

$$1 \text{ (a)} \quad e^{-\beta f} = \frac{1}{h^2} \int_0^L dx \int_{-\infty}^{\infty} dp e^{-\beta p^2/2m} e^{-\beta kL^2/4\pi^2 (1 - \cos \frac{2\pi x}{L})}$$

f : free energy per particle. $0 \leq x \leq L \rightarrow$ one period.

$$= \frac{1}{h^2} \left(\frac{2\pi m}{\beta} \right)^{\frac{1}{2}} \frac{1}{2} \int_0^{L/2} dx e^{-\beta kL^2/4\pi^2 (1 - \cos \frac{2\pi x}{L})}$$

$$= \frac{L}{h^2} \left(\frac{2\pi m}{\beta} \right)^{\frac{1}{2}} e^{-\beta kL^2/4\pi^2} \underbrace{\frac{1}{\pi} \int_0^{\pi} d\theta e^{\frac{\beta kL^2}{4\pi^2} \cos \theta}}_{I_0\left(\frac{\beta kL^2}{4\pi^2}\right)}$$

$F = N \cdot f \rightarrow$ free energy for N particles

$$\beta F(L) = -N \log \left[\frac{L}{h^2} \left(\frac{2\pi m}{\beta} \right)^{\frac{1}{2}} I_0\left(\frac{\beta kL^2}{4\pi^2}\right) \right] + \frac{N\beta kL^2}{4\pi^2}$$

$$\Delta F = F(L_2) - F(L_1) = \frac{Nk}{4\pi^2} (L_2^2 - L_1^2) - NkT \log \left(\frac{L_2}{L_1} \frac{I_0\left(\frac{\beta kL_2^2}{4\pi^2}\right)}{I_0\left(\frac{\beta kL_1^2}{4\pi^2}\right)} \right)$$

(b) From $d(\beta F) = Ed\beta - \beta p dL + \beta \mu dN$

$$-\beta p = \left(\frac{\partial \beta F}{\partial L} \right)_{\beta, N} = -\frac{N}{L} + \frac{N\beta k}{2\pi^2} L \left\{ 1 - \frac{I_0'\left(\frac{\beta kL^2}{4\pi^2}\right)}{I_0\left(\frac{\beta kL^2}{4\pi^2}\right)} \right\}$$

$$I_0'(x) = \frac{1}{2} [I_{-1}(x) + I_1(x)] = I_1(x)$$

$$\therefore \beta p = \frac{N}{L} - \frac{N\beta k}{2\pi^2} L \left(1 - \frac{I_1\left(\frac{\beta kL^2}{4\pi^2}\right)}{I_0\left(\frac{\beta kL^2}{4\pi^2}\right)} \right)$$

If $L \rightarrow \infty$

$$\frac{I_1\left(\frac{\beta kL^2}{4\pi^2}\right)}{I_0\left(\frac{\beta kL^2}{4\pi^2}\right)} \rightarrow 1$$

and $\beta p = \frac{N}{L} \rightarrow$ ideal gas equation

For $L \rightarrow \infty$

$$\mathcal{H} = \frac{p^2}{2m} + \frac{kL^2}{4\pi^2} \left(1 - \cos \frac{2\pi x}{L} \right) \rightarrow \frac{p^2}{2m}$$

$$2. E_r = -\sum_{i=1}^N n_i \mu_H \quad (n_i = \pm 1)$$

$$(a) Q = \sum_{\{n_i\}} e^{\beta \sum_{i=1}^N n_i \mu_H} = \prod_{i=1}^N \sum_{n_i=\pm 1} e^{\beta n_i \mu_H} = \prod_{i=1}^N (e^{\beta \mu_H} + e^{-\beta \mu_H})$$

$$F = -kT \ln Q = -NkT \ln(e^{\beta \mu_H} + e^{-\beta \mu_H})$$

$$\langle E \rangle = \left(\frac{\partial \ln Q}{\partial (-\beta)} \right)_{V,N} = \left(\frac{\partial \beta F}{\partial \beta} \right)_{V,N} = -N \mu_H \frac{e^{\beta \mu_H} - e^{-\beta \mu_H}}{e^{\beta \mu_H} + e^{-\beta \mu_H}}$$

$$= -N \mu_H \tanh(\beta \mu_H)$$

$$(b) dF = -SdT - pdV + \mu dN$$

$$\rightarrow S = -\left(\frac{\partial F}{\partial T} \right)_{V,N} = +Nk \ln(e^{\beta \mu_H} + e^{-\beta \mu_H}) + NkT \frac{e^{\beta \mu_H} - e^{-\beta \mu_H}}{e^{\beta \mu_H} + e^{-\beta \mu_H}} \left(-\frac{1}{kT^2} \right)$$

$$= Nk \ln(e^{\beta \mu_H} + e^{-\beta \mu_H}) - N \frac{\mu_H}{T} \tanh(\beta \mu_H)$$

(c) For $T \rightarrow 0$

$$\langle E \rangle = -N \mu_H$$

$$S = \frac{N}{T} \mu_H - N \frac{\mu_H}{T} \cdot 1 \rightarrow 0$$

$$(d) \langle M \rangle = \frac{\partial \ln Q}{\partial \beta \mu_H} = N \mu \left(\frac{e^{\beta \mu_H} - e^{-\beta \mu_H}}{e^{\beta \mu_H} + e^{-\beta \mu_H}} \right) = N \mu \tanh(\beta \mu_H)$$

$$(e) \langle (\delta M)^2 \rangle = \frac{\partial \langle M \rangle}{\partial \beta \mu_H} = N \mu^2 \frac{1}{\text{sech}^2 \beta \mu_H}$$

$$(f) \text{ as } T \rightarrow 0 \quad \langle M \rangle \rightarrow N \mu$$

$$\langle (\delta M)^2 \rangle \rightarrow 0$$

$$3. (a) \beta p = \frac{\rho}{1-b\rho} - \beta a \rho^2$$

$$dS = \frac{1}{T} dE + \frac{P}{T} dV - \frac{\mu}{T} dN \rightarrow \boxed{E d\beta + V d(\beta p) - N d(\beta \mu) = 0}$$

Gibbs-Duhem Eq.

$\therefore$ At constant β

$$\left(\frac{\partial \beta p}{\partial \rho} \right)_{\beta} = \rho \left(\frac{\partial \beta \mu}{\partial \rho} \right)_{\beta} \quad (*)$$

$$\Xi = e^{\beta p V} \rightarrow \left(\frac{\partial \ln \Xi}{\partial \beta \mu} \right)_{\beta, V} = \langle N \rangle = V \left(\frac{\partial \beta p}{\partial \beta \mu} \right)_{\beta, V} = \rho V$$

$$\left(\frac{\partial \langle N \rangle}{\partial \beta \mu} \right)_{\beta, V} = \langle (\delta N)^2 \rangle = V \left(\frac{\partial \rho}{\partial \beta \mu} \right)_{\beta, V} \stackrel{(*)}{=} V \rho \left(\frac{\partial \rho}{\partial \beta p} \right)_{\beta, V}$$

$$\therefore \langle (\delta p)^2 \rangle = \frac{\langle (\delta N)^2 \rangle}{V^2} = \frac{\rho}{V} \left(\frac{\partial \rho}{\partial \beta p} \right)_{\beta} = \frac{\rho}{V} \left[\frac{1-b\rho - \rho(-b)}{(1-b\rho)^2} - 2\beta a \rho \right]^{-1}$$

$$\frac{\langle (\delta p)^2 \rangle^{\frac{1}{2}}}{\rho} = \left\{ \frac{1}{L^3 \rho} \left[\frac{1}{(1-b\rho)^2} - 2\beta a \rho \right] \right\}^{\frac{1}{2}}$$

$$\text{as } L^3 \rightarrow \infty, \quad \boxed{\frac{\langle (\delta p)^2 \rangle^{\frac{1}{2}}}{\rho} \rightarrow 0}$$

(b) Critical point

$$\left(\frac{\partial \beta p}{\partial \rho} \right)_{\beta} = \left(\frac{\partial^2 \beta p}{\partial \rho^2} \right)_{\beta} = 0 = \frac{1}{(1-b\rho)^2} - 2\beta a \rho \Big|_{\rho=\rho_c, \beta=\beta_c} = 0 \quad (1)$$

$$\left(\frac{\partial \beta p}{\partial \rho} \right)_{\beta} = \frac{1-b\rho}{(1-b\rho)^4} - 2\beta a \rho = 0 \quad \rho=\rho_c, \beta=\beta_c$$

$$\left(\frac{\partial^2 \beta p}{\partial \rho^2} \right)_{\beta} = - \frac{2(1-b\rho)(-b)}{(1-b\rho)^4} - 2\beta a = 0 \Big|_{\rho=\rho_c, \beta=\beta_c} \quad (2)$$

$$\text{From (1) \& (2) } \rho_c = \frac{1}{3b} \quad \beta_c = \frac{27b}{8a}$$

$$\begin{aligned}
 (c) \left. \frac{\sqrt{\langle \delta \rho \rangle^2}}{\rho} \right|_{\substack{\rho=\rho_c \\ \beta=\beta_c}}^{\frac{1}{2}} &= \left[\frac{1}{L^3 \rho_c} \left(\frac{1}{(1-b\rho)^2} - 2\beta a \rho \right) \right]^{\frac{1}{2}} \\
 &= \left[\frac{1}{100b \cdot \frac{1}{3b}} \left(\frac{1}{(1-b \cdot \frac{1}{3b})^2} - 2 \cdot \underbrace{\phi \beta_c \cdot a \cdot \frac{1}{3b}}_{\beta_c = \frac{27b}{8a}} \right) \right]^{\frac{1}{2}} \\
 &= \left[\frac{3}{100} \left(\frac{1}{9/4(1-\phi)} \right) \right]^{\frac{1}{2}}
 \end{aligned}$$

$$= \frac{1}{\sqrt{75}} \frac{1}{(1-\phi)^{1/2}} = \begin{cases} \frac{1}{\sqrt{75}} \frac{1}{(0.1)^{1/2}} & \phi = 0.9 \text{ (10\%)} \\ \frac{1}{\sqrt{75}} \frac{1}{(0.001)^{1/2}} & \phi = 0.999 \text{ (0.1\%)} \\ \frac{1}{\sqrt{75}} \frac{1}{(0.00001)^{1/2}} & \phi = 0.99999 \text{ (0.001\%)} \end{cases}$$

as β approaches β_c $\frac{\sqrt{\langle \delta \rho \rangle^2}}{\rho}$ diverges further.

(d) For critical opalescence ($L = 100 \text{ nm}$, $b = (0.5)^3 \text{ nm}^3$)

$$\frac{\sqrt{\langle \delta \rho \rangle^2}}{\rho} \sim 1 \sim \left[\frac{3 \cdot (0.5)^3}{(100)^3} \left(\frac{1}{9/4(1-\phi)} \right) \right]^{1/2}$$

ϕ should be $\lesssim 0.9999999$ ($10^{-5}\%$ above T_c)
 then the density fluctuation is significant.

$$4. (a) \Xi = e^{\beta PV} = \prod_j \sum_{n_j=0}^1 e^{-\beta(\epsilon_j - \mu)n_j} = \prod_j [1 + e^{-\beta(\epsilon_j - \mu)}]$$

↑ Fermion

$$\beta PV = \ln \Xi = \sum_j \ln [1 + e^{-\beta(\epsilon_j - \mu)}]$$

$$= \int d\epsilon g(\epsilon) \log(1 + e^{-\beta(\epsilon - \mu)}) = \int d\epsilon g(\epsilon) \ln(1 + ze^{-\beta\epsilon})$$

$$g(\epsilon) = \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon^{1/2} \quad (\leftarrow \text{density of state for free particle})$$

$$\therefore \beta PV = \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\epsilon \epsilon^{1/2} \ln(1 + ze^{-\beta\epsilon})$$

$$= \beta^{-3/2} \int_0^\infty dx x^{1/2} \ln(1 + ze^{-x})$$

$$= 2\beta^{-3/2} \int_0^\infty dx x^2 \ln(1 + ze^{-x^2})$$

$$\therefore \beta P = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{3/2} \frac{4}{\sqrt{\pi}} \int_0^\infty dx x^2 \log(1 + ze^{-x^2})$$

$$\ln \Xi = \frac{V}{\lambda^3} f_{3/2}(z) = \frac{1}{\lambda^3} f_{5/2}(z) \quad \text{where } f_{5/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx x^2 \log(1 + ze^{-x^2})$$

$$\langle N \rangle = \left(\frac{\partial \ln \Xi}{\partial \beta \mu} \right)_{\beta, V} = z \left(\frac{\partial \ln \Xi}{\partial z} \right) = z \cdot \frac{V}{\lambda^3} \left(\frac{\partial f_{5/2}(z)}{\partial z} \right)$$

$$= \rho V$$

$$\therefore \rho = \frac{1}{\lambda^3} z \frac{\partial f_{5/2}(z)}{\partial z} = \frac{1}{\lambda^3} f_{3/2}(z)$$

$$(b) \langle E \rangle = \left(\frac{\partial \ln \Xi}{\partial (-\beta)} \right)_{\beta, \mu} = \frac{\partial}{\partial (-\beta)} \left[\frac{V}{\lambda^3} f_{5/2}(z) \right]_{\beta, \mu \text{ (or } z)}$$

$$= V f_{5/2}(z) \frac{\partial}{\partial (-\beta)} \left(\frac{2\pi\beta\hbar^2}{m} \right)^{-3/2} = \frac{1}{\beta} \left[\frac{V}{\lambda^3} f_{5/2}(z) \right] \cdot \frac{3}{2}$$

$$= \frac{3}{2} p V$$

$$(c) \rho\lambda^3 = \sum_{l=1}^{\infty} (-1)^{l+1} \frac{z^l}{l^{3/2}} = z - \frac{z^2}{2^{3/2}} + \dots$$

Set $z = (\rho\lambda^3) + a_1(\rho\lambda^3)^2 + \dots$ and insert then

$$\rho\lambda^3 = (\rho\lambda^3) + a_1(\rho\lambda^3)^2 + \dots - \frac{1}{2\sqrt{2}} \left((\rho\lambda^3)^2 + \dots \right)$$

$$\therefore a_1 = \frac{1}{2\sqrt{2}}$$

$$\therefore z = (\rho\lambda^3) + \frac{1}{2\sqrt{2}} (\rho\lambda^3)^2 + \dots$$

$$(d) \langle n_p \rangle = \frac{2 \ln \xi_j}{2\beta\mu} = \frac{2}{2\beta\mu} \ln(1 + e^{-\beta\varepsilon} e^{\beta\mu})$$

$$= \frac{z e^{-\beta\varepsilon}}{1 + z e^{-\beta\varepsilon}}$$

Since $\rho\lambda^3 \ll 1$ (which corresponds to $z \ll 1$ from (c))

$$\approx z e^{-\beta\varepsilon} \approx (\rho\lambda^3) e^{-\beta\varepsilon_p}$$

$$(e) \lambda = \left(\frac{2\pi\beta\hbar^2}{m} \right)^{1/2} = \left(\frac{\hbar^2}{2\pi m kT} \right)^{1/2} = \frac{\hbar}{\sqrt{2\pi m kT}}$$

$$\text{Since } p^2/2m = \frac{1}{2} kT \quad p = \sqrt{mkT}$$

$$\lambda \sim \frac{\hbar}{p}$$

$$\begin{aligned}
 (f) \quad \frac{\beta P}{g} &= \frac{f_{5/2}(z)}{f_{3/2}(z)} = \frac{\sum_{l=1}^{\infty} (-1)^{l+1} z^l / l^{5/2}}{\sum_{l=1}^{\infty} (-1)^{l+1} z^l / l^{3/2}} \\
 &= \frac{z - z^2/2^{5/2} + \dots}{z - z^2/2^{3/2} + \dots} \\
 &= 1 + \frac{1}{2^{5/2}} z + \dots
 \end{aligned}$$

$$\approx 1 + \frac{1}{2^{5/2}} (g \lambda^3) + \dots$$

If $g \lambda^3$ is finite, the system deviates from classical ideal gas law ($\frac{\beta P}{g} = 1$)

This results from the fact that at finite $g \lambda^3$ the number of accessible states is not large enough to accommodate the fermionic particles. ($\frac{N}{g} \sim 1$)

For classical behavior to be ~~not~~ observed $N/g \ll 1$ should be satisfied.

Quantum mechanical exclusion interaction is beginning to manifest itself for $N/g \sim 1$

$$(g). \quad f_{3/2}(z) \stackrel{p\lambda^3}{=} \frac{4}{\sqrt{\pi}} \int_0^\infty dx \, x^2 \left(\frac{1}{\frac{1}{2}e^{x^2} + 1} \right) \quad p\lambda^3 \gg 1 \Rightarrow z \gg 1$$

$$= \frac{4}{\sqrt{\pi}} \int_0^\infty dx \, x^2 \left(\frac{1}{e^{x^2 - \ln z} + 1} \right)$$

$$\left\{ \begin{array}{l} \text{Set } x^2 = t \quad 2x dx = dt \\ dx = \frac{1}{2\sqrt{t}} dt \end{array} \right.$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty dt \, \sqrt{t} \cdot \frac{1}{(e^{t - \ln z} + 1)}$$

$$= \frac{2}{\sqrt{\pi}} \left\{ \left[\frac{2}{3} t^{3/2} \frac{1}{e^{t - \ln z} + 1} \right]_0^\infty + \frac{2}{3} \int_0^\infty dt \frac{t^{3/2} e^{t - \ln z}}{(e^{t - \ln z} + 1)^2} \right\}$$

$$= \frac{4}{3\sqrt{\pi}} \int_0^\infty dt \frac{t^{3/2} e^{t - \ln z}}{(e^{t - \ln z} + 1)^2}$$

Expand $t^{3/2}$ around $\ln z$

$$t^{3/2} = (\ln z)^{3/2} + \frac{3}{2} (\ln z)^{1/2} (t - \ln z) + \frac{3}{8} (\ln z)^{-1/2} (t - \ln z)^2 + \dots$$

$$= \frac{4}{3\sqrt{\pi}} \int_{-\ln z}^\infty ds \left[(\ln z)^{3/2} + \frac{3}{2} (\ln z)^{1/2} s + \dots \right] \frac{e^s}{(e^s + 1)^2}$$

$$\approx \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} \int_{-\infty}^\infty ds \frac{e^s}{(e^s + 1)^2} \quad \left(\frac{3}{8} (\ln z)^{1/2} s^2 \right) \hookrightarrow \frac{e^s}{e^{s+1/2}}$$

$$= -2 \int_0^\infty ds \frac{d}{ds} \frac{1}{e^s + 1} = -2 \cdot \left(0 - \frac{1}{2} \right) = 1 \quad = \frac{1}{(e^{s/2} + e^{-s/2})^2}$$

$$\therefore f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} = p\lambda^3$$

$$4) \int_{-\infty}^\infty \frac{s^2 e^{-s}}{(e^s + 1)^2} ds = \frac{\pi^2}{8} \quad \text{even function}$$

hence $z = e^{\beta \epsilon_F} \rightarrow \frac{4}{3\sqrt{\pi}} (\log e^{\beta \epsilon_F})^{3/2} = g \lambda^3$

$\therefore \epsilon_F = \left(\frac{\hbar^2}{2m} \right) (6\pi^2 g)^{2/3}$

$\lambda = \left(\frac{2\pi \hbar^2}{m} \right)^{1/2}$

$= \frac{\hbar^2}{2m} \cdot \frac{(6\pi^2)^{2/3}}{a^2} = \left(\frac{\hbar^2}{m a_0^2} \right) \frac{a_0^2}{a^2} \frac{(6\pi^2)^{2/3}}{2}$

$\uparrow$ Hartree $\downarrow \left(\frac{0.5}{3} \right)^2$

$a_0 = 0.5 \text{ \AA}$

$a \approx 3 \text{ \AA} (\text{Cu})$

$= 27.2 \text{ eV} \cdot \left(\frac{1}{3} \right)^2 \cdot 7.5 \approx 5 \text{ eV}$

$z \approx e^{\epsilon_F / kT_F}$

$kT_F = \frac{1}{50} \text{ eV}$

$= e^{250} \gg 1$ Therefore low temperature and/or high density approximation ($g \lambda^3 \gg 1, z \gg 1$) is a good approximation to describe the electrons in Cu metal.

(h) Since we showed $E = \frac{3}{2} PV$ for arbitrary condition calculate $E(T)$ to compute $p(T)$

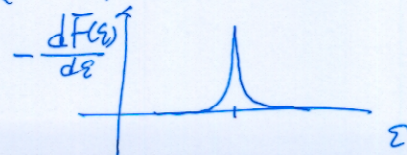
$E(T) = \int_0^\infty d\epsilon \epsilon g(\epsilon) F(\epsilon)$

$F(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} + 1} = \langle n \rangle$

$= \int_0^\infty d\epsilon \left[\left(\int_0^\epsilon d\epsilon' g(\epsilon') \epsilon' \right) F(\epsilon) \right] - \int_0^\infty d\epsilon \left(\int_0^\epsilon d\epsilon' \epsilon' g(\epsilon') \right) \left(\frac{dF(\epsilon)}{d\epsilon} \right)$

Evaluate this integral ---

Since $\left(-\frac{dF}{d\epsilon} \right)$ has peak at $\epsilon = \mu = \epsilon_F$



> all the contributions to the integral comes from $\epsilon \approx \mu (= \epsilon_F)$

→ Let us expand $\gamma(\epsilon)$ around ϵ_F

$$\gamma(\epsilon) = \gamma(\epsilon_F) + \gamma'(\epsilon_F)(\epsilon - \epsilon_F) + \frac{1}{2} \gamma''(\epsilon_F)(\epsilon - \epsilon_F)^2 + \dots$$

$$E(T) = \int_0^\infty d\epsilon \gamma(\epsilon_F) \left(-\frac{dF}{d\epsilon}\right) + \frac{1}{2} \int_0^\infty d\epsilon \gamma''(\epsilon_F) (\epsilon - \epsilon_F)^2 \left(-\frac{dF}{d\epsilon}\right)$$

$\left(-\frac{dF}{d\epsilon}\right)$ is an even function around $\epsilon - \epsilon_F$

$$\begin{aligned} \therefore E(T) &= \gamma(\epsilon_F) \int_{-\infty}^{\infty} dy \frac{\beta e^{\beta y}}{(e^{\beta y} + 1)^2} + \frac{1}{2} \gamma''(\epsilon_F) \int_{-\infty}^{\infty} dy \frac{\beta y^2 e^{\beta y}}{(e^{\beta y} + 1)^2} \\ &= \gamma(\epsilon_F) \underbrace{\int_{-\infty}^{\infty} dt \frac{e^t}{(e^t + 1)^2}}_1 + \frac{1}{2} \gamma''(\epsilon_F) (kT)^2 \underbrace{\int_{-\infty}^{\infty} dt \frac{t^2 e^t}{(e^t + 1)^2}}_{\pi^2/6} + \dots \end{aligned}$$

$$= \gamma(\epsilon_F) + \frac{\pi^2}{6} \gamma''(\epsilon_F) (kT)^2$$

$$\gamma(\epsilon_F) = \int_0^{\epsilon_F} d\epsilon \epsilon g(\epsilon)$$

$$= \frac{3}{5} N \epsilon_F$$

$$\gamma''(\epsilon_F) = \left. \frac{d^2 \gamma}{d\epsilon^2} \right|_{\epsilon=\epsilon_F} = \left. \frac{d}{d\epsilon} \epsilon g(\epsilon) \right|_{\epsilon=\epsilon_F}$$

$$= \frac{3}{4} N \cdot \frac{1}{\epsilon_F^{3/2}} \cdot \frac{3}{2} \cdot \frac{1}{2} \epsilon^{-1/2} \Big|_{\epsilon=\epsilon_F} \approx \frac{N}{2 \epsilon_F^2}$$

$$g(\epsilon) = \frac{V}{(2\pi)^3} \left(\frac{2m}{\hbar^2} \right)^{3/2} \epsilon^{1/2}$$

$$\left| \begin{aligned} 2 \int_0^{\epsilon_F} g(\epsilon) d\epsilon &= N \\ \rightarrow g(\epsilon) &= \frac{N}{\epsilon_F} \left(\frac{\epsilon}{\epsilon_F} \right)^{1/2} \end{aligned} \right.$$

$$= \frac{3}{5} N \epsilon_F + \frac{\pi^2}{6} \frac{N}{2 \epsilon_F^2} (kT)^2 + \dots$$

$$\frac{3}{2} PV = \frac{3}{5} N \epsilon_F \left(1 + O\left(\frac{k^2 T^2}{\epsilon_F^2}\right) \right)$$

$$\therefore P = \frac{2}{5} \epsilon_F \rho \left(1 + O\left(\frac{k^2 T^2}{\epsilon_F^2}\right) \right)$$

$$5. \quad E_{\text{rot}} = \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) = \frac{1}{2I} \left(P_\theta^2 + \frac{P_\phi^2}{\sin^2 \theta} \right)$$

$$\begin{aligned} (a) \quad Z_{\text{rot}} &= \frac{1}{h^2} \int dP_\theta \int d(P_\phi / \sin \theta) \int d\theta \int d\phi \sin \theta e^{-\beta(E_{\text{rot}} - \mu \cdot z)} \\ &= \frac{1}{h^2} \int_{-\infty}^{\infty} dP_\theta \int_{-\infty}^{\infty} d(P_\phi / \sin \theta) e^{-\frac{\beta}{2I} (P_\theta^2 + P_\phi^2 / \sin^2 \theta)} \int_0^\pi d\theta \int_{-\pi}^{\pi} d\phi \sin \theta e^{\beta \mu z \cos \theta} \\ &= \frac{1}{h^2} \left(\sqrt{\frac{2\pi I}{\beta}} \right)^2 2\pi \left(\frac{e^{\beta \mu z} - e^{-\beta \mu z}}{\beta \mu z} \right) \\ &= \frac{2I \sinh \beta \mu z}{h^2 \beta^2 \mu z} \end{aligned}$$

$$\begin{aligned} (b) \quad P &= \frac{N}{V} \mu \langle \cos \theta \rangle = \frac{N}{V} \mu \left(\frac{\partial \ln Z_{\text{rot}}}{\partial \beta \mu z} \right)_\beta \\ &= \frac{N}{V} \mu \left[\coth \beta \mu z - \frac{1}{\beta \mu z} \right] \\ &= \frac{N}{V} \left[\mu \coth \beta \mu z - \frac{1}{\beta z} \right] \end{aligned}$$

$$(c) \quad \text{for small } x \quad \coth x \approx \frac{1}{x} + \frac{x}{3} + \dots$$

$$\begin{aligned} \therefore P &= \frac{N}{V} \left[\mu \cancel{\frac{1}{\beta \mu z}} + \frac{1}{3} \mu^2 \beta z - \cancel{\frac{1}{\beta z}} + \dots \right] \\ &= z \mathcal{E} - \epsilon_0 \mathcal{E} = \frac{1}{3} \frac{N}{V} \mu^2 \beta \mathcal{E} \\ \therefore \quad \boxed{\epsilon - \epsilon_0 = \frac{N \beta \mu^2}{3V}} \end{aligned}$$