(2) Applications I (Reif chap. 9, Huang chap. 12.3)

~~① Ideal gas~~

- Fermions : conduction electrons in solids

- Ideal Bose gas : Bose-Einstein condensation

- photons : black-body radiation

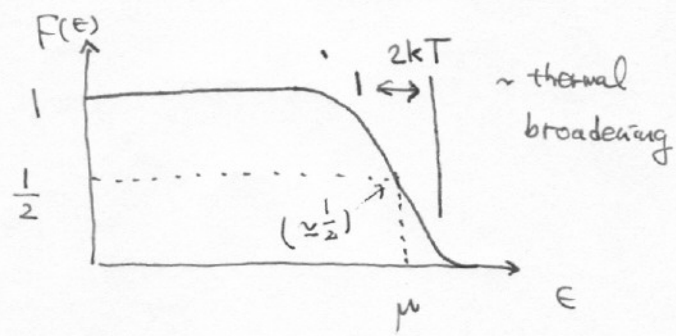
Fermion : conduction electrons in metals

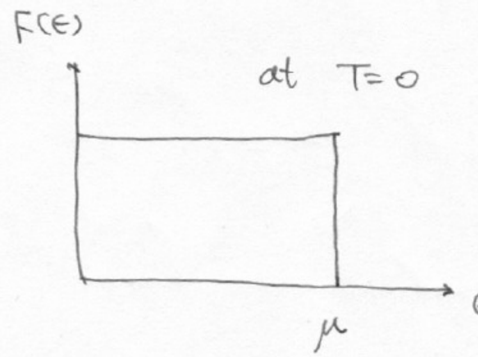
metal $\Rightarrow$ one can neglect electron - electron interactions as a first approximation. So, electrons in metals can be treated as an ideal gas. ( cf. Fermi liquid )

But, their density is so high that they can not be described by classical statistics.

$$\left\{\begin{array}{l} \bar{n}_s = \dfrac{1}{e^{\beta(\epsilon_s-\mu)}+1} \\ N = \sum_s \bar{n}_s \end{array}\right.$$

"Fermi function" $F(\epsilon) = \dfrac{1}{e^{\beta(\epsilon-\mu)}+1}$




$\Rightarrow$ $\mu$ is called as the Fermi energy in this case.

※ note : $\left\{\begin{array}{l} \epsilon \ll \mu \Rightarrow F(\epsilon) = 1 \\ \epsilon \gg \mu \Rightarrow F(\epsilon) = e^{\beta(\mu-\epsilon)} \end{array}\right.$

At $T=0$, each state with $\epsilon < \mu$ is occupied by one electron

( ~~Pat~~ Pauli exclusion )

◦ Fermi energy $\mu = \mu_0$ at $T=0$

$$\epsilon = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} . \qquad (\vec{p} = \hbar \vec{k})$$

$$\mu_0 \equiv \frac{p_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m}$$ → at $T=0$, all states with $k < k_F$ are filled.

$k > k_F$ empty.

the total number of states below $\mu_0$ :

To count the number of states, let's suppose that our system is translationally invariant.

$$\psi(x+L_x, y, z) = \psi(x, y, z)$$
$$\psi(x, y+L_y, z) = \psi(x, y, z)$$
$$\psi(x, y, z+L_z) = \psi(x, y, z)$$

$$\rightarrow \psi = e^{i\vec{k}\cdot\vec{r}}, \qquad k_x(x+L_x) = k_x x + 2\pi n_x$$

$$\Rightarrow k_x = \frac{2\pi}{L_x} n_x, \quad k_y = \frac{2\pi}{L_y} n_y, \quad k_z = \frac{2\pi}{L_z} n_z .$$

$$\Rightarrow \Delta n_x = \frac{L_x}{2\pi} dk_x$$

$$\Rightarrow \sum_{|\vec{k}|<k_F} 1 = \frac{L_x L_y L_z}{(2\pi)^3} \int_{|\vec{k}|<k_F} d\vec{k} = \frac{V}{(2\pi)^3} \frac{4}{3}\pi k_F^3$$

By taking into account of spin degeneracy,

$$N = \underbrace{2}_{} \cdot \frac{V}{(2\pi)^3} \cdot \frac{4}{3}\pi k_F^3$$

$$\Rightarrow k_F = \left(3\pi^2 \frac{N}{V}\right)^{1/3} \qquad \lambda_F = 2\pi / k_F$$

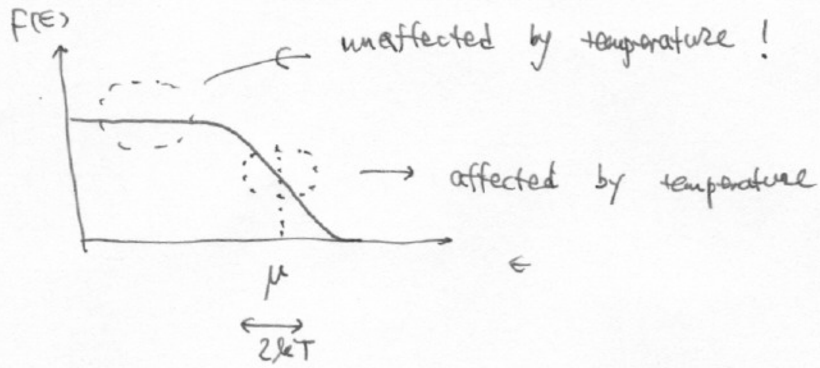
$$\mu_0 = \frac{\hbar^2}{2m} k_F^2 = \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V}\right)^{\frac{2}{3}}$$

◎ electronic specific heat

<u>estimation</u> : $C_V = \left( \frac{\partial \bar{E}}{\partial T} \right)_V$

classically , $\bar{E} = \frac{3}{2} NkT$ , $C_V = \frac{3}{2} Nk$ .

But , this is not the case in electronic systems .




$\Rightarrow N_{eff} \cong N \times \frac{kT}{\mu}$

$\rightarrow C_V^{(e)} \simeq \frac{3}{2} N_{eff}\, k \propto T.$

$C_V = \underbrace{C_V^{(e)}}_{\text{electron contribution}} + \underbrace{C_V^{(L)}}_{\text{phonon contribution}}$

$C_V^{(e)} = \gamma T .$ $\quad \underbrace{C_V^{(L)} = AT^3}$

We will see this later.

<u>detailed calculations</u> ( $\beta\mu \gg 1$ limit )

$$\bar{E} = \sum_r \frac{\epsilon_r}{e^{\beta(\epsilon_r - \mu)} + 1} = \underbrace{2}_{\text{spin degeneracy}} \int F(\epsilon)\, \epsilon\, \underbrace{\rho(\epsilon)}_{\text{densities of states}} d\epsilon$$

$$= 2 \int_0^\infty \frac{\epsilon}{e^{\beta(\epsilon - \mu)} + 1} \rho(\epsilon)\, d\epsilon$$

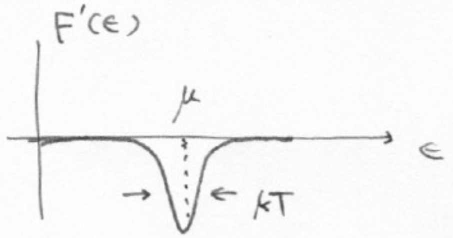
$$N = 2 \int F(\epsilon)\, \rho(\epsilon)\, d\epsilon$$

We will evaluate the integral of the form $\int_0^\infty F(\epsilon)\,\varphi(\epsilon)\,d\epsilon$

$$\int_0^\infty F(\epsilon)\varphi(\epsilon)\,d\epsilon = \cancel{F(\epsilon)\psi(\epsilon)\Big|_0^\infty} - \int_0^\infty F'(\epsilon)\psi(\epsilon)\,d\epsilon$$

$$\psi(\epsilon) = \int_0^\epsilon \varphi(\epsilon')\,d\epsilon' \qquad (\because\ F(\epsilon\to\infty)=0,\ \ \psi(\epsilon\to 0)=0)$$

Observe :




$\Rightarrow$ expand $\psi(\epsilon)$ near $\epsilon=\mu$

$$\psi(\epsilon) = \psi(\mu) + \frac{d\psi}{d\epsilon}\Big|_\mu (\epsilon-\mu) + \frac{1}{2}\frac{d^2\psi}{d\epsilon^2}\Big|_\mu (\epsilon-\mu)^2 + \cdots$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\left[\frac{d^m\psi}{d\epsilon^m}\right]_\mu (\epsilon-\mu)^m$$

$$\int_0^\infty F'(\epsilon)\,(\epsilon-\mu)^m\,d\epsilon = -\int_0^\infty \frac{\beta e^{\beta(\epsilon-\mu)}}{(e^{\beta(\epsilon-\mu)}+1)^2}\,(\epsilon-\mu)^m\,d\epsilon$$

$$= -\beta^{-m}\int_{-\beta\mu}^{\infty} \frac{x^m e^x}{(e^x+1)^2}\,dx \qquad x=\beta(\epsilon-\mu)$$

$\boxed{\beta\mu \gg 1 \text{ limit}}$

$$= -\beta^{-m} I_m$$

$$I_m \equiv \int_{-\infty}^{\infty} \frac{e^x x^m}{(e^x+1)^2}\,dx = \int_{-\infty}^{\infty} \frac{x^m}{(e^x+1)(e^{-x}+1)}\,dx$$

$$\begin{cases} I_0 = 1 \\ I_m = 0 \quad \text{if } m \text{ is odd} \end{cases}$$

$$\Rightarrow \int_0^\infty F(\epsilon)\varphi(\epsilon)\,d\epsilon = \psi(\mu) + I_2\frac{(kT)^2}{2}\left[\frac{d^2\psi}{d\epsilon^2}\right]_\mu + \cdots$$

$$I_2 = \frac{\pi^2}{3}$$ (see Problems 9.26, 9.27)

$$\Rightarrow \bar{E} = 2\int_0^{\mu} \epsilon\rho(\epsilon)\, d\epsilon + \frac{\pi^2}{3}(kT)^2 \left[\frac{d}{d\epsilon}(\epsilon\rho)\right]_{\mu}$$

$$2\int_0^{\mu} \epsilon\rho(\epsilon)\, d\epsilon = \underbrace{2\int_0^{\mu_0} \epsilon\rho(\epsilon)\, d\epsilon}_{\equiv \bar{E}_0} + 2\int_{\mu_0}^{\mu} \epsilon\rho(\epsilon)\, d\epsilon$$

$$\cong \bar{E}_0 + 2\mu_0\rho(\mu_0)(\mu-\mu_0)$$

$$\frac{d}{d\epsilon}(\epsilon\rho) = \rho + \epsilon\rho'.$$

$$\Rightarrow \bar{E} = \bar{E}_0 + 2\mu_0\rho(\mu_0)(\mu-\mu_0) + \frac{\pi^2}{3}(kT)^2 \left[\rho(\mu_0) + \mu_0\rho'(\mu_0)\right]$$

$$N = 2\int_0^{\mu} \rho(\epsilon)\, d\epsilon + \frac{\pi^2}{3}(kT)^2\rho'(\mu)$$

( from $N = 2\int f(\epsilon)\rho(\epsilon)\, d\epsilon$ )

$$= \underbrace{2\int_0^{\mu_0} \rho(\epsilon)\, d\epsilon}_{=N} + 2\int_{\mu_0}^{\mu} \rho(\epsilon)\, d\epsilon + \frac{\pi^2}{3}(kT)^2\rho'(\mu)$$

$$= N + 2\rho(\mu_0)(\mu-\mu_0) + \frac{\pi^2}{3}(kT)^2\rho'(\mu)$$

$$\Rightarrow (\mu-\mu_0) = -\frac{\pi^2}{6}(kT)^2\frac{\rho'(\mu_0)}{\rho(\mu_0)}$$

$$\Rightarrow \bar{E} = \bar{E}_0 - \frac{\pi^2}{3}(kT)^2\mu_0\rho'(\mu_0) + \frac{\pi^2}{3}(kT)^2\left(\rho(\mu_0) + \mu_0\rho'(\mu_0)\right)$$

$$= \bar{E}_0 + \frac{\pi^2}{3}(kT)^2\rho(\mu_0)$$

$$\Rightarrow C_V^{(e)} = \frac{\partial \bar{E}}{\partial T} = \frac{2\pi^2}{3}k^2\rho(\mu_0)\, T$$

$$\rho(\epsilon)\, d\epsilon = \frac{V}{(2\pi)^3}\cdot 4\pi k^2 \frac{dk}{d\epsilon}\, d\epsilon = \frac{V}{4\pi^2}\frac{(2m)^{3/2}}{\hbar^3}\sqrt{\epsilon}\, d\epsilon$$

$$\mu_0 = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{\frac{2}{3}}$$

$$\Rightarrow \rho(\mu_0) = V\frac{m}{2\pi^2\hbar^2}\left(3\pi^2\frac{N}{V}\right)^{\frac{1}{3}} = \frac{3}{4}\frac{N}{\mu_0}$$

$$\Rightarrow C_V = \frac{\pi^2}{2}kN\frac{kT}{\mu_0}$$

⊙ Ideal Bose gas : Bose - Einstein Condensation

(※ Classical ~~Blot~~ Boltzmann gas ⟵ Bose gas at high temperature (low density)

$$\frac{N}{V}\left(\frac{h^2}{2\pi m k T}\right)^{3/2} \ll 1$$

$$\frac{N}{V}\lambda^3 \ll 1$$

$$\ln \mathcal{Z}(T, V, \mu) = -\sum_k \ln\left(1 - \exp(-\beta(\epsilon_k - \mu))\right)$$

$$N = \sum_k \frac{1}{e^{-\beta(\epsilon_k - \mu)} - 1}$$

To evalute the above summations,

$$\sum_k \longrightarrow \frac{V}{(2\pi)^3}\int d^3k = \frac{2\pi V}{h^3}(2m)^{3/2}\int \sqrt{\epsilon}\, d\epsilon$$

$$\left(\epsilon_k = \frac{(\hbar k)^2}{2m}\right)$$

$$= \int \rho(\epsilon)\, d\epsilon$$

Note : $\beta(\epsilon_k - \mu) \geq 0$ for all $k$ $\Rightarrow$ $\mu \leq 0$ , $e^{\beta\mu} \in [0,1]$

BUT, this approximation is very bad for small energies $\epsilon \to 0$.

Notice : In a box with volume $V$ (periodic boundary condition),

there exists a state with $\epsilon = 0$ , $\psi_0(\vec{r}) = \frac{1}{\sqrt{V}}$ .

This state is not counted in the approximation.

One can not neglect this special state with the lowest energy.

$$\Rightarrow \ln \mathcal{Z}(T, V, \mu) = -\int_0^\infty \rho(\epsilon)\, d\epsilon \ln\left(1 - e^{-\beta(\epsilon - \mu)}\right) \underbrace{- \ln\left(1 - e^{\beta\mu}\right)}_{\text{contributed from } \psi_0}$$

$$= \frac{2\pi V}{h^3}(2m)^{3/2}\,\frac{2}{3}\beta \int_0^\infty d\epsilon\, \frac{\epsilon^{3/2}}{e^{\beta(\epsilon - \mu)} - 1} - \ln\left(1 - e^{\beta\mu}\right)$$

$$N(T, V, \mu) = \frac{2\pi V}{h^3}(2m)^{3/2}\int_0^\infty d\epsilon\, \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon - \mu)} - 1} + \frac{1}{e^{-\beta\mu} - 1}$$

$$N_0 = \frac{1}{e^{-\beta\mu} - 1}$$

Let's denote $z_f \equiv e^{+\beta\mu}$, which is called as "fugacity"

( use $g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\,dx}{z_f^{-1} e^x - 1}$, $0 \le z_f \le 1$

$n \in \mathbb{R}$

$x \leftarrow \beta\epsilon$ $\qquad \Gamma(n) \sim$ gamma function.

$$\Rightarrow \ln \mathcal{Z}(T.V.\mu) = \frac{V}{\lambda^3} g_{5/2}(z_f) - \ln(1 - z_f)$$

$$\Big( N(T.V.\mu) = \frac{V}{\lambda^3} g_{3/2}(z_f) + N_0(z_f)$$

$$\lambda \equiv \sqrt{\frac{h^2}{2\pi m k T}} \qquad \left(\frac{h^2}{2m}\frac{1}{\lambda}\right)^2 = \frac{h^2}{2m} \cdot \frac{2\pi m k T}{h^2} = \pi k T \Big)$$

$$\frac{1}{z_f^{-1}\exp(x) - 1} = z_f \exp(-x) \frac{1}{1 - z_f e^{-x}}$$

$$= \sum_{k=1}^{\infty} z_f^k \exp(-kx)$$

note: $z^{-1}\exp x = e^{\beta(\epsilon-\mu)} \gtrsim 1$

$z \in [0.1]$

$$\Rightarrow g_n(z_f) = \frac{1}{\Gamma(n)} \sum_{k=1}^{\infty} z_f^k \int_0^\infty x^{n-1} e^{-kx}\,dx$$

$$= \frac{1}{\Gamma(n)} \sum_{k=1}^{\infty} \frac{z_f^k}{k^n} \underbrace{\int_0^\infty y^{n-1} e^{-y}\,dy}_{=\Gamma(n)}$$

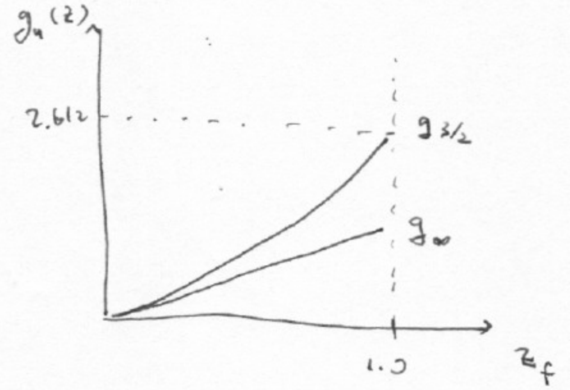
$$= \sum_{k=1}^{\infty} \frac{z_f^k}{k^n}, \quad z_f \in [0.1]$$

( $g_n(z_f)$ is finite for $n > 1$ for $z_f \in [0.1]$.

for $z_f = 1$ ($\mu = 0$), $g_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n)$ . $n > 1$

$\rightarrow$ Riemann's Zeta function $\zeta(n)$

$\zeta(1) \rightarrow \infty$. $\zeta(3/2) \simeq 2.612$



Study first $\quad N(T, V, \mu) = \underbrace{\frac{V}{\lambda^3} g_{3/2}(z_f)} + N_0(z_f).$

$$= \underbrace{N_\epsilon} + N_0$$

$$N_\epsilon^{max} = \frac{V}{\lambda^3} g_{3/2}(z_f = 1)$$

$$= \frac{V}{\lambda^3} \zeta\left(\tfrac{3}{2}\right)$$

Let's consider the thermodynamic limit ($N \to \infty$, $V \to \infty$, $N/V$ = constant).

$$N = N_\epsilon + N_0$$

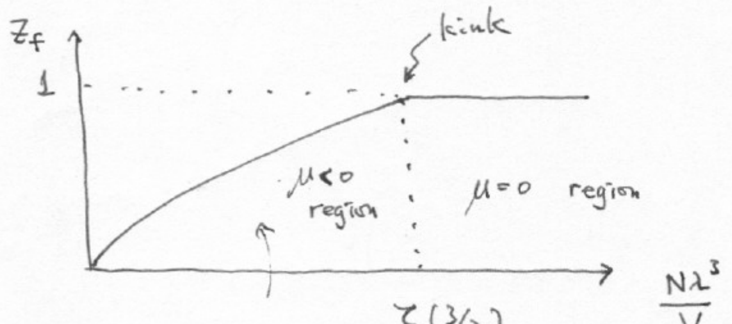
$z_f \neq 1 \quad \rightarrow \quad N_0/N \to 0$, all particles occupy excited states.

$z_f = 1 \quad \rightarrow \quad N_\epsilon = N_\epsilon^{max}$. $\quad N_0 = N - N_\epsilon^{max}$

Cross-over occurs at

$$N = N_\epsilon^{max} = \frac{V}{\lambda^3} \zeta(3/2)$$

$$\left(\because \; z \neq 1 \Rightarrow N = \frac{V}{\lambda^3} g_{3/2}(z_f) \right)$$




Notice $z = e^{+\beta\mu}$

Solution of

$$N = \frac{V}{\lambda^3} g_{3/2}(z_f)$$

(※ For finite $N$, $z_f$ is determined by

$N = N_\epsilon(z_f) + N_0(z_f)$. $\Rightarrow$ No kink.)

(But smooth curve.

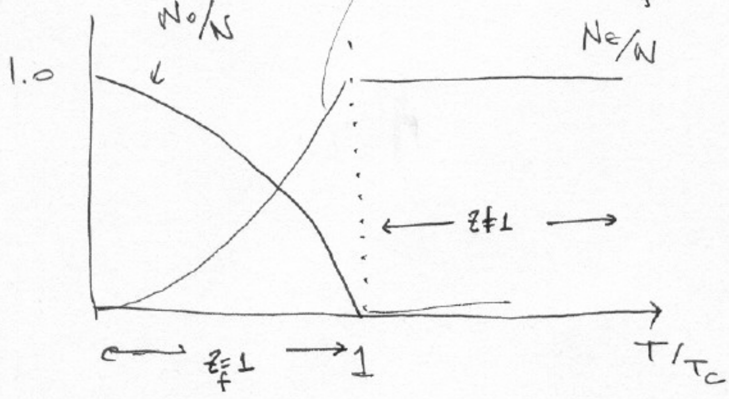
$$\frac{N\lambda^3}{V} > \zeta(3/2) \quad \Rightarrow \quad T < T_c \quad , kT_c = \left(\frac{N}{V}\right)^{2/3} \frac{h^2}{2\pi m (\zeta(3/2))^{2/3}}$$

$(N > N_\epsilon^{max})$

$$\frac{N_\epsilon^{max}}{N} = \left(\frac{T}{T_c}\right)^{3/2}$$

$$\left( \frac{V}{\lambda^3} g_{3/2}(z_f) = N_\epsilon^{max},\quad z_f = 1 \right) \Rightarrow \left(\frac{T}{T_c}\right)^{3/2}$$

$$\frac{N_0}{N} = 1 - \frac{N_\epsilon}{N}$$




$\hookrightarrow$ Bose - Einstein condensation. (Example: $^4He$)

* pressure:

$$E = TS - pV + \mu N$$

$$\Rightarrow -\frac{1}{\beta} \ln \mathcal{Z} = E - TS - \mu N = -pV$$

$$\ln \mathcal{Z} = \frac{V}{\lambda^3} g_{5/2}(z_f) - \ln(1 - z_f)$$

$$\Rightarrow p = \frac{kT}{\lambda^3} g_{5/2}(z_f) - \cancel{\underbrace{\frac{kT}{V} \ln(1 - z_f)}}$$

as $V \to \infty$, $N \to \infty$, $N/V$ = constant

if $z_f < 1$.

$$N_0 = \frac{z_f}{1 - z_f} \rightarrow 1 - z_f = \frac{1}{N_0 + 1}$$

$$\frac{\ln(1 - z_f)}{V} \sim \frac{\ln(N+1)}{V} \sim \frac{\ln N}{V} \to 0$$

as $V \to \infty$, $N \to \infty$, $N/V$ = constant.

$\Rightarrow \psi_0$ does not contribute to pressure.

$\Leftrightarrow$ $\psi_0$ has no kinetic energy.

at $T < T_c$,

$$p = \frac{kT}{\lambda^3} \zeta(5/2) \qquad (z_f = 1)$$

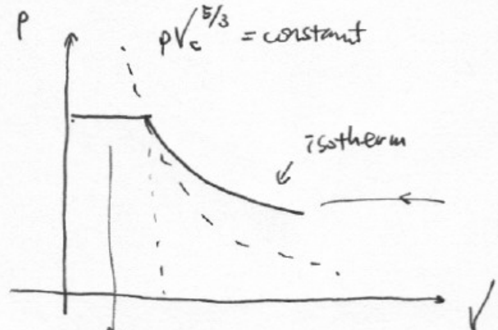
~ independent of $V$, $N$.

$\because$ If one adds one more particles into the system with $T < T_c$, it would sit in $\psi_0$.

Fix temperature and change volume $V$:

$$\left\{ \begin{aligned} \frac{N\lambda^3}{V_c} &= \zeta\left(\tfrac{3}{2}\right) \qquad \Rightarrow \quad V_c = \frac{N\lambda^3}{\zeta(3/2)} \\ p &= \frac{kT}{\lambda^3}\,\zeta\left(\tfrac{5}{2}\right) \end{aligned} \right.$$

By eliminating $T$, $\quad p V_c^{5/3} = \text{constant}$




$$p = \frac{kT}{\lambda^3}\, g_{5/2}(z_f)$$

$$p = \frac{kT}{\lambda^3}\, g_{5/2}(z_f = 1) = \frac{kT}{\lambda^3}\,\zeta(5/2)$$

We already know $z_f(T, V, \mu)$ from the equation of

$$N = \sum_r \frac{1}{e^{+\beta(\epsilon_r - \mu)} - 1}$$

※ classical limit: $\quad \dfrac{N\lambda^3}{V} \ll 1$

$$\frac{N\lambda^3}{V} = g_{3/2}(z) \quad \text{and} \quad g_{3/2}(z) \simeq z_f \quad \text{for } z_f \ll 1$$

$$\left( g_{3/2}(z_f) = z_f + \frac{z_f^2}{2^{3/2}} + \cdots \right)$$

$$\Rightarrow \frac{N\lambda^3}{V} = z_f \qquad g_{5/2}(z_f) = z_f + \frac{z_f^2}{2^{5/2}} + \cdots$$

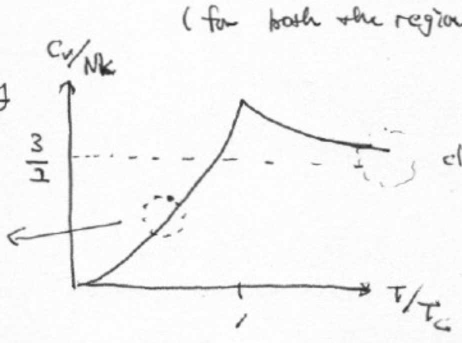
$$p = \frac{kT}{\lambda^3}\, g_{5/2}(z_f) \approx \frac{kT}{\lambda^3}\, z_f \simeq \frac{kT}{\lambda^3}\cdot\frac{N\lambda^3}{V} = \frac{1}{V} NkT.$$

* energy: $\quad E = -\dfrac{\partial}{\partial\beta}\ln \mathcal{Z} = \dfrac{3}{2}kT\,\dfrac{V}{\lambda^3}\, g_{5/2}(z_f)$

$$\Rightarrow \bar{E} = \frac{3}{2}pV \quad \longrightarrow \text{classical limit} \quad \bar{E} = \frac{3}{2}NkT$$

(for both the regions of $z_f = 1$ and $z_f \neq 1$)

* heat capacity



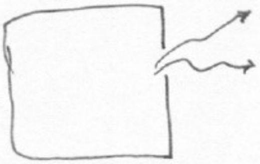

classical limit $\quad C_V \to \frac{3}{2}Nk$

$\rightarrow$ "$\lambda$-curve"

$$\frac{\partial}{\partial T}\bar{E}(z_f = 1) \propto \frac{\partial}{\partial T}\left(\frac{T}{\lambda^3}\right)$$

BE condensation does not occur in 1D and 2D systems without any bound states.

⊛ photon : black-body radiation



The atoms in the walls of the cavity will constantly emit and absorb electromagnetic radiation.

Cavity : volume $V$

The wall of cavity is heated to a given temperature $T$.

· quantize EM field $\Rightarrow$ photon.

· the total # of photons inside cavity (even in closed cavity) is NOT conserved.

$\Rightarrow$ Using canonical partition function.

$$Z = \sum_{n_1, n_2 \cdots} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}$$

← unrestricted summation

$$= \left(\sum_{n_1=0}^{\infty} e^{-\beta n_1 \epsilon_1}\right)\left(\sum_{n_2=0}^{\infty} e^{-\beta n_2 \epsilon_2}\right) \cdots$$

$$= \prod_{r=1} \frac{1}{1 - e^{-\beta\epsilon_r}}$$

$e^{-\beta\epsilon_r} < 1$

$(\epsilon_r > 0)$

$$\ln Z = -\sum_r \ln(1 - e^{-\beta\epsilon_r})$$

$$\bar{n}_s = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_s} = \frac{1}{e^{\beta\epsilon_s} - 1}$$

electric field $\vec{E}$

$$
\begin{cases}
\nabla^2 \vec{E} = \dfrac{1}{c^2}\dfrac{\partial^2 \vec{E}}{\partial t^2} \\
\nabla \cdot \vec{E} = 0
\end{cases}
$$

$$\vec{E} = \vec{A}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

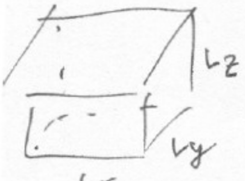
$\Rightarrow |\vec{k}| = \omega / c$ , $\underline{\vec{k}\cdot\vec{E} = 0}$ $\rightarrow$ for given $\vec{k}$,

there are two possible direction of $\vec{E}$. (polarization)

$\Rightarrow$ # of degeneracy $= 2$

quantization

$$
\begin{cases}
\epsilon = \hbar\omega \\
\vec{p} = \hbar\vec{k}
\end{cases}
$$




$L \gg \lambda = 2\pi / |\vec{k}|$

$$\sum_n \rightarrow \frac{V}{(2\pi)^3}\int d\vec{k}^3$$

$$\Rightarrow \underline{f(\vec{k})\, d^3\vec{k}} = \frac{d\vec{k}^3}{(2\pi)^3} \cdot \frac{1}{e^{\beta\hbar\omega} - 1}$$

mean # of photons (per unit volume) with $\vec{k} \in [\vec{k}, \vec{k} + d\vec{k})$

$\Rightarrow$ mean # of photons ( " ) of both direction of polarization

with $\omega \in [\omega, \omega + d\omega]$

$$= 2 \times f(k)\, 4\pi k^2 dk = \frac{8\pi}{(2\pi c)^3}\frac{\omega^2 d\omega}{e^{\beta\hbar\omega} - 1}$$

$$\bar{E} = \int \hbar\omega \cdot 2 f(k)\, 4\pi k^2 dk$$

$$= \frac{\hbar}{\pi^2 c^3}\left(\frac{kT}{\hbar}\right)^4 \int_0^\infty \frac{\eta^3 d\eta}{e^{\eta} - 1} \qquad \eta = \beta\hbar\omega$$

$\propto T^4$ ~ Stefan – Boltzmann law.

* radiation pressure

$$\bar{p} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V}$$

$$= \sum_s \bar{n}_s \left(-\frac{\partial \epsilon_s}{\partial V}\right)$$

$$\epsilon_s = \hbar\omega = \hbar c k = \hbar c\sqrt{k_x^2+k_y^2+k_z^2} = \hbar c\left(\frac{2\pi}{L}\right)\left(n_x^2+n_y^2+n_z^2\right)^{1/2}$$

$$= \boxed{\ } \, V^{-\frac{1}{3}}$$

$$\frac{\partial \epsilon_s}{\partial V} = -\frac{1}{3}\boxed{\ } \, V^{-\frac{4}{3}} = -\frac{1}{3}\frac{\epsilon_s}{V}$$

$$\Rightarrow \quad \bar{p} = \sum_s \bar{n}_s \epsilon_s \left(+\frac{V}{3}\right) = \frac{\bar{E}}{3V}$$

$$= \frac{1}{3}\bar{u}_0$$

$\bar{u}_0 \equiv \dfrac{\bar{E}}{V}$ ~ mean energy density.