◦ Pauli paramagnetism

An ideal Fermi gas of N electrons with magnetic moment

$$\mu_z = g\mu_B m_s\,, \qquad m_s = \pm\frac{1}{2}\,, \qquad g = 2 \quad (\text{g-factor})$$

(do not confuse μ_z with chemical potential μ)

external field $B\hat{z}$, low temperature $kT \ll E_F$ (Fermi energy)

→ calculate the susceptibility

Assume: the influence of the external magnetic field on the wave functions of the electrons is neglected.

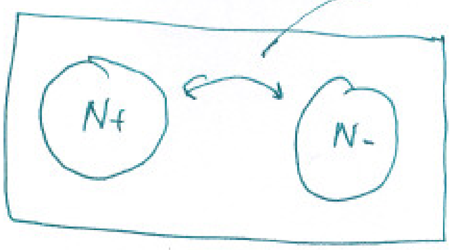
$$\rightarrow \quad E = \frac{p^2}{2m} - \mu_z B \qquad (\text{cf. Landau levels})$$

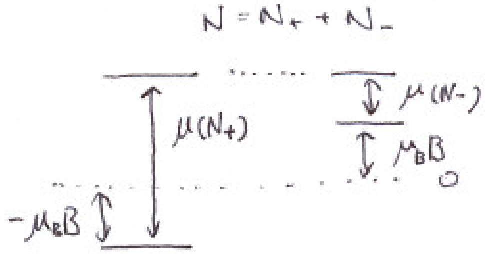
N_+ : # of electrons having $m = \frac{1}{2}$

N_- : " $m = -\frac{1}{2}$

$N_+ + N_- = N$

← exchange particles





∴ at equilibrium, $\mu_+(N_+) = \mu_-(N_-)$

$\mu_\pm$: chemical potential

$$\mu_+(N_+) = \underbrace{\mu(N_+)}_{\text{contributed from the kinetic energy}} - g\mu_B\cdot\frac{1}{2}B = \mu(N_+) - \mu_B B$$

$$\mu_-(N_-) = \mu(N_-) + \mu_B B$$

fermi gas $\Rightarrow$

$$N = \sum_k \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

$$= \int_0^\infty d\epsilon \, \rho(\epsilon) \frac{1}{z_f^{-1} \exp(\beta\epsilon) + 1}$$

$z_f = e^{\beta\mu}$, $\rho(\epsilon) = \boxed{g} \cdot \frac{2\pi V}{h^3} (2m)^{3/2} \cdot \sqrt{\epsilon}$

$0 \leq z_f \leq \infty$

(3 dimensional densities of states)

$\boxed{g}$: degeneracy factor.

Here , $\boxed{g} = 1$, since we consider spin-up and spin-down electrons , seperately.

(normally , $\boxed{g} = 2$)

$$\Rightarrow \quad N(T, V, z_f) = \frac{\boxed{g}\, V}{\lambda^3} f_{3/2}(z_f)$$

$$= \frac{V}{\lambda^3} f_{3/2}(z_f) .$$

~~$f_{3/2}(z)$~~

$$f_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1} dx}{z^{-1} e^x + 1}$$

$$\lambda = \sqrt{\frac{h^2}{2\pi m k T}}$$

This relation determines $\mu_\pm(N_\pm)$.

Now check energy scales :

$\mu_B = 0.578 \times 10^{-4}$ eV T^{-1}

$\rightarrow$ Even at several teslas of B , $\mu_B B \ll \underbrace{E_F}_{\text{order of several eV.}}$

$$\Rightarrow \quad N_\pm = \frac{1}{2} N (1 \pm r) \quad , \quad r \ll 1$$

($r = 0$ when $\mu_B B = 0$)

$$\begin{cases} N_+ - N_- = N r \\ \mu\left(\frac{N}{2}(1+r)\right) - \mu\left(\frac{N}{2}(1-r)\right) \simeq \left.\frac{\partial \mu(Nx)}{\partial x}\right|_{x = 1/2} r = 2\mu_B B \end{cases}$$

($\because \mu(N_+) - \mu(N_-) = 2\mu_B B$)

$$\Rightarrow N_+ - N_- = N r = \frac{2\mu_B B}{\left.\frac{\partial \mu(Nx)}{\partial x}\right|_{x=1/2}}$$

$$\Rightarrow \langle \mu_z \rangle = \mu_B (N_+ - N_-) = \frac{2\mu_B^2 B N}{\left.\frac{\partial \mu(Nx)}{\partial x}\right|_{x=1/2}}$$

Susceptibility $\quad \chi = \frac{\partial \langle \mu_z \rangle}{\partial B} = \frac{2\mu_B^2 N}{\left.\frac{\partial \mu(Nx)}{\partial x}\right|_{x=1/2}}$

Now, let's evaluate $\left.\frac{\partial \mu(Nx)}{\partial x}\right|_{x=1/2}$

(i) $kT \gg E_f$

One may use $\quad f_{3/2}(z_f) \simeq z_f - z_f^2/2^{3/2}$

(see K. Huang)

$$f_{3/2}(z_f) = \frac{N}{V}\lambda^3 = z_f - z_f^2/2^{3/2}$$

$$\Rightarrow z_f \simeq \frac{N}{V}\lambda^3 \left(1 + \frac{N}{V}\frac{\lambda^3}{2^{3/2}}\right)$$

$$\beta\mu = \ln z_f$$

$$\left.\frac{\partial \mu(Nx)}{\partial x}\right|_{x=1/2} = kT \frac{1}{z_f} \frac{\partial}{\partial x}\left(\frac{Nx}{V}\lambda^3\left(1 + \frac{Nx}{V}\frac{\lambda^3}{2^{3/2}}\right)\right)\Bigg|_{x=1/2}$$

$$\simeq kT \frac{1}{\frac{N}{2V}\lambda^3\left(1 + \frac{N}{2V}\frac{\lambda^3}{2^{3/2}}\right)}\left(\frac{N}{V}\lambda^3 + \frac{N^2}{V^2}\frac{\lambda^6}{2^{3/2}}\right)$$

$$\simeq 2kT\left(1 + \frac{N}{V}\frac{\lambda^3}{2^{5/2}}\right)$$

$$\Rightarrow \chi \simeq \frac{\mu_B^2 N}{kT}\left(1 - \frac{\lambda^3}{2^{5/2}}\frac{N}{V}\right)$$

※ compare the result with what we have learned in Course 3 :

$$\left\{ \begin{aligned} &\chi = N_0 \frac{g^2 \mu_B^2 J(J+1)}{3kT}, \quad J = \tfrac{1}{2},\ g = 2 \\ &\rightarrow \chi = N_0 \frac{\mu_B^2}{kT} \end{aligned} \right.$$

Consider : Why has χ the same value, although here we take into account of the kinetic term ?

(ii) $kT \ll E_f \quad \Rightarrow \quad z_f \gg 1$

$$\Rightarrow f_{3/2}(z_f) \simeq \frac{4}{3\sqrt{\pi}} (\ln z_f)^{3/2} \left(1 + \frac{\pi^2}{8} (\ln z_f)^{-2} + \cdots\right)$$

(see the book by K. Huang)

$$f_{3/2}(z_f) = \frac{N}{V} \lambda^3$$

$$\Rightarrow \frac{N}{V} = \frac{4\pi}{3} \left(\frac{2m}{h^2}\right)^{3/2} (kT \ln z_f)^{3/2} \left(1 + \frac{\pi^2}{8} (\ln z_f)^{-2} + \cdots\right)$$

$$\Rightarrow \mu(N) \simeq E_f(N) \left(1 - \frac{\pi^2}{12} \left(\frac{kT}{E_f(N)}\right)^2\right)$$

$$E_f(N) = \left(\frac{3}{4\pi} \frac{N}{V}\right)^{2/3} \frac{h^2}{2m}$$

$$\left.\frac{\partial \mu(Nx)}{\partial x}\right|_{x=1/2} = \frac{4}{3} E_f\!\left(\frac{N}{2}\right) \left(1 - \frac{\pi^2}{12} \left(\frac{kT}{E_f(N/2)}\right)^2\right)$$

$$- E_f\!\left(\frac{N}{2}\right) \frac{\pi^2}{12} \left(\frac{kT}{E_f(\frac{N}{2})}\right)^2 (-2) \cdot \frac{4}{3}$$

$$= \frac{4}{3} E_f\!\left(\frac{N}{2}\right) \left(1 + \frac{\pi^2}{12} \left(\frac{kT}{E_f(N/2)}\right)^2\right)$$

$$\Rightarrow \chi \simeq \frac{3}{2} \frac{\mu_0^2 N}{E_f} \left(1 - \frac{\pi^2}{12} \left(\frac{kT}{E_f}\right)^2\right)$$

(∵ $E_f(\frac{N}{2})$ ⇝ the fermi energy of the whole system with $N = \underbrace{\frac{N}{2}}_{\text{spin up}} + \underbrace{\frac{N}{2}}_{\text{spin down}}$)

→ compare it with the Curie's law at $T \to 0$. (see course 3)

<u>different!</u> $(\mu_z \to N \cdot g\mu \cdot S)$

The Curie's law is valid only in the classical limit.

(What is the missing component in the derivation of Curie's law? fermion. kinetic energy.)

Landau diamagnetism.

The circular current induction leads to dipole moments opposite to the direction of magnetic field.

$\Rightarrow$ it weakens magnetic field.

(cf. paramagnetism : enhances the magnetic field.)

$\vec{B} = B\hat{z}$

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} + \frac{|e|}{c}\vec{A}\right)^2$$

One ~~may~~ can choose the Landau gauge :

$$A_x = -By, \quad A_y = 0, \quad A_z = 0$$

(※ gauge invariance : $\mathcal{H}\psi = \epsilon\psi$

$$\vec{A}(\vec{r}) \to \vec{A}(\vec{r}) - \nabla\omega(\vec{r})$$

$$\psi(\vec{r}) \to \exp\left(-\frac{ie}{\hbar c}\omega(\vec{r})\right)\psi(\vec{r})\ )$$

$$\Rightarrow \mathcal{H} = \frac{1}{2m}\left\{\left(p_x - \left(\frac{|e|B}{c}y\right)\right)^2 + p_y^2 + p_z^2\right\}$$

$$\psi(\vec{r}) = e^{i(k_x x + k_z z)} f(y)$$

$$\Rightarrow \left(\frac{1}{2m}p_y^2 + \frac{1}{2}m\omega_c^2(y-y_0)^2\right)f(y) = \epsilon' f(y)$$

$$\epsilon' = \epsilon - \hbar^2 k_z^2/2m, \quad \omega_c = \frac{|e|B}{mc}, \quad y_0 = \frac{\hbar c}{|e|B}k_x$$

$\rightsquigarrow$ harmonic oscillator !

$$\epsilon = \frac{p_z^2}{2m} + \hbar\omega_c\left(j + \frac{1}{2}\right) \qquad (j = 0, 1, 2, \ldots)$$

To obtain magnetization, calculate first the partition fn.

$$\ln \mathcal{Z} = \sum_{p_z, j} \ln\left(1 + z_f\, e^{-\beta\left(\frac{p_z^2}{2m} + \hbar\omega_c (j+\frac{1}{2})\right)}\right)$$

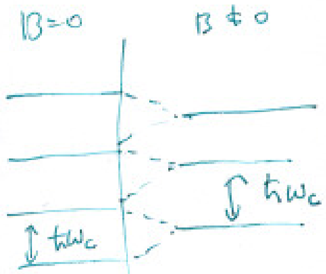
Landau level degeneracy : $g_L = \Phi / \Phi_0$.

$$\left(\begin{array}{ll} \Phi_0 = hc/e & \text{(flux quantum)} \\ \Phi = BA & A: \text{area (xy plane)} \end{array}\right.$$

$$g_L = \frac{|e|B}{hc} A .$$

(how to evaluate g_L ?

$$\cancel{g_L = \frac{1}{h^2} \int_{\substack{E \in [\hbar\omega_c j,\ \hbar\omega_c (j+1)] \\ E = \frac{1}{2m}(p_x^2 + p_y^2)}} dp_x dp_y dx dy}$$



$$g_L = \frac{1}{h^2} \int_{\substack{E \in [\hbar\omega_c j\ ,\ \hbar\omega_c (j+1)] \\ E = \frac{1}{2m}(p_x^2 + p_y^2)}} dp_x dp_y dx dy$$

$$= \frac{A}{h^2} \int_{p_j}^{p_{j+1}} 2\pi p\, dp . \qquad \frac{p_j^2}{2m} = \hbar\omega_c j$$

$$= \frac{2\pi A}{h^2} \frac{1}{2} \left(p_{j+1}^2 - p_j^2\right) = \frac{\pi A}{h^2} (2m\, \hbar\omega_c) = \frac{|e|B}{hc} A \Bigg)$$

$$\Rightarrow \ln \mathcal{Z} = \frac{L_z}{2\pi} \int dk_z \sum_{j=0}^{\infty} g_L \ln\left(1 + z_f \exp\left\{-\beta\left(\frac{\hbar^2 k^2}{2m} + \hbar\omega_c (j+\tfrac{1}{2})\right)\right\}\right)$$

$$\Bigg(\quad N = \frac{L_z}{2\pi} \int dk_z \sum_{j=0}^{\infty} g_L \cdot \frac{1}{z^{-1} \exp\left\{\beta\left(\frac{\hbar^2 k^2}{2m} + \hbar\omega_c (j+\frac{1}{2})\right)\right\} + 1}$$

Consider the regime of $E_F \gg kT \gg \hbar\omega_c$

: One can use the Euler - MacLaurin formula (see Arfken) eq. 5.168 a, b

$$\sum_{j=0}^{\infty} f(j+\tfrac{1}{2}) \simeq \int_0^{\infty} f(x)\,dx + \frac{1}{24} f'(0) + \cdots$$

(∵ Euler - MacLaurin formulai (useful sometimes for converting summation to integral)

$$\frac{1}{2}F(a) + \sum_{n=1}^{\infty} F(a+n) \simeq \int_a^{\infty} F(x)\,dx - \frac{1}{12}F'(a)$$

put $a = \frac{1}{2}$ and use $F(x) \simeq F(0) + x F'(0)$ for $x \in [0, \frac{1}{2}]$

$$\int_0^{\frac{1}{2}} F(x)\,dx \simeq \frac{1}{2}F(0) + \frac{1}{8}F'(0)$$

$$F(\tfrac{1}{2}) \simeq F(0) + \tfrac{1}{2}F'(0)$$

$$F'(\tfrac{1}{2}) \simeq F'(0)$$

$$\Rightarrow \sum_{n=0}^{\infty} F(\tfrac{1}{2}+n) \simeq \int_{\frac{1}{2}}^{\infty} F(x)\,dx + \frac{1}{2}F(0) + \frac{1}{4}F'(0) - \frac{1}{12}F'(0)$$

$$\simeq \int_{\frac{1}{2}}^{\infty} F(x)\,dx + \int_0^{\frac{1}{2}} F(x)\,dx + \frac{1}{24}F'(0)$$

$$\simeq \int_0^{\infty} F(x)\,dx + \frac{1}{24}F'(0) \quad)$$

Choose

$$f(x) = \ln\left[1 + z_f \exp\left\{-\beta\left(\frac{\hbar^2 k^2}{2m} + \hbar\omega_c x\right)\right\}\right] \qquad \left(\begin{array}{l} 2\mu_B B = \hbar\omega_c \\ p_z = \hbar k \end{array} \right.$$

$$\Rightarrow \ln \mathcal{Z} \simeq \frac{VeB}{h^2 c}\Bigg(\underbrace{\int_0^{\infty} dx \int_{-\infty}^{\infty} dp_z \ln\left[1 + z_f \exp\left\{-2\beta\mu_B B x - \frac{\beta p_z^2}{2m}\right\}\right]}_{\equiv \ln \mathcal{Z}_0}$$

$$\underbrace{- \frac{1}{12}\beta\mu_B B \int_{-\infty}^{\infty} dp_z \frac{1}{z_f^{-1}\exp\left\{\frac{\beta p_z^2}{2m}\right\} + 1}\Bigg)}_{\equiv \ln \mathcal{Z}_1} + O\left((\beta\mu_B B)^2\right)$$

$$\ln \mathcal{Z}_0 \equiv \frac{VeB}{h^2 c}\left(\int_0^\infty dx \int_{-\infty}^{\infty} dp_z \ln\left[1+z_f \exp\left\{-2\beta\mu_B B x - \frac{\beta p_z^2}{2m}\right\}\right]\right)$$

$$= \frac{VeB}{h^2 c}\left(\int_0^\infty dx \int_0^{\infty} 2dp_z \ln\left[1+z_f \exp\left\{-2\beta\mu_B B x - \frac{\beta p_z^2}{2m}\right\}\right]\right)$$

$$\epsilon = 2\mu_B B x + \frac{p_z^2}{2m}$$

$$= \frac{VeB}{h^2 c}\int_0^\infty \frac{d\epsilon}{2\mu_B B}\int_0^{\sqrt{2m\epsilon}} 2dp_z \ln\left[1+z_f \exp\{-\beta\epsilon\}\right]$$

$$\Rightarrow \frac{\partial \ln \mathcal{Z}_0}{\partial B} = 0 .$$

$$\ln \mathcal{Z}_1 = -\frac{Ve\beta\mu_B B^2}{12h^2 c}\int_0^\infty 2dp_z \frac{1}{z_f^{-1}\exp\left\{\frac{\beta p_z^2}{2m}\right\}+1}$$

$$y = \frac{p_z^2}{2m}, \quad \mu_B = \frac{e\hbar}{2mc}$$

$$= -\frac{\pi V (2m)^{3/2}}{6h^3}(\mu_B B)^2\sqrt{\beta}\int_0^\infty \frac{y^{-1/2}\,dy}{z_f^{-1}e^y+1}$$

$$= -\frac{\pi V (2m)^{3/2}}{6h^3}(\mu_B B)^2\sqrt{\beta}\,\sqrt{\pi}\, f_{1/2}(z_f)$$

$z_f \gg 1 \quad (\because \epsilon_f \gg kT)$

$$\Rightarrow f_{1/2}(z_f) \simeq (\ln z_f)^{1/2}\big/\Gamma(3/2) \simeq \frac{2}{\sqrt{\pi}}\sqrt{\frac{\epsilon_f}{kT}}$$

$$\ln \mathcal{Z}_1 \simeq -\frac{N}{4}\frac{(\mu_B B)^2}{\epsilon_f kT} \qquad \left(\text{use } \epsilon_f = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{\boxed{g}}\frac{N}{V}\right)^{2/3}\right)$$

$$M_z = \frac{1}{\beta}\frac{\partial}{\partial B}\ln \mathcal{Z}_1$$

$\boxed{g} = 1$ ∵ ~~magnetization~~ magnetic fields remove the spin degeneracy.

$$= -\frac{1}{2}N\mu_B\frac{\mu_B B}{\epsilon_f}$$

$$\chi = \frac{\partial M_z}{\partial B} < 0$$

negative!

Usually,

$$\chi = \chi^{dia} + \chi^{para}$$

$$= \frac{1}{2} \frac{N}{E_F} \left(-\mu_B'^2 + 3\mu_B^2 \right)$$

depend on effective mass m' (∵ conducting electrons in metals)

use bare mass (∵ not due to current but ~~due~~ due to spin)

※ classical limit (high temperature) → much more easier to evaluate the partition function
($kT \gg E_F \gg \hbar\omega_c$)

※ symmetric gauge

※ flux quantization

wave function encloses integer # of flux quanta.

Other examples : thermionic emission (Richardson effect)
— see H.W. (Reif 9.23)

relativistic $\begin{pmatrix}\text{Fermi} \\ \text{Bose}\end{pmatrix}$ gas

white dwarfs