The geometry of cluster varieties II.

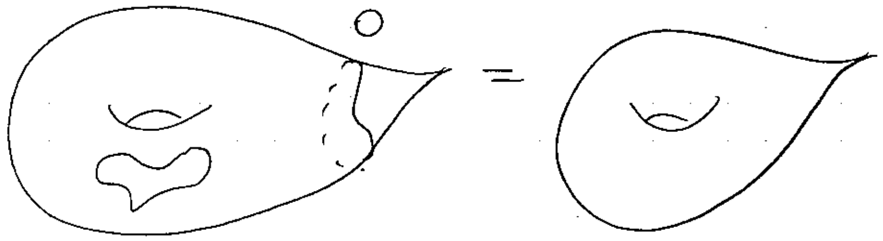
Laminations

S = decorated surface w/ no marked pts

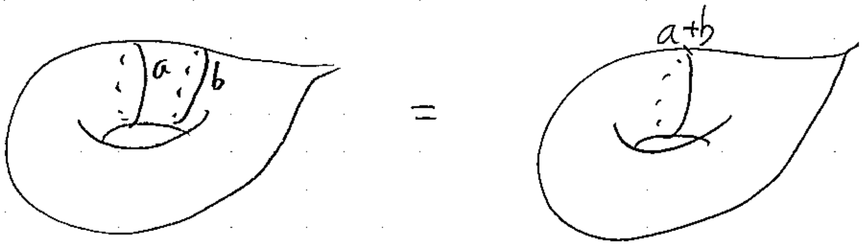
Def A (rational) bounded lamination on S is the homotopy class of a finite set of disjoint simple closed curves on S w/ rational weights, subject to

(1) weigt $< 0 \Rightarrow$ curve retracts to a puncture

(2) Can remove contractible and weight 0 curves:



(3) can "combine" homotopic curves:



Write $\mathcal{A}_L(S,\mathbb{Q}) = \{\text{rational bdd laminations}\}$

$\mathcal{A}_L(S,\mathbb{Z}) = \{ l \in \mathcal{A}_L(S,\mathbb{Q}) : l \text{ has integral weights} \}$

For $\lambda \in \mathcal{A}_L(S,\mathbb{Q})$ and $i \in I$, $= \{\text{edges of an ideal triangulation}\}$

let $a_i := \frac{1}{2}(\text{total weight of curves crossing } i)$

((when lamination crosses the edge in minimal # of pts))

<u>Prop</u> There is a bijection

$$\mathcal{A}_L(S,\mathbb{Q}) \xrightarrow{\cong} \mathbb{Q}^{|I|}$$

$$\lambda \longmapsto (a_i(\lambda))_{i \in I}$$

A (regular) flip changes the a_i to new coordinates

$$a_i' = \begin{cases} \max\left(\sum_{\varepsilon_{kj}>0} \varepsilon_{kj} a_j, \sum_{\varepsilon_{kj}<0} \varepsilon_{kj} a_j \right) - a_k & \text{if } i = k \\ a_i & \text{if } i \neq k \end{cases}$$

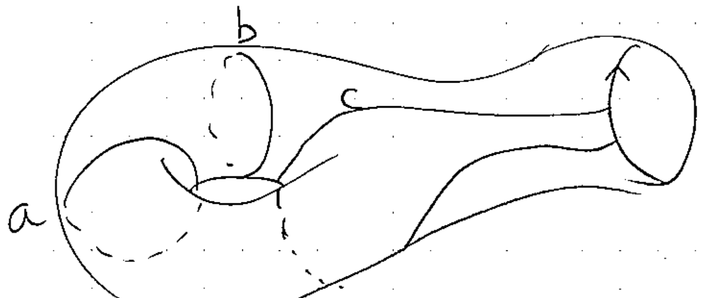
Remark This is a tropicalization of cluster transformation rules for coordinates on $\mathcal{A}^+(S)$, (decorated Teichmüller space)

i.e. $+ \rightsquigarrow \max$

$\cdot \rightsquigarrow +$

↑ first example

Def. A (rational) unbounded lamination on S is the homotopy class of a finite set of disjoint simple curves w/ positive rational weights



and an orientation for any component of ∂S that meets a curve. Moreover,

(1) Can remove contractible curves

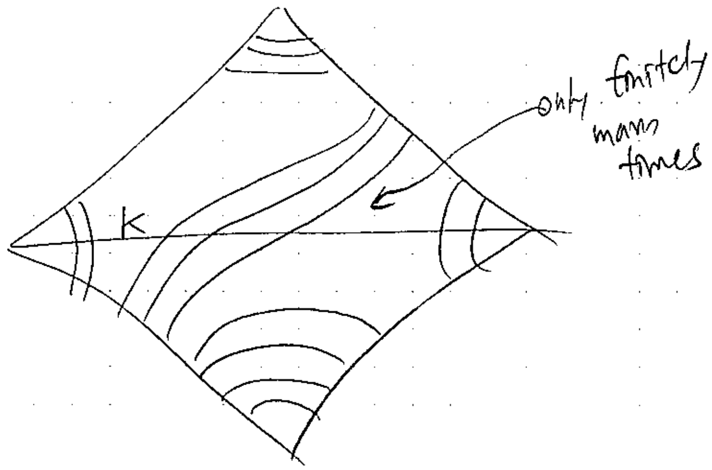
(2) Can "combine" homotopic curves

Write

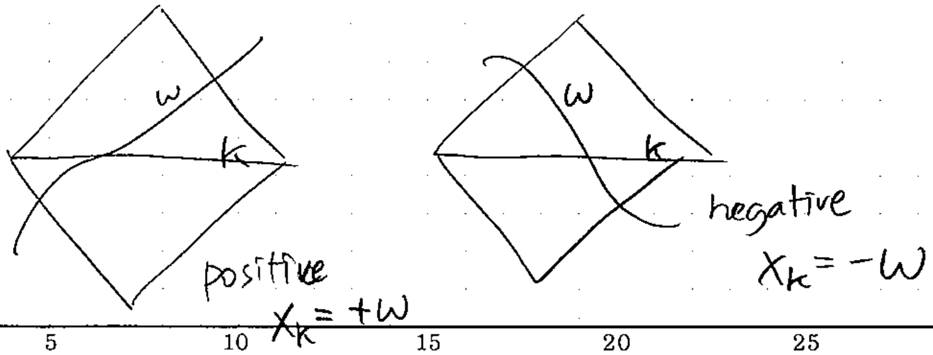
$$\mathcal{X}_L(S,\mathbb{Q}) = \{\text{rational unbounded laminations}\}$$

$$\mathcal{X}_L(S,\mathbb{Z}) = \{\ell \in \mathcal{X}_L(S,\mathbb{Q}) : \ell \text{ has integral weights}\}$$

Let $\lambda \in \mathcal{X}_L(S,\mathbb{Q})$, $k \in J$. Wind edges of triangulation infinitely many times around holes, get



Let x_k = total (signed) weight of curves crossing k diagonally



Prop. There is a bijection

$$\mathcal{X}_L(S,\mathbb{Q}) \xrightarrow{\cong} \mathbb{Q}^{|J|}$$

$$\lambda \longmapsto (x_j(\lambda))_{j\in J}$$

A (regular) flip changes the x_j to new coordinates

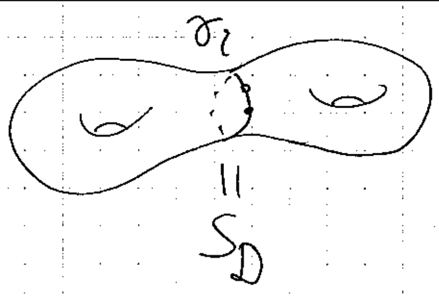
$$x_j' = \begin{cases} -x_k & \text{if } j=k \\ x_j + \varepsilon_{kj}\max(0, \operatorname{sgn}(\varepsilon_{kj})x_k) & \text{if } j\neq k. \end{cases}$$

Rmk This is a tropicalization of the transformation formula for $\mathcal{X}^+(S)$ (enhanced Teichmüller space)

second.

Def A (rational) D-lamination consists of

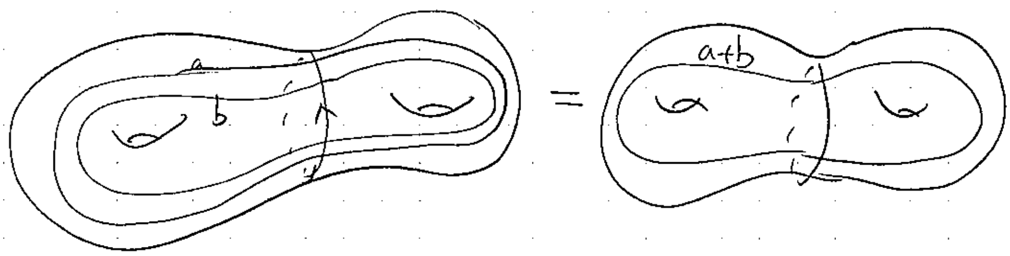
(1) the homotopy class of a finite set of disjoint simple nontrivial loops on S_D that do not retract to marked pts.



(2) A pos. weight for each curve

(3) An orientation for each γ_i that meets or is homotopic to a curve.

Can "combine" homotopic curves:

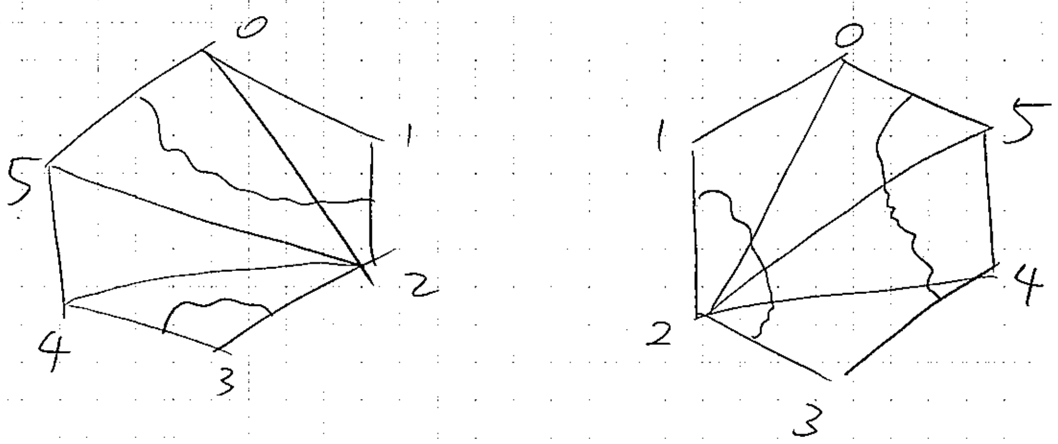


Write $\mathcal{D}_L(S,\mathbb{Q}) = \{\text{rational } \mathcal{D}\text{-laminations}\}$

$\mathcal{D}_L(S,\mathbb{Z}) = \{ l \in \mathcal{D}_L(S,\mathbb{Q}) : l \text{ has integral weights}\}$

<u>Example</u> S = disk w/ N marked pts,

$\lambda \in \mathcal{D}_L(S,\mathbb{Q})$. Cut along ∂S:



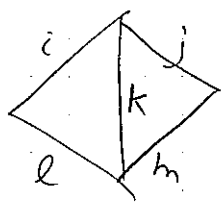
For $i \in I$, let

$$a_i := \frac{1}{2}\,(\text{number of intersections b/w } \lambda \text{ and } i)$$
(total wt)

$$a_i^\circ := \frac{1}{2}\,(\text{———— } \lambda \text{ and } i^\circ)$$

Then $b_k := a_k^\circ - a_k$, $x_k := a_i - a_j + a_m - a_l$

($k \in J$)



Proposition (A.) There is a bijection

$$\mathcal{D}_L(S, \mathbb{Q}) \xrightarrow{\cong} \mathbb{Q}^{2|J|}$$

$$\lambda \longmapsto (b_j(\lambda), x_j(\lambda))_{j \in J}$$

A (regular) flip at k changes the x_j, b_j to

(NP)

$$x_j' = \begin{cases} -x_k & \text{if } j=k \\ x_j + \varepsilon_{kj} \max(0, \operatorname{sgn}(\varepsilon_{kj}) x_k) & \text{if } j \neq k \end{cases}$$

$$b_j' = \begin{cases} \max\left(x_k + \sum_{\varepsilon_{ki}>0} \varepsilon_{ki} b_i, -\sum_{\varepsilon_{ki}<0} \varepsilon_{ki} b_i\right) - \max(0, x_k) - b_k & \text{if } j=k \\ b_j & \text{if } j \neq k \end{cases}$$

Rmk These tropicalize the formulas for $\mathcal{D}^+(S)$.

Cluster varieties

Def A seed consists of

(1) Set I (finite set)

(2) Subset $J \subseteq I$

(3) $\varepsilon_{ij} \in \mathbb{Z}$ $(i, j \in I)$ skew-symmetric matrix

For $\underline{\mathbf{i}} = (I, J, \varepsilon_{ij})$, put
(seed)

$$\mathcal{A}_{\underline{\mathbf{i}}} := \mathbb{G}_m^{|I|} \quad \text{(split algebraic torus)}$$

multiplicative gp

$$\mathcal{X}_{\underline{\mathbf{i}}} := \mathbb{G}_m^{|J|} \quad (\quad " \quad)$$

$$\mathcal{D}_{\underline{\mathbf{i}}} := \mathbb{G}_m^{2|J|} \quad (\quad " \quad)$$

$\{A_i\}_{i\in I}$, $\{X_j\}_{j\in J}$, $\{B_j, X_j\}_{j\in J}$ natural coordinates on $\mathcal{A}_{\underline{\mathbf{i}}}$, $\mathcal{X}_{\underline{\mathbf{i}}}$, $\mathcal{D}_{\underline{\mathbf{i}}}$.

<u>Def</u> $\underline{\mathbf{i}} = (I, J, \varepsilon_{ij})$ seed, $k \in J$. The seed $\mu_k(\underline{\mathbf{i}}) = (I', J', \varepsilon'_{ij})$ obtained by <u>mutation</u> has

$$I' = I, \quad J' = J,$$

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } k \in \{i, j\} \\ \varepsilon_{ij} + \dfrac{|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|}{2} & \text{if } k \notin \{i, j\} \end{cases}$$

(map b/w tori)

Define

$$\mu_k^* A_i' = \begin{cases} \dfrac{\prod_{\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}}}{A_k} & \text{if } i=k \\ A_i & \text{if } i\neq k \end{cases}$$

$$\mu_k^* X_i' = \begin{cases} X_k^{-1} & \text{if } i=k \\ X_i (1+X_k^{-\operatorname{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & \text{if } i\neq k \end{cases}$$

$$\mu_k^* B_i' = \begin{cases} \dfrac{X_k \prod_{\varepsilon_{kj}>0} B_j^{\varepsilon_{kj}} + \prod_{\varepsilon_{kj}<0} B_j^{-\varepsilon_{kj}}}{(1+X_k) B_k} & \text{if } i=k \\ B_i & \text{if } i\neq k \end{cases}$$

<u>Def</u> An isomorphism $\sigma : (I, J, \varepsilon_{ij}) \to (I', J', \varepsilon_{ij}')$ is a bijection $I \to I'$ that preserves all data.

Define

$$\sigma^* A'_{\sigma(i)} = A_i$$
$$\sigma^* X'_{\sigma(i)} = X_i$$
$$\sigma^* B'_{\sigma(i)} = B_i$$

Def A cluster transformation b/w seeds (or tori) is a composition of mutations and isomorphisms.

Write $\underline{\mathbf{i}} \sim \underline{\mathbf{i}}' \iff \underline{\mathbf{i}}, \underline{\mathbf{i}}'$ are related by a cluster transformation.

$|\underline{\mathbf{i}}|$ = equivalence class of $\underline{\mathbf{i}}$

Def (1) The cluster K_2-variety ($\mathcal{A}$-variety) $\mathcal{A} = \mathcal{A}_{|\underline{\mathbf{i}}|}$ is obtained by gluing all $\mathcal{A}_{\underline{\mathbf{i}}'}$ $(\underline{\mathbf{i}}' \in |\underline{\mathbf{i}}|)$ by these birational maps.

(2) The cluster Poisson variety ($\mathcal{X}$-variety) $\mathcal{X} = \mathcal{X}_{|\underline{\mathbf{i}}|}$ is glued from $\mathcal{X}_{\underline{\mathbf{i}}'}$ $(\underline{\mathbf{i}}' \in |\underline{\mathbf{i}}|)$.

(3) The cluster symplectic variety (symplectic double) $\mathcal{D} = \mathcal{D}_{|\underline{\mathbf{i}}|}$ glued from $\mathcal{D}_{\underline{\mathbf{i}}'}$ $(\underline{\mathbf{i}}' \in |\underline{\mathbf{i}}|)$

Def A <u>semifield</u> $\mathbb{P}$ is a set with binary operations $+, \cdot$ s.t.

(1) $(\mathbb{P}, \cdot)$ is an abelian group (not true in a field)

(2) $+$ is commutative and associative

(3) $(a+b)\cdot c = a\cdot c + b\cdot c$, $\forall a,b,c \in \mathbb{P}$

<u>Examples</u>

(1) $\mathbb{P} = \mathbb{R}_{>0}$ under usual $+, \cdot$

(2) $\mathbb{P} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ with
(tropical semifield)

$$a \oplus b = \max(a,b),$$
$$a \otimes b = a + b.$$

These are <u>tropical</u> <u>semifields</u> denoted $\mathbb{Z}^t, \mathbb{Q}^t, \mathbb{R}^t$.

Let $\mathbb{P}$ = semifield, $H \cong \mathbb{G}_m^n$ split algebraic torus.

Put $X_*(H) := \mathrm{Hom}(\mathbb{G}_m, H)$
(lattice of cocharacters)

$$H(\mathbb{P}) := X_*(H) \otimes_{\mathbb{Z}} \mathbb{P}$$

(as abel. gp)

$$\cong \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{P}$$

$$\cong \mathbb{P}^n$$

Define

$$\mathcal{A}(\mathbb{P}) := \coprod \mathcal{A}_{\underline{\mathbf{i}}}(\mathbb{P}) / \text{identificatoons}$$

$$\mathcal{X}(\mathbb{P}) := \coprod \mathcal{X}_{\underline{\mathbf{i}}}(\mathbb{P}) / \sim$$

$$\mathcal{D}(\mathbb{P}) := \coprod \mathcal{D}_{\underline{\mathbf{i}}}(\mathbb{P}) / \sim$$

<u>Corollary</u>.

$$\mathcal{A}^+(S) \cong \mathcal{A}(\mathbb{R}_{>0})$$

$$\mathcal{X}^+(S) \cong \mathcal{X}(\mathbb{R}_{>0})$$

$$\mathcal{D}^+(S) \cong \mathcal{D}(\mathbb{R}_{>0})$$

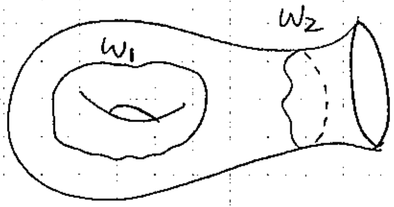
$\mathbb{R}^t \rightsquigarrow$ Thurston's measured geod. meas

$$\mathcal{A}_L(S, \mathbb{Q}) \cong \mathcal{A}(\mathbb{Q}^t)$$

$$\mathcal{X}_L(S, \mathbb{Q}) \cong \mathcal{X}(\mathbb{Q}^t)$$

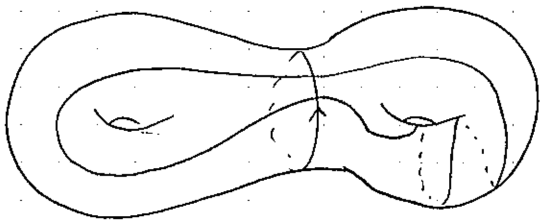
$$\mathcal{D}_L(S, \mathbb{Q}) \cong \mathcal{D}(\mathbb{Q}^t)$$

Tropical integral points :



$=: \ell \in \mathcal{A}_L(S, \mathbb{Z})$

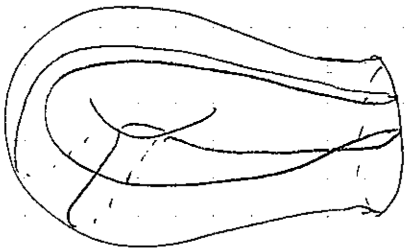
$\rightsquigarrow \sigma_\ell \in H_1(S, \mathbb{Z}/2\mathbb{Z})$



$=: \ell \in \mathcal{D}_L(S, \mathbb{Z})$

$\rightsquigarrow$

apply involution to right



$\rightsquigarrow \sigma_\ell \in H_1(S, \mathbb{Z}/2\mathbb{Z})$

<u>Theorem</u> (A.)

(1) $\mathcal{X}(\mathbb{Z}^t) = \mathcal{X}_L(S, \mathbb{Z})$

(2) $\mathcal{A}(\mathbb{Z}^t) = \{\ell \in \mathcal{A}_L(S, \mathbb{Z}) : \sigma_\ell = 0\}$

(3) $\mathcal{D}(\mathbb{Z}^t) = \{\ell \in \mathcal{D}_L(S, \mathbb{Z}) : \sigma_\ell = 0\}$