6-3. Wiener - Khintchine relations / Nyquist's theorem.

⊙ notation

$y(t)$: a random function of time

$$y_\Theta(t) \equiv \begin{cases} y(t) & \text{for } -\Theta < t < \Theta \\ 0 & \text{otherwise} \end{cases} \quad , \quad \Theta \to \infty$$

↓

Fourier transform.

$$C(\omega) = \frac{1}{2\pi} \int_{-\Theta}^{\Theta} y(t') e^{-i\omega t'} dt'$$

$$y_\Theta(t) = \int_{-\infty}^{\infty} C(\omega) e^{i\omega t} d\omega$$

If $y(t)$ is real, $C^*(\omega) = C(-\omega)$.

⊙ ensemble average and time average.

$$\overline{y(t)} \equiv \langle y(t) \rangle \equiv \frac{1}{N} \sum_{k=1}^{N} y^{(k)}(t) \quad \text{ensemble average}$$

$$\{ y^{(k)}(t) \} = \frac{1}{2\Theta} \int_{-\Theta}^{\Theta} y^{(k)}(t+t') dt' \quad \text{time average.}$$

$$\langle \{ y^{(k)}(t) \} \rangle = \sum \int = \int \sum = \{ \langle y(t) \rangle \}$$

ergodic assumption : $y^{(k)}(t)$ will pass through all the accessible values in a sufficiently long time interval.

$\Rightarrow \{ y^{(k)}(t) \} = \{ y \}$, independent of k.

for stationary system, $\langle y(t) \rangle = \langle y \rangle$ independent of t.

$$\Rightarrow \begin{cases} \langle \{ y^{(k)}(t) \} \rangle = \{ y \} \\ \{ \langle y(t) \rangle \} = \langle y \rangle \end{cases} \Rightarrow \boxed{\{ y \} = \langle y \rangle}$$

for a stationary ergodic ensemble.

⊙ Wiener – Khintchine relations

random function $y(t)$ in a stationary situation

$$K(s) \equiv \langle y(t)\, y(t+s) \rangle$$

~ independent of t ($\because$ stationary)

$$K(s) = \int_{-\infty}^{\infty} J(\omega)\, e^{i\omega s}\, d\omega$$

$J(\omega)$ ~ spectral density of y.

$$J(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(s)\, e^{-i\omega s}\, ds$$

→ Wiener – Khintchine relations

since $K(s) = K(-s)$ and $K(s)$ ~ real,

$$J^*(\omega) = J(\omega) \quad \text{and} \quad J(-\omega) = J(\omega)$$

Note :

$$\langle y^2 \rangle = K(0) = \frac{\pi}{\Theta} \int_{-\infty}^{\infty} |C(\omega)|^2\, d\omega$$ for stationary and ergodic $y(t)$.

proof.

$$K(s) = \langle y(0)\, y(s) \rangle = \{ y(0)\, y(s) \}$$

$$= \frac{1}{2\Theta} \int_{-\Theta}^{\Theta} dt'\, y(t')\, y(s+t')$$

$$= \frac{1}{2\Theta} \int_{-\infty}^{\infty} dt'\, y_\Theta(t')\, y_\Theta(s+t')$$

$$y_\Theta = \begin{cases} y(t') & t' \in [-\Theta, \Theta] \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{2\Theta} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d\omega'\, C(\omega') e^{i\omega' t'} \int_{-\infty}^{\infty} d\omega\, C(\omega)\, e^{i\omega(s+t')}$$

$$= \frac{1}{2\Theta} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'\, C(\omega')\, C(\omega)\, e^{i\omega s} \left[2\pi\, \delta(\omega' + \omega) \right]$$

$$= \frac{\pi}{\Theta} \int_{-\infty}^{\infty} d\omega\, C(-\omega)\, C(\omega)\, e^{i\omega s}$$

$\because C^*(-\omega) = C(\omega)$

$$\Rightarrow K(0) = \frac{\pi}{\Theta} \int_{-\infty}^{\infty} d\omega\, |C(\omega)|^2 .$$

⊙ Nyquist theorem.

$$\boxed{\begin{aligned} &\overline{J}_{V+}(\omega) = \frac{2}{\pi} kTR \quad \text{for} \quad \omega \ll \frac{1}{\tau^*} \\ &J_{V+}(\omega) : \text{spectral density of fluctuating voltage} \\ &R : \text{resistance} \\ &\tau^* : \text{correlation time of fluctuating voltage} \end{aligned}}$$

Consider an electrical circuit.

$$\Rightarrow L \frac{dI}{dt} = -RI + V_{ext}.$$

$$\Rightarrow \quad R = \frac{1}{2kT} \int_{-\infty}^{\infty} \langle V(0) V(s) \rangle_0 \, ds$$

$$\Rightarrow \quad R = \frac{1}{2kT} [2\pi J(0)] = \frac{\pi}{kT} J(0)$$

(※ $K(s) = \int_{-\infty}^{\infty} J(\omega) e^{i\omega s} d\omega$

$$\rightarrow K(0) = \int_{-\infty}^{\infty} J(\omega) d\omega = \int_0^{\infty} J_+(\omega) d\omega.$$

$\underset{\sim}{J_+(\omega)} \equiv 2J(\omega) \quad (\because J(-\omega) = J(\omega))$

⇝ spectral density for positive frequency.)

$$\Rightarrow \quad R = \frac{\pi}{2kT} J_+(0)$$

$$\Rightarrow \quad J_+(0) = \frac{2}{\pi} kTR.$$

Since $K(s)$ has a finite value for $s \lesssim \tau^*$,

$$\boxed{J_+(\omega) = \frac{2}{\pi} kTR \quad \text{for} \quad \omega \ll \frac{1}{\tau^*}}$$

$(\because J_+(\omega) = J_+(0))$

Nyquist's theorem.

~ It represents a connection between fluctuation and dissipation.