3) Gaussian distribution : distribution for large N

Start with the binomial distribution $W_N(n_1)$

and study the $N \to \infty$ limit.

At $\tilde{n}_1$: where $W_N(n_1)$ has a maximum

Taylor's expansion around $\tilde{n}_1$

$$\ln W(n_1) = \ln W(\tilde{n}_1) + B_1 \eta + \frac{1}{2} B_2 \eta^2 + \frac{1}{6} B_3 \eta^3 + \cdots$$

$$B_k = \frac{d^k \ln W_N(n_1)}{d n_1^k}$$

$$\eta = n_1 - \tilde{n}_1$$

※ $W(n_1)$ is a rapidly varying function in $N \to \infty$ limit. So we study $\ln W(n_1)$.

$B_1 = 0, \quad B_2 < 0.$

Use Stirling formula : $\ln n! \simeq n \ln n - n$

for large n.

$$\Rightarrow \tilde{n}_1 = Np \quad \left(\Leftarrow \left. \frac{d \ln W}{d n_1} \right|_{\tilde{n}_1} = 0 \right)$$

$$B_2 = -\frac{1}{Npq}, \quad B_3 = \frac{1}{N^2 p^2} - \frac{1}{N^2 q^2}$$

$$\Rightarrow |B_2| \eta^2 \sim O(1) \quad : \quad \eta \lesssim \sqrt{Npq}$$

$$B_3 \eta^3 \sim \frac{1}{N^2} \cdot N^{3/2} \sim \frac{1}{\sqrt{N}} \to 0 \quad \text{as } N \to \infty$$

$$\left(\text{or} \quad \left| B_3 \eta^3 / B_2 \eta^2 \right| \sim \frac{1}{N} \eta \sim \frac{1}{\sqrt{N}} \to 0 \right)$$

for $\eta \lesssim \sqrt{Npq}$

$$\Rightarrow W_{\text{Gaussian}}(n_1) = \sqrt{\frac{|B_2|}{2\pi}} \, e^{-\frac{1}{2}|B_2|(n_1 - \tilde{n}_1)^2}$$

$$\tilde{n}_1 = Np, \quad B_2 = -\frac{1}{Npq}$$

Central Limit Theorem (1D)

s_i : ith step (event) — step in random walk / spin direction in a set of $N \to \infty$ spins

$p(s_i)\,ds_i$: probability that the ith ~~step~~ displacement $\in [s_i, s_i+ds_i]$

$P(x)\,dx$: " that the total displacement x after N steps $\in [x, x+dx]$

If (i) $N \to \infty$

(ii) each step is statistically independent

(iii) $p(s)$ falls off very rapidly as $|s| \to \infty$

Then, $P(x)$ can be described by Gaussian distribution.

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proof.

$$x = \sum_{i=1}^{N} s_i$$

$$p(s_1, \cdots, s_N) = \prod_{j=1}^{N} p(s_j) \quad (\because \text{ statistically independent})$$

$$P(x) = \int \prod_{j=1}^{N} ds_j \; p(s_1 \cdots, s_N)\; \delta\left(x - \sum_{i=1}^{N} s_i\right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\; e^{ikx} \{Q(k)\}^N$$

$$Q(k) = \int ds\; e^{-iks}\, p(s)$$

$$= 1 - ik\langle s\rangle + \frac{1}{2!}(-ik)^2\langle s^2\rangle + \frac{1}{3!}(-ik)^3\langle s^3\rangle + \cdots$$

$\langle s^n\rangle \sim$ finite $\quad (\because |p(s)| \to 0$ as $|s| \to \infty)$

$$\delta(x) = \frac{1}{2\pi}\int_{-n}^{n} dk\; e^{ikx}, \quad n \to \infty$$

$$= \frac{\sin nx}{\pi x}$$

$$\int dx\, \delta(x) = 1$$

$$\int dx\, f(x)\,\delta(x) = f(0)$$

$\boxed{\text{Lorentzian}}$

$$\ln (Q(k))^N = N \ln Q(k)$$

$$= N\left(-ik\langle s\rangle - \left(\frac{k}{2}\right)^2(\langle s^2\rangle - \langle s\rangle^2) + \cdots\right)$$

$$\Rightarrow Q(k) = e^{-ik\langle s\rangle}\, e^{-k^2/2\,(\langle s^2\rangle-\langle s\rangle^2)}\, e^{O(k^3)}$$

$$P(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ik(x-N\langle s\rangle) - \frac{k^2}{2}(\langle s^2\rangle - \langle s\rangle^2)N}\; \cancel{e^{O(k^3)N}}$$

($\because$ In the range of k where $k^2N \lesssim O(1)$,

$k^3 N \sim \frac{1}{\sqrt{N}} \to 0$)

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}$$

$$\mu = N\bar{s}\ ,\quad \sigma^2 = N\overline{(\Delta s)^2}$$

< the most probable distance

$$\int_{-\infty}^{\infty}\cancel{x^2}(x-\mu)^2 P(x)\,dx = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} x^2 e^{-x^2/2\sigma^2}\,dx$$

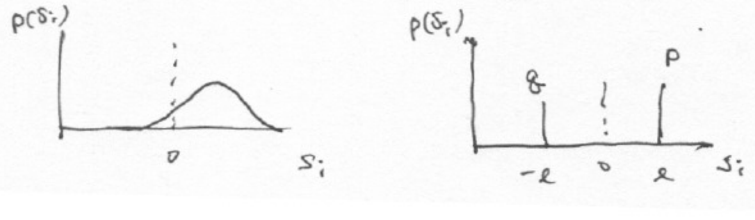
$$= \sqrt{\frac{a}{\pi}}\int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx$$

$$= \sqrt{\frac{a}{\pi}}\left(-\frac{d}{da}\right)\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{a}{\pi}}\cdot\left(\frac{d}{-da}\right)\sqrt{\frac{\pi}{a}}$$

$$= \frac{1}{2a} = \sigma^2$$

CLT →

※ quite general



(4) Poisson distribution

Start with the binomial $W_N(n_1)$

and study $p \to 0$ limit.

(ex) trick coins, radioactive decay of metastable nuclei (see the problem 1.12), vacuum diode

$$W_N(n_1) = \frac{N!}{n_1!\, n_2!}\, p^{n_1} q^{n_2}$$

$$p \to 0 \quad \Rightarrow \quad n_1 \ll N. \quad n_2 \approx N.$$

$$\frac{N!}{n_2!} = N(N-1)(N-2)\cdots(N-n_1+1) \simeq N^{n_1}$$

$$P(n_1) = \frac{1}{n_1!}(Np)^{n_1} q^{n_2}$$

$$\simeq \frac{1}{n_1!}(Np)^{n_1}(1-p)^N$$

$\lambda = Np$

$$= \frac{\lambda^{n_1}}{n_1!}\left(1-\frac{\lambda}{N}\right)^N = \frac{\lambda^{n_1}}{n_1!} e^{-\lambda}$$

$\sum P(n_1) = 1$ (still ~~sto~~ satisfied!)

Alternative derivation: Tunneling problem (vacuum diode)

find the probability that n_1 particles tunnel out after time t.

τ: mean time interval between tunneling.

$$P_{n_1}(t+dt) = P_{n_1-1}(t)\frac{dt}{\tau} + P_{n_1}(t)\left(1-\frac{dt}{\tau}\right)$$

$$\Rightarrow \frac{dP_{n_1}(t)}{dt} = \left(P_{n_1-1}(t) - P_{n_1}(t)\right)\frac{1}{\tau}$$

use $\left(P_{n_1} = \Pi_{n_1}(t)\, e^{-t/\tau}\right)$ & $\left(\frac{dP_0}{dt} = -\frac{P_0}{\tau}\right)$ $\Rightarrow$ $P_{n_1}(t)\ \frac{1}{n_1!}\left(\frac{t}{\tau}\right)^{n_1} e^{-t/\tau}$