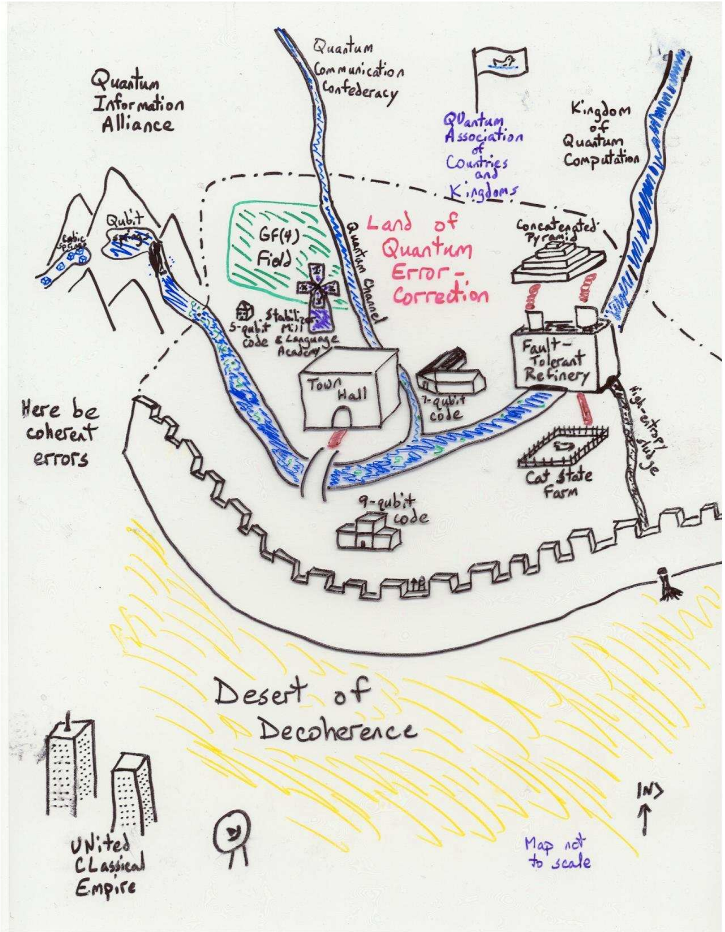

Errors ...:

Operational $\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0\rangle + \beta e^{i(\phi+\Delta)}|1\rangle$

Decoherence $\alpha|0\rangle + \beta|1\rangle \rightarrow |0\rangle$ Prob. $|\alpha|^2$

$|1\rangle$ Prob. $|\beta|^2$

Acts like a measurement!

Classical Repetition Code:

$0 \rightarrow 000$

$1 \rightarrow 111$

Correct 010 to 000

Quantum Repetition Code?

$$\alpha|0\rangle + \beta|1\rangle \rightarrow (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)$$

No-Cloning Theorem

$$|\psi\rangle \rightarrow |\psi\rangle|\psi\rangle$$
$$|\phi\rangle \rightarrow |\phi\rangle|\phi\rangle$$

But then

$$|\psi\rangle + |\phi\rangle \rightarrow \boxed{|\psi\rangle|\psi\rangle + |\phi\rangle|\phi\rangle}$$
$$\neq (|\psi\rangle + |\phi\rangle)(|\psi\rangle + |\phi\rangle)$$
$$=$$
$$\boxed{|\psi\rangle|\psi\rangle + |\phi\rangle|\phi\rangle + (|\psi\rangle|\phi\rangle + |\phi\rangle|\psi\rangle)}$$

Thus, no device can do this.

Bit flip $X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Phase flip $Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Bit & Phase $Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = iXZ$

Identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Tensor products form "Pauli group"

Note:

$XZ = -ZX$

$XY = -YX$

$YZ = -ZY$

X, Y, and Z "anticommute"

Also written $\{X, Z\} = 0$

$$(X \otimes Z)(Y \otimes Y) = (Y \otimes Y)(X \otimes Z)$$

These operators commute

$$[X \otimes Z, Y \otimes Y] = 0$$

9-qubit code

$|0\rangle \longrightarrow$

$|\bar{0}\rangle = (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$

$|1\rangle \longrightarrow$

$|\bar{1}\rangle = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$

Note: $\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle \neq [\alpha(|000\rangle + |111\rangle) + \beta(|000\rangle - |111\rangle)]^{\otimes 3}$

Correct bit flip:

$|010\rangle \pm |101\rangle \rightarrow |000\rangle \pm |111\rangle$

or phase flip:

$(|\cdot\rangle + |\cdot\rangle)(|\cdot\rangle - |\cdot\rangle)(|\cdot\rangle + |\cdot\rangle)$

$\rightarrow (|\cdot\rangle + |\cdot\rangle)(|\cdot\rangle + |\cdot\rangle)(|\cdot\rangle + |\cdot\rangle)$

or both

Another sort of error:

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \underset{\text{physically meaningless}}{e^{+i\theta/2}} \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

$$= \cos\theta/2 \; I - i \sin\theta/2 \; Z$$

Error Correction Procedure:

$I(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)\underset{\text{Ancilla}}{|0\rangle} \rightarrow I(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)|\text{no error}\rangle$

$\rightarrow (\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)|\text{no error}\rangle$

$Z(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)|0\rangle \rightarrow Z(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)|Z\rangle$

$\rightarrow (\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)|Z\rangle$

$(\cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Z)(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)|0\rangle$

$\rightarrow \cos\frac{\theta}{2} I(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)\boxed{|\text{no error}\rangle}$

$+ (-i\sin\frac{\theta}{2}) Z(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)\boxed{|Z\rangle}$

Measure

With prob. $\sin^2\frac{\theta}{2}$:

$Z(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)|Z\rangle$ ← can correct

With prob. $\cos^2\frac{\theta}{2}$:

$I(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)(|\text{no error}\rangle)$

In general:

If we can correct A, B, we can correct $A + B$

Decoherence:

Prob. P_1, error E_1

Prob. P_2, error E_2

$\vdots$

If we can correct $E_1, E_2, \ldots$, we can correct this

Small Error on Every Qubit?

$(I + \epsilon E) \otimes (I + \epsilon E) \otimes \ldots \otimes (I + \epsilon E)\; |\psi\rangle$

$= I|\psi\rangle$

$+ \epsilon(E \otimes I \otimes \ldots \otimes I + I \otimes E \otimes \ldots \otimes I + \ldots + I \otimes I \otimes \ldots \otimes E)|\psi\rangle$

$+ \epsilon^2(E \otimes E \otimes I \otimes \ldots \otimes I + \ldots)|\psi\rangle$

$+ \ldots$

To order ϵ, the error acts like a sum of 1-qubit errors

Correct $\leq k$-qubit errors $\Rightarrow$ correct this error to order ϵ^{k+1}.

$|000\rangle + |111\rangle \longrightarrow |100\rangle + |011\rangle$

To detect a bit flip, measure parities

$Z \otimes Z \otimes I$ — Detects X_1 or X_2

$I \otimes Z \otimes Z$ — Detects X_2 or X_3

Note:

X_1 and X_2 anticommute with $Z \otimes Z \otimes I$

$$Z\otimes Z\otimes I\,(\overbrace{|000\rangle + |111\rangle}^{|\psi\rangle}) = |\psi\rangle$$

(+1 eigenstate)

$$Z\otimes Z\otimes I\,(X_1|\psi\rangle) = -X_1 (Z\otimes Z\otimes I)|\psi\rangle = -X_1|\psi\rangle$$

(−1 eigenstate)

Anticommuting operators move us from +1 eigenstate to −1 eigenstate

Warning: only measure $Z\otimes Z\otimes I$, not Z_1 and Z_2!

Stabilizer for 9-qubit code

$$\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
Z & Z & & & & & & & \\
 & Z & Z & & & & & & \\
 & & & Z & Z & & & & \\
 & & & & Z & Z & & & \\
 & & & & & & Z & Z & \\
 & & & & & & & Z & Z \\
X & X & X & X & X & X & & & \\
 & & & X & X & X & X & X & X
\end{array}$$

Generate Abelian group

$$\begin{array}{rl}
9 & \text{physical qubits} \\
-8 & \text{generators of stabilizer} \\
\hline
1 & \text{logical qubit}
\end{array}$$

Codewords are +1-eigenstates of all generators $\Rightarrow$ only exist if group is Abelian

In general, given a stabilizer S, consider

$$N(S) = \{E \mid [E,M]=0 \ \forall M \in S\}$$

(Restrict attention to tensor products of Pauli matrices)

S detects an error E iff

$$E \notin N(S)\setminus S \iff E\in S \text{ or } \{E,M\}=0 \text{ for some } M\in S$$

S corrects a set $\mathcal{E}$ of errors iff S detects $E^\dagger F$ $\forall E,F \in \mathcal{E}$

Distance of a code given by S is the minimum weight of an operator in $N(S)\setminus S$

Converting Classical Codes:

(Calderbank and Shor 1995, Steane 1995)

e.g. [7,4,3] Hamming Code

Parity check matrix
$$\begin{pmatrix} 0&1&1&1&1&0&0 \\ 1&0&1&1&0&1&0 \\ 1&1&0&1&0&0&1 \end{pmatrix}$$

$\rightarrow$ Stabilizer

$$\left.\begin{array}{ccccccc}
I & Z_2 & Z_3 & Z_4 & Z_5 & I & I \\
Z_1 & I & Z_3 & Z_4 & I & Z_6 & I \\
Z_1 & Z_2 & I & Z_4 & I & I & Z_7
\end{array}\right\} \text{corrects amplitude errors } C_1$$

$$\left.\begin{array}{ccccccc}
I & X_2 & X_3 & X_4 & X_5 & I & I \\
X_1 & I & X_3 & X_4 & I & X_6 & I \\
X_1 & X_2 & I & X_4 & I & I & X_7
\end{array}\right\} \text{corrects phase errors } C_2$$

Subgroup must be abelian

$\iff$ Dual codes

$C_1^\perp \subseteq C_2$

5 qubit code:

$X_1 Z_2 Z_3 X_4 I$
$I X_2 Z_3 Z_4 X_5$
$X_1 I X_3 Z_4 Z_5$
$Z_1 X_2 I X_4 Z_5$

$$|0\rangle \rightarrow |00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle - |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle$$

$$|1\rangle \rightarrow |11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle - |00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle$$

Fault-tolerance:

Design protocols where 1 gate error only causes 1 wrong qubit per block of the code.

- Fault-tolerant error correction
- Fault-tolerant protocols exist for any stabilizer code
- 7 qubit code is particularly good for fault-tolerance
- If error rate per gate is below some threshold value, arbitrarily long fault-tolerant quantum computations are possible with polylogarithmic overhead

Summary
(Quantum Error Correction Sonnet)

We cannot clone, perforce; instead, we split
Coherence to protect it from that wrong
That would destroy our valued quantum bit
And make our computation take too long.

Correct a flip and phase – that will suffice.
If in our code another error's bred,
We simply measure it, then God plays dice,
Collapsing it to X or Y or zed.

We start with noisy seven, nine, or five
And end with perfect one. To better spot
Those flaws we must avoid, we first must strive
To find which ones commute and which do not.

With group and eigenstate, we've learned to fix
Your quantum errors with our quantum tricks.