Final exam solution 1.

1.

$$\bar{E} = -N\eta + \int_0^\infty \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\sigma(\omega)\,d\omega$$ (see Reif eq.(10.1.19))

$$\sigma(\omega)\,d\omega = \underbrace{g(d)}_{\text{degeneracy}} \underbrace{\frac{V}{(2\pi)^d}}_{V\equiv L^d} f_d(k)\,dk$$

$$f_d(k) = \begin{cases} 2 & (d=1) \\ 2\pi k & (d=2) \\ 4\pi k^2 & (d=3) \end{cases}$$

Use the dispersion relation $\omega = Ak^s \Rightarrow k = A^{-1/s}\omega^{1/s}$

$dk = A^{-1/s}\frac{1}{s}\omega^{(1/s-1)}d\omega$

$$f_d(k)\,dk = \begin{cases} \frac{2}{s}A^{-1/s}\omega^{(1/s-1)}\,d\omega & (d=1) \\ \frac{2\pi}{s}A^{-2/s}\omega^{(2/s-1)}\,d\omega & (d=2) \\ \frac{4\pi}{s}A^{-\frac{3}{s}}\omega^{(3/s-1)}\,d\omega & (d=3) \end{cases}$$

$$\Rightarrow \bar{E} = -N\eta + \int_0^\infty \frac{\hbar\omega^{d/s}}{e^{\beta\hbar\omega}-1}\underbrace{f'(d,s)}_{\text{independent of } \omega \text{ and } \beta.}\,d\omega$$

$$= -N\eta + \frac{\hbar f'(d,s)}{\beta^{d/s+1}}\underbrace{\int_0^\infty \frac{x^{d/s}dx}{e^x-1}}_{\text{constant. (independent of } \beta)}$$

$$C_V = \frac{\partial E}{\partial T} \propto \frac{\partial}{\partial T}\left(T^{d/s+1}\right) \propto T^{d/s}$$

Note: Here, we already use the fact that

ω_D (Debye frequency) is much larger than $\frac{kT}{\hbar}$.

($T\to 0$ limit)

check point: • calculation of energy using BE statistics

• d-dimensional density of states

• Debye model.

Final exam. solution 2.

2.

· partition fn. +2
· densities of state. +2
· full expression +2

(1) $$\ln \mathcal{Z} = \sum_{\vec{p}, J_z} \ln\left(1 + z_f e^{-\beta\left(\frac{p^2}{2m} - J_z g \mu_B B\right)}\right)$$

$$= \sum_{J_z} \frac{V}{\lambda^3} f_{5/2}\left(z_f e^{\beta J_z g \mu_B B}\right)$$

$$\lambda = \sqrt{\frac{h^2}{2\pi m k T}}$$

$$N = \sum_{J_z} \frac{V}{\lambda^3} f_{3/2}\left(z_f e^{\beta J_z g \mu_B B}\right)$$

$$z_f = e^{\beta \mu}$$

(2) $$M = kT \left.\frac{\partial \ln \mathcal{Z}}{\partial B}\right|_{z, V, T}$$

$$= kT \sum_{J_z} \frac{V}{\lambda^3} \left.\frac{d f_{5/2}(x)}{dx}\right|_{x = z_f e^{\beta J_z g \mu_B B}} \underbrace{\left.\frac{\partial\left(z_f e^{\beta J_z g \mu_B B}\right)}{\partial B}\right|_{z, V, T}}_{= \left(z_f e^{\beta J_z g \mu_B B}\right) \times (\beta J_z g \mu_B)}$$

$$= \frac{V}{\lambda^3} \sum_{J_z} f_{3/2}\left(z_f e^{\beta J_z g \mu_B B}\right) J_z g \mu_B$$

$$= \frac{V}{\lambda^3} g \mu_B \sum_{J_z} J_z f_{3/2}\left(z_f e^{\beta J_z g \mu_B B}\right)$$

$$x \frac{d f_{3/2}(x)}{dx} = f_{1/2}(x)$$

$$\chi_0 = \lim_{B \to 0} \frac{\partial M}{\partial B} \cdot \frac{1}{V}$$

In the limit of $B \to 0$, $f_{3/2}\left(z_f e^{\beta J_z g \mu_B B}\right) \simeq f_{3/2}\left(z_f (1 + \beta J_z g \mu_B B)\right)$

$$\simeq f_{3/2}(z_f) + z_f \beta J_z g \mu_B B \frac{\partial f_{3/2}}{\partial z_f}$$

$$= f_{3/2}(z_f) + \beta J_z g \mu_B B f_{1/2}(z_f)$$

$$\Rightarrow M = \frac{V}{\lambda^3} g \mu_B \left(\underbrace{\cancel{\left(\sum_{J_z} J_z\right) f_{3/2}(z_f)}} + \beta g \mu_B B \left(\sum_{J_z} J_z^2\right) f_{1/2}(z_f) \right)$$

$$\left(\because \sum_{J_z} J_z = 0\right)$$

$$= \frac{V}{\lambda^3} \beta g^2 \mu_B^2 f_{1/2}(z_f) \cdot \left(\sum_{J_z} J_z^2\right) \cdot B$$

$$\chi_0 = \lim_{B \to 0} \frac{\partial M}{\partial B} \cdot \frac{1}{V} = \lim_{B \to 0} \frac{M}{BV} = \frac{1}{\lambda^3} \beta g^2 \mu_B^2 f_{1/2}(z_f) \underbrace{\left(\sum_{J_z} J_z^2\right)}_{= \frac{J(J+1)(2J+1)}{3}}$$

(3) $T \to \infty \Rightarrow z_f \ll 1$ $\quad$ $\beta g \mu_B B \to 0$ $(\because B \to 0)$

$$N = \sum_{J_z} \frac{V}{\lambda^3} f_{3/2}\left(z_f e^{\beta J_z g \mu_B B}\right) \underset{\lim B \to 0}{=} \sum_{J_z} \frac{V}{\lambda^3} f_{3/2}(z_f) = \underbrace{\left(\sum_{J_z} 1\right)}_{= 2J+1} \cdot \frac{V}{\lambda^3} f_{3/2}(z_f)$$

Check point: · FD distribution · valid approximation $(B \to 0)$

· magnetization and susceptibility · $T \to \infty$ limit. · calculation of z_f.

2 (Continued)

$$\Rightarrow \frac{1}{2J+1}\frac{N}{V}\lambda^3 = f_{3/2}(z_f)$$

$$\simeq z_f - \frac{1}{2\sqrt{2}} z_f^2$$

$$\Rightarrow z_f \simeq \frac{1}{2J+1}\frac{N}{V}\lambda^3 \left(1 + \frac{1}{2\sqrt{2}}\frac{1}{2J+1}\frac{N}{V}\lambda^3 \right)$$

$$\Rightarrow f_{1/2}(z_f) \simeq z_f - \frac{1}{\sqrt{2}} z_f^2$$

$$\cong \frac{1}{2J+1}\frac{N}{V}\lambda^3 \left(1 + \frac{1}{2\sqrt{2}}\frac{1}{2J+1}\frac{N}{V}\lambda^3 - \frac{1}{\sqrt{2}}\frac{1}{2J+1}\frac{N}{V}\lambda^3 \right)$$

$$= \frac{1}{2J+1}\frac{N}{V}\lambda^3 \left(1 - \frac{1}{2\sqrt{2}}\frac{1}{2J+1}\frac{N}{V}\lambda^3 \right)$$

$$\Rightarrow \lim_{B\to 0}\chi = \chi_0 = \frac{1}{\lambda^3}(g\mu_B)^2 \beta \frac{J(J+1)(2J+1)}{3} f_{1/2}(z_f)$$

$$= \frac{N(g\mu_B)^2}{VkT}\frac{J(J+1)}{3}\left(1 - \frac{1}{2\sqrt{2}}\frac{1}{2J+1}\frac{N}{V}\lambda^3 \right)$$

→ Compare with Eq. (7.8.22) of the text (Reif)

※. $\sum J_z^2 = \sum_{J_z=-J}^{J} J_z^2 = 2\sum_{S_z=1}^{S}\left(\frac{2S_z-1}{2}\right)^2 = \frac{1}{2}\sum_{S_z=1}^{S}(2S_z-1)^2$

$J = \frac{2S-1}{2}$

$$= \frac{1}{2}\left[\left[1^2+2^2+\cdots+(2S)^2\right] - \left[2^2+4^2+\cdots(2S)^2\right] \right]$$

$$= \frac{1}{2}\cdot\left(\frac{2S(2S+1)(4S+1)}{6} - 4\frac{S(S+1)(2S+1)}{6} \right)$$

$$= \frac{1}{2}\cdot\frac{1}{6}\cdot 2S(2S+1)(4S+1-2S-2) = \frac{1}{12}2S(2S+1)(2S-1)$$

$$= \frac{1}{12}(2J+1)(2J+2)(2J) = \frac{J(J+1)(2J+1)}{3}$$

Final exam solution 3.

3. $\dfrac{\partial f}{\partial t} = 0$. $F_{ext} = 0$ (no external force)

$\Rightarrow$ Boltzmann eq $\boxed{v_x \dfrac{\partial f}{\partial x} = -\dfrac{f - f_0}{\tau_0}}$ 5 , $\boxed{f = f_0 + f_1}$ 3 , $f_1 \ll f_0$

($\because$ linear response regime of small $\dfrac{dn}{dx}$.)

$$\Rightarrow f \simeq f_0 - v_x \tau_0 \frac{df_0}{dx}$$

(1) MB statistics

$$f_0 = n(x) \left(\frac{m\beta}{2\pi}\right)^{3/2} e^{-\frac{1}{2}\beta m v^2}$$

τ_0 ~ independent of v

particle flux density $\boxed{J_x = n \langle v_x \rangle = \int f \, v_x \, dv^3}$ 5

$$= \cancel{\int f_0 v_x \, dv^3} - \tau_0 \frac{1}{n(x)} \frac{dn}{dx} \int v_x^2 f_0 \, d^3v$$

$$= -\tau_0 \frac{dn}{dx} \cdot \langle v_x^2 \rangle_0 = -\tau_0 \cdot \frac{1}{3} \langle v^2 \rangle_0 \frac{dn}{dx}.$$

$$\Rightarrow D = \frac{1}{3} \tau_0 \langle v^2 \rangle_0 , \quad \langle v^2 \rangle_0 = \frac{1}{n(x)} \int v_x^2 f_0 \, d^3v.$$

(2) FD statistics

$$f_0 = \frac{1}{e^{\beta(\epsilon - \mu(x))} + 1}$$

5 $\boxed{J_x = \dfrac{1}{V} \int f \, v_x \, \rho(\epsilon) \, d\epsilon}$

3 densities of states $\rho(\epsilon) = \underbrace{2}_{\text{spin degeneracy}} \times \dfrac{V}{4\pi^2} \dfrac{(2m)^{3/2}}{\hbar^3} \sqrt{\epsilon}$

$\uparrow$ $\dfrac{V}{(2\pi)^3} \cdot 4\pi k^2 \dfrac{dk}{d\epsilon}$

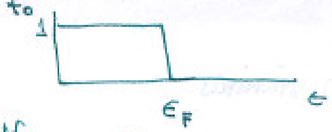
$$= \frac{1}{V} \int \cancel{f_0 v_x \rho(\epsilon) \, d\epsilon} - \frac{1}{V} \int v_x^2 \tau \frac{df_0}{dx} \rho(\epsilon) \, d\epsilon$$

$\left(\propto \int v_x f_0 \, d^3k = 0\right)$

$$= -\frac{1}{3V} \int v^2 \tau \frac{df_0}{dx} \rho(\epsilon) \, d\epsilon.$$

$$\boxed{\frac{df_0}{dx} = \frac{df_0}{d\mu} \cdot \frac{d\mu}{dx} = -\frac{df_0}{d\epsilon} \frac{d\mu}{dx} \simeq \delta(\epsilon - \mu) \frac{d\mu}{dx}}$$ 3

$\because$ In the limit of $T \to 0$,



, $\mu \simeq \epsilon_F$

$$\Rightarrow \frac{df_0}{dx} = -\delta(\epsilon - \epsilon_F)$$

$\epsilon_F = \dfrac{\hbar^2}{2m} (3\pi^2 n)^{\frac{2}{3}}$. $\boxed{\mu(x) = \dfrac{\hbar^2}{2m} (3\pi^2 n(x))^{\frac{2}{3}}}$ 3

$$\frac{d\mu}{dx} = \frac{\hbar^2}{2m}\cdot(3\pi^2 n)^{\frac{2}{3}} \cdot \frac{2}{3}\frac{1}{n}\frac{dn}{dx}$$

$$= \frac{2}{3}\frac{\epsilon_F}{n}\frac{dn}{dx}$$

$$\Rightarrow J_x = -\frac{1}{3V}\int v^2\cdot\tau\cdot\delta(\epsilon-\epsilon_F)\frac{d\mu}{dx}\rho(\epsilon)\,d\epsilon.$$

$$= -\frac{1}{3V}\cdot v_F^2\tau_F\left(\frac{2}{3}\frac{\epsilon_F}{n}\right)\cdot\frac{dn}{dx}\cdot\rho(\epsilon_F).$$

$$\rho(\epsilon_F) = 2\times\frac{V}{4\pi^2}\frac{(2m)^{3/2}}{\hbar^3}\sqrt{\epsilon_F}$$

$$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{\frac{2}{3}} \Rightarrow \frac{(2m)^{3/2}}{\hbar^3} = \left(\sqrt{\frac{(3\pi^2 n)^{\frac{2}{3}}}{\epsilon_F}}\right)^3 = \frac{3\pi^2 n}{\epsilon_F^{3/2}}$$

$$\Rightarrow \rho(\epsilon_F) = \frac{V}{2\pi^2}\cdot\frac{3\pi^2 n}{\epsilon_F} = \frac{3}{2}\frac{n}{\epsilon_F}\cdot V.$$

$$\Rightarrow J_x = -\frac{1}{3}v_F^2\tau\left(\frac{2}{3}\frac{\epsilon_F}{n}\right)\cdot\left(\frac{3}{2}\frac{n}{\epsilon_F}\right)\frac{dn}{dx}$$

$$= -\frac{1}{3}v_F^2\tau_F\frac{dn}{dx}$$

5.

$$\Rightarrow D = \frac{1}{3}v_F^2\tau_F$$

v_F : Fermi velocity $\quad \epsilon_F = \frac{1}{2}m v_F^2$

τ_F : relaxation time at $\epsilon = \epsilon_F$.

Check point: Write down electron flux density J_x using f.

approximation $f_1 \ll f_0$

FD statistics

2.

4.

(1) $\cancel{\sigma'(v, v_1}$ $\sigma'(v, v_1 \to v', v_1') = \sigma'(v', v_1' \to v, v_1)$.

(2) $$D_c f = \int_{v_1} \int_{v_1'} \int_{v'} \cancel{f' f_1' (1-f)} \left[f' f_1' (1-f)(1-f_1) - f f_1 (1-f')(1-f_1') \right] V \sigma'(v, v_1 \to v', v_1')\, d^3v_1\, d^3v'\, d^3v_1'$$

5. Final exam solution 5.

(1)

$\omega_{\pm}$: probabilities ~~of~~ that the electron is in state $|\pm\rangle$ in equilibrium.

$\omega_+ + \omega_- = 1.\quad \omega_+ \propto e^{+\beta\mu_B B} \quad \omega_- \propto e^{-\beta\mu_B B}.$

$$\Rightarrow \omega_+ = \frac{e^{+\beta\mu_B B}}{e^{\beta\mu_B B}+e^{-\beta\mu_B B}} \qquad \omega_- = \frac{e^{-\beta\mu_B B}}{e^{\beta\mu_B B}+e^{-\beta\mu_B B}}$$ +2

$$\langle M\rangle = -\mu_B(\omega_+ - \omega_-) = \mu_B \tanh(\beta\mu_B B)$$ +2 +1

$$\langle M^2\rangle = \mu_B^2(\omega_+ + \omega_-) = \mu_B^2.$$

$$\langle(\Delta M)^2\rangle = \langle M^2\rangle - \langle M\rangle^2 = \mu_B^2(1-\tanh^2(\beta\mu_B B))$$ +1

$$= \mu_B^2 / \cosh^2(\beta\mu_B B)$$

(2)

$$\begin{cases} \dfrac{\partial P_+(t)}{\partial t} = \sum_{s=\pm}\left[W_{s\to +}P_s(t) - W_{+\to s}P_+(t)\right] \\ \dfrac{\partial P_-(t)}{\partial t} = \sum_{s=\pm}\left[W_{s\to -}P_s(t) - W_{-\to s}P_-(t)\right] \end{cases}$$ 6

$$\frac{W_{+\to -}}{W_{-\to +}} = \frac{\omega_-}{\omega_+} = e^{-2\beta\mu_B B}, \qquad W_{+\to -} + W_{-\to +} \equiv W$$

$$\Rightarrow W_{-+} = \frac{W}{1+e^{-2\beta\mu_B B}} \qquad W_{+-} = \frac{W}{1+e^{+2\beta\mu_B B}}$$ 4

(3)

$P_{fi}(t') \equiv P(f.t'|i.0)$, $t' = t - t_0$

($P(f.t|i,t_0)$ is a function of $t' = t - t_0$ since we consider stationary processes.)

$$\frac{\partial P_{fi}(t')}{\partial t'} = \sum_s\left[W_{s\to f}P_{si}(t) - W_{f\to s}P_{fi}(t)\right]$$

initial condition of this equation is

$$P_{fi}(t) = \delta_{if} \quad \text{at } t \to 0$$

5 (continued) (5-(3))

Let's introduce the matrix $\hat{P}(t)$ with matrix elements $P_{if}(t)$

$i, f \in \{+, -\}$

$\Rightarrow$ The Master eq. becomes

$$\frac{\partial \hat{P}(t)}{\partial t} = \hat{W}\hat{P}(t).$$

$$\hat{W} = \left(\cancel{-W_{f\to s}} \quad \cancel{W_{s\to f}} \right.$$

$$\frac{\partial \hat{P}_{++}(t)}{\partial t} = -W_{+\to -}P_{++} + W_{-\to +}P_{-+}$$

$$\frac{\partial P_{+-}(t)}{\partial t} = -W_{+\to -}P_{+-} + W_{-\to +}P_{--}$$

$$\frac{\partial P_{-+}(t)}{\partial t} = -W_{-\to +}P_{-+} + W_{+\to -}P_{++}$$

$$\frac{\partial P_{--}(t)}{\partial t} = -W_{-\to +}P_{--} + W_{+\to -}P_{+-}$$

$$\begin{pmatrix} \frac{\partial P_{++}}{\partial t} & \frac{\partial P_{+-}}{\partial t} \\ \frac{\partial P_{-+}}{\partial t} & \frac{\partial P_{--}}{\partial t} \end{pmatrix} = \begin{pmatrix} -W_{+\to -} & W_{-\to +} \\ W_{+\to -} & -W_{-\to +} \end{pmatrix} \begin{pmatrix} P_{++} & P_{+-} \\ P_{-+} & P_{--} \end{pmatrix}$$

Now search the solution in the form of

$$\hat{P}(t) = \sum_k \hat{C}_k e^{-\lambda_k t}$$

$\hat{C}_k$ ~ constant matrix

$$\frac{d\hat{P}(t)}{dt} = \sum_k \hat{C}_k (-\lambda_k) e^{-\lambda_k t}$$

$$\Rightarrow \sum_k \hat{C}_k (-\lambda_k) e^{-\lambda_k t} = \sum_k \hat{W} \hat{C}_k e^{-\lambda_k t}$$

$\Rightarrow$ λ_k ~ eigenvalue of $-\hat{W}$.

$$\det(-\hat{W} - \lambda \hat{I}) = 0. \quad (\lambda - W_{+-})(\lambda - W_{-+}) - W_{+-}W_{-+} = 0.$$

$$\Rightarrow \lambda = 0 \quad \text{or} \quad W_{+-} + W_{-+} = W.$$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = W.$$

$$\Rightarrow \hat{P}(t) = \hat{C}_1 + \hat{C}_2 e^{-Wt}$$

initial condition $\Rightarrow$ $t \to 0$ $\quad \hat{P}(t) = \hat{I}$. $\quad \Rightarrow \hat{C}_1 + \hat{C}_2 = \hat{I}$

$t \to \infty \quad \Rightarrow \quad \hat{P}(t) = \begin{pmatrix} w_+ & w_- \\ w_+ & w_- \end{pmatrix}$. ($\because$ equilibrium)

$$\Rightarrow \boxed{P_{fi}(t) = w_f + (\delta_{fi} - w_f) e^{-W|t|}}$$

($\because$ time reversibility $t \to |t|$)

5 (continued)

5-(4).

$M_+ = -\mu_B$

$M_- = \mu_B$

$$\langle \Delta M(t) \Delta M(0) \rangle = \langle M(t) M(0) \rangle - \langle M \rangle^2$$

$$= \sum_{if} M_i M_f \left[P(f,t|i,0) - \omega_f \right] \omega_i$$

($\because$ $\langle M \rangle = \sum_i M_i W_i$, $\langle M(t) M(0) \rangle = \sum_{if} M_f P(f,t|i,0) \times M_i W_i$)

put the solution of 5-(3)

$$= e^{-|W|t} \sum_{if} M_i M_f (\delta_{if} - \omega_f) \omega_i$$

$$= \langle (\Delta M)^2 \rangle e^{-W|t|}$$

$$\langle (\Delta M)^2 \rangle \equiv \sum_{if} M_i M_f (\delta_{if} - \omega_f) \omega_i$$

$$= M_+ M_+ (1 - W_+) W_+$$

$$+ M_- M_- (1 - \omega_-) \omega_-$$

$$+ M_+ M_- (-W_+) \omega_- \times 2$$

$$= \mu_B^2 (1 - W_+) W_+ + \mu_B^2 (1 - W_-) W_-$$

$$+ 2\mu_B^2 W_+ W_-$$

$$= 4\mu_B^2 W_+ W_-$$

$$= \mu_B^2 \left((\omega_+ + \omega_-)^2 - (\omega_+ - \omega_-)^2 \right)$$

$$= \mu_B^2 \left(1 - \tanh^2 (\beta \mu_B B) \right)$$

$$= \mu_B^2 / \cosh^2 (\beta \mu_B B)$$

~ the same with the solution of 5-(1)

check point : Master equation

conditional probability

6.

$$\frac{dv}{dt} = -\gamma v + \frac{F'}{m}$$

$$\Rightarrow \quad v = v_0 e^{-\gamma t} + \frac{1}{m} e^{-\gamma t} \int_0^t e^{\gamma t'} F'(t')\, dt'$$

$\Rightarrow$ calculate $\langle v^2(t) \rangle$ and study $t \to \infty$ limit.

4 5

$$\lim_{t\to\infty} \langle v^2(t) \rangle = \underbrace{\cancel{\langle v_0^2 \rangle e^{-2\gamma t}}}_{\text{as } t\to\infty} + \frac{2v_0}{m} e^{-2\gamma t} \int_0^t e^{\gamma t'} \cancel{\langle F'(t') \rangle} dt'$$

$\langle v_0 F'(t') \rangle \simeq \langle v_0 \rangle \langle F'(t') \rangle$

$$+ \frac{e^{-2\gamma t}}{m^2} \int_0^t dt' \int_0^t dt'' \, e^{\gamma(t'+t'')} \langle F'(t') F''(t'') \rangle$$

$$= \frac{e^{-2\gamma t}}{m^2} \int_0^t dt' \int_{-t'}^{t-t'} e^{\gamma(2t'+s)} K'(s)\, ds \qquad t'' = s + t'$$

$$\downarrow \; \delta(s)$$

$$= \frac{e^{-2\gamma t}}{m^2} C \int_0^t dt' \, e^{2\gamma t'}$$

$$= \frac{e^{-2\gamma t}}{m^2} C \, \frac{1}{2\gamma} \left(e^{2\gamma t} - 1 \right)$$

$$= \frac{C}{2\gamma m^2} \left(1 - \underbrace{\cancel{e^{-2\gamma t}}}_{\text{as } t\to\infty} \right)$$

and $\displaystyle\lim_{t\to\infty} \langle v^2(t) \rangle = \frac{kT}{m}$.

5

$\Rightarrow \quad \underline{C = 2kT\gamma m}$