\\
&\quad=-\frac{1}{4\pi^2}\int'_{\mathcal{L}^1\times S^1}dv\wedge d\big(q^*\sigma^*\theta_{23}\big)=-\frac{1}{2\pi}\int'_{\mathcal{L}^1}d\big(\sigma^*\theta_{23}\big)\\
&\quad=-\frac{1}{2\pi}d_W\int'_{\mathcal{L}^1}\sigma^*\theta_{23}+\frac{1}{2\pi}\sum_{y\in\partial\mathcal{L}^1}\sigma^*\theta_{23}|_y.
\end{aligned}\tag{5.4}$$

Combining (5.3) and (5.4) proves the lemma. □

Lemma 5.2. *We have the following equality of 1-forms over W:*

$$\lim_{\epsilon\to 0}\bigg(\int'_{B^{2,\epsilon}}H^*\overline{C}+\frac{1}{2\pi}d_W\int'_{\mathcal{L}^{2,\epsilon}}\sigma^*(\theta_1+i\theta_{23})\bigg)=\lim_{\epsilon\to 0}\frac{1}{2\pi}\sum_{y\in\partial\mathcal{L}^{2,\epsilon}}\sigma^*(\theta_1+i\theta_{23})|_y.$$

Proof. We define the map $\tilde{H}:W\times\tilde{B}^2\to\tilde{B}^2_u$ by $\tilde{H}=\mathcal{A}^{-1}\circ H\circ(\mathrm{id},\mathcal{A})$ where $\mathcal{A}$ denotes the diffeomorphism introduced in Lemma 4.3. As in the proof of Propositions 4.5 and 4.6, by (4.10) and (4.15) and denoting $\mathcal{A}^{-1}(B^{2,\epsilon})$ by $\tilde{B}^{2,\epsilon}$,

$$\begin{aligned}
\int'_{B^{2,\epsilon}}H^*\overline{C}&=\int'_{\tilde{B}^{2,\epsilon}}(\mathrm{id},\mathcal{A})^*H^*\overline{C}=\int'_{\mathcal{L}^{2,\epsilon}\times S^1}\psi^*\tilde{H}^*\mathcal{A}^*\overline{C}\\
&=-\frac{1}{4\pi^2}\int'_{\mathcal{L}^{2,\epsilon}\times S^1}\psi^*\tilde{H}^*\big(d(\theta_1\wedge\theta_{23})\big)+i\psi^*\tilde{H}^*(\theta_{23}\wedge d\theta_{23})\\
&\quad-\frac{1}{4\pi^2}\int'_{\mathcal{L}^{2,\epsilon}\times S^1}\psi^*\tilde{H}^*d\bigg(\sum_{j=1}^{3}\theta_j\wedge(a_{j1}A_2\cdot dA_3+a_{j2}A_3\cdot dA_1+a_{j3}A_1\cdot dA_2)\bigg)\\
&\quad-\frac{1}{4\pi^2}\int'_{\mathcal{L}^{2,\epsilon}\times S^1}i\psi^*\tilde{H}^*d(\theta_{12}\wedge d\hat{A}_1\cdot\hat{A}_2+\theta_{13}\wedge d\hat{A}_1\cdot\hat{A}_3+\theta_{23}\wedge d\hat{A}_2\cdot\hat{A}_3)\\
&\quad-\frac{1}{4\pi^2}\int'_{\mathcal{L}^{2,\epsilon}\times S^1}\frac{i}{6}\psi^*\tilde{H}^*\,\mathrm{Tr}\big(\big(A^{-1}dA\big)^3\big).
\end{aligned}$$

By $\psi^*\tilde{H}^*(\theta_i)(\frac{\partial}{\partial v}) = 0$, $\psi^*\tilde{H}^*(\theta_{1j})(\frac{\partial}{\partial v}) = 0$, $\psi^*\tilde{H}^*(dA_j)(\frac{\partial}{\partial v}) = 0$, and $\psi^*\tilde{H}^*(d\hat{A}_k)(\frac{\partial}{\partial v}) = 0$, all the integrals vanish except the integral of $\psi^*\tilde{H}^* d(\theta_{23} \wedge d\hat{A}_2 \cdot \hat{A}_3)$ for the terms on the last three lines of the above equalities. But we have the following equality:

$$\begin{aligned} &\frac{1}{4\pi^2} \int\limits_{\mathcal{L}^{2,\epsilon}\times S^1}' \psi^*\tilde{H}^* d(\theta_{23} \wedge d\hat{A}_2 \cdot \hat{A}_3) \\ &\quad = \frac{1}{2\pi} d_W \int\limits_{\mathcal{L}^{2,\epsilon}}' \psi^*\tilde{H}^*(d\hat{A}_2 \cdot \hat{A}_3) - \frac{1}{2\pi} \sum_{y \in \partial\mathcal{L}^{2,\epsilon}} \psi^*\tilde{H}^*(d\hat{A}_2 \cdot \hat{A}_3)|_y. \end{aligned}$$

The first term in the second line can be shown to give the variation with respect to w of the sum of the total rotation angles of $\hat{A}_2$ around $\hat{A}_1$ along the components of $\mathcal{L}^{2,\epsilon}$. But by our assumptions on the framing, the matrix A defined in (3.7) limits to the identity at the boundary $\partial\mathcal{L}^2 \cap D$. Hence the limit of this term as $\epsilon \to 0$ gives an integer, which is invariant under the deformation. The last term is 0 since the contributions from boundary points in the interior of M cancel by an invariance under identification by the $L_r(u(w))$, and at the remaining boundary points, $\psi^*\tilde{H}^*(d\hat{A}_2 \cdot \hat{A}_3)|_y \to 0$ as $\epsilon \to 0$. From these equalities, we have

$$\lim_{\epsilon\to 0} \int\limits_{B^{2,\epsilon}}' H^*\overline{C} = \lim_{\epsilon\to 0} -\frac{1}{4\pi^2} \int\limits_{\mathcal{L}^{2,\epsilon}\times S^1}' \psi^*\tilde{H}^*\big(d(\theta_1 \wedge \theta_{23})\big) + i\psi^*\tilde{H}^*(\theta_{23} \wedge d\theta_{23}).$$

Now, repeating the derivation in (5.3) and (5.4) and recalling that ψ is an orientation reversing diffeomorphism in this case, completes the proof. □

Now we deal with the partial integral over $B^{0,\epsilon}$. Recall that D^ϵ is the subset in $\mathbf{D} = \mathbf{D}_0$ corresponding to X^ϵ. First we have

Lemma 5.3. *We have the following equality of 1-forms on W:*

$$\begin{aligned} \int\limits_{B^{0,\epsilon}}' H^*\overline{C} &= \int\limits_{D^\epsilon\setminus\mathcal{L}}' \sigma^*\overline{C} \\ &= \frac{1}{4\pi^2} \int\limits_{(D^\epsilon\setminus\mathcal{L})}' \big(d_D\omega_{23} \wedge (\chi_1 + i\chi_{23}) + \omega_{23} \wedge \big(d_D(\chi_1 + i\chi_{23}) + i d_W\omega_{23}\big)\big) \end{aligned}$$

where $d = d_W + d_D$ over $W \times D^\epsilon$. Here χ_1 and χ_{23} are defined by $\sigma^*\theta_1 = \omega_1 + \chi_1$ and $\sigma^*\theta_{23} = \omega_{23} + \chi_{23}$, where $\omega_1|_{TW} = \omega_{23}|_{TW} = 0$ and $\chi_1|_{TD^\epsilon} = \chi_{23}|_{TD^\epsilon} = 0$.

Note that ω_1, ω_{23}, χ_1 and χ_{23} depend on $w \in W$.

Proof of Lemma 5.3. By Proposition 4.2, it is easy to see

$$\sigma^*\overline{C} = -\frac{1}{4\pi^2}\big(d(\omega_{23}+\chi_{23})\wedge(\omega_1+i\omega_{23}+\chi_1+i\chi_{23})\big) + \frac{i}{4\pi^2}\big(d(\omega_1+\chi_1)\wedge(\omega_1+i\omega_{23}+\chi_1+i\chi_{23})\big).$$

Now, note that $d_D\omega_1 = 0$ by Lemma 4.4, and that ω_1 vanishes on tangent vectors of $(D^\epsilon \setminus \mathcal{L})$. Also note that

$$0 = d_W \int\limits_{D^\epsilon\setminus\mathcal{L}}' \omega_{23}\wedge\omega_1 = \int\limits_{D^\epsilon\setminus\mathcal{L}}' (d_W\omega_{23})\wedge\omega_1 - \int\limits_{D^\epsilon\setminus\mathcal{L}}' \omega_{23}\wedge(d_W\omega_1),$$

so $\int'_{D^\epsilon\setminus\mathcal{L}} \omega_{23}\wedge(d_W\omega_1) = 0$. Now, recalling that the orientation of D^ϵ is opposite to that of $B^{0,\epsilon}$, the result follows from direct computation. □

5.3. Limit of contribution over $B^{0,\epsilon}$

Now we want to push the expression of Lemma 5.3 down to the boundary $D \subset \hat{\mathbb{C}} \subset \partial H^3$. This will be accomplished in Proposition 5.7. First we need to prove some preliminary results, which will also be useful later.

By the uniformization of X by Γ, we identify X with $\Gamma\backslash\Omega$. Then the hyperbolic metric g_X of constant curvature -1 on X (or the flat metric g_X of area 1 in the case that X has genus 1) gives a metric $e^{\phi(z)}|dz|^2$ on Ω, invariant under the action of Γ. The invariance implies that

$$\phi(z) = \phi(\gamma z) + \log|\gamma'(z)|^2 \tag{5.5}$$

for all $z \in \Omega$ and $\gamma \in \Gamma$.

Proposition 5.4. *The set D^ϵ in H^3 is given by $D^\epsilon = \{(t,x,y) \in H^3 \mid t = \mathfrak{t}(\epsilon,x,y)\}$, where $\mathfrak{t}$ is a function satisfying*

$$\mathfrak{t}(\epsilon,x,y) = \epsilon e^{-\frac{\phi(x,y)}{2}} + k(\epsilon,x,y)\epsilon^3,$$

where k, k_x and k_y exist and are bounded on $\mathbf{D} \cup D$.

Proof. Let us recall that there is a unique defining function r over a collar neighborhood N of X in $\overline{M}$ such that the rescaled metric $\bar{g} := r^2 g_M$ extends smoothly to $\overline{M}$, its restriction to X is the hyperbolic metric g_X (or the flat metric g_X of area 1 in the case that X has genus 1) and $|dr|^2_{\bar{g}} = 1$. Let us denote the lifted defining function over the inverse image of N in H^3 by the same notation r, and write $\hat{r} := \frac{r}{t}$. Then the three conditions on $\bar{g}$ imply that $\hat{r}$ extends smoothly to $\mathbf{D} \cup D$, $\lim_{t\to 0}\hat{r}(t,x,y) = e^{\frac{\phi}{2}}$, and

$\hat{r}_t = -\frac{1}{2}(\hat{r}_t^2 + \hat{r}_x^2 + \hat{r}_y^2)\hat{r}^{-1}t$ respectively. Since $|\hat{r}_t| \leq Ct$ for a uniform constant C, we have $|(\hat{r} - e^{\frac{\phi}{2}})_t| \leq Ct$. Since $\hat{r}$ is smooth on $\mathbf{D} \cup D$, this means $|\hat{r} - e^{\frac{\phi}{2}}| \leq Ct^2$ for a uniform constant C, and therefore

$$\hat{r}(t,x,y) = e^{\frac{\phi(x,y)}{2}} + \alpha(t,x,y)t^2, \tag{5.6}$$

where α is uniformly bounded.

Similarly, since $\hat{r}_t = 0$ on D, we have $\hat{r}_{xt} = 0$ on D. Again, since $\hat{r}$ is smooth, we obtain $|(\hat{r} - e^{\frac{\phi}{2}})_x| \leq Ct^2$ for some uniform constant C. This implies

$$\hat{r}_x(t,x,y) = \left(e^{\frac{\phi(x,y)}{2}}\right)_x + \alpha_x(t,x,y)t^2$$

where α_x is uniformly bounded, and similarly for $\hat{r}_y$. This implies the claimed expression for $\mathfrak{t}$. Now, since $k = -\alpha e^{-\frac{\phi}{2}}\hat{r}^{-3}$, and $\hat{r}$ is nowhere zero on D, the result follows. $\square$

A holomorphic 1-form Φ with only simple zeroes over X is given by $h(z)dz$ over Ω with $h(\gamma z)\gamma'(z) = h(z)$ for $\gamma \in \Gamma$. The phase function $e^{i\theta(z)} := h(z)/|h(z)|$ is well-defined over $\Omega \setminus \bigcup_{\gamma\in\Gamma} \gamma Z$ where $Z := \{z_1, \ldots, z_{2g-2}\}$ denotes the zero set of $h(z)$ in a fixed fundamental domain D of Γ. The transformation law of $h(z)$ implies

$$i\theta(z) = i\theta(\gamma z) + \log \frac{\gamma'(z)}{|\gamma'(z)|} \tag{5.7}$$

for $\gamma \in \Gamma$. Note that θ is defined only up to an integer multiple of 2π. By (5.5), (5.7), it follows that $e^{\phi(z)/2+i\theta(z)}dz = \omega_2 + i\omega_3$ is invariant under the action of Γ; in particular,

$$\omega_2 = e^{\phi/2}(\cos\theta dx - \sin\theta dy), \qquad \omega_3 = e^{\phi/2}(\sin\theta dx + \cos\theta dy)$$

provides us with an orthonormal invariant co-frame (ω_2, ω_3) over $\Omega \setminus \bigcup_{\gamma\in\Gamma} \gamma Z$. Now we obtain an orthonormal framing

$$\mathcal{F}_\Phi = (f_2, f_3) \quad \text{where } f_2 = \omega_2^*,\ f_3 = \omega_3^*$$

over $D' := D \setminus Z$.

Near a zero $z_k \in Z$, $h(z)$ has an expression $h(z) = (z - z_k)\tilde{h}_k(z)$ such that $\tilde{h}_k(z)$ is non-vanishing at z_k. Now we put $e^{i\tilde{\theta}_k(z)} := \tilde{h}_k(z)/|\tilde{h}_k(z)|$. Since $\tilde{h}_k(z)$ is non-vanishing at $z = z_k$, $\tilde{\theta}_k(z_k)$ is well-defined only up to an integer multiple of 2π. As in (3.6), we define

$$\begin{aligned}\tilde{\omega}_2 &= e^{\phi/2}\big(\cos(\tilde{\theta}_k/2)dx - \sin(\tilde{\theta}_k/2)dy\big),\\ \tilde{\omega}_3 &= e^{\phi/2}\big(\sin(\tilde{\theta}_k/2)dx + \cos(\tilde{\theta}_k/2)dy\big)\end{aligned} \tag{5.8}$$

at $z_k \in Z$. Then the duals $(\tilde{f}_2, \tilde{f}_3)$ of $(\tilde{\omega}_2, \tilde{\omega}_3)$ define an orthonormal framing at $z_k \in Z$. That this orthonormal framing is well-defined up to sign follows from the fact that

$h(\gamma z)\gamma'(z) = h(z)$ and the following equality for $\gamma \in \Gamma$ and $z, z_k \in \Omega$:

$$(\gamma z - \gamma z_k) = (z - z_k)\gamma_z(z)^{\frac{1}{2}}\gamma_z(z_k)^{\frac{1}{2}}.$$

Proposition 5.5. *The one form ω_{23} on $\mathbf{D}'$ extends smoothly to a form on $\mathbf{D}' \cup D'$. We have*

$$\lim_{t\to 0} \omega_{23} = \frac{i}{2}\big((\phi - 2i\theta)_z dz - (\phi + 2i\theta)_{\bar{z}} d\bar{z}\big),$$

where the convergence in the global coordinate on H^3 is uniform on $\mathbf{D}' \cup D'$.

Note that the extension of ω_{23} to D' coincides with the connection form of the hyperbolic metric $e^{\phi}|dz|^2$, with respect to our choice of orthonormal frame $\mathcal{F}_{\Phi}$.

Proof of Proposition 5.5. By the Koszul formula, we have

$$\omega_{23} = g\big([e_2, e_3], e_2\big)\omega_2 + g\big([e_2, e_3], e_3\big)\omega_3$$

for an orthonormal frame (e_1, e_2, e_3) where e_1 is orthogonal to TD^{ϵ}. Here g denotes the hyperbolic metric over $\mathbf{D}'$ that is the lifting of g_M. By the asymptotics of the boundary defining function r in (5.6), we have

$$e_1 = t\big(1 + \tfrac{1}{4}t^2\big(\phi_x^2 + \phi_y^2\big)\big)^{-\frac{1}{2}}\big(\tfrac{1}{2}t\phi_x\partial_x + \tfrac{1}{2}t\phi_y\partial_y + \partial_t\big) + O\big(t^3\big),$$
$$e_2 = \alpha_{22}\bar{e}_2 + \alpha_{23}\bar{e}_3, \qquad e_3 = \alpha_{32}\bar{e}_2 + \alpha_{33}\bar{e}_3$$

with

$$\bar{e}_2 = t\big(1 + \tfrac{1}{4}t^2\phi_x^2\big)^{-\frac{1}{2}}\big(\partial_x - \tfrac{1}{2}t\phi_x\partial_t\big) + O\big(t^3\big),$$
$$\bar{e}_2 = t\big(1 + \tfrac{1}{4}t^2\phi_y^2\big)^{-\frac{1}{2}}\big(\partial_y - \tfrac{1}{2}t\phi_y\partial_t\big) + O\big(t^3\big).$$

Here and below, we use $O(t^k)$ to indicate a function of the form $a(t, x, y)t^k$ with respect to the global coordinate on H^3, where a is uniformly bounded in $\mathbf{D}' \cup D'$. To compute $g([e_2, e_3], e_2)$, $g([e_2, e_3], e_3)$, we consider $[e_2, e_3]$ first. By an elementary computation,

$$\begin{aligned}[e_2, e_3] = {} & (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})[\bar{e}_2, \bar{e}_3] \\ & + \big(\alpha_{22}\bar{e}_2(\alpha_{32}) - \alpha_{32}\bar{e}_2(\alpha_{22}) + \alpha_{23}\bar{e}_3(\alpha_{32}) - \alpha_{33}\bar{e}_3(\alpha_{22})\big)\bar{e}_2 \\ & + \big(\alpha_{22}\bar{e}_2(\alpha_{33}) - \alpha_{32}\bar{e}_2(\alpha_{23}) + \alpha_{23}\bar{e}_3(\alpha_{33}) - \alpha_{33}\bar{e}_3(\alpha_{23})\big)\bar{e}_3. \end{aligned} \tag{5.9}$$

Using Proposition 5.4, we have

$$[\bar{e}_2, \bar{e}_3] = \big(\tfrac{1}{2}t^2\phi_y\big)\partial_x - \big(\tfrac{1}{2}t^2\phi_x\big)\partial_y + O\big(t^3\big),$$

from which we also have

$$g\big([\bar{e}_2,\bar{e}_3],e_2\big)=\alpha_{22}\big(\tfrac{1}{2}t\phi_y\big)+\alpha_{23}\big(-\tfrac{1}{2}t\phi_x\big)+O\big(t^2\big)=\tfrac{1}{2}t(\cos\theta\,\phi_y+\sin\theta\,\phi_x)+O\big(t^2\big),$$
$$g\big([\bar{e}_2,\bar{e}_3],e_3\big)=\alpha_{32}\big(\tfrac{1}{2}t\phi_y\big)+\alpha_{33}\big(-\tfrac{1}{2}t\phi_x\big)+O\big(t^2\big)=\tfrac{1}{2}t(\sin\theta\,\phi_y-\cos\theta\,\phi_x)+O\big(t^2\big).$$

Here we used the fact $\alpha_{22}=\alpha_{33}=\cos\theta+O(t)$, $\alpha_{23}=-\alpha_{32}=-\sin\theta+O(t)$. Denoting by E the sum of the terms in the second and third lines on the right hand side of (5.9),

$$g(E,e_2)=-t(\cos\theta\,\theta_x-\sin\theta\,\theta_y)+O\big(t^2\big),$$
$$g(E,e_3)=-t(\sin\theta\,\theta_x+\cos\theta\,\theta_y)+O\big(t^2\big).$$

Finally we need

$$\omega_2=t^{-1}\big(\cos\theta dx-\sin\theta dy+O(t)\big),\qquad \omega_3=t^{-1}\big(\sin\theta dx+\cos\theta dy+O(t)\big).$$

Combining all the proved equalities, we have

$$\begin{aligned}\omega_{23}&=\left(\frac{1}{2}\cos\theta\,\phi_y+\frac{1}{2}\sin\theta\,\phi_x-\cos\theta\,\theta_x+\sin\theta\,\theta_y\right)(\cos\theta\,dx-\sin\theta\,dy)\\&\quad+\left(\frac{1}{2}\sin\theta\,\phi_y-\frac{1}{2}\cos\theta\,\phi_x-\sin\theta\,\theta_x-\cos\theta\,\theta_y\right)(\sin\theta\,dx+\cos\theta\,dy)+O(t)\\&=d\theta+\frac{1}{2}(\phi_y dx-\phi_x dy)+O(t).\end{aligned}$$

This completes the proof. □

Now, we define $c_1=\chi_1(\frac{\partial}{\partial w})$ and $c_{23}=\chi_{23}(\frac{\partial}{\partial w})$, where χ_1 and χ_{23} were defined in Lemma 5.3, and w is a local coordinate in W. We will write $'$ for the derivative with respect to w, for instance, $\phi'=\frac{\partial}{\partial w}\phi$.

Proposition 5.6. *The functions c_1,c_{23} on $W\times\mathbf{D}'$ extend smoothly to functions on $W\times(\mathbf{D}'\cup D')$. We have*

$$\lim_{t\to 0}c_1=-\frac{1}{2}\phi'\circ f,\qquad \lim_{t\to 0}c_{23}=\theta'\circ f+i\left(\frac{\phi}{2}-i\theta\right)_z f',$$

and the convergence in the global coordinate on H^3 is uniform on $\mathbf{D}'\cup D'$. We also have $\lim_{t\to 0}\chi_1(\frac{\partial}{\partial\bar{w}})=\bar{c}_1$, $\lim_{t\to 0}\chi_{23}(\frac{\partial}{\partial\bar{w}})=\bar{c}_{23}$.

Proof. Observe that c_1 is given by

$$\big(s\circ(u,\mathbf{f})\big)^*\theta_1\big(\tfrac{\partial}{\partial w}\big)=\theta_1\big(s_*u_*\tfrac{\partial}{\partial w}\big)+s^*\theta_1\big(\mathbf{f}_*u_*\tfrac{\partial}{\partial w}\big)=\omega_1\big(\mathbf{f}'\big),$$

where the second equality holds since $s_* u_* \frac{\partial}{\partial w}$ is vertical. Recall that the level surface D^ϵ is given by $\{(t,x,y) \in H^3 \mid t = \mathfrak{t}(\epsilon,x,y) = \epsilon e^{-\frac{\phi(x,y)}{2}} + O(\epsilon^3)\}$, and that the definition of $\mathbf{f}$ near the boundary given by (5.1) involves translation along gradient curves for r. Since translation from D to D^ϵ introduces an error of $O(\epsilon^2)$, and since f_z and f'_z are bounded on D, we have

$$\mathbf{f}\bigl(w,(t,z)\bigr) = \bigl(\mathfrak{t}\bigl(r(t,z), f(z)\bigr), f(z)\bigr) + O\bigl(t^2\bigr).$$

Here and below we understand $O(t^2)$ to be uniform as discussed in the previous proposition. Therefore we have

$$\mathbf{f}' = f' \tfrac{\partial}{\partial z} - \bigl(\tfrac{1}{2} t\bigl(\phi' \circ f + \phi_z f'\bigr)\bigr) \tfrac{\partial}{\partial t} + O\bigl(t^2\bigr).$$

The one form ω_1 is the dual of the first component e_1 of the orthonormal frame over the level surface D^ϵ so that

$$\omega_1 = \bigl(1 + \tfrac{1}{4} t^2 \bigl(\phi_x^2 + \phi_y^2\bigr)\bigr)^{-1/2} t^{-1} \bigl(\tfrac{1}{2} t \phi_z dz + \tfrac{1}{2} t \phi_{\bar z} d\bar z + dt\bigr) + O\bigl(t^3\bigr).$$

Hence, we have

$$\omega_1\bigl(\mathbf{f}'\bigr) = \frac{1}{2}\phi_z f' - \frac{1}{2}\phi' \circ f - \frac{1}{2}\phi_z f' + O(t) = -\frac{1}{2}\phi' \circ f + O(t),$$

from which it follows

$$\lim_{t\to 0} c_1 = \lim_{t\to 0} \omega_1\bigl(\mathbf{f}'\bigr) = -\frac{1}{2}\phi' \circ f. \tag{5.10}$$

As above, c_{23} is given by

$$\bigl(s \circ (u, \mathbf{f})\bigr)^* \theta_{23}\bigl(\tfrac{\partial}{\partial w}\bigr) = \theta_{23}\bigl(s_* u_* \tfrac{\partial}{\partial w}\bigr) + s^* \theta_{23}\bigl(\mathbf{f}_* u_* \tfrac{\partial}{\partial w}\bigr) = \theta_{23}\bigl(s'\bigr) + \omega_{23}\bigl(\mathbf{f}'\bigr).$$

Now, we have $\theta_{23} = \mathcal{L}^*_{g^{-1}}(-2(ih)^*)$ and $\lim_{t\to 0}(\mathcal{L}_{g^{-1}})_* s_* u_*(\frac{\partial}{\partial w}) = -\frac{1}{2}\theta_w(ih)$, where $h = \bigl(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\bigr) \in \mathfrak{sl}_2(\mathbb{C})$ and $\mathcal{L}_g$ is the left translation by $g \in PSL_2(\mathbb{C})$ (see Section 3 of [21]). By this and Proposition 5.5,

$$\lim_{t\to 0} c_{23} = \lim_{t\to 0}\bigl(\theta_{23}\bigl(s'\bigr) + \omega_{23}\bigl(\mathbf{f}'\bigr)\bigr) = \theta' \circ f + i\left(\frac{\phi}{2} - i\theta\right)_z f'. \tag{5.11}$$

The equalities (5.10) and (5.11) complete the proof of the first two equalities. Replacing $\frac{\partial}{\partial w}$ with $\frac{\partial}{\partial \bar w}$ in the computations above gives the last part of the statement. □

We denote by the same notations ω_{23}, c_1, c_{23}, the restriction to $W \times D'$ of the extensions of ω_{23}, c_1, c_{23} respectively, obtained in Propositions 5.5 and 5.6.

Now let us introduce some additional notation. The local coordinate expression for the members of the family $\{\Phi(w)\}$ can be identified with a map $h : \{(w, \Omega_{u(w)}) : w \in W\} \to \mathbb{C}$. We define $z_k : W \to \mathbb{C}$ to be the coordinates in $\Omega_{u(w)}$ of the zeroes of $\Phi(w)$, that is, $h(w, z_k(w)) = 0$ for all $w \in W$. Near each $z_k(w)$, we define $\tilde{h}_k$ by

$$h\big(w, f(w,z)\big) = \big(f(w,z) - f\big(w, z_k(w)\big)\big)\tilde{h}_k\big(w, f(w,z)\big) \tag{5.12}$$

for all $w \in W$.

Proposition 5.7. *The limit of the 1-form*

$$\lim_{\epsilon\to 0} \int\limits_{B^{0,\epsilon}}' H^*\overline{C} \tag{5.13}$$

over W is finite, and its $(1,0)$ part equals

$$\frac{1}{4\pi^2} \int\limits_{D'}' \big(d_D\omega_{23} \wedge (c_1 + ic_{23})dw + \omega_{23} \wedge \big(d_D(c_1 + ic_{23}) \wedge dw + i\partial_w\omega_{23}\big)\big) \tag{5.14}$$

where $d = d_W + d_D = \partial_w + \overline{\partial_w} + d_D$ over $W \times D$.

Proof. We have that

$$\lim_{\epsilon\to 0} \int\limits_{B^{0,\epsilon}}' H^*\overline{C} = \lim_{\epsilon\to 0} \int\limits_{s(P_\epsilon(D'))}' H^*\overline{C} = \lim_{\epsilon\to 0} \int\limits_{D'}' P_\epsilon^* s^* H^*\overline{C}.$$

Propositions 5.5 and 5.6, and the definition of admissible singularity, show that $s^*H^*\overline{C}$ extends continuously to D', and is uniformly bounded. Therefore we can exchange limit and integral in the last integral. Hence, the integral (5.14) equals the $(1,0)$ part of (5.13) by Lemma 5.3, Propositions 5.5 and 5.6. Now we prove the integral (5.14) is finite. By Eq. (5.12), near $z_k \in Z$ we have

$$\begin{aligned} (c_1 + ic_{23})(z) &= -\frac{1}{2}\big((\phi - 2i\theta)\circ f\big)'(z) \\ &= \frac{1}{2}\frac{f'(z) - f'(z_k) - f_z(z_k)(z_k)'}{z - z_k} - \frac{1}{2}\big((\phi - \log\tilde{h})\circ f\big)'(z). \end{aligned}$$

Note that $d_D\omega_{23}$ is a constant times the volume form and $c_1 + ic_{23}$ is singular at Z by the above equality, but its wedge product with the volume form is integrable. For the second term, we use the following formula,

$$\begin{aligned} &d_D(c_1 + ic_{23}) \wedge dw + i\partial_w\omega_{23} \\ &\quad = -\big(\phi'_{\bar\zeta}\circ f\overline{f}_z + \phi_{\zeta\bar\zeta}\circ f\overline{f}_z f'\big)dz \wedge dw - \big(\phi'_{\bar\zeta}\circ f\overline{f}_{\bar z} + \phi_{\zeta\bar\zeta}\circ f\overline{f}_{\bar z} f'\big)d\bar z \wedge dw \end{aligned}$$

where $\zeta = f(z)$, which can be derived from Propositions 5.5 and 5.6. Although ω_{23} is singular at Z, its wedge product with the expression above is integrable. This shows that the integral (5.14) is finite, hence $(1,0)$ part of (5.13) is finite. Similarly, the $(0,1)$ part of (5.13) is equal to the complex conjugate of (5.14) and is therefore also finite. $\square$

5.4. Holomorphic variation of $\overline{\mathbb{CS}}$

We begin this subsection with

Proposition 5.8. *Over $W \subset \mathbb{C}$, we have*

$$d\big(u^*\overline{\mathbb{CS}}\big) = \int_{B^0}' H^*\overline{C} - \frac{1}{2\pi}\sum_{y\in\partial\mathcal{L}^1}\sigma^*(\theta_1 + i\theta_{23})|_y + \frac{1}{2\pi}\sum_{y\in\partial\mathcal{L}^2}\sigma^*(\theta_1 + i\theta_{23})|_y + \sum_{r=1}^{g}\int_{T_r - L_rT_r}' H^*\overline{C}.$$

Here the sums over $\partial\mathcal{L}^1$ and $\partial\mathcal{L}^2$ are taken with signs inherited from the orientations on $\mathcal{L}^1$ and $\mathcal{L}^2$.

Proof. First, note that $\lim_{\epsilon\to 0} d_W(u^*\mathbb{CS}^\epsilon) = d_W(u^*\mathbb{CS})$ since the diverging term $\frac{2}{\pi}(1 - g)\log\epsilon$ in Definition 4.8 vanishes under d_W. By Proposition 5.7, the partial integral over $B^{0,\epsilon}$ in (5.2) converges to a finite limit as $\epsilon \to 0$. By Lemma 5.2 and a similar analysis in the proof of Proposition 5.6, the right hand side of the equality in Lemma 5.2 also converges as $\epsilon \to 0$. Hence this is also true for the last terms in (5.2) given by the sum of the partial integrals over $(T_r^\epsilon - L_rT_r^\epsilon)$. Taking $\epsilon \to 0$ on both sides of (5.2), and using Lemmas 5.1 and 5.2 we have the result. $\square$

The remainder of this section is devoted to finding an explicit expression for $d\,\overline{\mathbb{CS}}(\varpi)$ in the case that $\varpi \in T^{1,0}U$ at u_0 is a holomorphic tangent vector. The final result is given in Theorem 5.14.

Lemma 5.9. *For the holomorphic curve $u : W \to U$, we have*

$$\sigma^*(\theta_1 + i\theta_{23})|_{y_j(0)} - \sigma^*(\theta_1 + i\theta_{23})|_{L_{r(j)}(u(0))y_j(0)} = -(L_{r(j)}\circ u)^*(\theta_1 + i\theta_{23})$$

where $(L_{r(j)}\circ u)^*(\theta_1 + i\theta_{23})$ is a $(0,1)$-form on W for $j \in \mathcal{J}$.

Proof. For brevity we write $L_{r(j)}(w) := L_{r(j)}(u(w))$ and $y_j := y_j(0)$. The map $w \mapsto \sigma(w, L_{r(j)}(0)y_j) = L_{r(j)}(w)\sigma(w, y_j)$ is the composition of the maps

$$W \xrightarrow{L_{r(j)}\times\sigma(y_j)} PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C}) \xrightarrow{G} PSL_2(\mathbb{C})$$

where G denotes the multiplication map. Since $\theta_1 + i\theta_{23}$ is a bi-invariant 1-form on $PSL_2(\mathbb{C})$, we obtain $G^*(\theta_1 + i\theta_{23}) = p_1^*(\theta_1 + i\theta_{23}) + p_2^*(\theta_1 + i\theta_{23})$ where p_i denotes the projection onto i-th factor $PSL_2(\mathbb{C})$. It follows that

$$\begin{aligned}&\sigma\big(L_{r(j)}(0)y_j\big)^*(\theta_1 + i\theta_{23})\\ &\quad = \big((L_{r(j)} \circ u)\sigma(y_j)\big)^*(\theta_1 + i\theta_{23}) = (L_{r(j)} \circ u)^*(\theta_1 + i\theta_{23}) + \sigma(y_j)^*(\theta_1 + i\theta_{23}).\end{aligned}$$

Hence,

$$\sigma^*(\theta_1 + i\theta_{23})|_{y_j} - \sigma^*(\theta_1 + i\theta_{23})|_{L_{r(j)}(0)y_j} = -(L_{r(j)} \circ u)^*(\theta_1 + i\theta_{23}).$$

Since $L_{r(j)} \circ u : W \to PSL_2(\mathbb{C})$ is a holomorphic map, and $\theta_1 + i\theta_{23}$ is a $(0,1)$-form on $PSL_2(\mathbb{C})$ (see the section 3 of [21]), the statement follows. $\square$

Lemma 5.10. *The partial integral* $\sum_{r=1}^{g} \int'_{T_r - L_r T_r} H^*\overline{C}$ *is a* $(0,1)$*-form over* W*.*

Proof. For each $w \in W$ and $x \in D^r$,

$$\begin{aligned}H\big(w, s_0\big(L_r(0)x\big)\big) &= s\big(u(w), \mathbf{f}\big(w, L_r(0)x\big)\big) = s\big(u(w), L_r(w)\mathbf{f}(w,x)\big)\\ &= L_r(w)s\big(u(w), \mathbf{f}(w,x)\big) = L_r(w)H\big(w, s_0(x)\big),\end{aligned}$$

where $L_r(w) := L_r(u(w))$. Hence $H : W \times L_r T_r \to PSL_2(\mathbb{C})$ can be considered as the composition of the maps

$$W \times T_r \xrightarrow{L_r \times H} PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C}) \xrightarrow{G} PSL_2(\mathbb{C})$$

where $(L_r \times H)(w, s_0(x)) = (L_r(w), H(w, s_0(x)))$ and G denotes the multiplication map. The pull back of $\overline{C}$ by G is given by $G^*\overline{C} = p_1^*\overline{C} + (G^*\overline{C})^{2,1} + (G^*\overline{C})^{1,2} + p_2^*\overline{C}$, where $p_i : PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$, $i = 1, 2$, are the projections from the two factors, and where superscripts on a form indicate the degree in the two factors. Taking the pull back of $G^*\overline{C}$ by $L_r \times H$, we have

$$\big(G(L_r \times H)\big)^*\overline{C} = L_r^*\overline{C} + (L_r \times H)^*\big(G^*\overline{C}\big)^{2,1} + (L_r \times H)^*\big(G^*\overline{C}\big)^{1,2} + H^*\overline{C}.$$

Hence we have the following equality for the partial integrals:

$$\int'_{T^r} H^*\overline{C} - \int'_{L_r T^r} H^*\overline{C} = -\int'_{T^r} (L_r \times H)^*\big(G^*\overline{C}\big)^{1,2}.$$

Since the map $w \in W \mapsto L_r(w) \in PSL_2(\mathbb{C})$ is holomorphic, the dw term in $(L_r \times H)^*(G^*\overline{C})^{1,2}$ vanishes under the above partial integration. Hence the 1-form on

W obtained by the partial integration of $\int'_{T^r - L_r T^r} H^*\overline{C}$ does not involve dw, that is, it is of type $(0,1)$. $\square$

From now on, $\dot{}$ will denote the derivative with respect to w at $w = 0$, for instance, $\dot\phi = \frac{\partial}{\partial w}|_{w=0}\phi$. By the results on varying the hyperbolic metric in [1], we have, for all $z \in \Omega$,

$$\dot\phi + \phi_z \dot f + \dot f_z = 0. \tag{5.15}$$

(The same is true for the flat metric of area 1 in the case that the genus of X is 1.) From this, we also have

$$\dot\phi_z + \phi_{zz}\dot f + \phi_z \dot f_z + \dot f_{zz} = 0, \qquad \dot\phi_{\bar z} + \phi_{z\bar z}\dot f + \phi_z \dot f_{\bar z} + \dot f_{z\bar z} = 0. \tag{5.16}$$

Since $2i\theta = \log h - \log \bar h$,

$$2i\theta_z = \frac{h_z}{h}, \qquad 2i\theta_{\bar z} = -\frac{\bar h_{\bar z}}{\bar h}, \qquad \theta_{z\bar z} = 0. \tag{5.17}$$

Since Φ is a holomorphic family, we also have

$$\dot\theta_{\bar z} = 0. \tag{5.18}$$

It will be convenient in what follows to make the definition $\psi := \phi - 2i\theta$.

Lemma 5.11. *The following terms are invariant under the action of Γ,*

$$-\dot f_z - (2i\theta)^{\cdot} - (2i\theta)_z \dot f = \dot\psi + \psi_z \dot f.$$

Proof. The equality follows from $\dot\phi + \phi_z \dot f + \dot f_z = 0$. To see the invariance under the action of Γ, we note

$$(\phi - 2i\theta)^{\cdot}(z) = (\phi - 2i\theta)^{\cdot}(\gamma z) + (\phi - 2i\theta)_z(\gamma z)\dot\gamma(z),$$
$$(\phi - 2i\theta)_z(z) = (\phi - 2i\theta)_z(\gamma z)\gamma_z(z),$$

which follow from (5.5) and (5.7). Combining these and $\dot f \circ \gamma = \dot\gamma + \gamma_z \dot f$ completes the proof. $\square$

From now on, for convenience, we abbreviate $z_k(0)$ to z_k, and $\dot z_k(0)$ to $\dot z_k$.

Proposition 5.12. *For $\varpi \in T^{1,0}U$ at $u_0 \in U$, we have*

$$\partial\overline{\mathbb{CS}}(\varpi) = \frac{1}{4\pi^2}\int\limits_D d\omega_{23} \wedge (c_1 + ic_{23}) + \omega_{23} \wedge \big(d(c_1 + ic_{23}) - i\dot{\omega}_{23}\big)$$
$$-\frac{1}{4\pi}\sum_{z_k \in Z}\left(\dot{f}_z + \frac{1}{2}(\log\tilde{h}_k)^{\cdot} + \frac{1}{2}(\log\tilde{h}_k)_z\dot{f} - \left(\phi - \frac{1}{2}\log\tilde{h}_k\right)_z f_z\dot{z}_k\right)(z_k)$$

where Z denotes the set of zeros of Φ in the fundamental domain D of Γ and $\tilde{h}_k$ is defined by Eq. (5.12).

Proof. By Proposition 5.8 and Lemma 5.9, $\partial\overline{\mathbb{CS}}(\varpi)$ is equal to the evaluation of the one form

$$\int\limits_{B_0}{}' H^*\overline{C} + n(j)\frac{1}{2\pi}\sum_{j \in \mathcal{J}}(L_{r(j)} \circ u)^*(\theta_1 + i\theta_{23})$$
$$+\frac{1}{2\pi}\sum_{y \in (\partial\mathcal{L}^2 \cap D)}\sigma^*(\theta_1 + i\theta_{23})|_y + \sum_{r=1}^{g}\int\limits_{(T_r - L_rT_r)}{}' H^*\overline{C}$$

on $\frac{\partial}{\partial w}$. Here $n(j)$ is the index of the singularity at the corresponding component of $\mathcal{L}$, so $n(j) = 1$ or -1 if the points $y_j(0)$, $L_{r(j)}(0)y_j(0)$ are in $\partial\mathcal{L}^1$ or $\partial\mathcal{L}^2$ respectively. By Lemma 5.3 and Proposition 5.7, the evaluation of the first term on $\frac{\partial}{\partial w}$ is given by

$$\frac{1}{4\pi^2}\int\limits_D d\omega_{23} \wedge (c_1 + ic_{23}) + \omega_{23} \wedge \big(d(c_1 + ic_{23}) - i\dot{\omega}_{23}\big).$$

The second and fourth terms vanish on $\frac{\partial}{\partial w}$, since they are $(0,1)$-forms by Lemmas 5.9 and 5.10. Using Lemma 5.2, and following the proof of Proposition 5.6, we find that the third term evaluated on $\frac{\partial}{\partial w}$ is given by

$$\left(\frac{1}{2\pi}\sum_{y \in (\partial\mathcal{L}^2 \cap D)}\sigma^*(\theta_1 + i\theta_{23})|_y\right)\left(\frac{\partial}{\partial w}\right)$$
$$= \frac{1}{4\pi}\sum_{z_k \in Z}\big((\dot{\phi} - i\tilde{\theta}^{\cdot}) + (\phi - i\tilde{\theta})_z\dot{f} + (\phi - i\tilde{\theta})_z f_z\dot{z}_k\big)(z_k)$$
$$= \frac{1}{4\pi}\sum_{z_k \in Z}\left(-\dot{f}_z - \frac{1}{2}(\log\tilde{h}_k)^{\cdot} - \frac{1}{2}(\log\tilde{h}_k)_z\dot{f} + \left(\phi - \frac{1}{2}\log\tilde{h}_k\right)_z f_z\dot{z}_k\right)(z_k).$$

Here the last equality follows from (5.8) and (5.15). This completes the proof. □

Proposition 5.13. *The following equality holds:*

$$\begin{aligned}
&\frac{1}{4\pi^2}\int\limits_D d\omega_{23}\wedge(c_1+ic_{23})+\omega_{23}\wedge\big(d(c_1+ic_{23})-i\dot\omega_{23}\big)\\
&\quad=-\frac{1}{2\pi^2}\lim_{\delta\to 0}\int\limits_{D_\delta}\left(\phi_{zz}-\frac{1}{2}\phi_z^2-2\theta_z^2-2i\theta_{zz}\right)\mu d^2z\\
&\qquad-\frac{1}{4\pi}\sum_{z_k\in Z}\big(2\dot f_z+(\log\tilde h_k)^{\cdot}+(\log\tilde h_k)_z\dot f+(\phi-\log\tilde h_k)_z f_z\dot z_k\big)(z_k),
\end{aligned}$$

where D_δ is a subset of D whose δ-open neighborhoods of Z are removed and $d^2z = \frac{i}{2}dz\wedge d\bar z$.

Note that, since infinitesimal circles are preserved to first order under holomorphic change of coordinates, the limit as $\delta\to 0$ is independent of the choice of local coordinates.

Proof. By Proposition 5.6,

$$\begin{aligned}
c_1+ic_{23}&=-\frac{1}{2}(\psi\circ f)^{\cdot}\\
&=-\frac{1}{2}(\dot\phi+\phi_z\dot f-2i\dot\theta-2i\theta_z\dot f)=\left(i\dot\theta+i\theta_z\dot f+\frac{1}{2}\dot f_z\right)
\end{aligned}\tag{5.19}$$

where we used (5.15) for the third equality. From (5.19), we can also derive

$$d(c_1+ic_{23})=-\frac{1}{2}\big((\dot\psi_z+\psi_{zz}\dot f+\psi_z\dot f_z)dz+(\dot\psi_{\bar z}+\psi_{z\bar z}\dot f+\psi_z\dot f_{\bar z})d\bar z\big).\tag{5.20}$$

By Proposition 5.5,

$$-i\dot\omega_{23}=\frac{1}{2}\big((\dot\psi_z+\psi_{zz}\dot f+\psi_z\dot f_z)dz+(-\dot{\bar\psi}_{\bar z}-\bar\psi_{z\bar z}\dot f+\psi_z\dot f_{\bar z})\,d\bar z\big).\tag{5.21}$$

Again by Proposition 5.5 and (5.17),

$$d\omega_{23}=-i\phi_{z\bar z}dz\wedge d\bar z=-i\psi_{z\bar z}dz\wedge d\bar z.$$

Combining this and (5.19), (5.20), (5.21), we get

$$d\omega_{23}\wedge(c_1+ic_{23})=-i\psi_{z\bar z}\left(i\dot\theta+i\theta_z\dot f+\frac{1}{2}\dot f_z\right)dz\wedge d\bar z,$$

which is an invariant $(1,1)$-form under the action of Γ by Lemma 5.11 and

$$\omega_{23}\wedge\big(d(c_1+ic_{23})-i\dot\omega_{23}\big)=-\frac{i}{2}\psi_z(\dot\phi_{\bar z}+\phi_{z\bar z}\dot f)dz\wedge d\bar z=\frac{i}{2}\psi_z(\phi_z\dot f_{\bar z}+\dot f_{z\bar z})dz\wedge d\bar z$$

where we used (5.16) for the last equality.

By the above equalities and (5.17), (5.18),

$$\begin{aligned}
&\int_{D_\delta} d\omega_{23} \wedge (c_1 + ic_{23}) \\
&\quad = -i \int_{D_\delta} \psi_{z\bar{z}} \left(i\dot{\theta} + i\theta_z \dot{f} + \frac{1}{2}\dot{f}_z \right) dz \wedge d\bar{z} \\
&\quad = i \int_{D_\delta} \psi_z \left(i\theta_z \dot{f}_{\bar{z}} + \frac{1}{2}\dot{f}_{z\bar{z}} \right) dz \wedge d\bar{z} + i \int_{\partial D_\delta} \psi_z \left(i\dot{\theta} + i\theta_z \dot{f} + \frac{1}{2}\dot{f}_z \right) dz \\
&\quad = i \int_{D_\delta} \left(\psi_z i\theta_z - \frac{1}{2}\psi_{zz} \right) \dot{f}_{\bar{z}}\, dz \wedge d\bar{z} + i \int_{\partial D_\delta} \psi_z \left(i\dot{\theta} + i\theta_z \dot{f} + \frac{1}{2}\dot{f}_z \right) dz + \frac{1}{2}\psi_z \dot{f}_{\bar{z}} d\bar{z}
\end{aligned} \tag{5.22}$$

where ∂D_δ has the induced orientation from D_δ. In the integral over ∂D_δ, the contributions from C^r and $-L_r(0)C^r$ cancel, since the integrands concerned are invariant. We also have

$$\begin{aligned}
&\int_{D_\delta} \omega_{23} \wedge \big(d(c_1 + ic_{23}) - i\dot{\omega}_{23} \big) \\
&\quad = \frac{i}{2} \int_{D_\delta} \psi_z (\phi_z \dot{f}_{\bar{z}} + \dot{f}_{z\bar{z}})\, dz \wedge d\bar{z} \\
&\quad = \frac{i}{2} \int_{D_\delta} (\psi_z \phi_z - \psi_{zz}) \dot{f}_{\bar{z}}\, dz \wedge d\bar{z} + \frac{i}{2} \int_{\partial D_\delta} \psi_z \dot{f}_{\bar{z}} d\bar{z},
\end{aligned} \tag{5.23}$$

where once again the contributions from C^r and $-L_r(0)C^r$ cancel in the integral over ∂D_δ. By (5.19), (5.22) and (5.23),

$$\begin{aligned}
&\int_{D_\delta} d\omega_{23} \wedge (c_1 + ic_{23}) + \omega_{23} \wedge \big(d(c_1 + ic_{23}) - i\dot{\omega}_{23} \big) \\
&\quad = \frac{i}{2} \int_{D_\delta} (\psi_z \bar{\psi}_z - 2\psi_{zz}) \dot{f}_{\bar{z}}\, dz \wedge d\bar{z} - \frac{i}{2} \int_{\partial D_\delta} \psi_z (\psi \circ f)^{\cdot}\, dz + i \int_{\partial D_\delta} \psi_z \dot{f}_{\bar{z}} d\bar{z}.
\end{aligned} \tag{5.24}$$

For the last integral on the right hand side of (5.24), we have

$$\begin{aligned}
i \int_{\partial D_\delta} \psi_z \dot{f}_{\bar{z}}\, d\bar{z} &= -i \sum_{z_k \in Z} \int_{|z - z_k| = \delta} \left(-\frac{1}{z - z_k} + (\phi - \log \tilde{h}_k)_z \right) \dot{f}_{\bar{z}}\, d\bar{z} \\
&= -i \sum_{z_k \in Z} \int_{|z - z_k| = \delta} -\frac{1}{z - z_k} \dot{f}_{\bar{z}}\, d\bar{z} + O(\delta) = O(\delta).
\end{aligned} \tag{5.25}$$

To analyze the second integral on the right hand side of (5.24), we use (5.12). This implies that, near $z_k \in Z$, we have

$$\big((\phi - 2i\theta) \circ f\big)^{\cdot}(z) = -\frac{\dot{f}(z) - \dot{f}(z_k) - f_z(z_k)\dot{z}_k}{z - z_k} + \big((\phi - \log \tilde{h}_k) \circ f\big)^{\cdot}(z). \quad (5.26)$$

Therefore, we can rewrite the second integral of (5.24) as

$$\begin{aligned}
&-\frac{i}{2} \int\limits_{\partial D_\delta} \psi_z \big((\phi - 2i\theta) \circ f\big)^{\cdot} dz \\
&= \frac{i}{2} \sum_{z_k \in Z} \Bigg(\int\limits_{|z - z_k| = \delta} \left(-\frac{1}{z - z_k} + (\phi - \log \tilde{h}_k)_z \right) \left(-\frac{\dot{f}(z) - \dot{f}(z_k) - f_z(z_k)\dot{z}_k}{z - z_k} \right) dz \\
&\quad + \int\limits_{|z - z_k| = \delta} \left(-\frac{1}{z - z_k} + (\phi - \log \tilde{h}_k)_z \right) \big((\phi - \log \tilde{h}_k) \circ f\big)^{\cdot}(z)\, dz \Bigg) \\
&= -\pi \sum_{z_k \in Z} \big(\dot{f}_z(z_k) + (\phi - \log \tilde{h}_k)_z(z_k) f_z(z_k) \dot{z}_k - \big((\phi - \log \tilde{h}_k) \circ f\big)^{\cdot}(z_k) \big) + O(\delta) \\
&= -\pi \sum_{z_k \in Z} \big(2\dot{f}_z + (\log \tilde{h}_k)^{\cdot} + (\log \tilde{h}_k)_z \dot{f} + (\phi - \log \tilde{h}_k)_z f_z \dot{z}_k \big)(z_k) + O(\delta).
\end{aligned}$$

Combining this with (5.24) and (5.25), we conclude

$$\begin{aligned}
&\lim_{\delta \to 0} \int\limits_{D_\delta} d\omega_{23} \wedge (c_1 + ic_{23}) + \omega_{23} \wedge \big(d(c_1 + ic_{23}) - i\dot{\omega}_{23} \big) \\
&= \lim_{\delta \to 0} \int\limits_{D_\delta} \big(\phi_z^2 - 2\phi_{zz} + 4\theta_z^2 + 4i\theta_{zz} \big) \dot{f}_{\bar{z}}\, d^2 z \\
&\qquad - \pi \sum_{z_k \in Z} \big(2\dot{f}_z + (\log \tilde{h}_k)^{\cdot} + (\log \tilde{h}_k)_z \dot{f} + (\phi - \log \tilde{h}_k)_z f_z \dot{z}_k \big)(z_k).
\end{aligned}$$

Recalling that $\dot{f}_{\bar{z}} = \mu$ completes the proof. $\square$

Note that we have the formulae

$$\mathcal{S}\big(J^{-1}\big) = \phi_{zz} - \frac{1}{2}\phi_z^2, \qquad \mathcal{S}(h_\Phi) = \frac{h_{zz}}{h} - \frac{3}{2}\frac{h_z^2}{h^2} = 2\theta_z^2 + 2i\theta_{zz}$$

where $\mathcal{S}$ denotes the Schwarzian derivative, $J : H^2 \to \Omega$ is the universal covering map of Ω, (or $J : \mathbb{C} \to \Omega$ in the case of genus 1), and h_Φ is a multi-valued function such that $dh_\Phi = \Phi$. By these formulae and Propositions 5.12 and 5.13, we have the following theorem.

Theorem 5.14. *For $\varpi \in T^{1,0}U$ at $u_0 \in U$, and the corresponding $\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma)$,*

$$\partial\overline{\mathbb{CS}}(\varpi) = -\frac{1}{2\pi^2} \lim_{\delta\to 0} \int\limits_{D_\delta} \big(\mathcal{S}\big(J^{-1}\big) - \mathcal{S}(h_\Phi)\big)\mu \, d^2z$$
$$- \frac{1}{4\pi} \sum_{z_k \in Z} \left(3\dot{f}_z + \frac{3}{2}(\log \tilde{h}_k)^{\cdot} + \frac{3}{2}(\log \tilde{h}_k)_z \dot{f} - \frac{1}{2}(\log \tilde{h}_k)_z f_z \dot{z}_k\right)(z_k).$$

Corollary 5.15. *For $\varpi \in T^{1,0}U$ at $u_0 \in U$, and the corresponding $\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma)$,*

$$\partial CS(\varpi) = \frac{i}{4\pi^2} \lim_{\delta\to 0} \int\limits_{D_\delta} \mathcal{S}(h_\Phi)\mu \, d^2z$$
$$- \frac{i}{8\pi} \sum_{z_k \in Z} \left(3\dot{f}_z + \frac{3}{2}(\log \tilde{h}_k)^{\cdot} + \frac{3}{2}(\log \tilde{h}_k)_z \dot{f} - \frac{1}{2}(\log \tilde{h}_k)_z f_z \dot{z}_k\right)(z_k).$$

Proof. This follows from directly from Theorem 5.14, since we have

$$\mathbb{CS}(M_X, s_\Phi) = \frac{1}{\pi^2} W(M) + 2i\, CS(M_X, s_\Phi),$$

and, by [15,24], it is known that $\partial W = \frac{1}{4}\phi_z^2 - \frac{1}{2}\phi_{zz} = -\frac{1}{2}\mathcal{S}(J^{-1})$. □

6. Regularized Polyakov integral over *X*

In this section, we introduce a regularized integral defined in terms of the metric g_X and the holomorphic 1-form Φ over X. We assume that g_X is the hyperbolic metric if the genus of X is greater than 1, and the flat metric of area 1 if the genus equals 1. We assume that Φ has only simple zeroes and we denote by Z its zero set.

Now we define

$$I(X, \Phi) = \lim_{\delta\to 0} \left(\int\limits_{X_\delta} |\psi_z|^2 \, d^2z + \frac{i}{2} \sum_{p_k \in Z} \int\limits_{S_\delta(z_k)} \frac{(\phi - 2\log|h|)(z)}{\bar{z} - \bar{z}_k} d\bar{z} \right)$$
$$- \pi \sum_{p_k \in Z} \big(\phi - \log|\tilde{h}_k|\big)(z_k). \tag{6.1}$$

Here z, in the integral around p_k, represents a local coordinate near p_k, with $z_k = z(p_k)$. The set X_δ denotes the complement of δ-open discs $|z - z_k| < \delta$ centered at each $z_k \in Z$ in X, and $S_\delta(z_k)$ denotes a part of ∂X_δ which is the δ-circle centered at z_k with the induced orientation from X_δ. Note that each of the terms in (6.1) are independent of the choice of local coordinates, by the transformation laws given in Section 3.3. Note also that, since circles are preserved under change of coordinates, the limit as $\delta \to 0$ is independent of the choice of local coordinates. Hence I is a well-defined function on $\mathcal{H}_g$.

Suppose that ϖ is a tangent vector at $u_0 \in \mathcal{H}_g$, and that U is a neighborhood of u_0. We define a corresponding curve $u : W \to U$, for $W \subset \mathbb{C}$, and a corresponding deformation map $f(w, \cdot) : X \to X_w$ for each $w \in W$, in the same way as in Section 5.1. We also define the local coordinate expressions h, $\tilde{h}_k$ and z_k in the same way as the discussion before Eq. (5.12), except that here we do not assume a global uniformization coordinate, only local coordinates near the zeroes of Φ. For convenience we abbreviate $z_k(0)$ to z_k, and $\dot{z}_k(0)$ to $\dot{z}_k$.

Theorem 6.1. *For $\varpi \in T^{1,0}\mathcal{H}_g$ at the point (X, Φ) and the corresponding $\mu \in \mathcal{H}^{-1,1}(X)$,*

$$\partial I(\varpi) = 2 \lim_{\delta \to 0} \int_{X_\delta} \left(\phi_{zz} - \frac{1}{2}\phi_z^2 - 2\theta_z^2 - 2i\theta_{zz}\right) \mu \, d^2 z$$

$$+ \pi \sum_{p_k \in Z} \left(3\dot{f}_z + \frac{3}{2}(\log \tilde{h}_k)^{\cdot} + \frac{3}{2}(\log \tilde{h}_k)_z \dot{f} - \frac{1}{2}(\log \tilde{h}_k)_z f_z \dot{z}_k\right)(z_k).$$

Here $\phi_{zz} - \frac{1}{2}\phi_z^2 - 2\theta_z^2 - 2i\theta_{zz}$ is a meromorphic quadratic differential over X.

Proof. The domain $X_{w,\delta}$ is given by deleting the δ-discs centered at the $f(w, z_k(w))$ for $z_k \in Z$. Its boundaries are given by the circles $S_\delta(f(w, z_k(w)))$. Now we consider the pre-image domain, denoted by the same notation, of $X_{w,\delta}$ by f_w in X which has boundaries denoted by $B_\delta(z_k(w))$. Let us take δ_0 such that the δ_0-disc centered at z_k contains $B_\delta(z_k(w))$ for each $z_k \in Z$, and take w in an open neighborhood W of the origin in $\mathbb{C}$. Then $X_{w,\delta}$ in X decomposes into $X_{\delta_0} \cup A_{\delta_0,\delta}$. Here $A_{\delta_0,\delta} = \bigcup_{z_k \in Z} A_{\delta_0,\delta}(z_k)$ where the region $A_{\delta_0,\delta}(z_k)$ has two boundaries $S_{\delta_0}(z_k)$ and $B_\delta(z_k(w))$.

For the integral $|\psi_z|^2 d^2 z$ over $A_{\delta_0,\delta}$, we have

$$\begin{aligned}
&\int_{A_{\delta_0,\delta(z_k)}} |\psi_z|^2 d^2 z \\
&= \int_{A_{\delta_0,\delta(z_k)}} \left|(\phi - \log \tilde{h}_k)_z\right|^2 d^2 z - \int_{A_{\delta_0,\delta(z_k)}} \frac{(\phi - \log \bar{\tilde{h}}_k)_{\bar{z}}}{z - z_k(w)} d^2 z \\
&\quad - \int_{A_{\delta_0,\delta}(z_k)} \frac{(\phi - \log h)_z}{\bar{z} - \bar{z}_k(w)} d^2 z \\
&= \int_{A_{\delta_0,\delta(z_k)}} \left|(\phi - \log \tilde{h}_k)_z\right|^2 d^2 z + \frac{i}{2} \int_{\partial A_{\delta_0,\delta(z_k)}} \frac{(\phi - \log \bar{\tilde{h}}_k)}{z - z_k(w)} dz \\
&\quad - \frac{i}{2} \int_{\partial A_{\delta_0,\delta}(z_k)} \frac{(\phi - 2\log|h|)}{\bar{z} - \bar{z}_k(w)} d\bar{z}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int\limits_{X_{w,\delta}} |\psi_z|^2\, d^2z + \frac{i}{2}\sum_{p_k\in Z}\ \int\limits_{B_\delta(z_k(w))} \frac{(\phi - 2\log|h|)}{\bar z - \bar z_k(w)} d\bar z \\
&\quad = \int\limits_{X_{\delta_0}} |\psi_z|^2\, d^2z + \int\limits_{A_{\delta_0,\delta}} \big|(\phi - \log\tilde h_k)_z\big|^2\, d^2z + \frac{i}{2}\int\limits_{\partial A_{\delta_0,\delta}} \frac{(\phi - \log\bar{\tilde h}_k)}{z - z_k(w)}\, dz \\
&\qquad + \frac{i}{2}\sum_{p_k\in Z}\ \int\limits_{S_{\delta_0}(z_k)} \frac{(\phi - 2\log|h|)}{\bar z - \bar z_k(w)}\, d\bar z
\end{aligned} \tag{6.2}$$

where $B_\delta(z_k(w))$ and $S_{\delta_0}(z_k)$ have the orientation induced from $A_{\delta_0,\delta}(z_k)$ and X_{δ_0} respectively.

Now, we consider the holomorphic variation of each of the terms on the right hand side of (6.2). First, we deal with the term $I_{\delta_0} = \int_{X_{\delta_0}} |\psi_z|^2\, d^2z$. For this, observe that

$$\begin{aligned}
\delta_\mu(\psi_z dz) &= (\dot\psi_z + \psi_{zz}\dot f)dz + \psi_z(\dot f_z dz + \dot f_{\bar z} d\bar z), \\
\delta_\mu(\bar\psi_{\bar z} d\bar z) &= (\dot{\bar\psi}_{\bar z} + \bar\psi_{\bar z z}\dot f)d\bar z = (\dot\phi_{\bar z} + \phi_{\bar z z}\dot f)d\bar z.
\end{aligned}$$

Here, δ_μ denotes the Lie derivative. See Section 2.3 of [17] for details. Combining these facts with (5.16) and Lemma 5.11, we have

$$\begin{aligned}
\partial I_{\delta_0}(\varpi) = &-\frac{i}{2}\int\limits_{X_{\delta_0}} \psi_z(\phi_z \dot f_{\bar z} + \dot f_{z\bar z})\, dz\wedge d\bar z \\
&- \frac{i}{2}\int\limits_{X_{\delta_0}} \bar\psi_{\bar z}\big(\dot f_{zz} + (2i\theta)_z{}^{\cdot} + \big((2i\theta)_z \dot f\big)_z\big)\, dz\wedge d\bar z.
\end{aligned} \tag{6.3}$$

Let us denote the two terms on the right hand side of (6.3) by $(\partial I_{\delta_0}(\varpi))_i$ for $i = 1, 2$. Recalling that $\psi_z \dot f_{\bar z} d\bar z$ is an invariant $(0,1)$-form, we have

$$\big(\partial I_{\delta_0}(\varpi)\big)_1 = -\frac{i}{2}\bigg(\int\limits_{X_{\delta_0}} \psi_z\phi_z \dot f_{\bar z}\, dz\wedge d\bar z - \int\limits_{X_{\delta_0}} \psi_{zz}\dot f_{\bar z}\, dz\wedge d\bar z + \int\limits_{\partial X_{\delta_0}} \psi_z \dot f_{\bar z}\, d\bar z\bigg)$$

where ∂X_{δ_0} has the induced orientation from X_{δ_0}. For $(\partial I_{\delta_0}(\varpi))_2$, by Lemma 5.11 and (5.17), (5.18),

$$\begin{aligned}
\big(\partial I_{\delta_0}(\varpi)\big)_2 = \frac{i}{2}\bigg(&\int\limits_{X_{\delta_0}} \psi_{z\bar z}\big(\dot f_z + (2i\theta)^{\cdot} + (2i\theta)_z \dot f\big)\, dz\wedge d\bar z \\
&- \int\limits_{\partial X_{\delta_0}} \bar\psi_{\bar z}\big(\dot f_z + (2i\theta)^{\cdot} + (2i\theta)_z \dot f\big)\, d\bar z\bigg)
\end{aligned}$$

$$= -\frac{i}{2}\bigg(\int\limits_{X_{\delta_0}} \psi_z\big(\dot{f}_{z\bar{z}} + (2i\theta)_z\dot{f}_{\bar{z}}\big)\,dz\wedge d\bar{z}$$

$$+ \int\limits_{\partial X_{\delta_0}} \big(\dot{f}_z + (2i\theta)^{\cdot} + (2i\theta)_z\dot{f}\big)(\psi_z dz + \bar{\psi}_{\bar{z}}d\bar{z})\bigg).$$

Dealing with the term $\psi_z\dot{f}_{z\bar{z}}$ as before,

$$\big(\partial I_{\delta_0}(\varpi)\big)_2 = -\frac{i}{2}\bigg(\int\limits_{X_{\delta_0}} \psi_z(2i\theta)_z\dot{f}_{\bar{z}}\,dz\wedge d\bar{z} - \int\limits_{X_{\delta_0}} \psi_{zz}\dot{f}_{\bar{z}}\,dz\wedge d\bar{z}$$

$$+ \int\limits_{\partial X_{\delta_0}} \psi_z\dot{f}_{\bar{z}}\,d\bar{z} + \int\limits_{\partial X_{\delta_0}} \big(\dot{f}_z + (2i\theta)^{\cdot} + (2i\theta)_z\dot{f}\big)(\psi_z dz + \bar{\psi}_{\bar{z}}d\bar{z})\bigg).$$

Combining computations for $(\partial I_{\delta_0}(\varpi))_1$ and $(\partial I_{\delta_0}(\varpi))_2$, we get

$$\partial I_{\delta_0}(\varpi) = -\frac{i}{2}\bigg(\int\limits_{X_{\delta_0}} \psi_z\bar{\psi}_z\dot{f}_{\bar{z}}\,dz\wedge d\bar{z} - 2\int\limits_{X_{\delta_0}} \psi_{zz}\dot{f}_{\bar{z}}\,dz\wedge d\bar{z}$$

$$+ 2\int\limits_{\partial X_{\delta_0}} \psi_z\dot{f}_{\bar{z}}\,d\bar{z} + \int\limits_{\partial X_{\delta_0}} \big(\dot{f}_z + (2i\theta)^{\cdot} + (2i\theta)_z\dot{f}\big)(\psi_z dz + \bar{\psi}_{\bar{z}}d\bar{z})\bigg). \quad (6.4)$$

Now let us deal with the integrals over ∂X_{δ_0}. First, by (5.25) we have

$$\int\limits_{\partial X_{\delta_0}} \psi_z\dot{f}_{\bar{z}}\,d\bar{z} = \int\limits_{\partial X_{\delta_0}} (\phi - 2i\theta)_z\dot{f}_{\bar{z}}\,d\bar{z} = O(\delta_0). \quad (6.5)$$

For the other boundary integral given in the last line of (6.4), using (5.26), near $p_k \in Z$ we have

$$\begin{aligned}
&-\big(\dot{f}_z + (2i\theta)^{\cdot} + (2i\theta)_z\dot{f}\big)\big((\phi - 2i\theta)_z dz + (\phi + 2i\theta)_{\bar{z}}d\bar{z}\big)\\
&\quad = \big((\phi - 2i\theta)\circ f\big)^{\cdot}\big((\phi - 2i\theta)_z dz + (\phi + 2i\theta)_{\bar{z}}d\bar{z}\big)\\
&\quad = \bigg(-\frac{\dot{f}(z) - \dot{f}(z_k) - f_z(z_k)\dot{z}_k}{z - z_k} + \big((\phi - \log\tilde{h}_k)\circ f\big)^{\cdot}\bigg)\\
&\qquad \cdot\bigg(\bigg(-\frac{1}{z - z_k} + (\phi - \log\tilde{h}_k)_z\bigg)dz + \bigg(-\frac{1}{\bar{z} - \bar{z}_k} + (\phi - \log\bar{\tilde{h}}_k)_{\bar{z}}\bigg)d\bar{z}\bigg).
\end{aligned}$$

Using this and some computation as before, we obtain

$$
\begin{aligned}
&-\frac{i}{2}\int\limits_{\partial X_{\delta_0}} \big(\dot{f}_z + (2i\theta)^{\cdot} + (2i\theta)_z\dot{f}\big)\big((\phi-2i\theta)_z dz + (\phi+2i\theta)_{\bar{z}} d\bar{z}\big) \\
&= -\frac{i}{2}\sum_{p_k\in Z}\Bigg(\int\limits_{|z-z_k|=\delta_0}\bigg(-\frac{\dot{f}(z)-\dot{f}(z_k)-f_z(z_k)\dot{z}_k}{z-z_k}\bigg)\bigg(-\frac{1}{z-z_k}dz - \frac{1}{\bar{z}-\bar{z}_k}d\bar{z}\bigg) \\
&\quad + \bigg(-\frac{\dot{f}(z)-\dot{f}(z_k)-f_z(z_k)\dot{z}_k}{z-z_k}\bigg)\big((\phi-\log\tilde{h}_k)_z dz + (\phi-\log\bar{\tilde{h}}_k)_{\bar{z}} d\bar{z}\big) \\
&\quad + \big((\phi-\log\tilde{h}_k)\circ f\big)^{\cdot}(z)\bigg(-\frac{1}{z-z_k}dz - \frac{1}{\bar{z}-\bar{z}_k}d\bar{z}\bigg)\Bigg) + O(\delta_0) \\
&= \pi\sum_{p_k\in Z}(\phi-\log\tilde{h}_k)_z(z_k)f_z(z_k)\dot{z}_k + O(\delta_0). \qquad (6.6)
\end{aligned}
$$

By (6.4), (6.5) and (6.6),

$$
\begin{aligned}
&\partial\bigg(\int\limits_{X_{\delta_0}}|\psi_z|^2\, d^2z\bigg)(\varpi) \\
&= \int\limits_{X_{\delta_0}}\big(2\phi_{zz}-\phi_z^2-4\theta_z^2-4i\theta_{zz}\big)\dot{f}_{\bar{z}}\, d^2z + \pi\sum_{p_k\in Z}(\phi-\log\tilde{h}_k)_z(z_k)f_z(z_k)\dot{z}_k + O(\delta_0).
\end{aligned}
\tag{6.7}
$$

The holomorphic variation of the second term $\int_{A_{\delta_0,\delta}}|(\phi-\log\tilde{h}_k)_z|^2\, d^2z$ on the right hand side of (6.2) can be analyzed as above, but the integrand $|(\phi-\log\tilde{h}_k)_z|^2$ is regular over $A_{\delta_0,\delta}$ for any $\delta > 0$. Hence, we can see that

$$
\partial\bigg(\lim_{\delta\to 0}\int\limits_{A_{\delta_0,\delta}}\big|(\phi-\log\tilde{h}_k)_z\big|^2\, d^2z\bigg)(\varpi) = O(\delta_0). \tag{6.8}
$$

The limit of the third term on the right hand side of (6.2) as $\delta\to 0$ is given by

$$
\sum_{p_k\in Z}\frac{i}{2}\int\limits_{|z-z_k|=\delta_0}\frac{(\phi-\log\bar{\tilde{h}}_k)}{z-z_k(w)}\, dz + \pi(\phi_w-\log\bar{\tilde{h}}_{k,w})\big(f\big(w,z_k(w)\big)\big) \tag{6.9}
$$

where ϕ_w, $\tilde{h}_{k,w}$ denote (local) functions over X_w. For the holomorphic variation of the first term in (6.9), we have

$$
\begin{aligned}
&\partial\bigg(\frac{i}{2}\int\limits_{|z-z_k|=\delta_0}\frac{(\phi-\log\bar{\tilde{h}}_k)}{z-z_k(w)}\, dz\bigg)(\varpi) \\
&= \frac{i}{2}\int\limits_{|z-z_k|=\delta_0} -\frac{\dot{f}(z)-\dot{f}(z_k)-f_z(z_k)\dot{z}_k}{(z-z_k)^2}(\phi-\log\bar{\tilde{h}}_k)\, dz
\end{aligned}
$$

$$+\frac{\dot{\phi}+\phi_z\dot{f}}{z-z_k}\,dz+\frac{\phi-\log\bar{\tilde{h}}_k}{z-z_k}(\dot{f}_z dz+\dot{f}_{\bar{z}}d\bar{z})$$
$$=-\pi(\dot{\phi}+\phi_z\dot{f}+\phi_z f_z\dot{z}_k)(z_k)+O(\delta_0).$$

For the second term in (6.9), we have

$$\partial\big(\pi(\phi_w-\log\bar{\tilde{h}}_{k,w})\big(f\big(w,z_k(w)\big)\big)\big)(\varpi)=\pi(\dot{\phi}+\phi_z\dot{f}+\phi_z f_z\dot{z}_k)(z_k).$$

Hence,

$$\partial\left(\lim_{\delta\to 0}\frac{i}{2}\int\limits_{\partial A_{\delta_0,\delta}}\frac{(\phi-\log\bar{\tilde{h}}_k)}{z-z_k(w)}\,dz\right)(\varpi)=O(\delta_0). \tag{6.10}$$

In a similar way, we can show the following equality for the fourth term on the right hand side of (6.2),

$$\partial\left(\frac{i}{2}\sum_{p_k\in Z}\int\limits_{S_{\delta_0}(z_k)}\frac{(\phi-2\log|h|)}{\bar{z}-\bar{z}_k(w)}\,d\bar{z}\right)(\varpi)$$
$$=\pi\sum_{p_k\in Z}\big(\dot{f}_z-(\phi-\log\tilde{h}_k)^{\cdot}-(\phi-\log\tilde{h}_k)_z\dot{f}\big)(z_k)+O(\delta_0). \tag{6.11}$$

Combining the equalities (6.7), (6.8), (6.10), and (6.11), we have

$$\partial\left(\lim_{\delta\to 0}\left(\int\limits_{X_{w,\delta}}|\psi_z|^2\,d^2z+\frac{i}{2}\sum_{p_k\in Z}\int\limits_{B_\delta(z_k(w))}\frac{(\phi-2\log|h|)(z)}{\bar{z}-\bar{z}_k(w)}d\bar{z}\right)\right)(\varpi)$$
$$=\lim_{\delta_0\to 0}\int\limits_{X_{\delta_0}}\big(2\phi_{zz}-\phi_z^2-4\theta_z^2-4i\theta_{zz}\big)\dot{f}_{\bar{z}}\,d^2z$$
$$+\pi\sum_{p_k\in Z}\big(\dot{f}_z-(\phi-\log\tilde{h}_k)^{\cdot}-(\phi-\log\tilde{h}_k)_z\dot{f}+(\phi-\log\tilde{h}_k)_z f_z\dot{z}_k\big)(z_k). \tag{6.12}$$

Finally combining (6.2), (6.12) and the following equality

$$\partial\left(-\pi\sum_{p_k\in Z}(\phi-\log|\tilde{h}_k|)(z_k)\right)(\varpi)$$
$$=-\pi\sum_{p_k\in Z}\left(\left(\phi-\frac{1}{2}\log\tilde{h}_k\right)^{\cdot}+\left(\phi-\frac{1}{2}\log\tilde{h}_k\right)_z(\dot{f}+f_z\dot{z}_k)\right)(z_k)$$

completes the proof. □

The quantity $I(X,\Phi)$ in (6.1) is related to a Polyakov type formula which describes the difference of two regularized determinants of Laplacians defined by two different metrics. In [22] (see also [17]), it was shown that for $g>1$

$$\frac{\det\Delta_{\mathrm{hyp}}}{A_{\mathrm{hyp}}\det\langle\Phi_j,\Phi_k\rangle}=c\exp\left(-\frac{1}{12\pi}S\right)|F|^2 \quad \text{over } \mathfrak{S}_g \tag{6.13}$$

where Δ_{hyp} is the Laplacian in the unique metric of constant curvature -1 on X, A_{hyp} is the area of X in that metric, $\{\Phi_1,\dots,\Phi_g\}$ is a basis of holomorphic 1-forms normalized with respect to the marking, and S is the real valued classical Liouville action functional over $\mathfrak{S}_g$. Note that this is distinct from the usual expression of $\det\Delta_{\mathrm{hyp}}$ in terms of the Selberg zeta function; in particular, F is holomorphic in moduli. It is known that $S(X)=-4W(M_X)$, when M_X is related to X as above (see [14,20,15]). In [12], Kokotov and Korotkin showed that for $g\geq 1$

$$|\tau_B|^2=c\frac{\det\Delta_{\mathrm{flat}}}{A_{\mathrm{flat}}\det\langle\Phi_j,\Phi_k\rangle} \quad \text{over } \tilde{\mathcal{H}}_g \tag{6.14}$$

where Δ_{flat} is the Laplacian in the flat (singular) metric defined by Φ, and A_{flat} is the area of X in that metric. Combining (6.13) and (6.14), we have

$$|\tau_B|^{24}=c\exp\left(\frac{4}{\pi}W\right)\left(\frac{\det\Delta_{\mathrm{flat}}}{A_{\mathrm{flat}}}\cdot\frac{A_{\mathrm{hyp}}}{\det\Delta_{\mathrm{hyp}}}\right)^{12}|F|^{24} \quad \text{over } \tilde{\mathcal{H}}_g. \tag{6.15}$$

By Theorem 1.1 and (6.15), using the fact that I descends to $\mathcal{H}_g$, we have the following Polyakov type formula,

Corollary 6.2.

$$\frac{\det\Delta_{\mathrm{flat}}}{A_{\mathrm{flat}}}\cdot\frac{A_{\mathrm{hyp}}}{\det\Delta_{\mathrm{hyp}}}=c\exp\left(\frac{1}{12\pi}I\right) \quad \textit{over } \mathcal{H}_g,\ g>1.$$

Note that the usual argument proving the Polyakov formula for two smooth metrics does not apply in our case, since the domains of Δ_{flat} and Δ_{hyp} are different. This formula was first proved by Kokotov and Korotkin, in a slightly different form (combine results from [9] and [12]).

7. Proof of Theorem 1.1

In this section we collect the formulae proved in the previous sections to prove Theorem 1.1. Since we already proved the claimed equality in Theorem 1.1 for $g=1$ case by an elementary computation, we assume $g>1$ throughout this section. For this, first we recall a property of the Schwarzian derivative:

$$\mathcal{S}(h_1 \circ h_2) = \mathcal{S}(h_1) \circ h_2 \big(h_2'\big)^2 + \mathcal{S}(h_2). \tag{7.1}$$

Let $\pi_F : H^2 \to X$ and $\pi_S : \Omega \to X$ denote the Fuchsian and Schottky uniformization maps respectively. Then the universal covering map $J : H^2 \to \Omega$ satisfies $\pi_F = \pi_S \circ J$. Applying this to the composition of multi-valued functions $J^{-1} = \pi_F^{-1} \circ \pi_S$, we obtain

$$\mathcal{S}\big(J^{-1}\big) = \mathcal{S}\big(\pi_F^{-1}\big) \circ \pi_S \big(\pi_S'\big)^2 - \mathcal{S}\big(\pi_S^{-1}\big) \circ \pi_S \big(\pi_S'\big)^2.$$

Similarly applying (7.1) to the composition of multi-valued functions $h_\Phi = (\int^z \Phi) \circ \pi_S$ for a local coordinate z over X, we obtain

$$\mathcal{S}(h_\Phi) = \mathcal{S}\left(\int^z \Phi\right) \circ \pi_S \big(\pi_S'\big)^2 - \mathcal{S}\big(\pi_S^{-1}\big) \circ \pi_S \big(\pi_S'\big)^2.$$

Let us recall that $\mathcal{S}(\pi_F^{-1})$, $\mathcal{S}(\pi_S^{-1})$, $\mathcal{S}(\int^z \Phi)$ define the projective connections R_F, R_S, R_Φ over X respectively.

Lifting the coordinates ζ_i, $i = 1, \dots, 4g-3$ over a contractible open set containing (X, Φ), defined in (2.2), to $U \subset \tilde{\mathcal{H}}_g^*$, we have

Proposition 7.1. *The following holomorphic variation formulae hold for $i = 1, \dots, 4g-3$,*

$$\begin{aligned}
\frac{\partial(\frac{4}{\pi} W)}{\partial \zeta_i} &= \frac{i}{\pi} \int_{s_i} \frac{R_S - R_F}{h}\, dz, \\
\frac{\partial(8\pi i CS)}{\partial \zeta_i} &= \frac{i}{\pi} \int_{s_i} \frac{R_S - R_\Phi}{h}\, dz, \\
\frac{\partial(\frac{1}{\pi} I)}{\partial \zeta_i} &= \frac{i}{\pi} \int_{s_i} \frac{R_F - R_\Phi}{h}\, dz, \\
\frac{\partial(\log F^{24})}{\partial \zeta_i} &= \frac{2i}{\pi} \int_{s_i} \frac{R_B - R_S}{h}\, dz
\end{aligned} \tag{7.2}$$

where the functions W, I, F denote the lifted ones to $\tilde{\mathcal{H}}_g^*$ and $\Phi = h\, dz$. In particular, $\exp(4\pi \mathbb{CS} + \frac{1}{\pi} I)$ *is a holomorphic function on U.*

Proof. Since the proof is essentially same for each case, we prove the formula for $\frac{\partial(8\pi i CS)}{\partial \zeta_i}$. Given a point $u_0 \in U \subset \tilde{\mathcal{H}}_g^*$, with U a contractible open set, and a tangent vector $\varpi \in T^{1,0}U$ at u_0, we have a corresponding $\mu \in \mathcal{H}^{-1,1}(X)$, family of deformations $f_{w\mu}$ and holomorphic family of holomorphic 1-forms $\Phi(w)$. From Corollary 5.15, we have

$$
\begin{aligned}
&\partial(8\pi iCS)(\varpi)\\
&\quad= \frac{2}{\pi}\lim_{\delta\to 0}\int\limits_{X_\delta}(R_S-R_\Phi)\mu\,d^2z\\
&\qquad+\sum_{p_k\in Z}\left(3\dot f_z+\frac{3}{2}(\log\tilde h_k)^{\cdot}+\frac{3}{2}(\log\tilde h_k)_z\dot f-\frac{1}{2}(\log\tilde h_k)_z f_z\dot z_k\right)(z_k).
\end{aligned}
\tag{7.3}
$$

For the first integral in (7.3), we have

$$
\begin{aligned}
&\frac{2}{\pi}\int\limits_{X_\delta}(R_S-R_\Phi)\mu\,d^2z\\
&\quad=\frac{i}{\pi}\int\limits_{X_\delta}\frac{(R_S-R_\Phi)}{h}\,dz\wedge\big((\dot h+h_z\dot f+h\dot f_z)\,dz+(h\dot f)_{\bar z}\,d\bar z\big)\\
&\quad=-\frac{i}{\pi}\int\limits_{X_\delta}d\left(\int\limits_{z_1}^{z}(\dot h+h_z\dot f+h\dot f_z)\,dz+(h\dot f)_{\bar z}\,d\bar z\cdot\frac{(R_S-R_\Phi)}{h}\,dz\right).
\end{aligned}
$$

Here $(\dot h+h_z\dot f+h\dot f_z)\,dz+(h\dot f)_{\bar z}\,d\bar z$ is given by the holomorphic variation of the holomorphic family $\{\Phi(w)\circ f_{w\mu}\}$ so that its line integral defines a well-defined function. As in the proof of the Riemann's bilinear relation to the last line in the above equalities, we have

$$
\begin{aligned}
&\frac{2}{\pi}\int\limits_{X_\delta}(R_S-R_\Phi)\mu\,d^2z\\
&\quad=\frac{i}{\pi}\left(-\sum_{i=1}^{g}\int\limits_{b_i}\frac{(R_S-R_\Phi)}{h}\,dz\cdot\left(\int\limits_{a_i}(\dot h+h_z\dot f+h\dot f_z)\,dz+h\dot f_{\bar z}\,d\bar z\right)\right.\\
&\qquad+\sum_{i=1}^{g}\int\limits_{a_i}\frac{(R_S-R_\Phi)}{h}\,dz\cdot\left(\int\limits_{b_i}(\dot h+h_z\dot f+h\dot f_z)\,dz+h\dot f_{\bar z}\,d\bar z\right)\\
&\qquad\left.+\sum_{k=1}^{2g-2}\int\limits_{|z-z_k|=\delta}\left(\int\limits_{z_1}^{z}(\dot h+h_z\dot f+h\dot f_z)\,dz+h\dot f_{\bar z}\,d\bar z\right)\frac{(R_S-R_\Phi)}{h}\,dz\right).
\end{aligned}
\tag{7.4}
$$

To deal with the last term in (7.4), we consider the following expression $\frac{R_\Phi}{h}$ near $z_k\in Z$

$$
\begin{aligned}
\frac{R_\Phi}{h}(z)=&-\frac{3}{2}\frac{1}{\tilde h(z_k)}\frac{1}{(z-z_k)^3}+\frac{1}{2}\frac{\tilde h_z(z_k)}{\tilde h^2(z_k)}\frac{1}{(z-z_k)^2}\\
&+\left(3\frac{\tilde h_{zz}(z_k)}{\tilde h^2(z_k)}-\frac{\tilde h_z^2(z_k)}{\tilde h^3(z_k)}\right)\frac{1}{z-z_k}+\cdots
\end{aligned}
$$

where $\tilde{h} = \tilde{h}_k$ for $\Phi(z) = (z - z_k)\tilde{h}_k dz$. Putting $(\frac{R_\Phi}{h})_s := (-\frac{3}{2}\frac{1}{\tilde{h}(z_k)}\frac{1}{(z-z_k)^3} + \frac{1}{2}\frac{\tilde{h}_z(z_k)}{\tilde{h}^2(z_k)} \times \frac{1}{(z-z_k)^2})$ near z_k, the following equality

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{i}{\pi}\sum_{p_k\in Z}\int\limits_{|z-z_k|=\delta}\left(\int\limits_{z_1}^{z}(\dot{h} + h_z\dot{f} + h\dot{f}_z)\,dz + h\dot{f}_{\bar{z}}\,d\bar{z}\right)\left(\frac{R_\Phi}{h}\right)_s dz \\
&= \sum_{p_k\in Z}\left(3\dot{f}_z + \frac{3}{2}(\log\tilde{h}_k)^{\cdot} + \frac{3}{2}(\log\tilde{h}_k)_z\dot{f} - \frac{1}{2}(\log\tilde{h}_k)_z f_z\dot{z}_k\right)(z_k) \qquad (7.5)
\end{aligned}$$

follows from some elementary computations

$$\begin{aligned}
&\int\limits_{|z-z_k|=\delta}\left(\int\limits_{z_1}^{z}(\dot{h} + h_z\dot{f} + h\dot{f}_z)\,dz + h\dot{f}_{\bar{z}}\,d\bar{z}\right)\frac{1}{(z-z_k)^2}\,dz \\
&\quad = 2\pi i(\dot{h} + h_z\dot{f} + h\dot{f}_z)(z_k) + O(\delta) = -2\pi i(\tilde{h}_k f_z)(z_k)\dot{z}_k + O(\delta), \\
&\int\limits_{|z-z_k|=\delta}\left(\int\limits_{z_1}^{z}(\dot{h} + h_z\dot{f} + h\dot{f}_z)\,dz + h\dot{f}_{\bar{z}}\,d\bar{z}\right)\frac{1}{(z-z_k)^3}\,dz \\
&\quad = \pi i(\dot{h}_z + h_{zz}\dot{f} + 2h_z\dot{f}_z + h\dot{f}_{zz})(z_k) + O(\delta) \\
&\quad = \pi i\big(2\dot{f}_z\tilde{h}_k + \dot{\tilde{h}}_k + \tilde{h}_{kz}(\dot{f} - f_z\dot{z}_k)\big)(z_k) + O(\delta).
\end{aligned}$$

By (7.3), (7.4), and (7.5),

$$\begin{aligned}
&\partial(8\pi iCS)(\varpi) \\
&= \frac{i}{\pi}\left(-\sum_{i=1}^{g}\int\limits_{b_i}\frac{(R_S - R_\Phi)}{h}\,dz\cdot\left(\int\limits_{a_i}(\dot{h} + h_z\dot{f} + h\dot{f}_z)\,dz + h\dot{f}_{\bar{z}}\,d\bar{z}\right)\right. \\
&\quad + \sum_{i=1}^{g}\int\limits_{a_i}\frac{(R_S - R_\Phi)}{h}\,dz\cdot\left(\int\limits_{b_i}(\dot{h} + h_z\dot{f} + h\dot{f}_z)\,dz + h\dot{f}_{\bar{z}}\,d\bar{z}\right) \\
&\quad \left. + \sum_{k=1}^{2g-2}\int\limits_{|z-z_k|=\delta}\frac{(R_S - R_\Phi)}{h}\,dz\cdot\left(\int\limits_{z_1}^{z_k}(\dot{h} + h_z\dot{f} + h\dot{f}_z)\,dz + h\dot{f}_{\bar{z}}\,d\bar{z}\right)\right). \qquad (7.6)
\end{aligned}$$

Let us observe that the holomorphic variation of the coordinates A_i (given in (2.1)) is given by

$$\frac{\partial A_i}{\partial w} = \int\limits_{a_i}(\dot{h} + h_z\dot{f} + h\dot{f}_z)\,dz + h\dot{f}_{\bar{z}}\,d\bar{z},$$

and similar equalities hold for B_i, Z_k. Comparing this with (7.6) completes the proof for $\frac{\partial(8\pi iCS)}{\partial\zeta_i}$. □

Combining the equalities in (7.2) the holomorphic variation of the holomorphic function $\exp(4\pi\mathbb{CS} + \frac{1}{\pi}I)F^{24}$ is given by

$$\frac{\partial \log(\exp(4\pi\mathbb{CS} + \frac{1}{\pi}I)F^{24})}{\partial \zeta_i} = \frac{2i}{\pi}\int\limits_{s_i} \frac{R_B - R_\Phi}{h}\,dz \tag{7.7}$$

for $i = 1, \ldots, 4g-3$. Hence, by the definition of τ_B in (2.4), the two holomorphic functions τ_B^{24} (lifted to U) and $\exp(4\pi\mathbb{CS}+\frac{1}{\pi}I)F^{24}$ have the same holomorphic variations. Consequently, the liftings of τ_B^{24} and $\exp(4\pi\mathbb{CS}+\frac{1}{\pi}I)F^{24}$ to any connected component of $\tilde{\mathcal{H}}_g^*$ are equal up to a multiplicative constant. But the holomorphic function τ_B^{24} descends to $\tilde{\mathcal{H}}_g$, hence the holomorphic function $\exp(4\pi\mathbb{CS} + \frac{1}{\pi}I)F^{24}$ descends too. This proves Theorem 1.1. The constant c appearing in the theorem depends on our choice of a connected component of $\tilde{\mathcal{H}}_g^*$, which depends on the choice of the isotopy type of $\mathcal{L}^2$ curves as remarked in Remark 4.11.

Remark 7.2. By the equalities in (7.2), we have

$$\frac{\partial(4\pi\mathbb{CS} + \frac{1}{\pi}I)}{\partial \zeta_i} = \frac{2i}{\pi}\int\limits_{s_i} \frac{R_S - R_\Phi}{h}\,dz. \tag{7.8}$$

Note that the Fuchsian projective connection R_F—which is defined by the hyperbolic structure on X—does not appear on the right hand side of this equality. This suggests that, although we use the hyperbolic metric over X in our definitions of $\mathbb{CS}(M_X, s_\Phi)$ and $I(X,\Phi)$, the combined invariant $4\pi\mathbb{CS}(M_X, s_\Phi) + \frac{1}{\pi}I(X,\Phi)$ could be defined without it. Actually, we may define $\mathbb{CS}(M_X, s_\Phi)$ and $I(X,\Phi)$ using any other smooth metric in place of the hyperbolic metric on X, in which case simple modification of the variational arguments in Sections 5 and 6 then show that $4\pi\mathbb{CS}(M_X, s_\Phi) + \frac{1}{\pi}I(X,\Phi)$ is invariant under smooth conformal changes of the metric on X that we use. Consequently, the hyperbolic metric only plays a subsidiary role in our definitions and proofs; the final results actually depend only on the conformal class of smooth metric on X (which is determined by the metric on the 3-manifold M_X). Actually, it was suggested to us by D. Korotkin that we use the flat singular metric defined by Φ in place of the hyperbolic metric on X, since this would eliminate both an arbitrary choice of metric, and the awkward looking term $I(X,\Phi)$, which would simply equal zero in this case. However, we encounter additional technical difficulties in this case, which we were not yet able to resolve.

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