Mathematical Interlude

Practice with indices and how to differentiate the Oseen tensor

Oseen tensor:

$$G_{ij}(\underline{r}-\underline{\xi}) = \frac{1}{8\pi\eta}\left\{\frac{\delta_{ij}}{|\underline{r}-\underline{\xi}|} + \frac{(\underline{r}-\underline{\xi})_i(\underline{r}-\underline{\xi})_j}{|\underline{r}-\underline{\xi}|^3}\right\}$$

pt. where we are calculating the velocity field

pt. where force acts

want to find $\left.\dfrac{\partial G_{ij}}{\partial r_k}\right|_{\xi=0}$

$\underline{r}$ position vector

$|\underline{r}| = r$

components r_k

(x, y, z)

recall:

$$r^2 = x^2 + y^2 + z^2$$

$$\therefore 2r\frac{\partial r}{\partial x} = 2x, \qquad \frac{\partial r}{\partial x} = \frac{x}{r}$$

or, for a general component

$$\frac{\partial r}{\partial r_k} = \frac{r_k}{r}$$

$$\frac{\partial|\underline{r}-\xi|}{\partial r_k} = \frac{(r-\xi)_k}{|\underline{r}-\xi|}$$

$$|r-\xi|^2 = (x-\xi_x)^2 + (y-\xi_y)^2 + (z-\xi_z)^2$$

$$2|r-\xi|\frac{\partial|r-\xi|}{\partial r_k} = 2(r_k-\xi_k)$$

$$8\pi\eta\frac{\partial G_{ij}}{\partial r_k} = \frac{-\delta_{ij}}{|\underline{r}-\xi|^2}\frac{(r-\xi)_k}{|r-\xi|} - \frac{3}{|r-\xi|^4}\frac{(r-\xi)_k}{|\underline{r}-\xi|}(\underline{r}-\xi)_i(r-\xi)_j$$

$$+ \frac{\delta_{ik}(\underline{r}-\xi)_j}{|r-\xi|^3} + \frac{\delta_{jk}(\underline{r}-\xi)_i}{|\underline{r}-\xi|^3}$$

$$\therefore 8\pi\eta\left.\frac{\partial G_{ij}}{\partial r_k}\right|_{\xi=0} = -\frac{\delta_{ij}r_k}{r^3} - \frac{3}{r^5}r_ir_jr_k + \frac{\delta_{ik}r_j}{r^3} + \frac{\delta_{jk}r_i}{r^3}$$

if $j=k$ these terms cancel

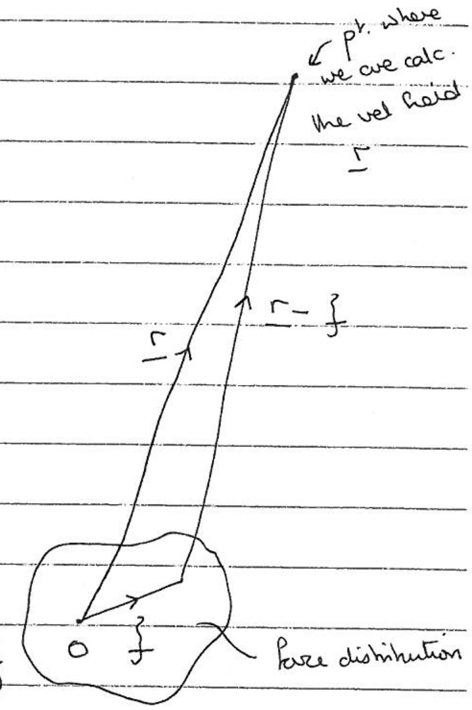
choosing a particular component

$$8\pi\eta \frac{\partial G_{iz}}{\partial z} = -\frac{3}{r^5} z^2 \cancel{x_k}^2 r_{\cancel{k}i} + \frac{r_i}{r^3}$$

$$(j = k = z) =$$

$$= \frac{1}{r^5}\left(-3z^2 + r^2\right) r_i$$

Multipole Expansion



$$v_i(\underline{r}) = \int G_{ij}(\underline{r}-\underline{\xi})\, f_j(\underline{\xi})\, d\underline{\xi}$$

where

$$G_{ij}(\underline{r}-\underline{\xi}) = \frac{1}{8\pi\eta}\left\{ \frac{\delta_{ij}}{|\underline{r}-\underline{\xi}|} + \frac{(\underline{r}-\underline{\xi})_i(\underline{r}-\underline{\xi})_j}{|\underline{r}-\underline{\xi}|^3} \right\}$$

Taylor expanding in $\underline{\xi}$ about the origin

$$v_i(\underline{r}) = \int \left\{ G_{ij}(\underline{r}) + \frac{\partial G_{ij}}{\partial r_k}(\underline{r})\, \xi_k + \frac{1}{2}\frac{\partial G_{ij}(\underline{r})}{\partial r_k \partial r_l}\xi_k \xi_l + \dots \right\} \times f_j(\underline{\xi})\, d\underline{\xi}$$

$$= G_{ij}(\underline{r}) \int f_j(\underline{\xi})\, d\underline{\xi} + \frac{\partial G_{ij}(\underline{r})}{\partial r_k} \int \xi_k f_j(\underline{\xi})\, d\underline{\xi} + \frac{1}{2}\frac{\partial G_{ij}(\underline{r})}{\partial r_k \partial r_l}\int \xi_k \xi_l f_j(\underline{\xi})\, d\underline{\xi} + \dots$$

$$= G_{ij}(\underline{r})\, F_j + \frac{\partial G_{ij}(\underline{r})}{\partial r_k} D_{jk} + \frac{1}{2}\frac{\partial^2 G_{ij}(\underline{r})}{\partial r_k \partial r_l} Q_{jkl} + \dots$$

G_{ij}: Stokeslet

F_j: monopole, net force, zero for swimmers

D_{jk}: dipole tensor, $D_{jk} = \int \xi_k f_j(\underline{\xi})\, d\underline{\xi}$

Q_{jkl}: quadrupole tensor

The dipole tensor (stresslet and rotlet):

conventional to write

$$D_{jk} - \frac{1}{3} D_{ii}\delta_{jk} = S_{jk} + T_{jk}$$

$-\frac{1}{3}D_{ii}\delta_{jk}$: subtracting $\frac{1}{3} \times \mathrm{Tr}$ off each diagonal element

S_{jk}: symmetric, traceless tensor $\boxed{\text{stresslet}}$

T_{jk}: antisymmetric tensor $\boxed{\text{rotlet}}$

OK because $\nabla \cdot G = 0$ $\quad \partial_i G_{ii} = 0$

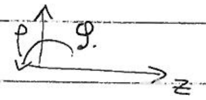
why can subst. Tr.
change of co-ords.
S_{ij}' $i \neq j$
cross check

stresslet:

$$S_{jk} = \frac{1}{2}\int \left(\xi_k f_j + \xi_j f_k\right) d\xi - \frac{1}{3}\int f_i \xi_i \delta_{jk}\, d\xi$$

roughly corresponds to straining flows $\longleftarrow \cdots \longrightarrow$

Example, for a force distribution that is symmetric around z:



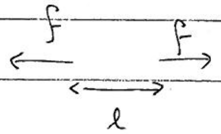
$$\underline{f} = f_z(\rho, z)\,\underline{e}_z + f_\rho(\rho, z)\,\underline{e}_\rho + f_\varphi(\rho, z)\,\underline{e}_\varphi$$

$$S_{zz} = \frac{1}{2}\int 2\xi_z f_z\, d\xi - \frac{1}{3}\int \left(\xi_x f_x + \xi_y f_y + \xi_z f_z\right) d\xi$$

$$= \frac{2}{3}\int z f_z\, d\xi - \frac{1}{3}\int \left\{\rho\cos\varphi\left(f_\rho\cos\varphi - f_\varphi\sin\varphi\right) + \rho\sin\varphi\left(f_\rho\sin\varphi + f_\varphi\cos\varphi\right)\right\} d\xi$$

$$= \frac{2}{3}\int z f_z\, d\xi - \frac{1}{3}\int \rho f_\rho\, d\xi$$

'ic's
simmer



$$S_{zz} = \frac{2}{3}\left(f \cdot \frac{l}{2} + (-f) \cdot \frac{-l}{2}\right) = \frac{2}{3} f l$$

similarly $\quad S_{xx} = S_{yy} = -\dfrac{S_{zz}}{2} \qquad S_{ij} = 0 \quad i \neq j$

velocity field

$$v_i(\underline{r}) = \frac{\partial G_{ij}(\underline{r})}{\partial r_k} S_{jk}$$

$$= \frac{\partial G_{iz}}{\partial z} S_{zz} + \frac{\partial G_{ix}}{\partial x} S_{xx} + \frac{\partial G_{iy}}{\partial y} S_{yy}$$

$$= S_{zz}\left\{\frac{\partial G_{iz}}{\partial z} - \frac{1}{2}\frac{\partial G_{ix}}{\partial x} - \frac{1}{2}\frac{\partial G_{iy}}{\partial y}\right\}$$

$$= \frac{S_{zz}}{8\pi\eta}\frac{r_i}{r^5}\left\{\left(r^2 - 3z^2\right) - \frac{1}{2}\left(r^2 - 3x^2\right) - \frac{1}{2}\left(r^2 - 3y^2\right)\right\}$$

$$= \frac{S_{zz}\, r_i}{8\pi\eta r^5}\,\frac{3\left(r^2 - 3z^2\right)}{2} = \frac{S_{zz}\,\hat{r}_i\, 3\left(1 - 3\cos^2\theta\right)}{8\pi\eta r^2\, 2}$$

$$\therefore \underline{v} = \frac{f l\left(1 - 3\cos^2\theta\right)\hat{\underline{r}}}{8\pi\eta r^2}$$

rotlet

$$T_{jk} = +\frac{1}{2}\int (\xi_k f_j - \xi_j f_k)\, d\xi$$

for uniaxial symmetry

$$T_{xy} = -T_{yx} = -\frac{1}{2}\int \rho f_\varphi \, d\xi$$

$$\Gamma_z = -\int \rho f_\varphi \, d\xi$$

all other components zero

swimmers are torque free so this term is zero on physical grounds

rotlet dipole

quadrupole swimmers