Dissipation and Semigroups

Def: A **semigroup G** is a set of elements which is closed under composition.

Note: The composition is associative, as for groups.

G may or may not have an identity element I, and some of its elements may or may not have inverses.

Example: The set {exp(-t): t>0} forms a *semigroup.*
Example: The set {exp(-t): ≥ 0 } forms a *semigroup with identity*.

Def: **Norm** of an operator: $||A|| = \sup \{|| A \psi || / || \psi ||, \ \psi \in H\}$

Def: **Bounded operator** The operator A in H is a bounded operator if $||A|| < K$ for some real K.

Examples: $X \psi(x) = x \psi(x)$ is not a bounded operator on H; exp(iX) IS a bounded operator.

Sets of **Bounded Operators** B(H) is the set of bounded operators on H.

Bounded Sets of operators:

Consider $S^-(A) = \{\exp(-t) A;$ A bounded, $t \geq 0 \}$.

Then, there exists a K such that $||B|| < K$ for all $B \in S^-(A)$ and $S^-(A)$ k B(H).

Clearly $S^+(A) = \{\exp(t) A;$ A bounded, $t \geq 0 \}$ does not have this (strong) property.

Dissipation and Semigroups

One-parameter semigroups

$T(t_1)*T(t_2)=T(t_1 + t_1)$

with identity, T(0)=I.

Important Example: If L is a (finite) matrix with ***negative*** eigenvalues, and T(t) = exp(Lt).

Then {T(t), t ≥ 0 } is a one-parameter semigroup, with Identity, and is a Bounded Set of Operators.

Evolution with Dissipation

Markovian Master Equation

1. **Hamiltonian Evolution: Operator X**

$$\dot{X} = i[H, X]$$

2. **Lindblad Form of Dissipation: Operator X**

$$L_D(X) = \frac{1}{2} \sum [V_j^\dagger, [X, V_j]]$$

3. **Evolution with Dissipation ()perator X)**

$$\dot{X} = i[H, X] + \frac{1}{2} \sum [V_j^\dagger, [X, V_j]]$$

Evolution with Dissipation

Markovian Master Equation

0. Dual maps (**X**, ρ) = **Trace(X** ρ)
(L X, ρ **) = (X, L*** ρ **)**

1. **Hamiltonian Evolution:** **State ρ** $\dot{\rho} = -i[H, \rho]$
2. **Lindblad Form of Dissipation: State ρ**

$$(1/2) \sum [V_j \rho, V_j^\dagger] + [V_j, \rho V_j^\dagger]$$

3. **Evolution with Dissipation (State ρ)**

$$\dot{\rho} = -i[H, \rho] + \frac{1}{2} \sum [V_j \rho, V_j^\dagger] + [V_j, \rho V_j^\dagger]$$

Properties: Trace-preserving, Positivity.

Bloch Sphere for Qubit

Qubit: $|0>=\begin{bmatrix}1\\0\end{bmatrix}$ $\qquad |1>=\begin{bmatrix}0\\1\end{bmatrix}$

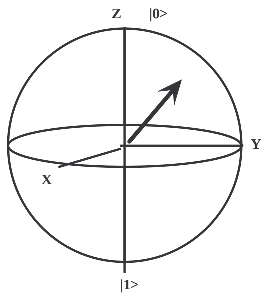
$$|\Psi>=\alpha\,|0>+\ \beta\ \ |1>=\begin{bmatrix}\alpha\\\beta\end{bmatrix}$$

$$\alpha\ =\ \cos(\ \theta\ /\ 2) \qquad \beta = e^{\,i\varphi}\ \sin(\ \theta\ /\ 2)$$

$$x=\sin\theta\ \cos\phi$$

$$y=\sin\theta\ \sin\phi$$

$$z=\cos\theta$$



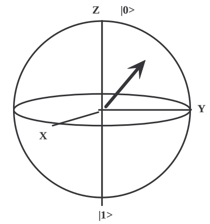
Bloch Ball

Qubit: $$|\Psi><\Psi|=\begin{bmatrix}\alpha\\\beta\end{bmatrix}[\alpha^*\quad\beta^*]=\begin{bmatrix}\alpha\alpha^* & \alpha\beta^*\\\beta\alpha^* & \beta\beta^*\end{bmatrix}$$

$$\rho=\begin{bmatrix}\rho_{11} & \rho_{12}\\\rho_{21} & \rho_{22}\end{bmatrix}=\begin{bmatrix}\cos^2(\theta/2) & (1/2)e^{i\phi}\sin\theta\\(1/2)e^{i\phi}\sin\theta & \sin^2(\theta/2)\end{bmatrix}$$

General 2-level state: $$\tau_0:=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\ \tau_1:=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\ \tau_2:=\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\ \tau_3:=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}tr(\tau_x\rho)\\tr(\tau_y\rho)\\tr(\tau_z\rho)\end{bmatrix}=\begin{bmatrix}\rho_{12}+\rho_{21}\\i(\rho_{12}-\rho_{21})\\\rho_{11}-\rho_{22}\end{bmatrix}$$



Example: Two-level System (a)

$$\dot{\rho}=-\,i\,[\,H\,,\,\rho\,]\,+\sum\ [V_j\rho,V_j^\dagger\,]+[V_j,\rho\,V_j^\dagger\ \ ]$$

Hamiltonian Part: fx and fy controls

$$H:=w\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}+fx\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}+fy\begin{bmatrix}0 & -I\\I & 0\end{bmatrix}$$

Dissipation Part:V-matrices

$$V1:=\begin{bmatrix}0 & 0\\\sqrt{\gamma_{21}} & 0\end{bmatrix}\qquad V2:=\begin{bmatrix}0 & \sqrt{\gamma_{12}}\\0 & 0\end{bmatrix}\qquad V3:=\begin{bmatrix}\sqrt{2}\sqrt{\Gamma_0} & 0\\0 & 0\end{bmatrix}$$

with $$\Gamma=\Gamma_0+(1/2)(\gamma_{12}+\gamma_{21})$$

Example: Two-level System (b)

(1) In Liouville form (4-vector $V\rho$)

$$V_\rho := [\rho_{1,\,1},\,\rho_{1,\,2},\,\rho_{2,\,1},\,\rho_{2,\,2}\,]$$

$$\dot{V}_\rho\ =\ (\,L_H\ +\ L_D\,)\,V_\rho$$

Where L_H has pure imaginary eigenvalues and LD real negative eigenvalues.

$$LD:=\begin{bmatrix}-\gamma_{2,\,1} & 0 & 0 & \gamma_{1,\,2}\\0 & -\Gamma & 0 & 0\\0 & 0 & -\Gamma & 0\\\gamma_{2,\,1} & 0 & 0 & -\gamma_{1,\,2}\end{bmatrix}$$

Example: Two-level System(c)

(1) In **Bloch vector** form (3-vector $S\rho$)

$$S\rho = [\rho_{1,2} + \rho_{2,1}, I(\rho_{1,2} - \rho_{2,1}), \rho_{1,1} - \rho_{2,2}]$$

$$(\partial / \partial t) S\rho = \{2\omega R_z + f_x R_x + f_y R_y + , C - (K_{12} - K_{21}) T\} S\rho$$

where

$2\omega R_z + f_x R_x + f_y R_y$ is a rotation in Bloch space

C is a contraction matrix

(M = (K_{12}-+K_{21}) / ,)

$$C := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\varepsilon \end{bmatrix}$$

and T is a translation.

'Conformal' Semigroup