No-cloning bound of non-Gaussian states and secure teleportation

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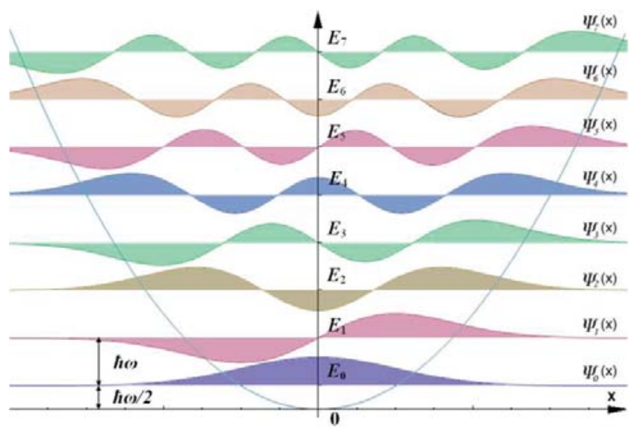
Outline

- Gaussian states and non-Gaussian states
- No-cloning bound and secure teleportation
  - Displaced Fock states input
  - Role of non-Gaussianity
  - Mixed states input

PART I.
Gaussian states and non-Gaussian states

Quantum harmonic oscillator

- Hamiltonian $\hat{H} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$
- Number operator $\hat{n}|n\rangle = n|n\rangle$
- Annihilation and creation operators
  $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$, $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$,
  $[\hat{a}, \hat{a}^\dagger] = 1$
- Quadrature operators
  $\hat{x} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a})$, $\hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a})$, $[\hat{x}, \hat{p}] = i$



Wavefunctions of a quantum harmonic oscillator
[https://en.wikipedia.org/wiki/Quantum_harmonic_oscillator
Author: AllenMcC.]

Quasi-probability distribution

- Characteristic function

  $$\chi(\xi) = \mathrm{Tr}\left[\rho \widehat{D}(\xi)\right]$$

  with $\widehat{D}(\alpha) = \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right)$: displacement operator

- Wigner function

  $$W(\alpha) = \frac{1}{\pi^2} \int d\xi^2 \chi(\xi) \exp(\alpha \xi^* - \alpha^* \xi)$$

  - Normalized, $\int d\alpha^2 W(\alpha) = 1$
  - Not always positive
  - With $\alpha = \frac{1}{\sqrt{2}}(x + ip)$, quadrature probability distribution is obtained by

    $P(x) = \frac{1}{2} \int dp W(\alpha)$, $P(p) = \frac{1}{2} \int dx W(\alpha)$

Gaussian states

- States with Gaussian characteristic function

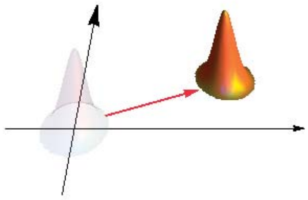
  $$\chi(\xi) = \exp\left[-\xi^T (\Omega \gamma \Omega^T) \xi - \sqrt{2} i (\Omega \bar{x})^T \xi\right]$$

  $$\Omega = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \mathbf{0} & \cdots \\ \mathbf{0} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

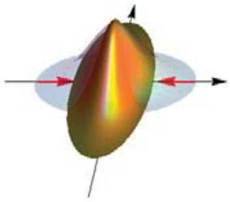
  - Let $\vec{x} = (\hat{x}_1 \quad \hat{p}_1 \quad \hat{x}_2 \quad \hat{p}_2 \quad \cdots)$
  - First moment (displacement) $\bar{x} = \mathrm{Tr}[\vec{x}\rho]$
  - Second moment (covariance matrix) $\gamma_{ij} = \langle \frac{1}{2} \left(\Delta\hat{x}_i \Delta\hat{x}_j + \Delta\hat{x}_j \Delta\hat{x}_i \right) \rangle$

- Uncertainty relation $\gamma + \frac{i}{2}\Omega \geq 0$

Single-mode Gaussian operations



- Displacement operation
  - Coherent states $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$



- Single-mode squeezing $\hat{S}(r) = \exp\left[\frac{r}{2}\left(\hat{a}^2 + \hat{a}^{\dagger 2}\right)\right]$
  - Squeezed states $|r\rangle = \hat{S}(r)|0\rangle$

- Phase rotation $\hat{R}(\theta) = \exp\left(-i\theta\hat{a}^\dagger\hat{a}\right)$

Two-mode Gaussian operations

- Two-mode squeezing $\hat{S}_2(r) = \exp\left[r\left(-\hat{a}^\dagger\hat{b}^\dagger + \hat{a}\hat{b}\right)\right]$
  - Two-mode squeezed states (TMSV)

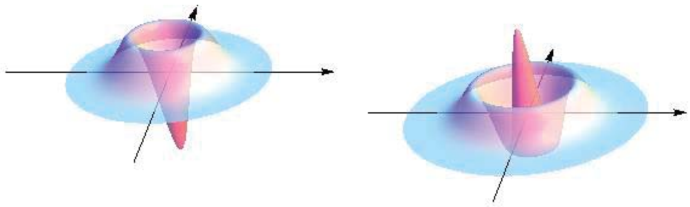
    $|\mathrm{TMSV}\rangle = \hat{S}_2(r)|0_A, 0_B\rangle = \mathrm{sech}\, r \sum_{n=0}^{\infty} \tanh^2 r\, |n_A, n_B\rangle$

    EPR correlation : $\langle\Delta^2(\hat{x}_A - \hat{x}_B)\rangle = \langle\Delta^2(\hat{p}_A + \hat{p}_B)\rangle = \frac{1}{2}e^{-2r}$

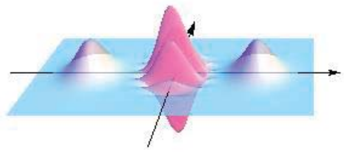
- Beam-splitter interaction $\hat{B}_2(\theta) = \exp\left[\theta\left(\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger\right)\right]$

Single-mode non-Gaussian states

- Fock states $|1\rangle, |2\rangle, \cdots$
- Superposition of finite Fock states
  $c_0|0\rangle + c_1|1\rangle, \cdots$



- Photon-added or subtraction
- Cat states
  $\frac{1}{\sqrt{N}}(|\alpha\rangle \pm |-\alpha\rangle)$



Quantifying non-Gaussianity

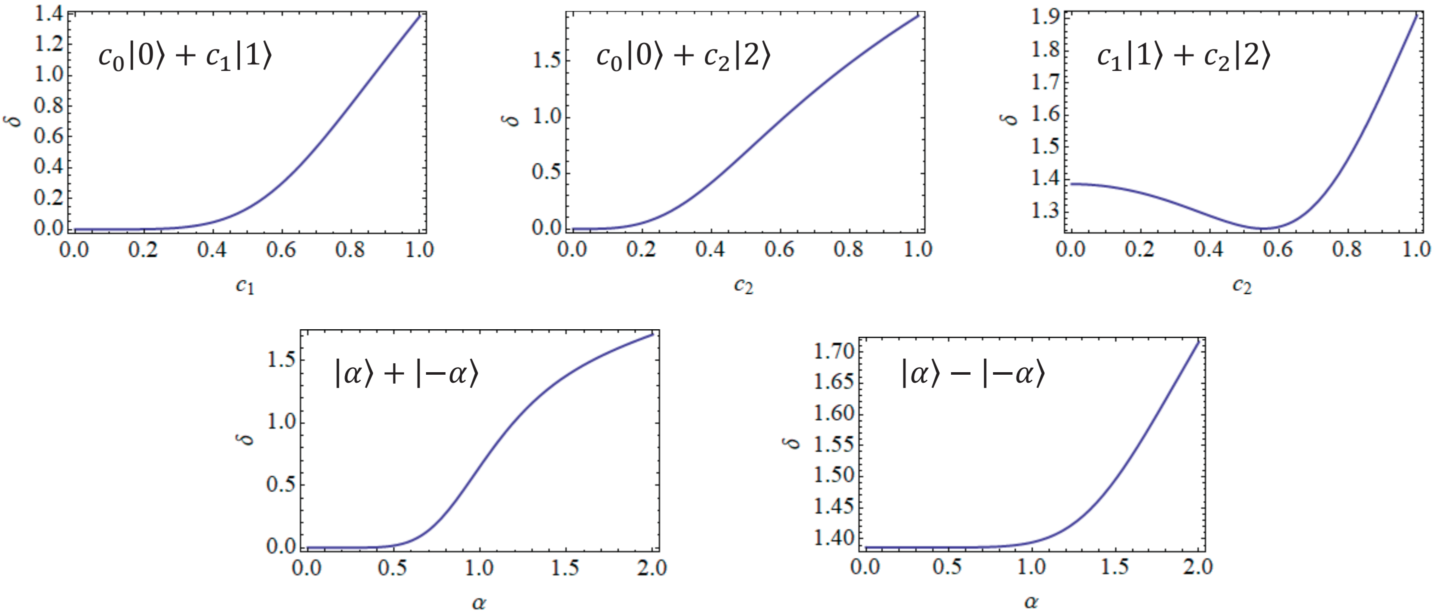
M. G. Genoni and M. G. A. Paris, Phys. Rev. A **82**, 052341 (2010).

- Relative entropy of non-Gaussianity
  $\delta(\rho) \equiv S(\rho||\rho_G) = S(\rho_G) - S(\rho)$

  Gaussian state with the same first and second moments

  - $\delta(\rho) = 0$ iff ρ is Gaussian.
  - Invariant under Gaussian unitary $\delta(U\rho U^{\dagger}) = \delta(\rho)$
  - Additive $\delta(\rho_A \otimes \rho_B) = \delta(\rho_A) + \delta(\rho_B)$
  - Monotonically decreases under Gaussian channels $\delta(\mathcal{E}_G(\rho)) \leq \delta(\rho)$

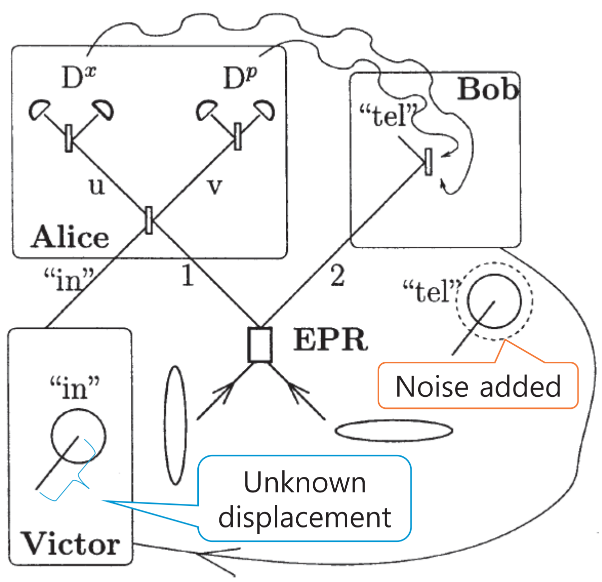
Quantifying non-Gaussianity



Quantum information processing with Gaussian states

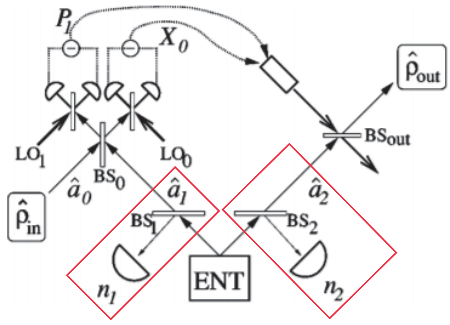
S. L. Braunstein and H. J. Kimble,
Phys. Rev. Lett. **80**, 869 (1998).

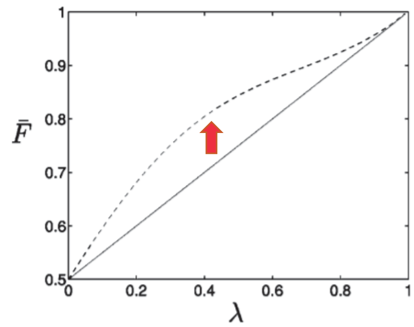
- CV teleportation
    - Input state $|\alpha_{\rm in}\rangle$
      : coherent state with unknown displacement
    - Resource state: TMSV
- Fidelity $F \equiv \langle \alpha_{\rm in}|\rho_{\rm out}|\alpha_{\rm in}\rangle$
    - Average output fidelity $F_{\rm av} = \frac{1}{1+e^{-2r}}$
    - Classical bound $F_{\rm cl} = \frac{1}{2}$



Quantum information processing with non-Gaussian states

- Improvement in CV teleportation
  - Photon subtraction $\hat{a}$
    T. Opatrný *et al.*, Phys. Rev. A **61**, 032302 (2000).
  - Photon addition $\hat{a}^\dagger$
    Yang Yang *et al.*, Phys. Rev. A **80**, 022315 (2009).
  - Coherent operation $t\hat{a} + r\hat{a}^\dagger$
    Su-Yong Lee *et al.*, Phys. Rev. A **84**, 012302 (2011).

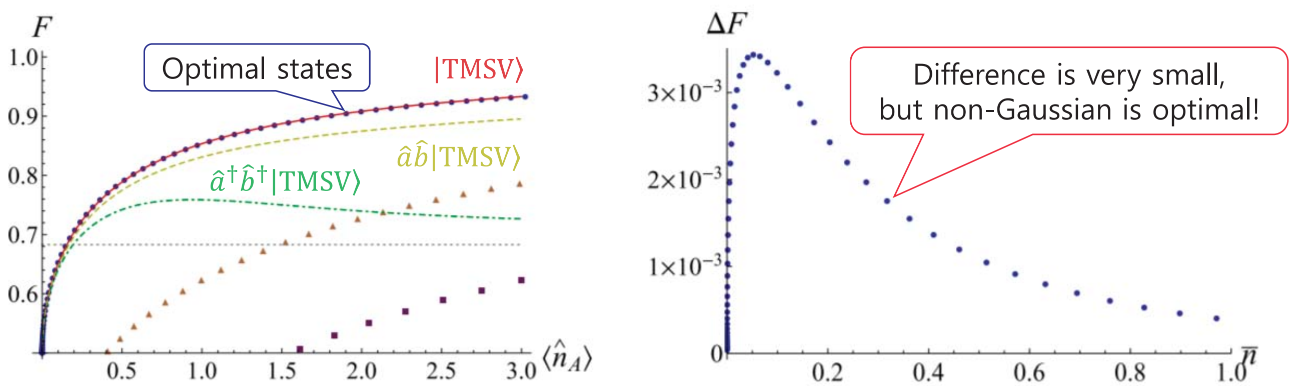




Gaussian vs non-Gaussian states

Jaehak Lee *et al.*, Phys. Rev. A **95**, 052343 (2017).

- CV teleportation with resources under the same energy constraint



No-go theorems on Gaussian states

- Impossible tasks with Gaussian states and operations
  - Entanglement distillation
  - Universal quantum computation
  - Quantum error correction

- Non-Gaussian operations are necessary.

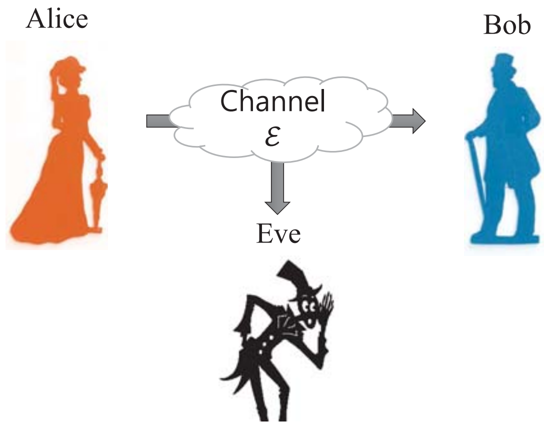
PART II.
No-cloning bound

No-cloning theorem

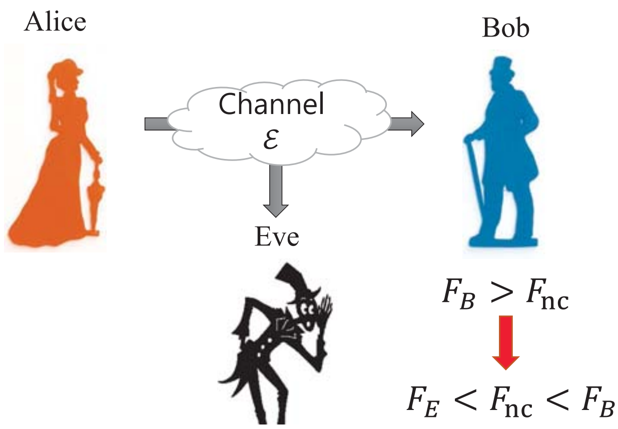
- Perfect cloning is impossible.

$$|\psi_j\rangle \not\rightarrow |\psi_j\rangle \otimes |\psi_j\rangle$$

- Secure communication
  - Eve cannot obtain information without disturbing the state.
  - How to assess security of the channel $\mathcal{E}$?



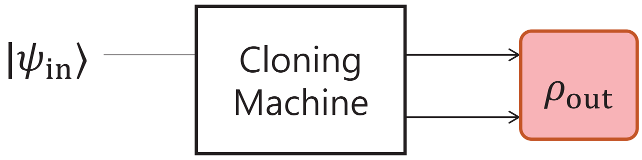
Secure communication



- Fidelity beyond the no-cloning bound guarantees secure communication!

No-cloning bound

- Optimal symmetric $1 \rightarrow 2$ cloning
  - $\rho_1 = \mathrm{tr}_2(\rho_{\mathrm{out}})$, $\rho_2 = \mathrm{tr}_1(\rho_{\mathrm{out}})$
  - Fidelity $F_1 = \langle\psi_{\mathrm{in}}|\rho_1|\psi_{\mathrm{in}}\rangle$, $F_2 = \langle\psi_{\mathrm{in}}|\rho_2|\psi_{\mathrm{in}}\rangle$
  - Find the maximum fidelity $F_{\mathrm{nc}} \equiv \max F_1 = \max F_2$

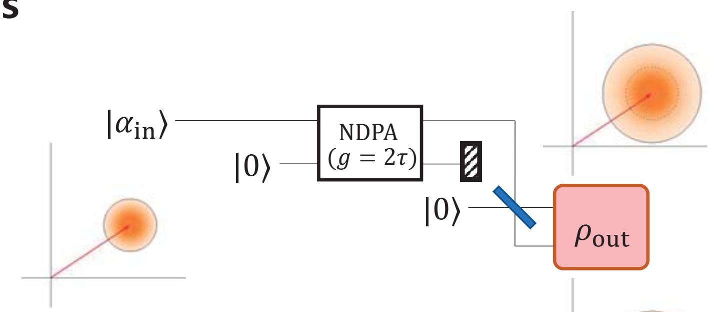


- No-cloning bound F_{nc}
  - For a given channel $\mathcal{E}$, if Bob obtains an output state beyond the no-cloning bound, $F_B > F_{\mathrm{nc}}$, then he has the best copy!

No-cloning bound for coherent states

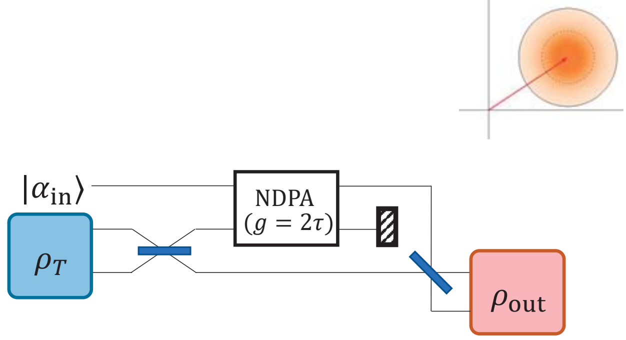
N. J. Cerf *et al.*, Phys. Rev. Lett. **85**, 1754 (2000).

- Gaussian cloning
  - $F_{\mathrm{nc}}^G = \frac{2}{3} \approx 0.6667$

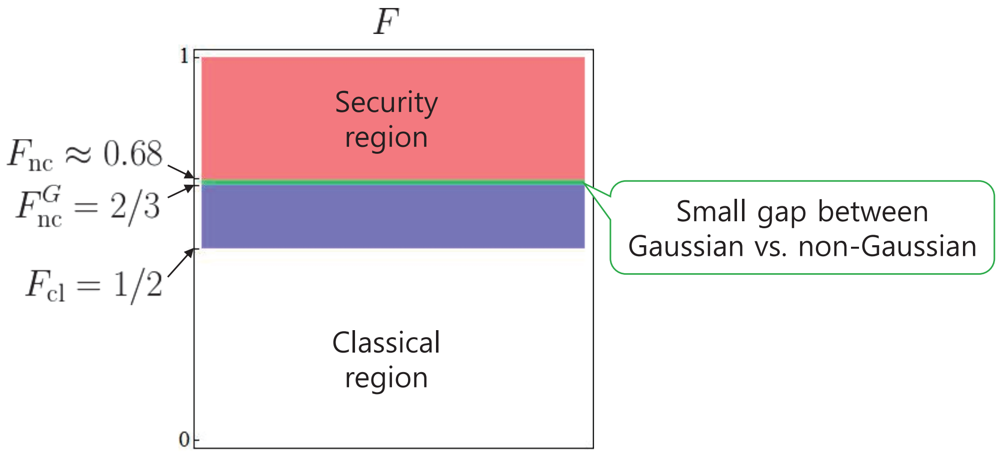


N. J. Cerf *et al.*, Phys. Rev. Lett. **95**, 070501 (2005).

- Non-Gaussian cloning
  - $F_{\mathrm{nc}} \approx 0.6826$
  - Employing non-Gaussian resource ρ_T



Fidelity region for coherent states

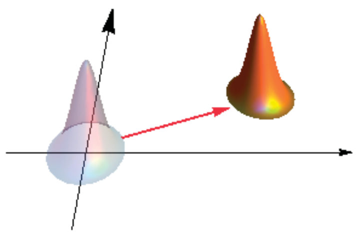


- What if we employ non-Gaussian input states?

Input states with unknown displacement

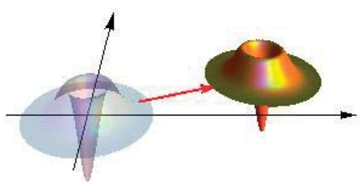
- Coherent states

  $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$



- Displaced Fock states (DFS)

  $|k,\alpha\rangle = \hat{D}(\alpha)|k\rangle$



- Several non-Gaussian states with unknown displacement

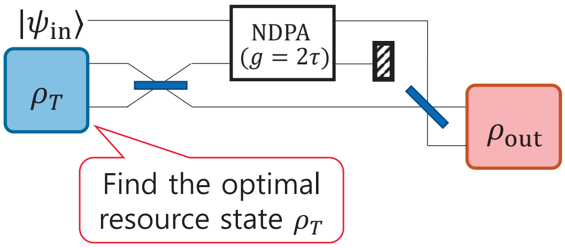
Covariant cloner

- $1 \rightarrow 2$ symmetric cloner $\rho_{\mathrm{out}} = T(|\psi_{\mathrm{in}}\rangle\langle\psi_{\mathrm{in}}|)$

- When distribution of α is completely unknown, covariant cloner is optimal.

$$\widehat{D}^{\otimes 2}(\alpha) T(|\psi_{\mathrm{in}}\rangle\langle\psi_{\mathrm{in}}|) \widehat{D}^{\dagger\otimes 2}(\alpha) = T\left(\widehat{D}(\alpha)|\psi_{\mathrm{in}}\rangle\langle\psi_{\mathrm{in}}|\widehat{D}^{\dagger}(\alpha)\right)$$

- Implementation



Displaced-Fock-state input

- Single-copy fidelity

$$f^{(i)} = \left\langle \hat{f}_k^{(i)} \right\rangle_{\rho_T} = \left\langle \left[L_k(\hat{O}_i) \right]^2 e^{-\hat{O}_i} \right\rangle_{\rho_T}, \ i = 1,2$$

$$\hat{O}_1 \equiv \frac{1}{2}(\hat{p}_1^2 + \hat{x}_2^2),\ \hat{O}_2 \equiv \frac{1}{2}(\hat{p}_2^2 + \hat{x}_1^2)$$

- Gaussian cloning
  - Among Gaussian states ρ_T, $\rho_T = |0\rangle\langle 0| \otimes |0\rangle\langle 0|$ is optimal.

- Non-Gaussian cloning
  - Find the maximum value of $\frac{1}{2}\left\langle \hat{f}_k^{(1)} + \hat{f}_k^{(2)} \right\rangle_{\rho_T}$

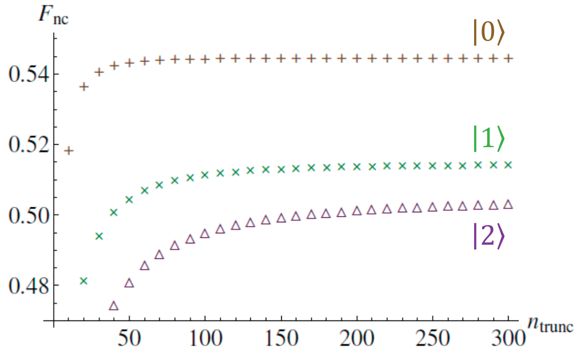
Numerical calculation of no-cloning bound

- Eigenvalues of $\frac{1}{2}\left(\hat{f}_k^{(1)} + \hat{f}_k^{(2)}\right)$ in truncated Fock basis
  - Matrix elements in Fock basis

$$\langle i,j|\hat{f}_k^{(1)}|l,m\rangle = \int dp_1 dx_2 \langle i|p_1\rangle\langle j|x_2\rangle \left[L_k\left(\frac{p_1^2+x_2^2}{2}\right)\right]^2 \exp\left(-\frac{p_1^2+x_2^2}{2}\right)\langle p_1|l\rangle\langle x_2|m\rangle$$

$$\langle x_\theta|n\rangle = \frac{1}{\sqrt{\pi^{1/2}2^n n!}} e^{-x_\theta^2/2} H_n(x_\theta) e^{-in\theta},$$

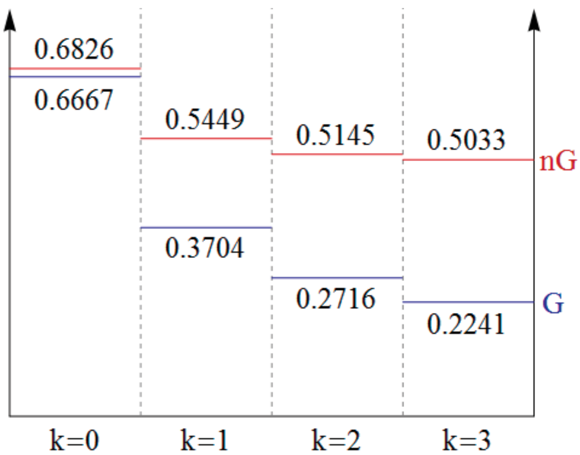
$$\text{with } \hat{x}_\theta = \hat{x}\cos\theta + \hat{p}\sin\theta$$



Eigenvalue saturates as dimension increases

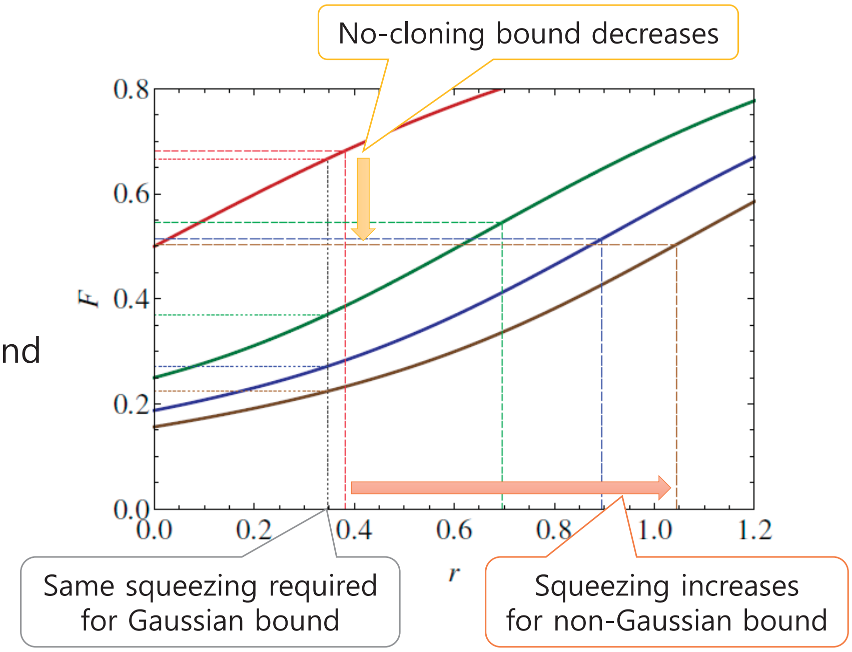
No-cloning bound of DFS

- No-cloning bound decreases with k increasing
- Gap between Gaussian and non-Gaussian increases with k increasing
  - Non-Gaussian resources become essential.

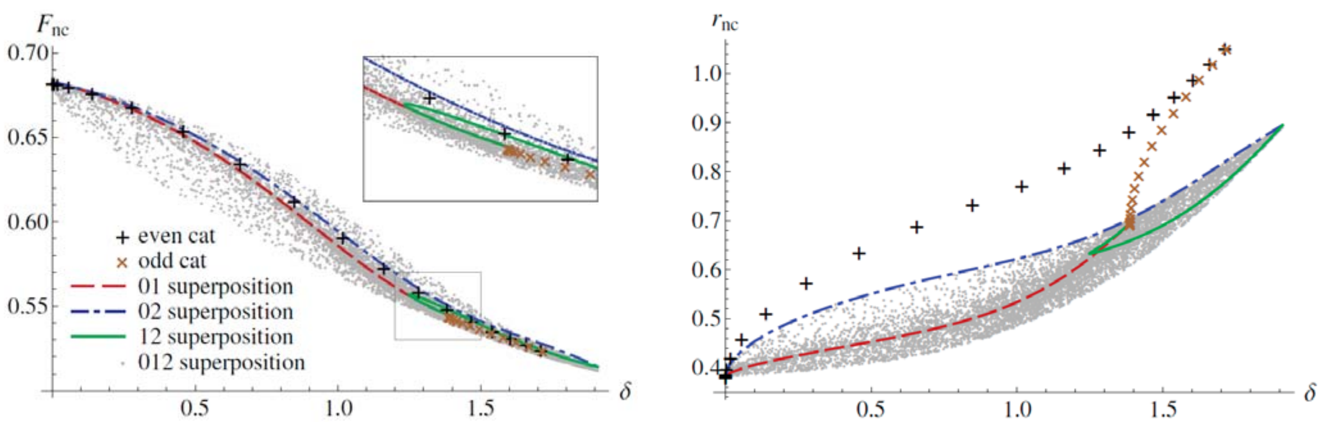


Secure teleportation

- Teleportation with TMSV
- Find the critical squeezing r_{nc} achieving the no-cloning bound
- Hard to achieve no-cloning bound for non-Gaussian input states!

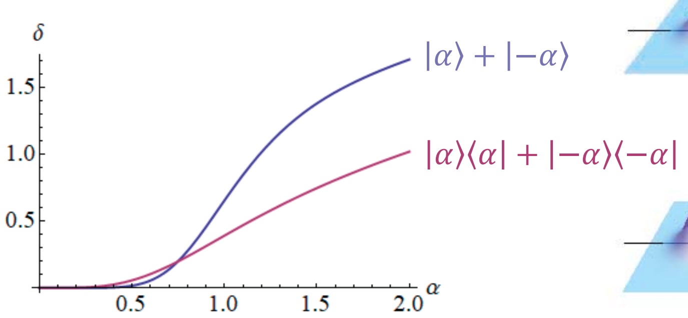


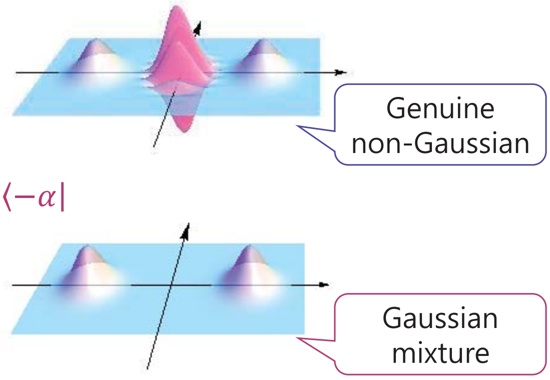
Several non-Gaussian input states



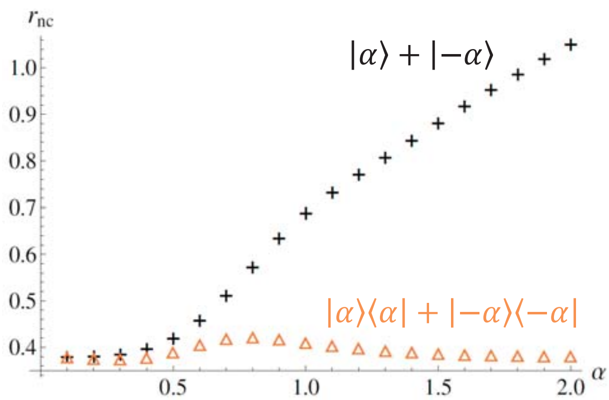
- Hard to achieve secure teleportation for non-Gaussian states

Genuine non-Gaussianity





Mixed input states



- Gaussian mixture does not increase critical squeezing.

Summary

- Optimal cloning of non-Gaussian input states requires non-Gaussian resource states.
- No-cloning bound decreases as non-Gaussianity increases.
- Secure teleportation is hard to achieve for non-Gaussian input states.
- Mixture of Gaussian states does not increases the critical squeezing.