‘Low Re Swimming: Experiments’: Problems (1)

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1. Physics: Understanding Reynold’s number

(a) Show that the viscosity η and the density ρ can be combined to form a characteristic force, f_0. Calculate f_0 for water near room temperature ($\eta \approx 10^{-3}$ Pa.s, $\rho \approx 10^3$ kgm^{-3}).

(b) Estimate the viscous force needed to keep a sphere of radius a stationary in a fluid flow of speed u in the Stokes flow regime. (Hint: use the approximate formula developed in the lecture for estimating the viscous force on a fluid parcel of size ℓ and set $\ell \sim a$). Calculate the ratio of this force to f_0: do you recognise this quantity? Thus, when forces in a fluid are $\ll f_0$, the flow is viscosity dominated.

(c) To appreciate how *big* the value of f_0 you have calculated for water actually is, calculate the work done by this force moving 1 nm. Express your answer in units of k_BT, the thermal energy, in eV, and in kJ/mol. Compare this to the kind of energy needed to break a covalent bond (a few hundred kJ/mol $\equiv$ a few eV).

(d) Another way of thinking about what it means to say ‘inertia doesn’t matter’: When you stop pushing in a swimming pool, you carry on moving for a bit before coming to a halt. This is inertia. At low Re, when inertia is negligible, the absence of pushing brings things to a halt immediately. To see this quantitatively, Take typical numbers from an *E. coli* bacterium: model it as a sphere of radius $a \sim 1\mu$m moving at a speed of $v \sim 10\mu$m/s. Calculate how much force is needed to drag such a sphere along at this speed in water (and compare it to f_0). Now write the equation of motion for this sphere. Calculate how long it will take for the sphere to stop if the driving force is removed. That is what it means to say that inertia doesn’t matter at low Re.

2. Dimensional analysis: making sure you can do it

The flow equation we obtained in the lecture was:

$$\rho a = -\frac{dP}{dx} + \eta\frac{d^2u}{dz^2} + f.$$

Show that the dimensionless variables suggested in the lecture, $\{\bar{x}, \bar{z}, \bar{a}, \bar{P}, \bar{u}, \bar{f}\}$, indeed gives rise to

$$\mathrm{Re} \times \bar{a} = -\frac{d\bar{P}}{d\bar{x}} + \frac{d^2\bar{u}}{d\bar{z}^2} + \bar{f} = 0,$$

and that once we've taken the $\mathrm{Re} \to 0$ limit, the dimensional form of the Stokes flow equation is indeed

$$-\frac{dP}{dx} + \eta\frac{d^2u}{dz^2} + f = 0.$$

Extra: In the above exercise I used $\eta U/L$ to non-dimensionalise the pressure. Show that there exist an alternative non-dimensionalising scheme in which $\bar{P} = P/(\rho U^2)$. Obtain the flow equation in this case. Notice that the pressure now drops out. In full 3D form the flow equation is now $\nabla^2\mathbf{v} = 0$. This represents a special sub-class of Stokes flow.

3. Maths: for those who are good at vector calculus

The Stokes flow problem with no external force is completely specified by

$$-\nabla P + \eta\nabla^2\mathbf{v} = 0 \text{ subject to } \nabla \cdot \mathbf{v} = 0.$$

Show that this is equivalent to

$$\nabla^2 P = 0,$$

i.e. the pressure field is 'harmonic', and

$$\nabla^4\mathbf{v} = 0 \text{ subject to } \nabla \cdot \mathbf{v} = 0,$$

where ∇^4 is the 'biharmonic operator'. Many formal treatments of Stokes flow take this as the starting point, since a lot is known about solving harmonic and bioharmonic PDEs.

(Hint: take the divergence of the Stokes equation; easiest for both parts if you use the full apparatus of suffix notation and summation convention).

Separately, consider the consequence of the fact that the pressure is harmonic. By comparing $\nabla^P = 0$ with the equation of the pressure field in a sound wave, i.e.

$$\nabla^p = \frac{1}{v_s^2}\frac{\partial^P}{\partial t^2},$$

explore another consequence of Stokes flow, namely, its instantaneity. This property can already be seen directly by noting that the Stokes flow equations contain no time, so that flow fields are determined by the instantaneous positions of all boundaries (at infinity as well as at surfaces of all particles).

4. Physics: thinking through reversibility

Revisit the argument we used in the lecture to show that a sphere moving parallel to a wall cannot experience any lift or attraction from the presence of the wall at $\mathrm{Re} \to 0$.

(a) Explain why this argument does not apply in the inertia regime. (Hint: how does force scale with v in the viscous and inertial regimes?)

(b) Explain why this argument does not apply even at $\mathrm{Re} \to 0$ to an *E. coli* swimming next to a wall. (Hint: draw a similar diagram to the one used in the lecture for a sphere. Draw everything, and reverse everything!)

5. The importance of local anisotropy and global non-reversibility

(a) Derive the expression given in the lecture for the propulsive component in the motion of a thin rod, viz.

$$\mathbf{f}_{\mathrm{prop}} = (\xi_\perp - \xi_\parallel)(u \sin\theta \cos\theta)\,\hat{\mathbf{x}}.$$

(Hint: rotate the coordinate system.)

(b) Sketch a propagating waveform down a filament (like a sperm tail) and convince yourself that this satisfies the non-reciprocating requirement.