Quantum Control:

Implementation and Dissipation

Allan I. Solomon and Sonia G. Schirmer

Quantum Processes Group

The Open University, Milton Keynes, UK

Korea, June, 2002

Implementation of quantum control

Many different approaches have been proposed for implementing quantum control:

- Open-loop control: No feedback from measurements.
  - Local optimization (gradient techniques)
  - Global optimization (simulated annealing, genetic or evolutionary algorithms)
  - Global & local optimal control (variational calculus)
  - Adiabatic passage via light-induced potentials.
  - Geometric control using Lie group techniques. Ideal for constructing arbitrary unitary operators (quantum gates)
- Closed-loop control: Feedback from measurements required.
  - Learning approach using genetic algorithms and experiments. Originally developed for quantum chemistry [Judson, Rabitz]
  - Continuous feedback using weak measurements. Originally developed for quantum optics [Wiseman, Milburn] Ideal for protecting qubits from decoherence.

Basic Model Description

- Internal Hamiltonian of the system: $\hat{H}_0 = \sum_{n=1}^{N} E_n |n\rangle\langle n|$
  - Energy levels: E_n, transition frequencies $\omega_n \equiv E_{n+1} - E_n$
  - Complete orthonormal set of energy eigenstates: $\{|n\rangle : 1 \le n \le N\}$
- Control fields:

$$\begin{aligned} f_m(t) &= A_m(t)\left[e^{i(\omega_m t+\phi_m)} + e^{-i(\omega_m t+\phi_m)}\right] \\ &= 2A_m(t)\cos(\omega_m t + \phi_m) \end{aligned}$$

  - $A_m(t)$: envelope function, slowly varying compared to ω_m
  - ϕ_m: initial pulse phase, ω_m: pulse frequency
- Interaction:

$$\begin{aligned} \hat{H}_m(f_m) = A_m(t)[&e^{i(\omega_m t+\phi_m)} d_m |m\rangle\langle m+1| \\ &+ e^{-i(\omega_m t+\phi_m)} d_m |m+1\rangle\langle m|] \end{aligned}$$

  - $\omega_m^2 \ne \omega_n^2$ for $m \ne n$, off-resonant effects negligible
  - Rotating Wave Approximation applies, d_m: transition dipole moment

Implementation of Unitary Operators using Geometric Control

- Given a target evolution operator $\hat{U}_T$, to be realized at some time $T > 0$, find product decomposition of this operator where the factors are elementary generators of the dynamical Lie group $\hat{V}_k$, e.g.:

$\hat{x}_m = |m\rangle\langle m+1| - |m+1\rangle\langle m|$
$\hat{y}_m = i[|m\rangle\langle m+1| + |m+1\rangle\langle m|]$
C_k, ϕ_k: constants, $C_k \ge 0$
σ: $[1, 2, \ldots, K] \mapsto [1, 2, \ldots, M]$

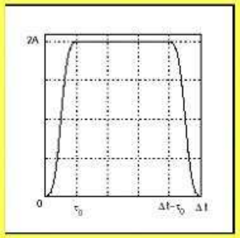
$$\hat{U}_T = \hat{U}(T) = e^{i\hat{H}_0 T}\hat{V}_K \hat{V}_{K-1} \cdots \hat{V}_1$$

$$\hat{V}_k = \exp\left[C_k\left(\sin\phi_k \hat{x}_{\sigma(k)} - \cos\phi_k \hat{y}_{\sigma(k)}\right)\right]$$

- The order of the factors in this decomposition determines the pulse sequence; the constants C_k and ϕ_k determine the (effective) area and initial phase of the pulse, respectively.
- Partition time-interval into K subintervals $[t_{k-1}, t_k]$, $t_0 = 0$, $t_K = T$ and apply sequence of K pulses $f_k(t) = 2A_k(t)\cos(\omega_{\sigma(k)} t + \phi_k)$ such that $A_k(t) = 0$ for $t \notin [t_{k-1}, t_k]$ and $d_{\sigma(k)} \int_{t_{k-1}}^{t_k} A_k(\tau)d\tau = C_k$

Possible Choices for Control Pulses

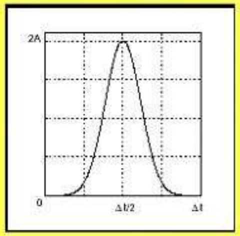
Square wave pulses with rise/decay time τ_0:



- Effective pulse area: $A_k d_{\sigma(k)}(\Delta t_k - \tau_0) \equiv \hbar C_k$

+ Easily derived from CW lasers using Pockel cells.
+ Time optimal property, smooth transfer.
− High intensities / fast switching challenging.

Gaussian wavepackets with $q_k = 4/\Delta t_k$



- Effective pulse area: $A_k d_{\sigma(k)} \Delta t_k \sqrt{\pi}/4 \equiv \hbar C_k$

+ Easily derived from standard pulsed laser systems.
+ Minimal frequency dispersion, high intensities possible.
− Precise control of pulse area and phase challenging.

In both cases, the amplitude $2A_k$ of the pulse can be adjusted by changing the pulse length Δt_k, which allows us to minimize off-resonant effects by limiting the frequency dispersion of the pulse and the Rabi frequency of the driven transitions.

Application: Inversion of Ensemble Populations

1. Arbitrary Init. Ensemble:

$$\hat{U}_T = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Phases important!

2. Ensemble of Energy Eigenstates:

$$\hat{U}_T = \begin{pmatrix} 0 & 0 & \cdots & 0 & e^{i\theta_1} \\ 0 & 0 & \cdots & e^{i\theta_2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & e^{i\theta_{N-1}} & \cdots & 0 & 0 \\ e^{i\theta_N} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Phase factors $e^{i\theta_n}$ arbitrary!

Case 2: Decomposition

$$\hat{U}_T = e^{i\hat{H}_0 T} \prod_{\ell=N-1}^{1} \left(\prod_{m=1}^{\ell} \hat{V}_m \right)$$

$$\hat{V}_m = \exp\left[\tfrac{\pi}{2}(\sin\phi \hat{x}_m - \cos\phi \hat{y}_m)\right]$$

T, ϕ arbitrary!

Sequence of $K = \frac{1}{2}N(N-1)$ pulses:

- alternating frequencies $\omega_k = \omega_{\sigma(k)}$
- total pulse area π
- arbitrary initial phases ϕ_k
- arbitrary pulse length Δt_k

Example 1: Inversion of Ensemble Populations for 4-level System with distinct transition frequencies

Three distinct transition frequencies

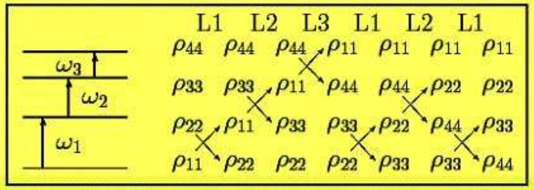
$$\omega_1 = E_2 - E_1$$
$$\omega_2 = E_3 - E_2$$
$$\omega_3 = E_4 - E_3$$

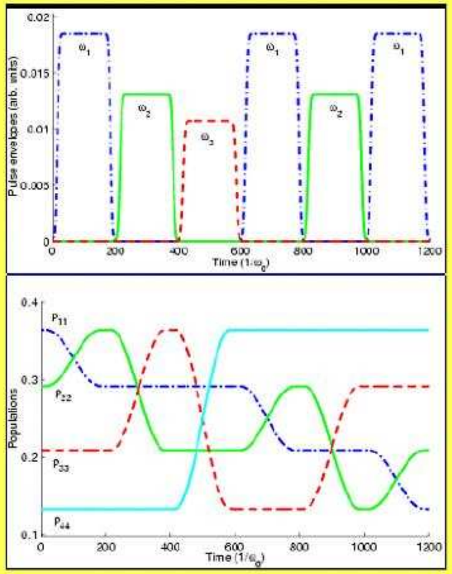
⇒ three CW lasers required

Initial Populations: ρ_{11}, ρ_{22}, ρ_{33}, ρ_{44}, if

$$\rho_{nn} \neq \rho_{mm}, \quad n \neq m$$

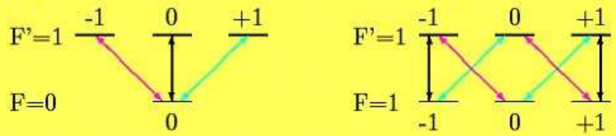
⇒ Six pulses are required





Systems with degenerate energy levels

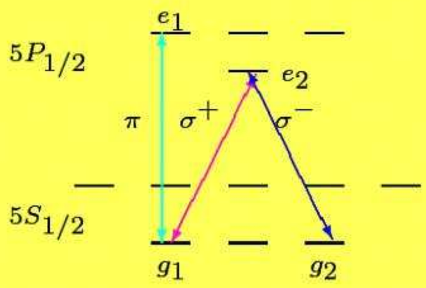
- Standard decomposition assumes selective addressability of each transition.
  ⇒ not directly applicable to transitions between degenerate energy levels



- Atomic selection rules: only transitions with $\Delta F = 0, \pm 1$ are permitted. Linearly polarized light couples only sublevels with $\Delta m = 0$; left and right circularly polarized light couple only sublevels with $\Delta m = \pm 1$, respectively.
- Controllability: Assuming all three polarizations are available
  - Transitions with $\Delta F = \pm 1$ are completely controllable
  - Transitions with $\Delta F = 0$ are only pure-state controllable
- Constructive control schemes can be derived using geometric control techniques by using more general decompositions.

Four-level Quantum Memory Scheme

We consider a four-level atom with two degenerate ground states $|g_1\rangle$ and $|g_2\rangle$ and two non-degenerate excited states $|e_1\rangle$ and $|e_2\rangle$.



We assume the $|g_1\rangle \to |e_1\rangle$ transition is an optical transition that forms part of a quantum information processing scheme.

After performing a series of quantum logical operations, the quantum information stored in the $|g_1\rangle, |e_1\rangle$ subsystem is to be protected by mapping it to the decoherence-free $|g_1\rangle, |g_2\rangle$ subsystem by applying a series of simple control pulses derived from optical fields.

For instance, the ground states could be Zeeman sublevels of the $|5S_{1/2}\rangle$ ground state for laser-cooled ^{87}Rb and the excited states could be $|e_1\rangle = |5\mathrm{P}_{3/2}F = 1, m = -1\rangle$ and $|e_2\rangle = |5\mathrm{P}_{3/2}F = 0, m = 0\rangle$.

The Quantum Memory Map

Aim: Map the (initial) state $\hat\rho^{(0)}$ onto the target state $\hat\rho^{(1)}$, where

$$\hat\rho^{(0)} = \begin{pmatrix} \rho_{g_1g_1} & 0 & \rho_{g_1e_1} & 0 \\ 0 & 0 & 0 & 0 \\ \rho^*_{g_1e_1} & 0 & \rho_{e_1e_1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat\rho^{(1)} = \begin{pmatrix} \rho_{g_1g_1} & \rho_{g_1e_1} & 0 & 0 \\ \rho^*_{g_1e_1} & \rho_{e_1e_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis $|g_1\rangle$, $|g_2\rangle$, $|e_1\rangle$, $|e_2\rangle$.

1st Step: Find a unitary operator $\hat U$ such that $\boxed{\hat\rho^{(1)} = \hat U \hat\rho^{(0)} \hat U^\dagger.}$

Note that $\hat U$ is not unique. In fact, any unitary operator of the form

$$\hat U = \begin{pmatrix} e^{i\theta_1} & 0 & 0 & 0 \\ 0 & 0 & e^{i\theta_3} & 0 \\ 0 & e^{i\theta_2} & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta_4} \end{pmatrix} \text{ with } \theta_1 = \theta_3 \text{ works.}$$

2nd Step: Find a suitable product decomposition of the target operator.

To find a suitable factorization, we note that applying a control pulse of the form

$$\boxed{\begin{aligned} f_1(t) &= A_1(t)\left[e^{i(\omega_{g_2e_2}t+\phi_1)} + e^{-i(\omega_{g_2e_2}t+\phi_1)}\right] \\ &= 2A_1(t)\cos(\omega_{g_2e_2}t + \phi_1) \end{aligned}}$$

which is resonant with the transition frequency $\omega_{g_2e_2}$ and suitably polarized to drive only the transition $|g_2\rangle \to |e_2\rangle$, gives rise to a unitary evolution operator

$$\boxed{\hat V_1(C_1, \phi_1) = \exp\left[\tfrac{1}{2}C_1\left(\sin\phi_1 \hat x_{g_2e_2} - \cos\phi_1 \hat y_{g_2e_2}\right)\right]}$$

where the constant C_1 is the area of the pulse

$$\boxed{C_1 = \int_{\Delta t_1} \Omega_1(t)\,dt = \int_{\Delta t_1} d_{g_2e_2} 2A_1(t)\,dt.}$$

Note that $\hat x_{g_2e_2} = |g_2\rangle\langle e_2| - |e_2\rangle\langle g_2|$, $\hat y_{g_2e_2} = i\left(|g_2\rangle\langle e_2| + |e_2\rangle\langle g_2|\right)$ and $d_{g_2e_2}$ is the transition dipole moment of the transition $|g_2\rangle \to |e_2\rangle$.

Similarly, applying a control pulse

$$\boxed{\begin{aligned} f_2(t) &= A_2(t)\left[e^{i(\omega_{g_1e_2}t+\phi_2)} + e^{-i(\omega_{g_1e_2}t+\phi_2)}\right] \\ &= 2A_2(t)\cos(\omega_{g_1e_2}t + \phi_2) \end{aligned}}$$

which is resonant with the transition frequency $\omega_{g_1e_2}$ and suitably polarized to drive only the transition $|g_1\rangle \to |e_2\rangle$, gives rise to a unitary evolution operator

$$\boxed{\hat V_2(C_2, \phi_2) = \exp\left[\tfrac{1}{2}C_2\left(\sin\phi_2 \hat x_{g_1e_2} - \cos\phi_2 \hat y_{g_1e_2}\right)\right]}$$

where the constant C_2 is the area of the pulse

$$\boxed{C_2 = \int_{\Delta t_2} \Omega_2(t)\,dt = \int_{\Delta t_2} d_{g_1e_2} 2A_2(t)\,dt.}$$

Note that $\hat x_{g_1e_2} = |g_1\rangle\langle e_2| - |e_2\rangle\langle g_1|$, $\hat y_{g_1e_2} = i\left(|g_1\rangle\langle e_2| + |e_2\rangle\langle g_1|\right)$ and $d_{g_1e_2}$ is the transition dipole moment of the transition $|g_1\rangle \to |e_2\rangle$.

Finally, applying a control pulse

$$\begin{array}{rcl} f_3(t) & = & A_3(t)\left[e^{i(\omega_{g_1e_1}t+\phi_3)} + e^{-i(\omega_{g_1e_1}t+\phi_3)}\right] \\ & = & 2A_3(t)\cos(\omega_{g_1e_1}t+\phi_3) \end{array}$$

which is resonant with the transition frequency $\omega_{g_1e_1}$ and drives only the transition $|g_1\rangle \to |e_1\rangle$, gives rise to a unitary evolution operator

$$\hat{V}_3(C_3,\phi_3) = \exp\left[\tfrac{1}{2}C_3\left(\sin\phi_3\hat{x}_{g_1e_1} - \cos\phi_3\hat{y}_{g_1e_1}\right)\right]$$

where the constant C_3 is the area of the pulse

$$C_3 = \int_{\Delta t_3}\Omega_3(t)\,dt = \int_{\Delta t_3} d_{g_1e_1}2A_3(t)\,dt$$

Note that $\hat{x}_{g_1e_1} = |g_1\rangle\langle e_1| - |e_1\rangle\langle g_1|$, $\hat{y}_{g_1e_1} = i\left(|g_1\rangle\langle e_1| + |e_1\rangle\langle g_1|\right)$ and $d_{g_1e_1}$ is the transition dipole moment of the transition $|g_1\rangle \to |e_1\rangle$.

Note that setting $C_1 = C_2 = C_3 = \pi/2$ gives

$$\hat{V}_1(\phi_1) = \hat{V}_1(\pi/2,\phi_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -ie^{i\phi_1} \\ 0 & 0 & 1 & 0 \\ 0 & -ie^{-i\phi_1} & 0 & 0 \end{pmatrix}$$

$$\hat{V}_2(\phi_2) = \hat{V}_2(\pi/2,\phi_2) = \begin{pmatrix} 0 & 0 & 0 & -ie^{i\phi_2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -ie^{-\phi_2} & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{V}_3(\phi_3) = \hat{V}_3(\pi/2,\phi_3) = \begin{pmatrix} 0 & 0 & -ie^{i\phi_3} & 0 \\ 0 & 1 & 0 & 0 \\ -ie^{-i\phi_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Set $\hat{V}_4(\phi_4) = \hat{V}_2(\pi/2,\phi_4)$, $\hat{V}_5(\phi_5) = \hat{V}_1(\pi/2,\phi_5)$ and define

$$\hat{U} \equiv \hat{V}_5(\phi_5)\hat{V}_4(\phi_4)\hat{V}_3(\phi_3)\hat{V}_2(\phi_2)\hat{V}_1(\phi_1)$$

Note that the operator $\hat{U}$ is of the form

$$\hat{U} = \begin{pmatrix} e^{i\theta_1} & 0 & 0 & 0 \\ 0 & 0 & e^{i\theta_3} & 0 \\ 0 & e^{i\theta_2} & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta_4} \end{pmatrix}$$

where the elements θ_n are phase factors depending on the pulse phases ϕ_k: $\theta_1 = \pi - \phi_2 + \phi_4$, $\theta_2 = \pi/2 - \phi_3 + \phi_2 - \phi_1$, $\theta_3 = \pi/2 + \phi_5 - \phi_4 + \phi_3$ and $\theta_4 = \pi - \phi_5 + \phi_1$. Moreover $\hat{U}$ maps the initial state $\hat{\rho}^{(0)}$ to

$$\hat{U}\hat{\rho}^{(0)}\hat{U}^\dagger = \begin{pmatrix} \rho_{g_1g_1} & e^{i\Phi}\rho_{g_1e_1} & 0 & 0 \\ e^{-i\Phi}\rho^*_{g_1e_1} & \rho_{e_1e_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, $\hat{U}$ maps the initial state onto the target state if $\Phi \equiv \pi/2 + 2\phi_4 - \phi_2 - \phi_3 - \phi_5 = 0 (\text{mod } 2\pi)$ which can be achieved for instance by setting $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5 = \pi/2$.

Pulse Scheme A

$\hat{U}$ can be dynamically generated by applying a sequence of five π-pulses:

1. a pulse $f_1(t) = -2A_1(t)\sin(\omega_{g_2e_2}t)$ that drives the transition $|g_2\rangle \to |e_2\rangle$ and has pulse area π;
2. a pulse $f_2(t) = -2A_2(t)\sin(\omega_{g_1e_2}t)$ that drives the transition $|g_1\rangle \to |e_2\rangle$ and has pulse area π;
3. a pulse $f_3(t) = -2A_3(t)\sin(\omega_{g_1e_1}t)$ that drives the transition $|g_1\rangle \to |e_1\rangle$ and has pulse area π;
4. a pulse $f_4(t) = -2A_4(t)\sin(\omega_{g_1e_2}t)$ that drives the transition $|g_1\rangle \to |e_2\rangle$ and has pulse area π;
5. a pulse $f_5(t) = -2A_5(t)\sin(\omega_{g_2e_2}t)$ that drives the transition $|g_2\rangle \to |e_2\rangle$ and has pulse area π.

Computer Simulations

Figure 1 (A) shows the control pulse sequence and corresponding evolution of a system initially in the superposition state $(|g_1\rangle + 2|e_1\rangle)/\sqrt{5}$. The time unit in all plots is $1/\Omega$, where Ω is the Rabi frequency of the transition $|g_1\rangle \to |e_1\rangle$ and the pulse envelopes have been chosen to be Gaussian.

Note that the pulse sequence maps the qubit $(|g_1\rangle + 2|e_2\rangle)/\sqrt{5}$ onto the equivalent qubit $(|g_1\rangle + 2|g_2\rangle)/\sqrt{5}$, as desired.

Initial populations and coherences:

$$\rho^{(0)}_{g_1g_1} = 0.2,$$
$$\rho^{(0)}_{e_1e_1} = 0.8,$$
$$\rho^{(0)}_{g_1e_1} = 0.4,$$

and all other values of $\hat{\rho}^{(0)}$ zero.

Final populations and coherences:

$$\rho^{(1)}_{g_1g_1} = 0.2,$$
$$\rho^{(1)}_{g_2g_2} = 0.8,$$
$$\rho^{(1)}_{g_1g_2} = 0.4$$

and all other values of $\hat{\rho}^{(1)}$ zero.

Shortcomings of Pulse Scheme A

A potential shortcoming of the pulse scheme above is that the auxiliary excited state $|e_2\rangle$, is substantially populated at intermediate times. Since the meta-stable state $|e_2\rangle$ has a finite lifetime, the quantum information would be subject to degradation through spontaneous decay.

Although this decay can be minimized by choosing short control pulses, it would be advantageous to minimize the transient population of state $|e_2\rangle$.

Fortunately, using similar techniques as above, one can derive alternative control schemes involving concurrent application of multiple fields, which reduce the transient population of the intermediate excited state $|e_2\rangle$ considerably.

Pulse Scheme B: An Improved Control Scheme

An alternative pulse scheme that achieves the same information transfer but limits the transient population of state $|e_2\rangle$ involves the application of four π-pulses:

Pulse Scheme B

1. apply a pulse $f_1(t) = -2A_1(t)\sin(\omega_{g_1e_1}t)$ that drives the transition $|g_1\rangle \leftrightarrow |e_1\rangle$ and has pulse area π;
2. simultaneously apply two "congruent" π-pulses

$$f_2(t) = -2A_2(t)\sin(\omega_{g_1e_2}t)$$
$$f_2'(t) = -2A_2(t)\sin(\omega_{g_2e_2}t)$$

which are identical except for different polarizations;

3. apply a pulse $f_3(t) = -2A_3(t)\sin(\omega_{g_1e_1}t)$ that drives the transition $|g_1\rangle \leftrightarrow |e_1\rangle$ and has pulse area π.

A control simulation for this scheme is shown in figure 1 (B). Observe that the meta-stable excited state $|e_2\rangle$ is not significantly populated using this scheme.

Figure 1: Control pulse sequence and evolution of the populations and coherences for control scheme A (left) and control scheme B (right)