5-(3) 對 Boltzmann eq without relaxation time approximation

⊙ scattering cross section

$\sigma'(\vec{v}_1, \vec{v}_2 \to \vec{v}_1', \vec{v}_2')\, d^3\vec{v}_1' d^3\vec{v}_2' \equiv$ # of molecules

per unit time (per unit flux of type 1 molecules incident with relative velocity $\vec{v}_{rel}$ on type 2 molecules) emerging after scattering with final velocities $\in [\vec{v}_1', \vec{v}_1' + d\vec{v}_1']$ $[\vec{v}_2', \vec{v}_2' + d\vec{v}_2']$

$\vec{v}_{rel} = \vec{v}_1 - \vec{v}_2$

$\vec{v}_{rel}' = \vec{v}_1' - \vec{v}_2'$

$\vec{v}_c = \frac{m_1}{M}\vec{v}_1 + \frac{m_2}{M}\vec{v}_2$ $\quad (M = m_1 + m_2)$

$\vec{v}_c' = \frac{m_1}{M}\vec{v}_1' + \frac{m_2}{M}\vec{v}_2'$

$\left(\begin{aligned} \vec{v}_1 &= \vec{v}_c + \frac{\mu}{m_1}\vec{v}_{rel} \\ \vec{v}_2 &= \vec{v}_c - \frac{\mu}{m_2}\vec{v}_{rel} \end{aligned} \right.$

$M\vec{v}_c = M\vec{v}_c'$ = total momentum.

~ conserved. $\vec{v}_c = \vec{v}_c'$

We will consider elastic scatterings only : total energy is conserved.

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}M v_c^2 + \frac{1}{2}\mu v_{rel}^2$$

$$\Rightarrow |\vec{v}_{rel}| = |\vec{v}_{rel}'|$$

$\Rightarrow$ σ' must vanish unless $\vec{v}_c = \vec{v}_c'$ and $|\vec{v}_{rel}| = |\vec{v}_{rel}'|$.

$\sigma(\vec{v}_{rel}')\, d\Omega' \equiv$ # of molecules per unit time (per unit flux of type 1 molecules with $\vec{v}_{rel}$ onto type 2 molecules) emerging after scattering with $\vec{v}_{rel}' \in [\Omega', \Omega' + d\Omega']$

$$\sigma(\vec{v}_{rel}')\, d\Omega' = \int_{\vec{v}_c'} \int_{|\vec{v}_{rel}'|} \sigma'(\vec{v}_1, \vec{v}_2 \to \vec{v}_1', \vec{v}_2')\, d^3\vec{v}_1' d^3\vec{v}_2'$$

※ $d^3\vec{v}_1 d^3\vec{v}_2 = |J|\, d^3\vec{v}_c\, d^3\vec{v}_{rel}$, $|J| = 1$

$= d^3\vec{v}_c'\, d^3\vec{v}_{rel}'$

$= d^3\vec{v}_1'\, d^3\vec{v}_2'$

(i) symmetry properties :

① time reversal (invariant under $t \to -t$)

$$\sigma'(\vec{v}_1 , \vec{v}_2 \to \vec{v}_1{}', \vec{v}_2{}') \, d^3\vec{v}_1{}' d^3\vec{v}_2{}'$$

$$= \sigma'(-\vec{v}_1{}' , -\vec{v}_2{}' \to -\vec{v}_1 , -\vec{v}_2) \, d^3\vec{v}_1 d^3\vec{v}_2$$

$$\Rightarrow \quad \sigma'(\vec{v}_1 , \vec{v}_2 \to \vec{v}_1{}' \vec{v}_2{}') = \sigma'(-\vec{v}_1{}' , -\vec{v}_2{}' \to -\vec{v}_1 , -\vec{v}_2)$$

② inversion (invariant under $\vec{r} \to -\vec{r}$)

$$\sigma'(\vec{v}_1 , \vec{v}_2 \to \vec{v}_1{}', \vec{v}_2{}') = \sigma'(-\vec{v}_1 , -\vec{v}_2 \to -\vec{v}_1{}' -\vec{v}_2{}')$$

time reversal & inversion

$$\sigma'(\vec{v}_1 , \vec{v}_2 \to \vec{v}_1{}', \vec{v}_2{}') = \sigma'(\vec{v}_1{}', \vec{v}_2{}' \to \vec{v}_1 , \vec{v}_2)$$

⊙ Boltzmann equation

a set of assumption :

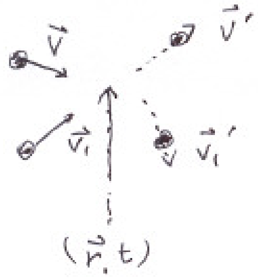
- only two-particle collisions (dilute enough)
- σ_0 does not depend on $\vec{F}$ (external force).
- $f(\vec{r}, \vec{v}, t)$ does not vary during collision ($d\vec{r} > d \approx \sqrt{\sigma_0}$
  $dt >$ duration of collision)
- No correlation between the initial velocities of two molecules prior to their collision. ("molecular chaos" assumption
  $\Leftarrow \quad l \gg d \approx \sqrt{\sigma_0}$)

$D_c f(\vec{r}, \vec{v}, t)\, d^3\vec{r}\, d^3\vec{v}\, dt$: a net change in the number of molecules with $\left(\begin{array}{l}\vec{v} \in [\vec{v}, \vec{v}+d\vec{v}] \\ \vec{r} \in [\vec{r}, \vec{r}+d\vec{r}]\end{array}\right.$ due to collisions at $t \in [t, t+dt]$

$$D_c f = \underbrace{-D_c^{(-)} f}_{\substack{\text{scattered out of} \\ [\vec{v}, \vec{v}+d\vec{v}]}} + \underbrace{D_c^{(+)} f}_{\substack{\text{scattered into} \\ [\vec{v}, \vec{v}+d\vec{v}]}}$$

$D_c^{(-)} f(\vec{r}, \vec{v}, t)\, d^3\vec{r}\, d^3\vec{v}\, dt$

$$= \int_{\vec{v}_1'} \int_{\vec{v}'} \int_{\vec{v}_1} \underbrace{\left[|\vec{v}-\vec{v}_1| f(\vec{r}, \vec{v}, t) d^3\vec{v} \right]}_{\text{relative flux}} \left[f(\vec{r}, \vec{v}_1, t)\, d^3\vec{r}\, d^3\vec{v}_1 \right] \left[\sigma(\vec{v}, \vec{v}_1 \rightarrow \vec{v}', \vec{v}_1') \, d^3\vec{v}'\, d^3\vec{v}_1' \right]$$

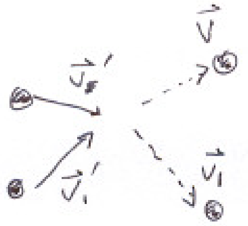


no correlation between $\vec{v}_1$ and $\vec{v}$
~ "molecular chaos" assumption

similarly,

$$D_c^{(+)} f(\vec r,\vec v,t)\, d^3\vec r\, d^3\vec v\, dt$$

$$= \int_{\vec v_1}\int_{\vec v_1'}\int_{\vec v'} \left[|\vec v' - \vec v_1'|\, f(\vec r,\vec v',t)\, d^3\vec v'\right]\left[f(\vec r,\vec v_1',t)\, d^3\vec r\, d\vec v_1'\right]\left[\sigma'(\vec v',\vec v_1' \to \vec v,\vec v_1)\, d^3\vec v\, d^3\vec v_1\right]$$



Use symmetric properties;

(inversion ⊕ time reversal)

$$\sigma'(\vec v',\vec v_1' \to \vec v,\vec v_1) = \sigma'(\vec v,\vec v_1 \to \vec v',\vec v_1')$$

And use

$$\begin{cases} \vec V_{rel} = \vec v - \vec v_1, \quad \vec V_{rel}' = \vec v' - \vec v_1' \\ |\vec V_{rel}| = |\vec V_{rel}'| = V_{rel} \\ f = f(\vec r,\vec v,t),\ f_1 = f(\vec r,\vec v_1,t),\ f' = f(\vec r,\vec v',t),\ f_1' = f(\vec r,\vec v_1',t) \end{cases}$$

$$\Rightarrow D_c f = \int_{\vec v_1}\int_{\vec v'}\int_{\vec v_1'} (f'f_1' - ff_1)\, V_{rel}\, \sigma'(\vec v,\vec v_1 \to \vec v',\vec v_1')\, d^3\vec v_1\, d^3\vec v_1'\, d^3\vec v'$$

$$= \int_{\vec v_1}\int_{\Omega'} (f'f_1' - ff_1)\, V_{rel}\,\sigma\, d\Omega'\, d^3\vec v_1 \qquad \sigma = \sigma(\vec V_{rel}')$$

$$\frac{\partial f}{\partial t} + \vec v\cdot\frac{df}{d\vec r} + \frac{\vec F}{m}\cdot\frac{\partial f}{\partial \vec v} = D_c f$$

6-25

⊙ Equation of change ~~f~~ for mean values

a function $\chi(\vec{r}, \vec{v}, t)$

$$\langle \chi(\vec{r}, t) \rangle = \frac{1}{n(\vec{r}, t)} \int \chi(\vec{r}, \vec{v}, t) f(\vec{r}, \vec{v}, t) \, d^3\vec{v}$$

$\Rightarrow$ Equation for $\langle \chi \rangle$?

$$Df = D_c f$$

$$\rightarrow \int d^3\vec{v} \, Df \chi = \underbrace{\int d^3\vec{v} \, D_c f \chi}_{= C(\chi)}$$

$$\int d^3\vec{v} \, Df \chi = \int d^3\vec{v} \frac{\partial f}{\partial t} \chi + \int d^3\vec{v} \, \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} \chi + \int d^3\vec{v} \, \frac{\vec{F}}{m} \cdot \frac{\partial f}{\partial \vec{v}} \chi$$

$$\int d^3\vec{v} \frac{\partial f}{\partial t} \chi = \int d^3\vec{v} \left[\frac{\partial (f\chi)}{\partial t} - f \frac{\partial \chi}{\partial t} \right] = \frac{\partial}{\partial t} \left(\int d^3\vec{v} \, f\chi \right) - \int d^3\vec{v} f \frac{\partial \chi}{\partial t}$$

$$= \frac{\partial}{\partial t} (n \langle \chi \rangle) - n \left\langle \frac{\partial \chi}{\partial t} \right\rangle$$

$$\int d^3\vec{v} \, \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} \chi = \int d^3\vec{v} \, \vec{v} \cdot \frac{\partial}{\partial \vec{r}} (f\chi) - f \vec{v} \cdot \frac{\partial \chi}{\partial \vec{r}} = \frac{\partial}{\partial \vec{r}} (n \langle \vec{v} \chi \rangle) - n \left\langle \vec{v} \cdot \frac{\partial \chi}{\partial \vec{r}} \right\rangle$$

$$\int d^3\vec{v} \, \frac{\vec{F}}{m} \cdot \frac{\partial f}{\partial \vec{v}} \chi = \int d^3\vec{v} \, \frac{\partial}{\partial \vec{v}} \left(\frac{\vec{F}}{m} f \chi \right) - f \frac{\vec{F}}{m} \cdot \frac{\partial \chi}{\partial \vec{v}} = - n \frac{\vec{F}}{m} \cdot \left\langle \frac{\partial \chi}{\partial \vec{v}} \right\rangle$$

$$\left(\because \quad \frac{F_\alpha}{m} f \chi \Big|_{v_\alpha = -\infty}^{v_\alpha = \infty} = 0 \right)$$

$$(f(v_\alpha = \pm\infty) = 0)$$

$$\Rightarrow \int d^3\vec{v} \, Df \chi = \frac{\partial}{\partial t} (n \langle \chi \rangle) + \frac{\partial}{\partial \vec{r}} (n \langle \vec{v} \chi \rangle) - n \langle D \chi \rangle = C(\chi)$$

$$C(\chi) = \iiiint d^3\vec{v} \, d^3\vec{v}' \, d^3\vec{v}_1 \, d^3\vec{v}_1' \, (f' f_1' - f f_1) V_{rel} \sigma' \chi(\vec{r}, \vec{v}, t)$$

$\downarrow$ $\vec{v} \leftrightarrow \vec{v}_1$

Here, σ' is unchanged.

$$= \iiiint d^3\vec{v} \, d^3\vec{v}' \, d^3\vec{v}_1 \, d^3\vec{v}_1' \, (f' f_1' - f f_1) V_{rel} \sigma' \chi(\vec{r}, \vec{v}_1, t)$$

$$= \frac{1}{2} \iiiint \_\_\_\_\_\_\_\_ (\chi + \chi_1)$$

$\chi_1 = \chi(\vec{r}, \vec{v}_1, t)$

$\chi = \chi(\vec{r}, \vec{v}, t)$

use

$$\iiiint d^3\vec{v}\, d^3\vec{v}'\, d^3\vec{v}_1 d^3\vec{v}_1'\ f' f_1'\ V_{rel}\ \underbrace{\sigma'(\vec{v}_1\vec{v}_1 \to \vec{v}', \vec{v}_1')}_{\shortparallel}\ (\chi + \chi_1)$$

$\swarrow$ $\vec{v} \leftrightarrow \vec{v}'$, $\vec{v}_1 \leftrightarrow \vec{v}_1'$

$$= \iiiint \text{———}\ f f_1\ V_{rel}\ \underbrace{\sigma'(\vec{v}', \vec{v}_1' - \vec{v}, \vec{v}_1)}\ (\chi' + \chi_1')$$

$$\Rightarrow \quad C(\chi) = \frac{1}{2} \iiiint d^3\vec{v}\, d^3\vec{v}_1 d^3\vec{v}' d^3\vec{v}_1'\ f f_1\, V\sigma' \Delta\chi$$

$$(\Delta\chi \equiv \chi' + \chi_1' - \chi - \chi_1)$$

$$= \frac{1}{2} \iiint d^3\vec{v}\, d^3\vec{v}_1\ d\Omega'\ f f_1\, V\sigma\, \Delta\chi$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\langle n\chi\rangle = n\langle D\chi\rangle - \frac{\partial}{\partial \vec{r}} \cdot \langle n\vec{v}\chi\rangle + C(\chi)$$

⊙ Conservation eq.

Conserved quantities during collisions : $\chi = \begin{cases} \text{mass} \\ \text{total momentum} \\ \text{total energy} \end{cases} \Rightarrow C(\chi) = 0$

$(\because \Delta\chi = \chi' + \chi_1' - \chi - \chi_1 = 0\ )$

(ex) $\chi = m$ (mass)

$$\Rightarrow \quad \frac{\partial}{\partial t}(nm) = -\frac{\partial}{\partial \vec{r}} \cdot \langle nm\vec{v}\rangle \qquad (Dm = 0)$$

$\rho(\vec{r}.t) \equiv m\, n(\vec{r}.t)$. mass density

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 . \qquad \vec{u} = \langle\vec{v}\rangle$$

$\sim$ eq. of continuity.

(ex2) $\chi = m v_\gamma$ $(\gamma = x, y, z)$

χ = total momentum $\rightarrow$ $C(\chi) = 0$.

$$\frac{\partial}{\partial t}\langle n(m v_{1,\gamma} + m v_{2,\gamma})\rangle + \frac{\partial}{\partial \vec{r}}\langle n\vec{v}\,(m v_{1,\gamma} + m v_{2,\gamma})\rangle$$

$$= n\langle m D\,(m v_{1,\gamma} + m v_{2,\gamma})\rangle$$

We will study mean values.

$\Rightarrow$

$$\left(\frac{\partial}{\partial t}\langle n m v_{1,\gamma}\rangle + \frac{\partial}{\partial \vec{r}}\langle n\vec{v}\; m v_{1,\gamma}\rangle = n\langle m D\, v_{1,\gamma}\rangle\right)\times 2$$

drop the index 1.

$$\frac{\partial}{\partial t}\langle n m v_\gamma\rangle + \frac{\partial}{\partial \vec{r}}\langle n\vec{v}\; m v_\gamma\rangle = n\langle m D\, v_\gamma\rangle$$

$$D v_\gamma = \frac{\vec{F}}{m}\cdot\frac{\partial v_\gamma}{\partial \vec{v}} = \frac{F_\gamma}{m}$$

$nm = \rho$, $\langle v_\gamma\rangle = u_\gamma$

$$\frac{\partial}{\partial t}(\rho u_\gamma) + \frac{\partial}{\partial \vec{r}}(\rho\langle\vec{v}\, v_\gamma\rangle) = \rho F_\gamma/m$$

$$\vec{v} = \vec{u} + \vec{U}$$

$$\langle v_\alpha v_\gamma\rangle = \langle(u_\alpha + U_\alpha)(u_\gamma + U_\gamma)\rangle = u_\alpha u_\gamma + \langle U_\alpha U_\gamma\rangle$$

$(\because \langle\vec{U}\rangle = 0)$

$$P_{\alpha\gamma} \equiv \rho\langle U_\alpha U_\gamma\rangle = P_{\gamma\alpha}$$

$$\frac{\partial}{\partial t}(\rho u_\gamma) + \frac{\partial}{\partial x_\alpha}(\rho u_\alpha u_\gamma) = -\frac{\partial P_{\alpha\gamma}}{\partial x_\alpha} + \rho F_\gamma'$$ $F_\gamma' = F_\gamma/m$

$\rightarrow$ Euler eq. of motion

② Approximation methods for solving Boltzmann eq.. (possible to Skip 參) 6-28

$$f = f^{(0)}(1+\Phi) \qquad (\Phi \ll 1).$$

$$f^{(0)}(\vec{r},\vec{v},t) = n\left(\frac{m\beta}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{1}{2}\beta(\vec{v}-\vec{u})^2}$$

$n, \beta, \vec{u}$ ~ can be a function of $(\vec{r}, t)$.

$$D_c f^{(0)} = 0$$

(∵ $f^{(0)}(\vec{v}) f^{(0)}(\vec{v}_1) = f^{(0)}(\vec{v}') f^{(0)}(\vec{v}_1')$

⇐ total momentum, energy conservation)

see Reif Eq. 14.7.5

$$Df = D(f^{(0)}(1+\Phi))$$

$$\simeq D f^{(0)}$$

$$D_c f = \iint d^3\vec{v}_1 \, d\Omega' \, (f'f_1' - ff_1) V_{rel}\,\sigma$$

$$ff_1 = f^{(0)} f_1^{(0)} (1+\Phi+\Phi_1)$$

$$f'f_1' = f^{(0)} f_1^{(0)} (1+\Phi'+\Phi_1')$$

參 $f^{(0)\prime} f_1^{(0)\prime} = f^{(0)} f_1^{(0)}$

$$\Rightarrow \quad Df^{(0)} = \iint d^3\vec{v}\, d\Omega' \, f^{(0)} f_1^{(0)} V_{rel}\,\sigma\, \Delta\Phi$$

~ Integral eq. for $\Delta\Phi$ $\qquad \Delta\Phi = \Phi' + \Phi_1' - \Phi - \Phi_1$

One can assume :

$$\Phi = \sum_{\lambda=1}^{3} a_\lambda U_\lambda + \sum_{\lambda,\mu=1}^{3} a_{\lambda\mu} U_\lambda U_\mu + \cdots$$

(Ex) viscosity (Reif chap. 14.8)

$$f^{(0)}(\vec{r},\vec{v},t) = n\left(\frac{m\beta}{2\pi}\right)^{3/2} e^{-\frac{1}{2}\beta m\{[v_x - u_x(z)]^2 + v_y^2 + v_z^2\}}$$

$$= g(U) = n\left(\frac{m\beta}{2\pi}\right)^{3/2} e^{-\frac{1}{2}\beta m \vec{U}^2}$$

$$\Phi = A U_z U_x$$

$$Df^{(0)} = v_z \frac{\partial f^{(0)}}{\partial z} = \beta m g(U) U_z U_x \frac{\partial u_x}{\partial z} = \iint d^3\vec{U}_1 \, d\Omega' \, g(\vec{U}) g(\vec{U}_1) V\sigma\,\Delta\Phi$$

$$P_{zx} = m\int d^3\vec{v}\, f U_z U_x = m\int d^3\vec{U}\, g(U)(1+\Phi) U_z U_x = Am\int d^3\vec{U}\, g\, U_z^2 U_x^2.$$