Solving Flash Ratchet Problem

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Optimal switching time for a flash ratchet: At $t = 0$, system switches from surface 1 to surface 2, and starts diffusing outwards. Surface 1 is an assymetric sawtooth potential with distances a and b from the minima to the maxima, with $a > b$. State 2 is flast, so the probability density in state 2 diffuses outwards as

$$\rho(x,t) = \frac{1}{\sqrt{4\pi dt}} e^{-x^2/4dt} \tag{1}$$

where d is the diffusion constant in the state 2.

Upon switching back (and ignoring double steps) the motor might move right by one step with probability p_r, left with one step with probability p_l, or stay in the same place with probability p_0. The expectation value for step size is thus

$$\langle x \rangle \quad = \ell p_r - \ell p_l + (0)p_0 \quad = \ell(p_r - p_l) \tag{2}$$

where $\ell = a + b$ is the pitch of the sawtooth.

The probability of a rightward step at time τ is

$$p_r(\tau) = \int_b^\infty \rho(x,\tau)dx \tag{3}$$

and the probability of a leftward step is

$$p_l(\tau) = \int_{-a}^{-\infty} \rho(x,\tau)dx = \int_a^\infty \rho(x,\tau)dx \tag{4}$$

To maximize the velocity, we want to optimize $p_r - p_l$ with respect to the switching time τ:

$$\frac{d}{d\tau}(p_r - p_l) \quad = 0 \tag{5}$$

$$\frac{d}{d\tau}\left[\int_b^a \rho(x,\tau)dx\right] \quad = 0 \tag{6}$$

$$\tag{7}$$

At this point there is a trick: We rewrite the integral to make it dimensionless with the substitution $z = x/\sqrt{4dt}$. Care must be taken with the integral limits, which still have a time-dependence:

$$\int_b^a \rho(x,\tau)dx = \int_{b/\sqrt{4Dt}}^{a/\sqrt{4Dt}} \frac{1}{\sqrt{\pi}} e^{-z^2} dz \tag{8}$$

Note the leading $1/\sqrt{4dt}$ dissappears with the variable change.

Then you use the following identity of calculus (or ask Mathematica to do the integral!):

$$\frac{d}{dt}\int_{f(t)}^{g(t)} z[x]dx = z[g(t)]g'(t) - z[f(t)]f'(t) \tag{9}$$

Applying this gives

$$\frac{d}{d\tau}\int_{b/\sqrt{4Dt}}^{a/\sqrt{4Dt}} \frac{1}{\sqrt{\pi}} e^{-z^2} dz = \frac{1}{4\tau^{3/2}\sqrt{\pi d}}\left(be^{-b^2/4dt} - ae^{-a^2/4dt}\right) = 0 \tag{10}$$

This can be solved for τ:

$$\tau = \frac{a^2 - b^2}{4d \ln a/b} \tag{11}$$

This is the optimal switching time that maximizes forward velocity of the system.