Course #1. ~~Introduction~~ Statistics

(1) Definition of Statistics

(2) Binomial distribution / Random Walk

(3) Gaussian distribution / Central Limit Theorem

(4) Poisson distribution

* (5) Connection to Statistical Mechanics : multinomial distribution

* (6) Others

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(1) Definition of Statistics

a. ensemble : an assembly consisting of a very large # of similarly prepared systems
($\# \to \infty$)

b. probabilities : the fraction of systems in the ensemble which characterized by a given specified event.

c. statistics

u_i : a variable of the ith event

statistics $\equiv$ $\overline{u_i}$, $\overline{(\Delta u)^2} = \overline{(u_i - \bar{u})^2}$, ... $\overline{(\Delta u)^n}$

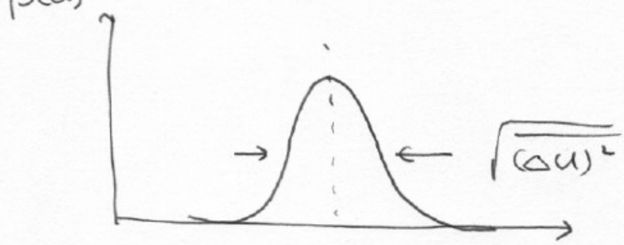
mean. dispersion

$$\bar{u} = \frac{\sum_{i=1}^{M} u_i \, P(u_i)}{\sum_{i=1}^{M} P(u_i)} \qquad , \qquad \sum_{i=1}^{M} P(u_i) = 1 \quad \text{(normalization)}$$

M: total # of accessible values of ~~the~~ u

In general, $\overline{f(u)} = \sum_i P(u_i) f(u_i)$

Knowledge of $P(u)$ $\equiv$ Knowledge of all moments



* provide an example : N ~~number~~ tosses of coin

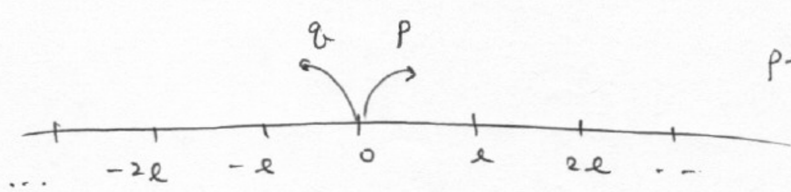
(p : probability having the head for each toss
(q : back

$p+q=1$

calculate the probability ~~of~~ that one has n_1 heads among N tosses. $\longrightarrow$ binomial distribution, random walk

(2) Binomial distribution : random walk

1D random walk :



$p+q=1$

Q. After N steps, what is the probability that the particle is located at $x = m\ell$?

$N = n_1 + n_2$ n_1: # of steps to the right

$m = n_1 - n_2$ n_2 left

Here, we have a fundamental assumption:

successive steps are statistically independent to each other (~~Markov process~~).

Answer:
$$W_N(n_1) = \frac{N!}{n_1!\, n_2!} p^{n_1} q^{n_2}$$

~ binomial distribution

check. $\sum_{n_1=0}^{N} W_N(n_1) = (p+q)^N = 1$.

($0 \le W_N(n_1) \le 1$

$$\rightarrow P_N(m) = W_N(n_1) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!} p^{\frac{N+m}{2}} q^{\frac{N-m}{2}}$$

. "Random walk" appears in nature:

(ex) Diffusion of a molecule in a gas

. Calculate moments:

$$\bar{n}_1 = \sum_{n_1=0}^{N} n_1 W_N(n_1) = p \frac{\partial}{\partial p} (p+q)^N = pN$$

similarly, $\overline{n_1^2} = \bar{n}_1^2 + Npq$

$$\overline{(\Delta n_1)^2} = Np(1-p)$$

$$\Rightarrow \bar{m} = \bar{n}_1 - \bar{n}_2 = (2p-1)N \quad \rightarrow 0 \text{ if } p = \tfrac{1}{2}$$

$$\overline{(\Delta m)^2} = 4Npq \quad \rightarrow N \text{ if } p = \tfrac{1}{2}$$

$$\sqrt{\overline{(\Delta m)^2}} = \sqrt{N} \quad \Rightarrow \text{1-D diffusion} \propto \sqrt{t}$$

※ Simple derivation of (diffusion $\propto \sqrt{t}$) in general d-dim.

$\vec{r}_n$: position after n steps from the origin

$(n \to \infty)$

$\vec{S}_n = \vec{r}_n - \vec{r}_{n-1}$

Assume : (i) a given step length $\lambda^2 = \langle \vec{S}_n^{\,2} \rangle$

(ii) a perfect random direction $\Rightarrow \langle \vec{S}_n \rangle = 0$

(iii) each step is uncorrelated with those preceding it.

$\Rightarrow$

$$\langle r_N^2 \rangle = \langle (\Sigma_n \vec{S}_n) \cdot (\Sigma \vec{S}_m) \rangle$$

$$= N\lambda^2 + \sum_{n \neq m} \underbrace{\langle \vec{S}_n \cdot \vec{S}_m \rangle}$$

$$= \langle \vec{S}_n \rangle \cdot \langle \vec{S}_m \rangle = 0$$

(∵ lack of correlation)

$$\Rightarrow R = \sqrt{\Delta r_N^2} \equiv \sqrt{\langle r_N^2 \rangle - \langle r_N \rangle^2} = \lambda \sqrt{N} \propto \sqrt{t}$$

(assuming constant magnitude of velocity)

classical diffusion $R \propto \sqrt{t}$

(cf) ballistic motion $R \propto t$