Assume that $f(x)$ is a periodic function of period 2π that can be represented by a trigonometric series,

$$f(x) = a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx);$$

that is, we assume that this series converges and has $f(x)$ as its sum. By a easy calculation, we have the so-called **Euler formulas**

$$(1) \qquad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,$$

$$(2) \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \;\; n = 1, 2, \cdots,$$

$$(3) \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \;\; n = 1, 2, \cdots$$

These numbers given by (1) $\sim$ (3) are called **Fourier coefficients** of $f(x)$. The trigonometric series

$$a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with coefficients given in (1) $\sim$ (3) is called the **Fourier series** of $f(x)$ (regardless of convergence)

Theorem 1. *(Representation by a Fourier series) If a periodic function $f(x)$ with period 2π is* *piecewise continuous in the interval* $-\pi \leq x \leq \pi$ *and has a left-hand derivative and right-hand derivative at each point of that interval, then the Fourier series*

$$a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with coefficients given in (1) $\sim$ (3) *is convergent. Its sum is* $f(x)$*, except at a point* x_0 *at which* $f(x)$ *is discontinuous and the sum of the series is the average of the left- and right-hand limits of* $f(x)$ *at* x_0*.*

Remark 1. (Functions of any period $p = 2L$) If a periodic function $f(x)$ with period $2L$ has a **Fourier series**, we claim that this series is

$$f(x) = a_0 + \sum_{1}^{\infty} (a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x)$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx, \ n = 1, 2, \cdots,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx, \ n = 1, 2, \cdots$$

⌈ Note that $f(x+p) = f(x)$.

Let $x = \frac{p}{2\pi}t$ and $g(t) = f(\frac{p}{2\pi}t)$.

Then $g(t+2\pi) = f(\frac{p}{2\pi}(t+2\pi)) = f(\frac{pt}{2\pi}+p) = f(\frac{pt}{2\pi}) = g(t)$. So $g(t)$ has period 2π. ⌋

Theorem 2. *(Fourier cosine series, Fourier sine series) The Fourier series of an* **even** *function of period* $2L$ *is a* **"Fourier cosine series"**

$$f(x) = a_0 + \sum_{1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

with coefficients

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx, \ n = 1, 2, \cdots.$$

The Fourier series of an **odd** *function of period* $2L$ *is a* **"Fourier sine series"**

$$f(x) = \sum_{1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

with coefficients

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x dx, \;\; n = 1, 2, \cdots .$$

Example 1. Show that $\displaystyle\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Let $f(x) = x + x^2$, where $-\pi < x < \pi$. Then the Fourier series of $f(x)$ is

$$\frac{\pi^2}{3} + \sum_{1}^{\infty} (-1)^n \left(\frac{4}{n^2} \cos nx - \frac{2}{n} \sin nx\right)$$

Observe that

$$f(\pi - 0) = \pi + \pi^2, \;\; f(\pi + 0) = f(-\pi + 0) = -\pi + \pi^2.$$

So

$$\frac{1}{2}(f(\pi - 0) + f(\pi + 0)) = \pi^2.$$

Thus $\pi^2 = \frac{\pi^2}{3} + \sum_{1}^{\infty} \frac{4}{n^2}$

Theorem 3. *If $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and left-hand derivative at every point and f is absolutely integrable*
(i.e. $lim_{a\to-\infty} \int_a^0 \mid f(x) \mid dx + lim_{b\to\infty} \int_0^b \mid f(x) \mid dx$ exists), then $f(x)$ can be represented by a **Fourier integral**

$$f(x) = \int_0^\infty A(\omega)\cos\omega x + B(\omega)\sin\omega x\, d\omega,$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv, \;\; B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$$

At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right- hand limits of $f(x)$ at that point.