Mid-exam
Solution 1

1-(1)

$$P(r; N, d) = \left(\frac{1}{2\pi\sigma_d^2}\right)^{d/2} e^{-r^2/2\sigma_d^2}$$

$$\sigma_d^2 = N/d$$

proof

Way 1 : Binomial $\rightarrow$ Gaussian by using Stirling formula for one-dimension.

1D

$$\boxed{P_{d=1}(r) = \sqrt{\frac{1}{2\pi\sigma_{d=1}^2}}\; e^{-\frac{r^2}{2\sigma_{d=1}^2}}, \quad \sigma_{d=1}^2 = N}$$

2D : for the details, see the text book or Lecture note.

$$P_{d=2}(\vec{r}) = \underline{P_{d=1}(x, N_x)} \cdot \underline{P_{d=1}(y, N_y)}$$

one can simply multiply the two probabilities since we have no correlation between steps.

$$\langle N_x\rangle = \langle N_y\rangle, \quad \langle N_x\rangle + \langle N_y\rangle = N$$

$$\Rightarrow \langle N_x\rangle = \langle N_y\rangle = N/2.$$

$$\Rightarrow \sigma_{d=2}^2 = N/2$$

$$\Rightarrow P_{d=2}(\vec{r}) = \left(\sqrt{\frac{1}{2\pi\sigma_{d=2}^2}}\right)^2 e^{-\frac{1}{2\sigma_{d=2}^2}(x^2+y^2)}$$

3D :

$$\langle N_x\rangle = \langle N_y\rangle = \langle N_z\rangle = \frac{N}{3}, \quad \sigma_{d=3}^2 = N/3$$

$$P_{d=3}(\vec{r}) = \left(\sqrt{\frac{1}{2\pi\sigma_{d=3}^2}}\right)^3 e^{-\frac{1}{2\sigma_{d=3}^2}(x^2+y^2+z^2)}$$

Way 2

$$\vec{x} = \sum_{i=1}^{N} \vec{s_i}$$

$$P(\vec{x}) = \int \prod_{j=1}^{N} [d\vec{s_j}\, P(\vec{s_j})]\, \delta(\vec{x} - \sum_{i=1}^{N} \vec{s_j})$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\vec{k}\, e^{i\vec{k}\cdot\vec{x}} \{Q(\vec{k})\}^N$$

$$Q(\vec{k}) = \int d\vec{s}\, e^{-i\vec{k}\cdot\vec{s}}\, P(\vec{s})$$

$$= 1 - i\langle\vec{k}\cdot\vec{s}\rangle + \frac{1}{2!}\langle(-i\vec{k}\cdot\vec{s})^2\rangle + \frac{1}{3!}\langle(-i\vec{k}\cdot\vec{s})^3\rangle + \cdots$$

(ex) for 3D , $p(\vec{s}) = \frac{1}{6}\Big[\delta(s_x=1)\,\delta(s_y=0)\,\delta(s_z=0) + \delta(\vec{s}=(-1,0,0))$

$+ \delta(\vec{s}=(0,1,0)) + \delta(\vec{s}=(0,-1,0)) + \delta(\vec{s}=(0,0,1))$

$+ \delta(\vec{s}=(0,0,-1))\Big]$

$$\Rightarrow \langle \vec{k}\cdot\vec{s}\rangle = \int p(\vec{s})\,(k_x x + k_y s_y + k_z s_z)\,d\vec{s} = 0$$

$$\langle(\vec{k}\cdot\vec{s})^2\rangle = \frac{1}{3}(k_x^2 + k_y^2 + k_z^2)$$

$$\Rightarrow Q(\vec{k}) \simeq e^{-\frac{1}{6}(k_x^2+k_y^2+k_z^2)}$$

$$p(\vec{r}) = \frac{1}{(2\pi)^3}\int d\vec{k}\; e^{i(k_x x + k_y y + k_z z)}\, e^{-\frac{N}{6}(k_x^2+k_y^2+k_z^2)}$$

$$\frac{1}{2\pi}\int dk_x\, e^{ik_x x}\, e^{-\frac{N}{6}k_x^2} = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-x^2/2\sigma_x^2}$$

$$\sigma_x^2 = N/3$$

$$\Rightarrow p(\vec{r}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/2} e^{-r^2/2\sigma^2}$$

$$r^2 = x^2+y^2+z^2, \quad \sigma^2 = \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = N/3$$

2D → skip.

(2) $\langle r^2\rangle = \int r^2\, p(\vec{r})\, d^d\vec{r}$

$$A_d = \begin{cases} 2 & d=1 \\ 2\pi & d=2 \\ 4\pi & d=3 \end{cases}$$

$$= \left(\frac{1}{2\pi\sigma_d^2}\right)^{\frac{d}{2}} A_d \int_0^\infty r^{d+1}\, e^{-r^2/2\sigma_d^2}\, dr$$

$$= \left(\frac{1}{2\pi\sigma_d^2}\right)^{\frac{d}{2}} A_d \begin{cases} \frac{1}{4}\sqrt{\pi}\,(2\sigma_d^2)^{\frac{3}{2}} & \text{for } d=1 \\ \frac{1}{2}(2\sigma_d^2)^2 & 2 \\ \frac{3}{8}\sqrt{\pi}\,(2\sigma_d^2)^{\frac{5}{2}} & 3 \end{cases}$$

$$= (2\sigma_d^2)\cdot\left\{\begin{matrix} \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\sqrt{\pi}} \\ 1 \\ \frac{3}{2} \end{matrix}\right\}$$

$$= d\,\sigma_d^2$$

$$= d\cdot\frac{N}{d} = N$$ //

2. -(1) $\begin{cases} N = N_+ + N_- \\ E = \epsilon(N_+ - N_-) \end{cases}$

$$\Rightarrow \quad N_+ = \frac{1}{2}\left(N + \frac{E}{\epsilon}\right), \quad N_- = \frac{1}{2}\left(N - \frac{E}{\epsilon}\right)$$

⊙ $$\Omega(E,N) = \frac{N!}{N_+!\,N_-!} = \frac{N!}{\left[\frac{1}{2}\left(N+\frac{E}{\epsilon}\right)\right]!\left[\frac{1}{2}\left(N-\frac{E}{\epsilon}\right)\right]!}$$

$$S(E,N) = k\ln\Omega(E,N)$$

$$= k\left[N\ln N - N - \frac{1}{2}\left(N+\frac{E}{\epsilon}\right)\ln\left\{\frac{1}{2}\left(N+\frac{E}{\epsilon}\right)\right\} + \frac{1}{2}\left(N+\frac{E}{\epsilon}\right) - \frac{1}{2}\left(N-\frac{E}{\epsilon}\right)\ln\left\{\frac{1}{2}\left(N-\frac{E}{\epsilon}\right)\right\} + \frac{1}{2}\left(N-\frac{E}{\epsilon}\right)\right]$$

$$= k\left[N\ln N - \frac{1}{2}\left(N+\frac{E}{\epsilon}\right)\ln\left\{\frac{1}{2}\left(N+\frac{E}{\epsilon}\right)\right\} - \frac{1}{2}\left(N-\frac{E}{\epsilon}\right)\ln\left\{\frac{1}{2}\left(N-\frac{E}{\epsilon}\right)\right\}\right].$$

⊙ $$\boxed{\frac{1}{T} = \beta k = \left.\frac{\partial S}{\partial E}\right|_N} = \frac{k}{2\epsilon}\ln\left\{\frac{N-\frac{E}{\epsilon}}{N+\frac{E}{\epsilon}}\right\}$$

(2) system A: $$\frac{2\epsilon}{kT} = 1 \Rightarrow \frac{N-\frac{E}{\epsilon}}{N+\frac{E}{\epsilon}} = e$$

$$\Rightarrow N - \frac{E}{\epsilon} = e\left(N+\frac{E}{\epsilon}\right) \Rightarrow E = \epsilon\,\frac{1-e}{1+e}\,N$$

system B: $\begin{cases} N' = aN \\ \frac{2\epsilon}{kT} = b \end{cases}$ $$\Rightarrow \frac{aN - \frac{E'}{\epsilon}}{aN + \frac{E'}{\epsilon}} = e^b$$

$$\Rightarrow E' = \epsilon\cdot aN\cdot\frac{1-e^b}{1+e^b} = \epsilon\cdot N'\cdot\frac{1-e^b}{1+e^b}$$

$$\begin{cases} E_{tot} = E + E' \\ S^{tot}(E,N,N') = k\left[N\ln N - \frac{1}{2}\left(N+\frac{E}{\epsilon}\right)\ln\left\{\frac{1}{2}\left(N+\frac{E}{\epsilon}\right)\right\} - \frac{1}{2}\left(N-\frac{E}{\epsilon}\right)\ln\left\{\frac{1}{2}\left(N-\frac{E}{\epsilon}\right)\right\}\right. \\ \left. + N'\ln N' - \frac{1}{2}\left(N'+\frac{E_{tot}-E}{\epsilon}\right)\ln\left\{\frac{1}{2}\left(N'+\frac{E_{tot}-E}{\epsilon}\right)\right\} - \frac{1}{2}\left(N'-\frac{E_{tot}-E}{\epsilon}\right)\ln\left\{\frac{1}{2}\left(N'-\frac{E_{tot}-E}{\epsilon}\right)\right\}\right] \end{cases}$$

$$\left.\frac{\partial S^{tot}}{\partial E}\right|_{N,N'} = 0$$

$$\Rightarrow \frac{\partial S^{tot}}{\partial E}\bigg|_{N,N'} = \frac{k}{2\epsilon}\left[\ln \frac{(N-\frac{E}{\epsilon})}{(N+\frac{E}{\epsilon})} - \ln \left\{ \frac{N'-\frac{E_{tot}-E}{\epsilon}}{N'+\frac{E_{tot}-E}{\epsilon}} \right\} \right] = 0$$

$$\Rightarrow \left(N-\frac{E}{\epsilon}\right)\left(N'+\frac{E_{tot}-E}{\epsilon}\right) = \left(N'-\frac{E_{tot}-E}{\epsilon}\right)\left(N+\frac{E}{\epsilon}\right)$$

$$\Rightarrow -N'\frac{E}{\epsilon} + \frac{E_{tot}-E}{\epsilon}N - \cancel{\frac{E(E_{tot}-E)}{\epsilon^2}} = N'\frac{E}{\epsilon} - N\frac{E_{tot}-E}{\epsilon} - \cancel{\frac{(E_{tot}-E)E}{\epsilon^2}}$$

$$\Rightarrow 2N'\frac{E}{\epsilon} = 2N\frac{E_{tot}-E}{\epsilon}$$

put $N' = aN$ $\quad \Rightarrow \frac{aE}{\epsilon} = \frac{E_{tot}-E}{\epsilon} \Rightarrow \underline{E_{eq} = \frac{E_{tot}}{a+1}}$

$$\Rightarrow E_{eq} = \frac{1}{a+1}\left(\frac{\epsilon N(1-e)}{1+e} + \frac{\epsilon Na(1-e^b)}{1+e^b}\right) = \frac{\epsilon N}{a+1}\left(\frac{1-e}{1+e} + \frac{(1-e^b)a}{1+e^b}\right)$$

$$\frac{1}{T_{eq}} = \frac{k}{2\epsilon}\ln\left\{\frac{N-\frac{E_{eq}}{\epsilon}}{N+\frac{E_{eq}}{\epsilon}}\right\}$$

$$= \frac{k}{2\epsilon}\ln\left\{\frac{1-\frac{1}{a+1}\left(\frac{1-e}{1+e}+\frac{(1-e^b)a}{1+e^b}\right)}{1+\frac{1}{a+1}\left(\frac{1-e}{1+e}+\frac{(1-e^b)a}{1+e^b}\right)}\right\}$$

$$= \frac{ae^b(1+e)+e(1+e^b)}{a(1+e)+1+e^b}$$

(i) $(a=1,\ b=-1)$

$$\Rightarrow \frac{1}{T_{eq}} = \frac{k}{2\epsilon}\ln\left\{\frac{1-\frac{1}{2}\left(\frac{1-e}{1+e}+\frac{e-1}{e+1}\right)}{1+\frac{1}{2}\left(\frac{1-e}{1+e}+\frac{e-1}{e+1}\right)}\right\} = 0$$

$$T_{eq} = \infty$$

(ii) $(b=1)$

$$\Rightarrow \frac{1}{T_{eq}} = \frac{k}{2\epsilon}\ln\left\{\frac{1-\frac{1}{2}\cdot 2\cdot\frac{1-e}{1+e}}{1+\frac{1}{2}\cdot 2\cdot \frac{1-e}{1+e}}\right\}$$

$$= \frac{k}{2\epsilon}\ln\left\{\frac{1+e-1+e}{1+e+1-e}\right\} = \frac{k}{2\epsilon} = \frac{1}{T}$$

Mid - Exam

Solution 3

3.

$$Z = \sum_r e^{-\beta E_r}$$

$$= g_1 e^{-\beta\epsilon_1} + g_2 e^{-\beta\epsilon_2} \quad , \; \epsilon_2 > \epsilon_1$$

$$\ln Z = \ln\left(g_1 e^{-\beta\epsilon_1}\right)\left(1 + \frac{g_2}{g_1} e^{-\beta(\epsilon_2-\epsilon_1)}\right)$$

$$F = E - TS = -kT\ln Z$$

$$\bar{E} = -\frac{\partial \ln Z}{\partial\beta} = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = \frac{g_1\epsilon_1 e^{-\beta\epsilon_1} + g_2\epsilon_2 e^{-\beta\epsilon_2}}{g_1 e^{-\beta\epsilon_1} + g_2 e^{-\beta\epsilon_2}}$$

$$TS = E + kT\ln Z$$

$$\Rightarrow S = k(\ln Z + \beta\bar{E})$$

$$= k\left(\ln\left(1 + \frac{g_2}{g_1} e^{-\beta(\epsilon_2-\epsilon_1)}\right) + \ln g_1 - \beta\epsilon_1 + \beta\frac{g_1\epsilon_1 e^{-\beta\epsilon_1} + g_2\epsilon_2 e^{-\beta\epsilon_2}}{g_1 e^{-\beta\epsilon_1} + g_2 e^{-\beta\epsilon_2}}\right)$$

$$= k\left(\ln\left(1 + \frac{g_2}{g_1} e^{-x}\right) + \ln g_1 + \frac{x}{1 + \frac{g_1}{g_2} e^{x}}\right)$$

$$x = \beta(\epsilon_2 - \epsilon_1)$$

$$T \to 0, \Rightarrow x \to \infty \Rightarrow S \to k\ln g_1$$

The system goes to ground states as $T \to 0$.

The generacy of ground states is g_1

$$\Rightarrow \Omega(E = \epsilon_1) = g_1.$$

$$S = k\ln\Omega = k\ln g_1$$

Zero-temperature entropy is finite due to ~~a~~ degeneracy.

Mid-Exam 英

Solution 4.

4. (1)

$$Z_1 = \sum e^{-\beta\epsilon_i} = \sum e^{-\beta(\epsilon_{i,tr} + \epsilon_{i,rot})} = Z_{1,trans}\, Z_{1,rot}$$

$$Z_{1,trans} = \frac{V}{\lambda^3} \quad . \quad \lambda = \sqrt{\frac{h^2}{2\pi m kT}}$$

$$Z_{1,rot} = \sum_{J=0}^{\infty} (2J+1)\, e^{-\beta\hbar^2 J(J+1)/2I}$$

$$x = \beta\hbar^2 J(J+1)/2I \quad \sim \text{small.}$$

$$dx = (2J+1)\frac{\beta\hbar^2}{2I}\, dJ$$

$$\Rightarrow \quad Z_{1,rot} \to \frac{2I}{\beta\hbar^2}\int_0^\infty e^{-x}\,dx = \frac{2I}{\beta\hbar^2}$$

$$Z = \frac{1}{N!} Z_1^N = \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N (Z_{1,rot})^N = \frac{1}{N!}\left(VI\,\frac{1}{h^5}\cdot 8\pi^2\left(\sqrt{2\pi m}\right)^3 \beta^{-5/2}\right)^N$$

(2)

$$\langle \epsilon_{rot} \rangle = -\frac{\partial \ln Z_{rot}}{\partial\beta} = \frac{\partial \ln\beta}{\partial\beta} = \frac{1}{\beta} = kT$$

$$\langle \epsilon_{kinetic} \rangle = \frac{3}{2}kT$$

$$\langle \epsilon \rangle = \frac{5}{2}kT$$

(3)

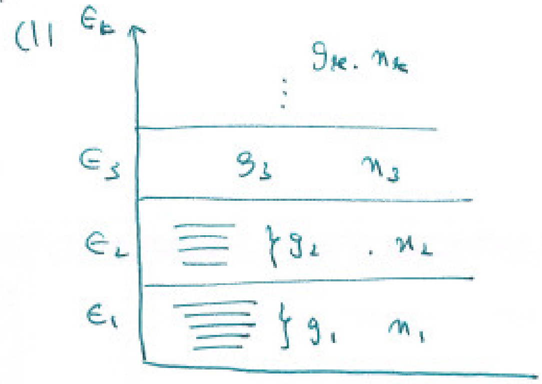
$$F = -kT\ln Z = -NkT\ln\left(\frac{V}{\lambda^3}\cdot\frac{2I}{\beta\hbar^2}\right) + kT(N\ln N - N)$$

$$\mu = \left.\frac{\partial F}{\partial N}\right|_{T,V} = -kT\ln\left(\frac{V}{\lambda^3}\cdot\frac{2I}{\beta\hbar^2}\right) + kT(\ln N + 1 - 1)$$

$$= kT\left(\ln N - \ln\left(\frac{V}{\lambda^3}\,\frac{2I}{\beta\hbar^2}\right)\right)$$

5.

(1)



$N = \sum n_k$

$E = \sum n_k \epsilon_k$.

Any degeneracy is counted in g_k.

$\omega(n_k, g_k)$: total # of ways to distribute n_k particles over the g_k states in the cell ϵ_k.

indistinguishable fermions :

$$\omega_{FD} = \binom{g_k}{n_k} = \frac{g_k!}{n_k!\,(g_k - n_k)!}$$

$$W_{FD}(\{n_k\}) = \prod_k \frac{g_k!}{n_k!\,(g_k - n_k)!}$$

indistinguishable bosons :

$$\omega_{BE} = \binom{n_k + g_k - 1}{n_k} = \frac{(n_k + g_k - 1)!}{n_k!\,(g_k - 1)!}$$

$$W_{BE}(\{n_k\}) = \prod_k \frac{(n_k + g_k - 1)!}{n_k!\,(g_k - 1)!}$$

Boltzmann particles (distinguishable)

$$\omega_{MB}(n_k, g_k) = g_k^{\,n_k}$$

$$W_{MB}(\{n_k\}) = \frac{N!}{n_1!\,n_2!\cdots} \prod_k g_k^{\,n_k}$$

(2) $W(\{n_k\}) \to$ maximum

$$\delta \ln W(\{n_k\}) - \alpha \sum_k \delta n_k - \beta \sum_k \epsilon_k \delta n_k = 0$$

$$\ln W(n_k) = \sum_k \left[n_k \ln\left(\frac{g_k}{n_k} - a\right) - \frac{g_k}{a} \ln\left(1 - a\frac{n_k}{g_k}\right) \right] + \boxed{\ }$$

independent of $\{n_k\}$

$a = 1$ (FD) . $a = -1$ (BE) . $a = 0$ (MB)

$$\left(N\ln N - \cancel{N} - \sum n_k \ln n_k + \cancel{\sum n_k} + \sum n_k \ln g_k \right) +$$

$$\delta \ln W(\{n_k\}) = \left(\ln\left(\frac{g_k}{n_k} - a\right) + \frac{n_k \cdot \left(-\frac{g_k}{n_k^2}\right)}{\frac{g_k}{n_k} - a} - \frac{g_k}{a} \frac{-a \cdot \frac{1}{g_k}}{1 - a\frac{n_k}{g_k}} \right) \delta n_k$$

$$= \delta n_k \left(\ln\left(\frac{g_k}{n_k} - a\right) + \frac{-g_k}{g_k - a n_k} + \frac{g_k}{g_k - a n_k} \right)$$

$$\delta \ln W(\{n_k\}) - \alpha \sum \delta n_k - \beta \sum \epsilon_k \delta n_k = 0$$

$$\Rightarrow \delta n_k \left(\ln\left(\frac{g_k}{n_k} - a\right) - \alpha - \beta \epsilon_k \right) = 0$$

$$\Rightarrow \frac{g_k}{n_k} - a = e^{\alpha + \beta \epsilon_k}$$

$$\Rightarrow \bar{n}_k = \frac{g_k}{a + e^{+\alpha + \beta \epsilon_k}}$$

Mid-Exam 6.

Solution 6.

two-dimension

(1) $\sum_k \to \int \rho(\epsilon)\, d\epsilon$.

$$\rho(\epsilon) = \left(\frac{L}{2\pi}\right)^2 \cdot 2\pi k \cdot \frac{dk}{d\epsilon} \qquad d\epsilon = \frac{\hbar^2 k}{m} dk.$$

$$= \left(\frac{L}{2\pi}\right)^2 \cdot 2\pi \cdot \frac{m}{\hbar^2} \qquad \Rightarrow \frac{dk}{d\epsilon} = \frac{m}{\hbar^2}\frac{1}{k}$$

$$= \frac{L^2 m}{2\pi \hbar^2}$$

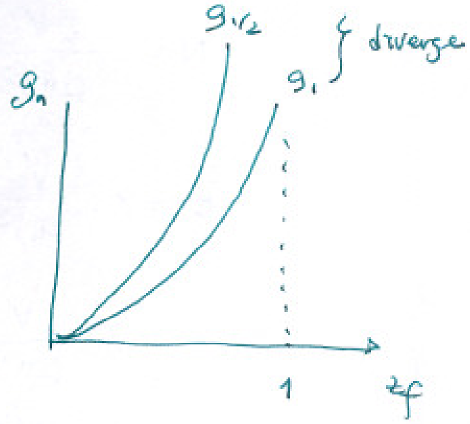
notice: $\rho(\epsilon)$ vanishes at the band edge, $\epsilon \to 0^+$, for $d > 2$ whereas it does not for $d \le 2$.

(2) BE.

$$N_\epsilon = \left(\frac{L}{2\pi}\right)^2 \cdot 2\pi \cdot \frac{m}{\hbar^2}\frac{1}{\beta}\int_0^\infty dx \frac{1}{z_f^{-1} e^x - 1}$$

$$g_n(z_f) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}\,dx}{z_f^{-1} e^x - 1}$$

$\Gamma(1) = 1$



$$\Rightarrow \quad N_\epsilon = \frac{L^2 m}{2\pi\hbar^2}\frac{1}{\beta} g_1(z_f)$$

T_c: $N_\epsilon^{max} = N. = \frac{L^2 m}{2\pi\hbar^2} k_B T_c\, g_1(z_f = 1)$

$$\Rightarrow \quad T_c = \frac{2\pi\hbar^2 N}{L^2 m k_B} \cdot g_1^{-1}(z_f = 1)$$

$\frac{N}{L^2}$ = finite. (density)

but $g_1(z_f = 1) \to \infty$

$\Rightarrow T_c = 0$

$\to$ no Bose-Einstein condensation in 2D.

※ $d = 1$ (1D) $\Rightarrow$ $N_\epsilon = N = g_{1/2}(z)\frac{1}{\sqrt{\beta_c}}\frac{L}{h}\sqrt{2m\pi}$ $\Rightarrow$ $T_c \to \infty$ as well.

(3) $\bar{E} = -\frac{\partial}{\partial \beta} \ln \mathcal{Z} \Big|_{z_f, V} = kT^2 \frac{\partial}{\partial T} \ln \mathcal{Z} \Big|_{z_f, V}$

$= \frac{3}{2} kT \frac{gV}{\lambda^3} f_{5/2}(z_f)$ g: degeneracy factor (eg. spin)

$f_n(z_f) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\,dx}{z_f^{-1} e^x + 1}$, $0 \le z_f \le \infty$

$\bar{p} V = kT \ln \mathcal{Z} = kT \frac{gV}{\lambda^3} f_{5/2}(z)$

$\Rightarrow \bar{p} V = \frac{2}{3} \bar{E}$

Other ways:

(i.e. $\bar{p}_r = -\frac{\partial \epsilon_r}{\partial V}$, $\bar{p} = \frac{\partial \ln Z}{\beta \partial V}$)

(4)

at $T = 0$,

$N = \frac{gV}{(2\pi)^3} \frac{4\pi}{3} k_f^3 \Rightarrow k_f^3 = \frac{6\pi^2 N}{gV} = \frac{6\pi^2}{g} n$

$n = \frac{N}{V}$.

$\bar{E} = g \cdot \frac{V}{(2\pi)^3} 4\pi \int_0^{k_f} \cdot k^2 \cdot \frac{\hbar^2 k^2}{2m} dk$

$= g \cdot \frac{V}{(2\pi)^3} \cdot 4\pi \cdot \frac{\hbar^2}{2m} \cdot \frac{1}{5} k_f^5$

$= g \cdot \frac{V \hbar^2}{20\pi^2 m} \left(\frac{6\pi^2}{g} n \right)^{5/3}$

in equilibrium $\bar{p}_1 = \bar{p}_2$

$\Rightarrow \frac{\bar{E}_1}{V_1} = \frac{\bar{E}_2}{V_2}$

$\Rightarrow n_1^{5/2} \cdot g_1^{-3/2} = n_2^{5/2} g_2^{-3/2}$

$\Rightarrow \left(\frac{n_2}{n_1} \right)^{5/3} = \left(\frac{g_2}{g_1} \right)^{2/3} \Rightarrow \frac{n_2}{n_1} = \left(\frac{g_2}{g_1} \right)^{\frac{2}{5}}$.

$g_i = 2s_i + 1$