

On Varieties with Large Fundamental Group

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We will discuss some techniques to construct sections of adjoint bundles, especially on varieties with the so-called large fundamental group which is introduced by Kollár.

Fujita's conjecture predicts that for a projective manifold X and an ample line bundle L on X , the adjoint linear system $|K_X + mL|$ is base point free as soon as $m > n = \dim X$. One major achievement is due to Angehrn-Siu, who show that $|K_X + mL|$ is base point free for $m > n(n + 1)/2$. For the moment, Fujita's conjecture is out of reach. But if we add an additional structure, even weaker arguments sometimes give interesting consequences. Instead of explaining the result of Angehrn-Siu, or possible approach to Fujita's conjecture, we apply an effective base point freeness argument for the study of varieties with large fundamental group. A projective manifold X is said to have large fundamental group, if the universal cover has no compact subvariety of positive dimension. For example if the universal cover is Stein, then X has large fundamental group. In the lecture we would like to discuss the following type of effective existence theorems.

Theorem 0.1. *Let X be a projective manifold with large fundamental group, and L an ample line bundle on it. Then $H^0(X, K_X + L) \neq 0$.*

Theorem 0.2. *Let X be an n -dimensional projective manifold of general type with large fundamental group. Then*

- (1) $\dim H^0(X, mK_X) \geq 1$ for every $m \geq 2$,
- (2) $\dim H^0(X, mK_X) \geq 2$ for every $m \geq 4$,
- (3) $|mK_X|$ gives a birational map for every $m \geq 10^n$.

We will start with recalling basic notions: singular Hermitian metric, multiplier ideal sheaf, Nadel vanishing, and will recall an effective base point freeness argument. We also recall the so-called Shafarevich map due to Kollár. The Shafarevich map $\text{sh}_X : X \dashrightarrow \text{Sh}(X)$ is a fibre space structure, whose fibre has “small” fundamental group in $\pi_1(X)$, and that is a non-abelian analogue of the Albanese map. Then the main part of these lectures is to explain how the property of fundamental groups come into the picture of an effective base point freeness argument.

These lectures are based on a book of Kollár: [Kollár95] *Shafarevich Maps and Automorphic Forms*, Princeton University Press, 1995, especially around Chapter 3 and 14. While we will use the complex analytic method. [Demailly96] *L^2 vanishing theorem for positive line bundles and adjunction theory*, Transcendental Methods in Alg, Geom., Springer LNM, vol. 1646, 1996, pp. 1–97, will be a standard reference for our method.