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Non-vanishing mod p of special L-values

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To Sunhee

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Abstract of the Dissertation

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Professor Haruzo Hida, Chair

In this thesis, the non-vanishing mod p of special values of various L-functions is studied. We study the classical theorem of Washington about the non-vanishing mod p of the special values of Dirichlet L-functions from two different points of view. First, we study the topological way of establishing the theorem, so-called homological independence, which is deduced from the abelian modular symbol method. Secondly, the geometrical approach which was developed by Sinnott and Hida using Zariski density on the torus is studied. The homological independence is tested numerically in terms of modular symbols on the modular curve and the modular L-values with cyclotomic twists are studied. The geometrical point of view is extended to the case of modular L-values with anti-cycltotomic twists by utilizing the Zariski density of CM points on the modular curves.

CHAPTER 1

Introduction

Although they are complex analytic, the L-functions reflect an arithmetic nature. Dirichlet's Theorem, which asserts that there are infinitely many prime numbers in arithmetic progressions, is one of the famous examples which relate an arithmetic problem to the study on the L-functions. The class number formula establish a connection between the special values of L-functions and several invariants of the number fields including the class numbers. The land of number theory has been enriched since Iwasawa introduced his theory on the $\mathbb{Z}_p$-extensions of number fields which is developed as an analogue of similar phenomenons in the function fields.

For a prime number p, the $\mathbb{Z}_p$-*extension* of a number field K is a tower of number fields

$$K = K_0 \subset K_1 \subset \cdots \subset K_\infty$$

such that $\mathrm{Gal}(K_{n+1}/K_n) \simeq \mathbb{Z}/p^n\mathbb{Z}$ and $\mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p$. The celebrated theorem of Iwasawa ([15]) asserts that if e_n is the exact power of the prime divisor p of the class number h_n of the number field K_n, then for sufficiently large n one has $e_n = \lambda n + \mu p^n + \nu$ for some integers λ, μ, and ν which are independent of n. Let Λ be the power series ring $\mathbb{Z}_p[[T]]$. To the $\mathbb{Z}_p$-extensions, Iwasawa associates a Λ-module X and a power series in Λ where the degree and gcd of the coefficients are λ and p^μ respectively. The p-adic L-function on $\mathbb{Z}_p$ is defined to be a certain simple change of variable of this power series.

A $\mathbb{Z}_p$-extension K_∞/K is called *cyclotomic* if one has $K_\infty \subseteq K(\mu_{p^\infty})$ for the union μ_{p^∞} of all p-power roots of unity in $\overline{\mathbb{Q}}$. Based on the strong analogy between the number fields and the function field, the invariant μ of a cyclotomic $\mathbb{Z}_p$-extension is conjectured to be zero by Iwasawa. This conjecture of long standing was finally verified for the case that K is abelian over $\mathbb{Q}$ by Ferrero and Washington ([7]). Later this result was reproven by Sinnott ([29]) in a conceptual way using p-adic measure theory on $\mathbb{Z}_p^\times$ and some manipulations of power series (*the independence result*). Several attempts have been made to generalize the theorem and only partial results were obtained. For example, Barsky claims the vanishing of the μ-invariant of a totally real field in [2] using p-adic analytic function theory. However the validity of this argument is not yet confirmed.

Along with the development of the theory on the μ-invariant, one can also consider the exact power of other prime divisors ℓ of the class number of K_n. A statement similar to the vanishing of the μ-invariant can be made so that the exact ℓ-power of the class number of K_n is bounded. As discussed in [35], this statement is equivalent to the non-vanishing mod p of special values of Dirichlet L-functions with characters with ℓ-power conductors. The non-vanishing result for Dirichlet L-values for the abelian number field is obtained by Washington ([36]). Again this was reproven by Sinnott ([28]) using a variant of the measure theoretic argument.

Contrary to the lack of success for the cyclotomic case beyond the result of Ferrero and Washington, there have been major developments due to Hida ([10]) and Vatsal ([33]) for the so-called *anti-cyclotomic* case of modular L-functions. However the cyclotomic case for the modular L-values are still open and in this thesis we shall present a possible approach to this problems.

1.1 Homological Point of View

The special L-values with cyclotomic twists of prime power conductors have been posing difficulties to researchers mainly due to the absence of the appropriate machinery. In [31], we get a nice homlogical interpretation of the equi-distribution argument of Washington using abelian modular symbols, which are an cylindrical version of modular symbols. In section 3.6, we shall observe that Eisenstein primes are involved in the discussion as expected by Greenberg. He conjectured that the cyclotomic $\mu_p(E)$-invariant of an elliptic curve E vanishes if $E[p]$ is an irreducible Galois module.

We now illustrate roughly the homological argument based on the equidistribution property originated by Washington and Ferrero. We consider a Riemann surface X, a finite subset $\mathfrak{b}$ of X, a subset Ξ of $H_1(X, \mathfrak{b}, \mathbb{Z})$ with an action of $\mathbb{Z}_\ell^\times$, and a differential form ω_f on X. Let Δ be a certain finite subset of $\mathbb{Z}_\ell^\times$ consisting of linearly independent elements over $\mathbb{Q}$. There is an integral representation of L-values which is interpretable as a sum of (cup-product) pairings between homology classes in Ξ and a cohomology class on X. This is the modular symbol approach. Let $\mathcal{B}$ be a prime ideal in $\overline{\mathbb{Q}}$ over p. Following the process due to Sinnott in [28], we expect to arrive at the vanishing:

$$\sum_{\eta \in \Delta} \langle \eta \cdot v, \omega_f \rangle \equiv 0 \bmod \mathcal{B} \text{ for each } v \in \Xi. \tag{1.1.1}$$

Hence, if the image of the map $j : \Xi \to H_1(X, \mathfrak{b}, \mathbb{Z})^\Delta$, $v \mapsto (\eta \cdot v)_\eta$ generates the whole homology group $H_1(X, \mathbb{Z})^\Delta$, or in other words, if the image has an equidistribution property, then we expect to get a contradictory statement such that the differential form ω_f vanishes mod $\mathcal{B}$.

Two particular instances shall be presented in the next two chapters. First in Section 2.2, we are going to consider Dirichlet L-values with the setting $X =$

$(\mathbb{C}/\mathbb{Z} - S_N)\bigcup\{\pm i\infty\}$ for $S_N = \left\{\frac{j}{N} \,\middle|\, 0 \le j < N\right\}$, $\mathfrak{b} = \{\pm i\infty\}$, and $\Xi = \{v(r) = r + i\mathbb{R} \,|\, r \in \ell^{-\infty}\mathbb{Z}/\mathbb{Z}\}$. In Section 3, we deal with the complex modular L-values $L(m, f, \chi)$ for a modular form f of $\Gamma_0(N)$, which are the cyclotomic twists of the Dirichlet characters χ with ℓ-power conductors. In this context, we set $X = X_0(N)$, $\mathfrak{b} = \{cusps\}$, and $\Xi = \{v(r) = r + i\mathbb{R}_+ \,|\, r \in \ell^{-\infty}\mathbb{Z}/\mathbb{Z}\}$. We hope to get the residual non-vanishing result in the near future.

1.2 Geometrical Point of View

Twists by anti-cyclotomic characters χ (of imaginary quadratic fields) have been dealt with quite differently from the cyclotomic case, and some generalizations have been given by Hida([10]), Vatsal([32],[33]) and other researchers. This anti-cyclotomic case involves more sophisticated tools, but is philosophically more well-understood than the cyclotomic case. It is Vatsal([32]) who successfully transferred the original equidistribution idea of [7] to the case of modular L-values for a modular form of weight 2 under the twists by ring class characters. As an application of non-vanishing, a conjecture of Mazur on the special values of elliptic L-function is established. He has used the equidistribution of Heegner points on Gross curve associated with definite quaternion algebra and Ratner's ergodic theory on p-adic Lie group. On the other hand, even though Sinnott's algebraic proof of the theorem was elegant and simple, a generalization was not made until Hida interpreted the proof as a geometric one. He extended it to abelian Hecke L-series using the Zariski density of CM points on a Hilbert modular Shimura variety.

Let us now sketch the geometric argument for the non-vanishing result. Let $X_{/\overline{\mathbb{F}}_p}$ be an algebraic variety over $\overline{\mathbb{F}}_p$, K a number field, Ξ a collection of special points in X, and f a section of a line bundle on X. Let Z be a profinite group which is the projective limit of (ring or standard) class groups of K with ℓ-power conductors. It has the decomposition $Z = \Delta \times \Gamma$, where Δ is finite and Γ is a torsion-free subgroup. Suppose that we have an action of Z on Ξ. The (square root of) special L-values are expressible as a finite sum of the characters against evaluations of f over Z. In these examples, the value of f at $x \in \Xi$ is a partial L-value. Similarly as the last vanishing 1.1.1, Sinnott's procedure enables us to derive

$$\sum_{\eta \in \Delta} f(\eta \cdot \mathfrak{a}) \equiv 0 \mod \mathcal{B} \text{ for each } \mathfrak{a} \in \Xi. \tag{1.2.1}$$

Then the Zariski-density of the image of the map $j : \Xi \to X^{\Delta}$, $\mathfrak{a} \mapsto (\eta \cdot \mathfrak{a})_{\eta \in \Delta}$ often enables us to conclude that $f \equiv 0 \mod \mathcal{B}$, which is a contradiction.

For the Dirichlet L-values, in addition to the homological argument we also have the geometric one due to Sinnott and Hida. In Section 2.4, we take $K = \mathbb{Q}$, $Z \cong \mathbb{Z}_\ell^\times$, $X = \mathbb{G}_m$. And f is a suitable rational function on $\mathbb{G}_m$ and Ξ is the set μ_{ℓ^∞} of ℓ-power roots of unity in $\overline{\mathbb{F}}_p$. The anti-cyclotomic case is considered with the setting $K = \mathbb{Q}(\sqrt{-d})$, $d > 0$, $Z = \varprojlim_n \mathrm{Pic}(\mathcal{O}_{\ell^n})$ (see Chapter 4). We consider the case that X is a modular curve $X_0(N)$, f is an eigencusp form of level N, and Ξ is a set of Heegner (CM) points associated to K of ℓ-power conductors. This case deals with the modular L-value $L(1, f, \theta_\chi)$, which is a Rankin product of f and the theta series θ_χ associated with χ. In Chapter 4, we consider rather the special case that f is an eigen cusp form for the group $\mathrm{SL}_2(\mathbb{Z})$ of arbitrary weight $k \geq 2$.

CHAPTER 2

Dirichlet L-values

In this chapter, we discuss a homological and geometrical interpretations of Washington's theorem. To state the result, let us consider two different odd primes, p and ℓ, and two Dirichlet characters $\lambda : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}$ of conductor N, and $\chi : (\mathbb{Z}/\ell^n\mathbb{Z})^\times \to \mu_{\ell^\infty}$ of ℓ-power conductor respectively. Here N is relatively prime to both p and ℓ and $\mu_{\ell^\infty} = \cup_{n\geq 1}\mu_{\ell^n}$ with the group μ_{ℓ^n} of ℓ^n-th roots of unity. It is well-known that the special value of the Dirichlet L-function $L(0, \lambda\chi)$, after excluding its Euler ℓ-factor, is integral. Let $\mathcal{B}$ be a prime in $\overline{\mathbb{Q}}_p$ over p. Washington proved in [36] that

Theorem 2.0.1. *Let λ be an odd Dirichlet character of conductor N. Then for only finitely many Dirichlet characters $\chi : (\mathbb{Z}/\ell^n\mathbb{Z})^\times \to \mu_{\ell^\infty}$, we have $L(0, \lambda\chi) \equiv 0 \mod \mathcal{B}$.*

Note that this implies the finiteness of exponents of prime divisor ℓ in the number h_n^-, which is the relative class number of each subfield K_n of the cyclotomic $\mathbb{Z}_\ell$-extension K_∞ of $\mathbb{Q}$. Using a Kummer type argument studied in [35], we also get the finiteness of h_n, the class number of K_n. The group $\mathbb{Z}_\ell^\times = \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q})$ has decomposition $\mathbb{Z}_\ell^\times = \mu_{\ell-1} \times \Gamma$ and a profinite subgroup, $\Gamma \cong \mathbb{Z}_\ell$ is the Galois group $\mathrm{Gal}(K_\infty/\mathbb{Q})$. A factor $\mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$ is the $(\ell-1)$-th roots of unity in $\mathbb{Z}_\ell^\times$. One of the main ingredients in [36] is that the maximal subset of $\mu_{\ell-1}$ which is linearly independent over $\mathbb{Q}$ generates a subset of $(\ell^{-\infty}\mathbb{Z}/\mathbb{Z})^{\frac{\ell-1}{2}}$ so that it is uniformly distributed on $(0,1)^{\frac{\ell-1}{2}}$. This will be discussed briefly in Section 2.1.

In [29], Sinnott invented a totally different approach to the proof of this problem. He studied a suitable measure on $\mathbb{Z}_\ell^\times$, which is defined by a rational function twisted by elements in $\mu_{\ell-1}$, called formal functions. Instead of uniform distribution property, he made use of the algebraic independence of such formal functions. It was Hida who realized that Sinnott's method can be written in an algebro-geometric language in [9] and this (modular) version of Sinnott's proof is explained in the appendix of present paper. Let U be a maximal linearly independent subset of $\mu_{\ell-1}$ and consider the map

$$j : \mu_{\ell^\infty} \to \mathbb{G}_m(\overline{\mathbb{F}}_p)^U, \ \zeta \mapsto (\zeta^\eta)_{\eta \in U}.$$

Here $\mathbb{G}_m(\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p^\times$ is regarded as an algebraic group over the algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$. Then the algebraic independence result can be reformulated as follows.

Proposition 2.0.2. *The image of j is Zariski dense in $\mathbb{G}_m(\overline{\mathbb{F}}_p)^U$.*

By an integral representation of L-functions, we can think of special L-values as a (cup product) pairing of a homology class and a cohomology class defined by a power series which is actually a rational function whose poles are roots of unity. This is an abelian version of the Mazur's modular symbol method, which is discussed in great detail in [12]. The crucial point of this discussion, we call it homological independence, is a mixture of Washington's original uniform distribution result and Hida's Zariski density argument. Let us explain the situation briefly. We consider a punctured cylinder $\mathbb{C}/\mathbb{Z} - S_N \cong \mathbb{G}_m(\mathbb{C}) - \mu_N(\mathbb{C})$, where $S_N = \{j/N | 0 \le j < N\}$ for an integer N. We add two points $\{\pm i\infty\}$ to the cylinder in order to get X_N°. Let $v(r)$ be the vertical line $r + it$, $-\infty \le t \le \infty$ for $0 < r < 1$, $r \notin S_N$. Hence $v(r)$ are in the relative homology $H_1(X_N^\circ, \{\pm i\infty\}, \mathbb{Z})$.

We consider a map

$$j : \ell^{-\infty}\mathbb{Z}/\mathbb{Z} \to H_1(X_N^\circ, \{\pm i\infty\}, \mathbb{Z})^{\frac{\ell-1}{2}}.$$

We will prove an independence result related to above map j in the Section 2.1 from this homological setting and we shall give a new proof of Theorem 2.0.1 in Section 2.2, interpreting Washington's argument in a homological way. Although the proof itself is not much new, it seems to be simple enough to be extended to the case of elliptic modular L-values

In Section 2.5 using the speical class of rational functions (see 2.5.1), one can formulate the Zariski density result on the multi-dimensional setting over a totally real field.

In the last section, we shall apply the homological independence to obtain the vanishing of μ-invariant for the Dirichlet p-adic L-functions after defining a suitable measure using abelian moudlar symbols.

2.1 Homological Independence: Abelian modular symbols

As mentioned in the introduction, we consider X_N°, which is the cylinder $\mathbb{C}/\mathbb{Z}$ punctured at points S_N and with two points $\{\pm i\infty\}$ added. Each vertical line $v(r)$ is in the relative homology $H_1(X_N^\circ, \{\pm i\infty\}, \mathbb{Z})$. Let $c(k/N) \in H_1(X_N^\circ, \mathbb{Z})$ be the closed path on X_N° starting from a fixed base point $z_0 \in \mathbb{C}$, then turning around k/N counterclockwise and returning to z_0. The homology $H_1(X_N^\circ, \mathbb{Z})$ is generated by such $c(\frac{k}{N})$. For any $r \in (\frac{k-1}{N}, \frac{k}{N})$, and $s \in (\frac{k}{N}, \frac{k+1}{N})$, we have $c(\frac{k}{N}) = v(s) - v(r)$. The main idea is that given any partition of $(0,1)$, say $(0, \frac{1}{N}) \cup \cdots \cup (\frac{N-2}{N}, \frac{N-1}{N})$, we can find two vertical lines from $\{v(r) | r \in \ell^{-\infty}\mathbb{Z}/\mathbb{Z}\}$, which are on the two consecutive partitions $(\frac{k-1}{N}, \frac{k}{N})$ and $(\frac{k}{N}, \frac{k+1}{N})$ to get the homology class $c(k/N)$.

To make this idea working, the uniform distribution property should be brought into the present picture.

We consider a $\mathbb{Q}$-linearly independent maximal subset U of $\mu_{\ell-1}$. In [7], it is shown that the collection $\{\,(\frac{a\xi}{\ell^n})_{\xi\in U}\,|\,n \geq 1\,\}$ is uniformly distributed on $[0,1)^U$ for almost all $a \in \mathbb{Z}_\ell^\times$; i.e., for all a outside of a measure zero subset of $\mathbb{Z}_\ell^\times$. Here the ℓ-adic numbers $\frac{a\xi}{\ell^n}$ are thought of as the element $\frac{s_n(a\xi)}{\ell^n}$ in X_N° for n-th partial sum s_n of ℓ-adic integers.

From the definition of uniform distribution, one can easily deduce that when a subset $Z \subseteq \mathbb{Z}_\ell^\times$ with non-zero measure, a point $x_\xi \in (0,1)$, and $\epsilon_\xi > 0$ are given for each $\xi \in U$, then for all sufficiently large n, we can find $a \in Z$ so that $|x_\xi - \frac{a\xi}{\ell^n}| < \epsilon_\xi$ for each $\xi \in U$. Roughly speaking, for a given vector $(x_\xi)_{\xi\in U}$, we can find a point $(\frac{a\xi}{\ell^n})_{\xi\in U}$ which is close enough to $(x_\xi)_{\xi\in U}$.

Set $U = \{\eta_1 = \eta, \eta_2, \cdots, \eta_t\}$ and $\mu_{\ell-1}/\{\pm 1\} = \{\eta_1, \cdots, \eta_t, \tau_1, \cdots, \tau_s\}$. We have an integral $s \times t$ matrix A such that

$$(\tau_1, \cdots, \tau_s) = (\eta_1, \cdots, \eta_t)A.$$

Since $\mu_{\ell-1}/\{\pm 1\}$ is $\mathbb{Z}$-multiplicatively independent i.e. there are no integers m, n such that $m\eta \neq n\eta'$ for distinct η, $\eta' \in \mu_{\ell-1}/\{\pm 1\}$, all the columns of A have at least two non-zero entries. Now let us state a lemma.

Lemma 2.1.1. *Let $\alpha_1, \cdots, \alpha_t$ be any real numbers and set*

$$P(\alpha_1, \cdots, \alpha_t) := (\alpha_1, \cdots, \alpha_t)\,(I|A)$$

for a $t\times t$ identity matrix I. For each $k \in \mathbb{Z}$, we can find numbers α_1', α_1'', and α_2^o, $\cdots$, α_t^o such that (1) $\alpha_1' \in (\frac{k-1}{N}, \frac{k}{N})$, $\alpha_1'' \in (\frac{k}{N}, \frac{k+1}{N})$, and (2) $P(\alpha_1', \alpha_2^o, \cdots, \alpha_t^o)$, $P(\alpha_1'', \alpha_2^o, \cdots, \alpha_t^o)$ are away from $\left(\frac{1}{N}\mathbb{Z}\right)^{\frac{\ell-1}{2}}$. In other words, no coordinate of them are in $\frac{1}{N}\mathbb{Z}$.

Proof. Consider the function $P(\frac{k}{N}, \alpha_2, \cdots, \alpha_t)$ of $\alpha_2, \cdots, \alpha_t$. Since A is an integral matrix, $P(\frac{k}{N}, 0, \cdots, 0)$ is in $\left(\frac{1}{N}\mathbb{Z}\right)^{\frac{\ell-1}{2}}$. The $\mathbb{Z}$-multiplicative independence of $\mu_{\ell-1}/\{\pm 1\}$ implies that no coordinate in $P(\alpha_1, \cdots, \alpha_t)$ is $\mathbb{Z}$-multiple of α_1 except itself. Hence, $P(\frac{k}{N}, \alpha_2, \cdots, \alpha_t)$ is an open map; i.e., no coordinates are constant except the first one and we can find $(\alpha_2^o, \cdots, \alpha_t^o)$ around $(0, \cdots, 0)$ such that each coordinate of $P(\frac{k}{N}, \alpha_2^o, \cdots, \alpha_t^o)$ is not in $\frac{1}{N}\mathbb{Z}$ except the first one that is k/N. Since $P(\alpha_1, \alpha_2^o, \cdots, \alpha_t^o)$ is continuous in the variable α_1, we can find α_1' $\in (\frac{k-1}{N}, \frac{k}{N})$, $\alpha_1'' \in (\frac{k}{N}, \frac{k+1}{N})$ satisfying the condition (2). □ □

We define $j : \ell^{-\infty}\mathbb{Z}/\mathbb{Z} \to H_1(X_N^\circ, \{\pm i\infty\}, \mathbb{Z})^{\frac{\ell-1}{2}}$ for $\ell^{-\infty}\mathbb{Z} = \cup_n \ell^{-n}\mathbb{Z}$ by

$$j\left(\frac{a}{\ell^n}\right) = \left(v\left(\frac{a\xi}{\ell^n}\right)\right)_{\xi \in \mu_{\ell-1}/\{\pm\}}.$$

The following proposition is a homological variant of Proposition 2.0.2.

Proposition 2.1.2. $H_1(X_N^\circ, \mathbb{Z})^{\frac{\ell-1}{2}}$ $(\subseteq H_1(X_N^\circ, \{\pm i\infty\}, \mathbb{Z})^{\frac{\ell-1}{2}})$ *is generated by the image (under the map j) of*

$$\{\frac{a}{\ell^{n_j}} \,|\, j \geq 0,\, a \in Z\}$$

for any increasing sequence $\{n_j\}$ of positive integers and any subset Z of $\mathbb{Z}_\ell^\times$ with non-zero measure. In particular, $H_1(X_N^\circ, \mathbb{Z})^{\frac{\ell-1}{2}}$ is generated by the image of $j : \ell^{-\infty}\mathbb{Z}/\mathbb{Z} \to H_1(X_N^\circ, \{\pm i\infty\}, \mathbb{Z})^{\frac{\ell-1}{2}}$.

Proof. As discussed above, the subset $\{\,(\frac{a\xi}{\ell^n})_{\xi \in U} \,|\, n \geq 1\}$ are uniformly distributed on $(0,1)^U$ for all $a \in Z$. Hence it is possible to find suitable n, m in $\{n_j\}$ and a, b in Z such that vectors $(\frac{a\xi}{\ell^n})_{\xi \in U}$, $(\frac{b\xi}{\ell^m})_{\xi \in U}$ are close enough to two vectors $(\alpha_1', \alpha_2^o, \cdots, \alpha_t^o)$, $(\alpha_1'', \alpha_2^o, \cdots, \alpha_t^o)$ previously chosen in the lemma respectively and, therefore, all coordinates of $P((\frac{a\xi}{\ell^n})_{\xi \in U})$, $P((\frac{b\xi}{\ell^m})_{\xi \in U})$ are inside of the same partitions except the first ones that are in consecutive partitions $(\frac{k-1}{N}, \frac{k}{N})$,

$(\frac{k}{N}, \frac{k+1}{N})$ respectively. Due to this result, we have

$$j(\frac{b}{\ell^n}) - j(\frac{a}{\ell^m}) = (\,[0], \cdots, [0], c(k/N), [0], \cdots, [0]\,) \in H_1(X_N^\circ, \mathbb{Z})^{\frac{\ell-1}{2}},$$

where the position of non-zero homology class corresponds to η. Since η and k are chosen arbitrarily, we prove the proposition. □

2.2 Special L-values and a theorem of L. Washington

For a Dirichlet character ψ, we define

$$f_\psi(z) = \begin{cases} \sum_{n\geq 1} \psi(n) q^n & Im(z) > 0, \\ -\sum_{n \geq 1} \psi(-n) q^{-n} & Im(z) < 0 \end{cases}$$

for $q = \exp(2\pi i z)$. As shown in [12, §4], the differential $f_\psi(z)dz$ gives a cohomology class $\omega(f_\psi)$ in $H_c^1(X_N^\circ, \mathbb{C})$, and we have

$$\int_{-i\infty}^{i\infty} f_{\lambda\chi}(z)dz = -G(\chi)\frac{L(1, \lambda\chi)}{2\pi i} = -L(0, \lambda^{-1}\chi^{-1}).$$

Here $G(\chi)$ is the Gauss sum defined by χ. We define a pairing between v in $H_1(X_N^\circ, \{\pm i\infty\}, \mathbb{Z})$ and ω in $H_c^1(X_N^\circ, \mathbb{C})$ as follows:

$$\langle v, \omega \rangle = \int_v \omega.$$

There are operators $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ acting on X_N° for each $r \in \ell^{-\infty}\mathbb{Z}/\mathbb{Z}$ defined by

$$z \mapsto \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \cdot z := z + r.$$

For any $\omega \in H_c^1(X_N^\circ, \mathbb{C})$, we define $\omega\Big|\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ by the natural action of $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ on $H_c^1(X_N^\circ, \mathbb{C})$ induced from the one on X_N°.

From the decomposition

$$f_{\lambda\chi}(z) = \frac{1}{G(\chi)} \sum_{a\in(\mathbb{Z}/\ell^n\mathbb{Z})^\times} \chi(a) f_\lambda(z+\frac{a}{\ell^n}),$$

we have

$$\begin{aligned} -L(0,\lambda^{-1}\chi^{-1}) &= \frac{1}{G(\chi)} \sum_{a\in(\mathbb{Z}/\ell^n\mathbb{Z})^\times} \chi(a) \int_{-i\infty}^{i\infty} f_\lambda(z+\frac{a}{\ell^n}) dz & (2.2.1)\\ &= \frac{1}{G(\chi)} \sum_a \chi(a) \int_{v(\frac{a}{\ell^n})} f_\lambda(z) dz & (2.2.2)\\ &= \frac{1}{G(\chi)} \sum_a \chi(a) \left\langle v\Big(\frac{a}{\ell^n}\Big), \omega(f_\lambda)\right\rangle. & (2.2.3)\end{aligned}$$

By the inversion formula we obtain

$$\left\langle v\Big(\frac{a}{\ell^n}\Big), \omega(f_\lambda)\right\rangle = \frac{-1}{\ell^{n-1}(\ell-1)} \sum_{\psi \bmod \ell^n} \psi^{-1}(a) L(0,\lambda^{-1}\psi^{-1}),$$

and hence it is algebraic. The pairing is an element of $\mathbb{Q}(\lambda)$ because it is fixed under the action of $\mathrm{Gal}(\mathbb{Q}(\lambda,\psi)/\mathbb{Q}(\lambda))$.

Let k be the finite field over $\mathbb{F}_p$ generated by λ and the ℓ-th root of unity. There is an integer m such that $\mu_{\ell^\infty} \cap k = \mu_{\ell^m}$. Setting $k_j = k(\mu_{\ell^{m+j}})$ and $k_\infty = \bigcup_{j\geq 1} k_j$, we have $\mathrm{Gal}(k_\infty/k) = \Gamma_m := 1+\ell^m\mathbb{Z}_\ell$ with the action on μ_{ℓ^∞} given by $\zeta \mapsto \zeta^t$ for $t\in\Gamma_m$.

Recall that $\mathcal{B}$ is a prime in $\overline{\mathbb{Q}}_p$ over p. Now assume that $L(0,\lambda^{-1}\chi^{-1}) \equiv 0 \bmod \mathcal{B}$ for a Dirichlet character χ of conductor ℓ^n and $n \geq 2m$. Multiplying $\chi(a)$ for $a \in \Gamma_m$ and taking modulus $\mathcal{B}$ and $\mathrm{Tr}_{k_n/k}$, and using the fact

$$\mathrm{Tr}_{k_n/k}(\chi(a)) = \begin{cases} [k_n:k]\chi(a) & \text{if } \chi(a)\in\mu_{\ell^m},\\ 0 & \text{otherwise}\end{cases}$$

the formula (2.2.3) is reduced to

$$[k_n:k]\chi(a) \sum_{\eta\in\mu_{\ell-1}} \sum_{b\in a\frac{\Gamma_{n-m}}{\Gamma_n}} \chi^{-1}(b) \left\langle v\left(\frac{b\eta}{\ell^n}\right), \omega(f_\lambda)\right\rangle \equiv 0 \bmod \mathcal{B}$$

for all $a \in \Gamma_m$. Furthermore, writing down the representatives of the quotient $a\Gamma_{n-m}/\Gamma_n$ explicitly, we get

$$\sum_{\eta} \sum_{c \in \mathbb{Z}/\ell^m\mathbb{Z}} \chi^{-1}(1+\ell^{n-m}c) \left\langle v\left(\frac{a\eta}{\ell^n} + \frac{\eta c}{\ell^m}\right), \omega(f_\lambda) \right\rangle \equiv 0 \bmod \mathcal{B}.$$

Since $2(n-m) \geq n$ and $(1+\ell^{n-m})^c \equiv 1+\ell^{n-m}c \bmod \ell^n$, setting $\chi(1+\ell^{n-m}) = \zeta_m$, a primitive ℓ^m-th root of unity, we have

$$\sum_{\eta} \left\langle v\left(\frac{a\eta}{\ell^n}\right), \sum_{c \in \mathbb{Z}/\ell^m\mathbb{Z}} \zeta_m^c \omega(f_\lambda)\Big| \begin{pmatrix} 1 & \frac{\eta c}{\ell^m} \\ 0 & 1 \end{pmatrix} \right\rangle \equiv 0 \bmod \mathcal{B}. \qquad (2.2.4)$$

Define for any $b \in \mathbb{Z}_\ell^\times$

$$f_{\lambda,b}(z) = \frac{\displaystyle\sum_{r=0, r\equiv b(\ell^m)}^{\ell^m N-1} \lambda(r) q^r}{q^{\ell^m N} - 1}.$$

Above congruence (2.2.4) is written as

$$\sum_{\eta} \left\langle v\left(\frac{a\eta}{\ell^n}\right), \omega(f_{\lambda,\eta^{-1}}) \right\rangle \equiv 0 \bmod \mathcal{B} \qquad (2.2.5)$$

for all $a \in \Gamma_m$ because

$$f_{\lambda,b}(z) = \sum_{c \in \mathbb{Z}/\ell^m\mathbb{Z}} \zeta_m^c f_\lambda\Big| \begin{pmatrix} 1 & \frac{bc}{\ell^m} \\ 0 & 1 \end{pmatrix}(z) = \sum_{\substack{n=1 \\ n\equiv b^{-1}(\ell^m)}}^{\infty} \lambda(n) q^n.$$

The function $f_{\lambda,b}(z)$ has poles only at $z = \frac{k}{\ell^m N}$ for $0 \leq k < \ell^m N$ and $(k, N) = 1$ with residues

$$\frac{\lambda(k) G(\lambda^{-1}) (\zeta_m^{b+1} \eta_N)^k}{N \ell^m}.$$

Here η_N is a primitive N-th root of unity. Note that the residues are algebraic and p-integral. The definition of f shows that it goes to 0 as q goes to infinity. In other words, there is no holomorphic term in $f_{\lambda,b}(q)$. Hence, we have the partial fraction expansion

$$f_{\lambda,b}(q) = \sum_{(k,N)=1} \frac{\lambda(k) G(\lambda^{-1}) (\zeta_m^{b+1} \eta_N)^k}{N \ell^m (q - (\eta_N \zeta_m)^k)}.$$

Since the pairing is a $\mathbb{Z}$-linear combination of residues of $f_{\lambda,b}$, above partial fraction expansion of $f_{\lambda,b}(q)$ guarantees p-integrality of the pairing $\langle u, \omega(f_{\lambda,b})\rangle$ for each $u \in H_1(X^\circ_{N'}, \mathbb{Z})$, and $N' = N\ell^m$, and thus $\omega(f_{\lambda,b}) \in H^1(X^\circ_{N'}, \mathbb{Z}_{(p)})$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p.

Note also that vanishing mod $\mathcal{B}$ of the pairing on the whole homology group

$$\langle H_1(X^\circ_{N'}, \mathbb{Z}), \omega(f_{\lambda,b})\rangle \equiv 0 \bmod \mathcal{B}$$

implies that all residues are congruent to 0 modulo $\mathcal{B}$. Therefore non-existence of holomorphic part in $f_{\lambda,b}(q)$ forces us to conclude that $f_{\lambda,b} \equiv 0 \bmod \mathcal{B}$.

Now we prove Theorem 2.0.1. Let us assume the contrary that $L(0, \lambda\chi) \equiv 0 \bmod \mathcal{B}$ for infinitely many χ's. It implies that for infinitely many integers n and all $a \in \Gamma_m$ we have

$$\sum_{\eta \in \mu_{\ell-1}} \left\langle v\left(\frac{a\eta}{\ell^n}\right), \omega(f_{\lambda,\eta^{-1}})\right\rangle \equiv 0 \bmod \mathcal{B}.$$

From the identity $f_{\lambda,-b}(q) = f_{\lambda,b}(q^{-1})$; i.e., $f_{\lambda,-b}(z) = f_{\lambda,b}(-z)$, we have

$$\left\langle v\left(\frac{a\eta}{\ell^n}\right), \omega(f_{\lambda,\eta^{-1}})\right\rangle = \left\langle v\left(\frac{-a\eta}{\ell^n}\right), \omega\left(f_{\lambda,(-\eta)^{-1}}\right)\right\rangle,$$

and

$$\sum_{\xi \in \mu_{\ell-1}/\{\pm 1\}} \langle v(\frac{a\xi}{\ell^n}), \omega(f_{\lambda,\xi^{-1}})\rangle \equiv 0 \bmod \mathcal{B}.$$

By Proposition 2.6.2, we know that the linear map $H_1(X^\circ_{N'}, \mathbb{Z})^{\frac{\ell-1}{2}} \to \overline{\mathbb{F}}_p$, which is given by

$$(v_\xi)_\xi \mapsto \sum_{\xi \in \mu_{\ell-1}/\{\pm 1\}} \langle v_\xi, \omega(f_{\lambda,\xi^{-1}})\rangle \bmod \mathcal{B}$$

is zero. For a fixed η, we choose $(v_\xi)_\xi \in H_1(X^\circ_{N'}, \mathbb{Z})^{\frac{\ell-1}{2}}$ such that $v_\xi = [0]$ if $\xi \neq \eta$ and v_η is arbitrary. Then we have vanishing $\langle H_1(X^\circ_{N'}, \mathbb{Z}), \omega(f_{\lambda,\eta})\rangle \equiv 0 \bmod \mathcal{B}$ and, hence, vanishing $\bmod \mathcal{B}$ of $f_{\lambda,\eta}$ for each $\eta \in \mu_{\ell-1}$, which is contradiction. So we have proved the theorem. $\square$

Remark 2.2.1. Actually, in [36] Washington chose a_1, $a_2 \in \Gamma_m$ such that

$$\mathrm{Tr}_{k_n/k}(\chi(a_1)L(0,\lambda^{-1}\chi^{-1})) \neq \mathrm{Tr}_{k_n/k}(\chi(a_2)L(0,\lambda^{-1}\chi^{-1})).$$

As expressed before, this can be rewritten as

$$\sum_{\eta}\left\langle v\left(\frac{a_1\eta}{\ell^n}\right), \omega(f_{\lambda,\eta^{-1}})\right\rangle - \sum_{\eta}\left\langle v\left(\frac{a_2\eta}{\ell^n}\right), \omega(f_{\lambda,\eta^{-1}})\right\rangle \not\equiv 0 \bmod \mathcal{B}.$$

Two numbers a_1, a_2 are chosen specially so that above difference of the sums over $\mu_{\ell-1}$ is equal to

$$\left\langle v\left(\frac{a_1\eta_0}{\ell^n}\right), \omega(f_{\lambda,\eta_0^{-1}})\right\rangle - \left\langle v\left(\frac{a_2\eta_0}{\ell^n}\right), \omega(f_{\lambda,\eta_0^{-1}})\right\rangle$$

for a fixed $\eta_0 \in \mu_{\ell-1}$ and the last difference is nothing but a residue of $f_{\lambda,\eta_0^{-1}}$. It is not a coincidence that the proof of Lemma 2.1.1 is similar to the proof of the Proposition 2 in [36] in choosing suitable a_1, a_2.

2.3 Zariski density on the torus $\mathbb{G}_m$

In this section we are going to study the Zariski density property on the torus $\mathbb{G}_m$ to apply it to the Washington's theorem. This method will give us another way of getting non-vanishing in addition to the homological method.

Let A be a free $\mathbb{Z}$-module of finite rank and $A^* = \mathrm{Hom}(A,\mathbb{Z})$. There is a torus $\mathbb{G}_m \otimes A = \mathrm{Spec}(\mathbb{Z}[t^{A^*}])$. Given the pairing $\langle\cdot,\cdot\rangle : A \times A^* \to \mathbb{Z}$ defined by $\langle a, f\rangle = f(a)$ and a basis $\{w_i\} \subseteq A$, $\{w_i^*\} \subseteq A^*$ the dual basis, we have the isomorphism, $\mathbb{G}_m \otimes A \cong \mathbb{G}_m^{rkA}$. More explicitly, the isomorphism is induced from

$$\mathbb{Z}[t^{A^*}] \to \mathbb{Z}[t_1,\cdots,t_n],\ t^{w_i^*} \mapsto t_i,\ n = rkA$$

and for a commutative ring R, we have an isomorphism $\mathbb{G}_m(R) \otimes A \cong \mathbb{G}_m(R)^{\mathrm{rk}A}$ defined by $\alpha \otimes x \to (x^{\langle \alpha, w_i^*\rangle})_{i=1}^n$, and $w_i \otimes x \leftarrow (1,\cdots,x,\cdots,1)$ with x in i-th position. We have a commutative diagrams in which all the maps are injective :

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{x\mapsto 1\otimes x} & (\mathbb{Z}[\mu_{\ell-1}]\otimes \mathbb{G}_m)^{\mu_{\ell-1}} \\ \uparrow & & \uparrow {\scriptstyle i\,:\, x\mapsto (\epsilon\otimes x)_\epsilon} \\ \mu_{\ell^\infty} & \xrightarrow{j} & \mathbb{G}_m^{\mu_{\ell-1}} \end{array}$$

First row comes from the surjection

$$\mathbb{Z}[t^{\alpha_1^*},\cdots,t^{\alpha_n^*}] \to \mathbb{Z}[t,t^{-1}],\ t^{\alpha_i^*}\mapsto 1 \text{ for } i\neq 1 \text{ and } t^{\alpha_1^*}\mapsto t$$

and

$$R^\times \to \mathbb{Z}[\mu_{\ell-1}]\otimes R^\times,\ x\mapsto 1\otimes x.$$

Second column comes from the surjection

$$\mathbb{Z}[\{t_\epsilon^{\alpha_i^*}\}] \to \mathbb{Z}[t,t^{-1}],\ t_\epsilon^{\alpha_i^*}\mapsto t^{<\epsilon,\alpha_i^*>}$$

and

$$R^\times \to (\mathbb{Z}[\mu_{\ell-1}]\otimes R^\times)^{\mu_{\ell-1}}, x\mapsto (\epsilon\otimes x)_{\epsilon\in\mu_{\ell-1}},$$

where $\{\alpha_i^*\}$ is the dual basis of a fixed basis $\{w_i\}$ of $\mathbb{Z}[\mu_{\ell-1}]$. This diagram is commutative since $1\otimes\zeta^\epsilon = \epsilon\otimes\zeta$ for $\zeta\in\mu_{\ell^\infty}$ and $\epsilon\in\mu_{\ell-1}$. Let us make a remark that j is not an algebraic map and there is no morphism from $\mathbb{G}_m$ to $\mathbb{G}_m^{\mu_{\ell-1}}$ of which restriction to μ_{ℓ^∞} is j. But after introducing i and having in mind the commutativity of above diagram, we may regard j as a restriction of an algebraic map. If V is the subscheme of $(\mathbb{Z}[\mu_{\ell-1}]\otimes\mathbb{G}_m)^{\mu_{\ell-1}}$ defined by the equations

$$\Big\{\prod_\epsilon t_\epsilon^{n_\epsilon w} - 1 \,\Big|\, \sum n_\epsilon \epsilon = 0, w\in\mathbb{Z}[\mu_{\ell-1}]^*\Big\},$$

then each row is factorized through V and $V\cap\mathbb{G}_m^{\mu_{\ell-1}}$; i.e., the image is contained in V and we have the diagram

$$\begin{array}{ccccc}
\mathbb{G}_m & \longrightarrow & V & \longrightarrow & (\mathbb{Z}[\mu_{\ell-1}] \otimes \mathbb{G}_m)^{\mu_{\ell-1}} \\
\uparrow & & \uparrow & & i\uparrow \\
\mu_{\ell^\infty} & \xrightarrow{j} & V \cap \mathbb{G}_m^{\mu_{\ell-1}} & \longrightarrow & \mathbb{G}_m^{\mu_{\ell-1}}
\end{array}.$$

Here, $V \cap \mathbb{G}_m^{\mu_{\ell-1}}$ is the subscheme of $\mathbb{G}_m^{\mu_{\ell-1}}$ defined by the equation

$$S = \Big\{ \prod_\epsilon t_\epsilon^{n_\epsilon} - 1 \,\Big|\, \sum n_\epsilon \epsilon = 0 \Big\},$$

and it is isomorphic to $\mathbb{G}_m^n$ by the isomorphism

$$\mathbb{Z}[\{t_\epsilon, t_\epsilon^{-1}\}]/\langle S \rangle \to \mathbb{Z}[t_1, t_1^{-1} \cdots, t_n, t_n^{-1}],\ t_\epsilon \mapsto \prod_i t_i^{c_i(\epsilon)},$$

where $\epsilon = \sum_i c_i(\epsilon)\alpha_i$, i.e. $c_i(\epsilon) = \langle \epsilon, \alpha_i^* \rangle$. Then we restate the proposition 1.2 :

Theorem 2.3.1. *The image of* $j : \mu_{\ell^\infty} \to V \cap \mathbb{G}_m^{\mu_{\ell-1}} \cong \mathbb{G}_m^n$, $\zeta \mapsto (\zeta^{\alpha_i})_{i=1}^n$ *is Zariski dense in* $\mathbb{G}_m^n$.

As mentioned in the introduction, the proof is an immediate consequence of the Theorem 2.2 in [29]. The proof also can be established using the Dirichlet's Theorem on the linear independence of characters by considering each formal monomial t^α, $\alpha \in \mathbb{Z}_\ell$ as a character on μ_{ℓ^∞}.

2.4 Geometrical Argument: Modular version of Sinnott's proof

In this section, we review the modification of Sinnott's proof [29] of the theorem 1.1 due to Hida [9].

For a commutative ring R, let $\lambda : (\mathbb{Z}/N\mathbb{Z})^\times \to R^\times$ be an odd Dirichlet character, $\chi : \mathbb{Z}_\ell^\times \to \mu_{\ell^\infty}(R)$ and

$$\Phi_\lambda(t) = \sum_{n=1}^\infty \lambda(n) t^n = \frac{\sum_{a=1}^N \lambda(a) t^a}{1 - t^N}.$$

Considering the group scheme $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$, we have $\Phi_\lambda(t) \in \mathcal{O}_{\mathbb{G}_m, 1/R}$. Note that from the well-known result of Euler, we obtain

$$\Phi_\lambda(1) = L(0, \lambda) \tag{2.4.1}$$

Define an R-valued measure $d\mu_\lambda$ on $\mathbb{Z}_\ell$ by

$$\int_{\mathbb{Z}_\ell} f d\mu_\lambda = (\ell\lambda(\ell))^{-n} \sum_{x \in \mathbb{Z}/\ell^n\mathbb{Z}} f(x)\Phi(\zeta_n^x),$$

where $f : \mathbb{Z}/\ell^n\mathbb{Z} \to R$; i.e., f is a locally constant function on $\mathbb{Z}_\ell$ and ζ_n is the primitive ℓ^n-th root of unity . Note that this definition is independent of $\mathbb{Z}/\ell^n\mathbb{Z}$ where f is defined. For a primitive Dirichlet character $\chi : (\mathbb{Z}/\ell^n\mathbb{Z})^\times \to R^\times$, we have

$$\begin{aligned} \int_{\mathbb{Z}_\ell} \chi d\mu_\lambda(t) &= (\ell\lambda(\ell))^{-n} \sum_x \sum_m \chi(x)\lambda(m)(\zeta_n^x t)^m \\ &= (\ell\lambda(\ell))^{-n} \sum_m \Big(\sum_{x \in \mathbb{Z}/\ell^n\mathbb{Z}} \chi(x)\zeta_n^{mx} \Big) \lambda(m) t^m \\ &= \frac{G(\chi)}{(\ell\lambda(\ell))^n} \sum_m \chi^{-1}(m)\lambda(m)t^m \end{aligned}$$

Evaluating this sum at t=1, we have

$$\int_{\mathbb{Z}_\ell} \chi d\mu_\lambda = \frac{G(\chi)}{(\ell\lambda(\ell))^n} L(0, \chi^{-1}\lambda).$$

Since $\chi : \mathbb{Z}_\ell \to \mu_{\ell^\infty}$ is trivial on $\mu_{\ell-1}$, we have

$$\int_{\mathbb{Z}_\ell} \chi d\mu_\lambda = \int_{\mathbb{Z}_\ell{}^\times} \chi d\mu_\lambda = \int_\Gamma \chi d\mu_\Psi,$$

where $\Gamma = \mathbb{Z}_\ell{}^\times/\mu_{\ell-1}$, and $d\mu_\Psi$ is a measure on Γ associated with the formal function Ψ on μ_{ℓ^∞} such that

$$\Psi(t) = \sum_{\epsilon \in \mu_{\ell-1}} \Phi(t^\epsilon) = \sum_{\epsilon \in \mu_{\ell-1}/\{\pm 1\}} (\Phi(t^\epsilon) + \Phi(t^{-\epsilon})).$$

In other words,

$$\int_{\Gamma} \chi d\mu_{\Psi} = \sum_{x \in (\mathbb{Z}/\ell^n\mathbb{Z})^{\times}/\mu_{\ell-1}} \chi(x)\Psi(\zeta_n^x).$$

Now we consider the case that R is the algebraic closure $\overline{\mathbb{F}}_p$ of the finite field of characteristic p. Since Γ has a topological generator g, we have an 1-1 correspondence between the set of Dirichlet characters $\{\ \chi : \Gamma \to \mu_{\ell^\infty}\}$ and μ_{ℓ^∞}. More explicitly, the correspondence is $\chi \mapsto \chi(g)$ and if the conductor of χ is ℓ^n; i.e., $\ker(\chi) = 1 + \ell^n\mathbb{Z}_\ell =: \Gamma_n$ in Γ, then after fixing the primitive root of unity in μ_{ℓ^n} for each n, the character χ corresponds to a unique ζ_χ, which is an ℓ^n-th root of unity and $\chi(g)$=ζ_χ^ℓ. We have isomorphisms

$$1 + \ell\mathbb{Z}_\ell \xrightarrow{\log} \ell\mathbb{Z}_\ell,\ 1 + \ell\mathbb{Z}_\ell \xleftarrow{\exp} \ell\mathbb{Z}_\ell$$

and $g = \exp(\ell)$. As before, let m be the integer such that $k \cap \mu_{\ell^\infty} = \mu_{\ell^m}$. If $n \geq 2m$ and $x = \exp(\ell^{n-m}z) \in \Gamma_{n-m}$, then setting $v = n - m$ we have $2v \geq n$ and

$$\chi(x) = \chi(\exp(\ell))^{\ell^{v-1}z} = \zeta_\chi^{\ell^v z} = \zeta_\chi^{\log(x)} = \zeta_\chi^{x-1}$$

since $\log(x) = (x-1)$+$O((x-1)^2)$ and $\ell^{2v} \mid (x-1)^2$. Here we define

$$\Phi_a(t) = \frac{\sum_{r \equiv a^{-1}(\ell^m)} \lambda(r)t^r}{1 - t^{N\ell^m}},\ \text{where } 0\ \leq r\ < \ell^m N$$

for $a \in \mathbb{Z}_\ell{}^\times$. Following the similar process to obtain the formula (2.2.5), we have the following proposition.

Proposition 2.4.1. *Let n be the conductor of χ and $n \geq 2m$. Then we have*

$$\int_{\Gamma} \chi d\mu_{\Psi} = 0 \textit{ if and only if } \sum_{\epsilon \in \mu_{\ell-1}} \Phi_{\epsilon y}(\zeta_\chi^{\epsilon y}) = 0 \textit{ for all } y \in \Gamma.$$

Now we consider the regular function

$$F(\{t_\epsilon\}) = \sum_{\epsilon} \Phi_{\lambda,\epsilon}(t_\epsilon^{\alpha_1^*})$$

on $(\mathbb{Z}[\mu_{\ell-1}] \otimes \mathbb{G}_m)^{\mu_{\ell-1}}$ and

$$i^*F(\{t_\epsilon\}) = \sum_\epsilon \Phi_{\lambda,\epsilon}(t_\epsilon)$$

on $\mathbb{G}_m^{\mu_{\ell-1}}$. Suppose that $\mathbb{G}_m \subseteq V(F)$; i.e., F is identically zero on $\mathbb{G}_m$. Then i^*F is identically zero on μ_{ℓ^∞}. Since $j(\mu_{\ell^\infty})$ is Zariski dense in $V \cap \mathbb{G}_m^{\mu_{\ell-1}}$, i^*F is identically zero on it. Since i^*F is of the form in $\mathbb{G}_m^n$

$$\sum_{\epsilon \in \mu_{\ell-1}/\{\pm 1\}} \Phi_{\lambda,\epsilon}\left(\prod_i t_i^{c_i(\epsilon)}\right) + \Phi_{\lambda,-\epsilon}\left(\prod_i t_i^{-c_i(\epsilon)}\right),$$

and $\prod_i t_i^{c_i(\epsilon)}$ are pairwise $\mathbb{Z}$-mutiplicatively independent, they cannot be identically zero on $\mathbb{G}_m^n$. Hence, $\mathbb{G}_m \not\subseteq V(F)$. We know that the equality $L(0, \chi\lambda) = 0 \bmod \mathfrak{B}$ implies that $\sum_\epsilon \Phi_{\lambda,\epsilon}(\zeta_\chi^\epsilon) = 0$. Since these ζ_χ's are in the Zariski closed set $\mathbb{G}_m \cap V(F)$, we can conclude that there are finite number of such ζ_χ's.

Remark 2.4.1. The geometrical argument can be generalized to

2.5 Zariski density on the multi-dimensional torus

In this section, we try to generalize the previous discussions to totally real field and to get the analogue of many proposition. Following [16], one is able to develop a similar argument as before. Unfortunately, to get the non-vanishing result it seems that there should be more research on a quotient measure obtained by the unit group of the totally real field. There is an p-adic analytical approach which is discussed in [2] but it is not yet reported for the proof to be settled completely.

First we consider a special rational function on a multi-dimensional torus which is closely related to Hecke L-function over the totally real field F. Let F be a totally real field of degree d over $\mathbb{Q}$ and $\mathcal{O}_F$ the ring of integer of F. Denote by $(F \otimes \mathbb{R})_+$ the subset of $F \otimes \mathbb{R} \simeq \mathbb{R}^d$ which is $\mathbb{R}_+^d$ and by A_+ for any $A \subset F \subset F \otimes \mathbb{R}$ we mean $A_+ = A \cap (F \otimes \mathbb{R})_+$, the totally positive element of A.

By a the Shintani decomposition of F we mean a finite collection $\mathfrak{S}$ of finite subsets $V = \{v_1, \cdots, v_r\} \subset F_+$ such that (1) $\{v_i\}$ is linearly independent over $\mathbb{Q}$ (2) if we define $C(V) = \mathbb{R}_+$-span of $V \subset (F \otimes \mathbb{R})_+$, then we have a disjoint union

$$(F \otimes \mathbb{R})_+ = \bigcup_{e \in \mathcal{O}_+^\times} \bigcup_{V \in \mathfrak{S}} eC(V) \text{ or } (F \otimes \mathbb{R})_+/\mathcal{O}_F{}_+^\times = \bigcup_{V \in \mathfrak{S}} C(V).$$

For an integral ideal $\mathfrak{a}$, we have

$$\mathfrak{a}_+ = \bigcup_{e \in \mathcal{O}_+^\times} \bigcup_{V \in \mathfrak{S}} e(\mathfrak{a} \cap C(V)) \text{ or } \mathfrak{a}_+/\mathcal{O}_F{}_+^\times = \bigcup_{V \in \mathfrak{S}} (\mathfrak{a} \cap C(V)).$$

For each $V \in \mathfrak{S}$, define

$$R(V, \mathfrak{a}) = \mathfrak{a}_+ \cap \left\{ \sum_{v \in V} x_v v \in C(V) \;\middle|\; x_v \in \mathbb{Q}_+,\ 0 < x_v \leq 1 \right\}.$$

Then each element of $\mathfrak{a} \cap C(V)$ can be written uniquely in the form $r + \sum_{v \in V} n_v v$ for some $r \in R(V, \mathfrak{a})$ and integers $n_v \geq 0$. Since the set $R(V, \mathfrak{a})$ is finite, for a ring A, we can define a rational function on $\mathfrak{a} \otimes \mathbb{G}_{m/A} = \mathrm{Spec}(A[t^{\mathfrak{a}}])$ such that

$$\Phi(\mathfrak{a}, \xi; t) = \sum_{V \in \mathfrak{S}} \sum_{r \in R(V,\mathfrak{a})} \frac{\xi(r) t^r}{\prod_{v \in V} (1 - \xi(v) t^v)} \tag{2.5.1}$$

for $\xi \in \mathrm{Hom}(\mathfrak{a}, A^\times)$. One of important feature of this rational function on the multidimensional torus is that the evaluation at the point $1 \in \mathbb{G}_m \otimes \mathcal{O}_F$ is the special value of Hecke L-function of F. In other words, one defines a Dirichlet series for $\xi \in$ as $\mathrm{Hom}(\mathfrak{a}, \mathbb{C}^\times)$,

$$\zeta(\mathfrak{a}, \xi; s) = \sum_{\alpha \in \mathfrak{a}_+/\mathcal{O}_F^\times} \xi(\alpha) N(\alpha)^{-s}$$

and it can be showed that the Dirichlet series is defined in a right half plane of $\mathbb{C}$ and can be analytically continued to entire $\mathbb{C}$. Thanks to Shintani ([27]), one obtains

Theorem 2.5.1 ([27]). *$\zeta_{\xi,\mathfrak{a}}$ converges for some right half region of $\mathbb{C}$, extends to an entire function of s, and*

$$\zeta(\xi, \mathfrak{a}; 0) = \Phi(\mathfrak{a}, \xi; 1).$$

In order to explain the multi-dimensional analogue of Zariski density result (Theorem 2.3.1), we now consider the simplest case i.e. $h_F = 1$ and ℓ is inert in F. In this case, the class group $\mathrm{Cl}_F(\ell^\infty)$ is nothing but the quotient $\mathcal{O}_\ell{}^\times/\overline{\mathcal{O}_F^\times}$ of unit group and $d\mu_\lambda$ is a measure on $\mathcal{O}_\ell$. As we have done in abelian case, let $A = \overline{\mathbb{F}}_p$. Let k be the finite field over $\mathbb{F}_p$ generated by λ and the ℓ-th root of unity and $\mu_{\ell^\infty} \cap k = \mu_{\ell^{n_0}}$, $k_j = k(\mu_{\ell^{n_0+j}})$, and $K = \bigcup_{j \geq 1} k_j$.

Since $\Gamma = 1 + \ell\mathcal{O}_\ell \simeq \ell\mathcal{O}_\ell$ has topological generators $g_1, \cdots, g_d$, $d = [F : Q]$, we have an 1-1 correspondence between $\{ \chi : \Gamma \to \mu_{\ell^\infty} \}$ and $\mu_{\ell^\infty}^d \simeq \mathcal{O}_F \otimes \mu_{\ell^\infty}$. More explicitly, the correspondence is $\chi \mapsto (\chi(g_1), \cdots, \chi(g_d))$ and if the conductor of χ is ℓ^m i.e. $\ker(\chi) = 1 + \ell^m\mathcal{O}_\ell =: \Gamma_m$ in Γ, then χ correspond to $\alpha \otimes \zeta_\chi$ for some $\alpha \in \mathcal{O}_F$, which is in $\mathcal{O}_F \otimes \mu_{\ell^m}$. In fact, we have the isomorphism

$$1 + \ell\mathcal{O}_\ell \xrightarrow{\log} \ell\mathcal{O}_\ell,\ 1 + \ell\mathcal{O}_\ell \xleftarrow{\exp} \ell\mathcal{O}_\ell$$

and $g_i = \exp(\ell w_i)$, where $\{ w_i \}$ is a $\mathbb{Z}_\ell$-basis of $\mathcal{O}_\ell$. If $m \geq 2n_0$, $n = m - n_0$, and $x = \exp(\ell^n z) \in \Gamma_n$, $z = \sum_i a_i w_i$, $a_i \in \mathbb{Z}_\ell$, then

$$\chi(x) = \prod_i \chi(\exp(\ell w_i))^{\ell^{n-1}a_i} = \zeta_\chi^{\mathrm{Tr}(\alpha \log(x))} = \zeta_\chi^{\mathrm{Tr}(\alpha(x-1))}$$

since $\log(x) = (x-1) + O((x-1)^2)$ and $\ell^{2n} \mid (x-1)^2$.

We can prove similar one as in abelian case.

To obtain similar result as in abelian case, we need following proposition, which is an analogue of theorem in Sinnott[3].

Proposition 2.5.2. *Let $f(t) = \sum_i a_i t^{\alpha_i}$, $a_i \in \overline{\mathbb{F}}_p$, $\alpha_i \in \mathcal{O}_\ell$ be a formal function, and $Z(f)$ be the set of zeros of f in $\mathcal{O}_F \otimes \mu_{\ell^\infty}$. Then we have*

$$Z(f) \subseteq \left(\bigcup_{i \neq j} Z\left(t^{\alpha_i - \alpha_j} - 1\right) \right) \cup \mathcal{O}_F \otimes \mu_{\ell^{n_0}},$$

where $\mu_{\ell^{n_0}} = k \cap \mu_{\ell^\infty}$ and k is the finite field over $\mathbb{F}_p$ generated by a_i's and ℓ-th roots of unity.

Proof. Let $k{=}\mathbb{F}_p(a_i, \mu_\ell)$, $\mu_{\ell^{n_0}} = k{\cap}\mu_{\ell^\infty}$ and $k_n = k(\mu_{\ell^n})$. If $\alpha \otimes \zeta_n {\in} \mathcal{O}_F \otimes \mu_{\ell^\infty}$, $n{\geq}n_0{+}1$, satisfies that for all $i{\neq}j$, $\mathrm{Tr}(\alpha\alpha_i){\neq}\mathrm{Tr}(\alpha\alpha_j) \mod \ell^n$, then taking $\mathrm{Tr}_{k_n/k}$ after multiplying $\alpha_i \otimes \zeta_n$, we have

$$0 = \sum_j a_j \mathrm{Tr}_{k_n/k}(\zeta_n^{\mathrm{Tr}(\alpha\alpha_i - \alpha\alpha_j)}) = a_i[k_n : k].$$

Hence we have $b_i = 0$ for all i. This means that

$$\{\alpha \otimes \zeta_n \mid \forall i \neq j, \mathrm{Tr}(\alpha\alpha_i) \neq \mathrm{Tr}(\alpha\alpha_j)(\ell^n)\} \setminus \mathcal{O}_F \otimes \mu_{\ell^{n_0}} \subseteq \mathcal{O}_F \otimes \mu_{\ell^\infty} \setminus \mathrm{Z}(f).$$

Then we have

$$\begin{aligned} \mathrm{Z}(f) &\subseteq \{\alpha \otimes \zeta_n \mid \exists i \neq j,\ \mathrm{Tr}(\alpha\alpha_i - \alpha\alpha_j) = 0(\ell^n)\} \cup \mathcal{O}_F \otimes \mu_{\ell^{n_0}} \\ &= \left(\bigcup_{i \neq j} \{\alpha \otimes \zeta_n \mid \ \mathrm{Tr}(\alpha\alpha_i - \alpha\alpha_j) = 0(\ell^n)\} \right) \cup \mathcal{O}_F \otimes \mu_{\ell^{n_0}} \\ &= \left(\bigcup_{i \neq j} \mathrm{Z}(t^{\alpha_i - \alpha_j} - 1) \right) \cup \mathcal{O}_F \otimes \mu_{\ell^{n_0}}. \end{aligned}$$

□

Let W be the set of roots of unity in $\mathcal{O}_\ell^\times$ and $\mathcal{O}_F\{W\}$ be the group ring over W with coefficients in $\mathcal{O}_F$. Then we have natural homomorphism. $\pi : \mathcal{O}_F\{W\} \to \mathcal{O}_F[W]$ and this induces that $\mathcal{O}_F[W] \otimes \mathbb{G}_m \to \mathcal{O}_F\{W\} \otimes \mathbb{G}_m$. We have

$$(\mathcal{O}_F \otimes \mathbb{G}_m)^W \cong \mathcal{O}_F\{W\} \otimes \mathbb{G}_m,\ (\alpha_\epsilon \otimes \zeta_\epsilon)_\epsilon \mapsto \prod_\epsilon \alpha_\epsilon \cdot \epsilon \otimes \zeta_\epsilon.$$

Consider the map $\mathcal{O}_F \otimes \mu_{\ell^\infty} \to \mathcal{O}_F\{W\} \otimes \mathbb{G}_m$ defined by $\alpha \otimes \zeta \mapsto (\epsilon\alpha \otimes \zeta)_\epsilon$. Then the image is in the subvariety Y of $\mathcal{O}_F\{W\} \otimes \mathbb{G}_m$ defined by functions $\{t^\gamma - 1 | \gamma \in \ker(\pi)\}$, which is isomorphic to $\mathcal{O}_F[W] \otimes \mathbb{G}_m$. Let $a_1, \cdots, a_n$ be the maximal $\mathcal{O}_F$-linearly independent elements in W. Then the map $\mathcal{O}_F \otimes \mu_{\ell^\infty} \to Y$ is just the map $\mathcal{O}_F \otimes \mu_{\ell^\infty} \to (\mathcal{O}_F \otimes \mathbb{G}_m)^n$, $\alpha \otimes \zeta \mapsto (a_i\alpha \otimes \zeta)_{i=1}^n$. From above discussions, we are able to deduce the multi-dimensional version of the Theorem 2.3.1 as follows.

Proposition 2.5.3. *the map $\mathcal{O}_F \otimes \mu_{\ell^\infty} \to (\mathcal{O}_F \otimes \mathbb{G}_m)^n$, $\alpha \otimes \zeta \mapsto (a_i\alpha \otimes \zeta)_{i=1}^n$ has a Zariski dense image.*

Proof. First, we prove that for distinct $\beta_j \in \mathcal{O}_\ell \setminus \{0\}$, $j = 1, \cdots, m$, there exists $\alpha \in \mathcal{O}_F$ such that $\mathrm{Tr}(\alpha\beta_i) \neq 0$ for all i. Let $\{w_i\} \subseteq \mathcal{O}_F$ be a $\mathbb{Z}_\ell$-basis of $\mathcal{O}_\ell$ and $\alpha = \sum_i x_i w_i$, $x_i \in \mathbb{Z}_\ell$. Then $\mathrm{Tr}(\alpha\beta_i) = \sum_j x_j \mathrm{Tr}(\beta_i w_j)$. Set $v_i = (\mathrm{Tr}(\beta_i w_j))_{j=1}^d$, $d = [\mathcal{O}_\ell : \mathbb{Z}_\ell]$. Then $v_i \neq 0$. Now set $V_i = \left\{w \in \mathbb{Q}^d \mid v_i \cdot w = 0\right\}$. V_i's are proper subspace of $\mathbb{Q}^d$. Since $\mathbb{Q}^d$ is not the union of finite number of proper subspaces, there exits $(x_i) \in \mathbb{Q}^d$, $(x_i) \notin V_i$, $\forall\ i$. Then $\alpha = \sum_j x_j\beta_j$ works.

Now, let the image be the zero set of a polynomial $F(\{t_i\})$. Then all elements in $\mathcal{O}_F \otimes \mu_{\ell^\infty}$ are zeros of the formal function $f(t) = F(\{t^{a_i}\})$. Set $f(t) = \sum_i b_i t^{\alpha_i}$. We can choose $\alpha \in \mathcal{O}_F$ such that $\mathrm{Tr}(\alpha(\alpha_i - \alpha_j)) \neq 0$ for all $i \neq j$. Then for sufficiently large n, $\alpha \otimes \zeta_n$ is not a zero of f by above proposition. Hence, $f(t)$ has to be zero. But since a_i's are $\mathcal{O}_F$ linearly independent, each monomial in $F(\{t_i\})$ corresponds to different term in $f(t)$. Hence, $F(\{t_i\}) \equiv 0$. □

2.6 Vanishing of μ-invariant

In this section, we apply the homological idea to determine μ-invariant.

Let $\overline{\mathbb{Z}}_p$ be the integer ring of $\overline{\mathbb{Q}}_p$ and π be a uniformizer. Let X_N° be the punctured cylinder $\mathbf{C}/\mathbb{Z} - S_N$ and $X_N = X_N^\circ \cup \{\pm i\infty\}$, where $S_N = \{r/N \,|\, 1 \le r \le N-1\}$. Let $v(r)$ be the vertical line $r + i\mathbb{R}$ for $r \in \mathbb{Q}/\mathbb{Z} - S_N$ and $\mathcal{D}_0$ be the free abelian group generated by $v(r)$. Then we have the surjective map

$$\mathcal{D}_0 \to H_1(X_N, \{\pm i\infty\}, \mathbb{Z}\} \to 0,$$

where $H_1(X_N, \{\pm i\infty\}, R)$ is the first relative homology group, which is generated by classes $[v(r)]$. For a primitive Dirichlet character ψ of conductor N, we define

$$f_\psi(z) = \begin{cases} \sum_{n\ge 1} \psi(n) q^n & Im(z) > 0, \\ -\sum_{n \ge 1} \psi(-n) q^{-n} & Im(z) < 0 \end{cases}$$

for $q = \exp(2\pi i z)$. As shown in [12, Section 4], the differential $f_\psi(z)dz$ gives a cohomology class $\omega(f_\psi)$ in $H^1(X_N^\circ, \mathbf{C})$ and furthermore $G(\psi^{-1})\omega(f_\psi) \in H^1(X_N^\circ, \mathbb{Z}[\psi])$. Hence $\omega(f_\psi)$ is p-integral as long as $(p, N) = 1$. From the partial fraction expansion of f_ψ,

$$f_\psi(q) = \frac{\sum_{r=1}^N \psi(r) q^r}{q^N - 1} = \sum_{r \in (\mathbb{Z}/N\mathbb{Z})^\times} \frac{(N\psi(r))^{-1} G(\psi)}{q - \zeta_N^r},$$

we can think of $\omega(f_\psi)$ as an element in $H_1(X_N, \overline{\mathbb{Z}}_p)$.

Now we have

$$\sum_{r=0}^{p-1} f_\psi\big(z + \frac{r}{p}\big) = \sum_{n=1}^{\infty} \psi(n) \Big(\sum_{r=0}^{p-1} \exp(\frac{2\pi n i}{p})\Big) q^n = p\psi(p) f_\psi(z).$$

Now we rewrite above sum using the Hecke operator $T(p)$ as $f_\psi | T(p) = \psi(p) f_\psi$. Define a pairing $\langle -, - \rangle : H_1(X_N, \{\pm i\infty\}, \mathbb{Z}) \times H^1(X_N, \overline{\mathbb{Z}}_p) \to \overline{\mathbb{Z}}_p$ such that $\langle v, w \rangle = \int_v w$. We define a distribution σ_ψ on $\mathbb{Z}_p$ such that

$$\sigma_\psi(a + p^m \mathbb{Z}_p) = \frac{1}{\psi(p)^m} \langle v\big(\frac{a}{p^m}\big), \omega(f_\psi) \rangle.$$

Proposition 2.6.1. *σ_ψ is a measure on $\mathbb{Z}_p$. i.e. it is a bounded distribution.*

Proof. First, consider the sum

$$\langle v(\frac{a}{p^m}), \omega(f_\psi)|T(p)\rangle = \langle v(\frac{a}{p^m})|T(p), \omega(f_\psi)\rangle = \sum_{r=0}^{p-1} \langle v(\frac{a+rp^m}{p^{m+1}}), \omega(f_\psi)\rangle.$$

This can be interpreted as the distribution relation:

$$\sigma_\psi(a+p^m\mathbb{Z}_p) = \sum_{r=0}^{p-1} \sigma_\psi(a+rp^m+p^{m+1}\mathbb{Z}_p).$$

From the definition, we have $|\sigma_\psi(a+p^m\mathbb{Z}_p)|_p = |\langle v\big(\frac{a}{p^m}\big), \omega(f_\psi)\rangle|_p \leq |\omega(f_\psi)|_p$, where $|\omega(f_\psi)|_p$ is the maximum of $|\langle v, \omega(f_\psi)\rangle|_p$ for $v \in \mathcal{D}_0$. Hence σ_ψ is bounded. This finishes the proof. □

A power series $G(\sigma; T) \in \overline{\mathbb{Z}}_p[[T-1]]$ is associated to a p-adic measure σ. The correspondence is that

$$G(\sigma; T) = \int_{\mathbb{Z}_p} T^x d\sigma(x).$$

Then we define its μ-invariant $\mu(\sigma)$ of a measure σ as $\mu(G(\sigma;T))$, the minimum of the p-adic valuations of all coefficients of $G(\sigma;T)$. From this definition, we can easily deduce that $\mu(\sigma) > 0$ implies that $\sigma \equiv 0 \mod \pi$. Let $\mathbb{Z}_p^\times = \mu_{p-1} \times \Lambda$ be the decomposition into a finite subgroup μ_{p-1} and a subgroup Λ without a torsion, whose topological generator is γ. As did in [29], we define a gamma transform of σ such that

$$\Gamma(\sigma, s) = \int_{\mathbb{Z}_p^\times} \langle x\rangle^{-s} d\sigma(x) = \int_\Lambda x^{-s} d\widetilde{\sigma}(x),$$

where $\widetilde{\sigma} = \sum_{\eta \in \mu_{p-1}} \sigma \circ \eta$. We have $\Gamma(\sigma, s) = G(\widetilde{\sigma} \circ \gamma^x; \gamma^{-s})$ where $\gamma^x : \Lambda \xrightarrow{\sim} \mathbb{Z}_p$. Define the μ-invariant of the Gamma transform as $\mu(\Gamma(\sigma, s)) := \mu(G(\widetilde{\sigma} \circ \gamma^x; T))$.

We define $j : p^{-\infty}\mathbb{Z}/\mathbb{Z} \to H_1(X_N, \{\pm i\infty\}, \mathbb{Z})^{\frac{p-1}{2}}$ for $p^{-\infty}\mathbb{Z} = \cup_n p^{-n}\mathbb{Z}$ by

$$j\big(\frac{a}{p^n}\big) = \big(v\big(\frac{a\xi}{p^n}\big)\big)_{\xi \in U}.$$

Let $Z \subset \mathbb{Z}_p^\times$ and $M_n(Z)$ be the submodule of V generated by the image under j of $\big\{\frac{a}{p^n} \,\big|\, j \geq 0,\, a \in Z\big\}$. Now we state the homological independence.

Proposition 2.6.2. *For all sufficiently large n and any subset Z of $\mathbb{Z}_p^\times$ with non-zero measure, we have*

$$M_n(Z) \supseteq H_1(X_N, \mathbb{Z})^{\frac{p-1}{2}}.$$

In particular, $H_1(X_N, \mathbb{Z})^{\frac{p-1}{2}}$ is generated by the image of j.

Proof. See [31] □

For any $\omega \in H_1(X_N, \overline{\mathbb{Z}}_p)$, the vanishing $\langle H_1(X_N, \mathbb{Z}), \omega \rangle \equiv 0 \mod \pi$ implies that $\omega \equiv 0 \mod \pi$.

Need discuss basic facts about the p-adic L-function from [15].

Theorem 2.6.3. *Let ψ be an odd character. We have $\mu(\Gamma(\sigma_\psi, s)) = 0$.*

Proof. Assume on the contrary that we have $\mu(\Gamma(\sigma_\psi, s)) > 0$. This is equivalent to say that $\widetilde{\sigma}_\psi \circ \gamma^x \equiv 0 \mod \pi$. Now for each $a \in \Lambda$ and $m \geq 1$, we have

$$\widetilde{\sigma}_\psi(a + p^m\mathbb{Z}_p) = \sum_{\eta \in \mu_{p-1}} \sigma_\psi(\eta a + p^m\mathbb{Z}_p) = \sum_{\eta} \langle \omega(f_\psi), v(\frac{a\eta}{p^m}) \rangle \equiv 0 \mod \pi.$$

By Proposition 2.6.2, we can choose an arbitrary $\left(\frac{p-1}{2}\right)$-tuple $(v_\eta)_\eta \in H_1(X_N, \mathbb{Z})^{\frac{p-1}{2}}$ and, therefore get the vanishing $\langle H_1(X_N, \mathbb{Z}), \omega(f_\psi) + (-1)^*\omega(f_\psi) \rangle \equiv 0 \mod \pi$. So we have $2\omega(f_\psi) \equiv 0 \mod \pi$ since ψ is an odd character and from the partial fraction expansion of f_ψ we conclude that $f_\psi \equiv 0 \mod \pi$, which is the contradiction. □

Corollary 2.6.4. *The Iwasawa μ-invariant is 0.*

Proof. This comes from the formula [12, Theorem 1]

$$L_p(s, \psi\omega) = \langle N \rangle^{-s}\Gamma(\sigma_\psi, s).$$

Then $\mu = 0$ if and only if $\mu(\Gamma(\sigma_\psi, s)) = 0$ for all odd characters ψ. □

CHAPTER 3

Cyclotomic Modular L-values

It is remarkable that unlike the case of anti-cyclotomic twists of modular L-values there has been almost no successful achievement on this subject for about thirty years since Washington proved the theorem on the non-vanishing of Dirichlet L-values. As Hida and Vatsal have successfully attacked the anti-cyclotomic case with the machinery of Zariski density on the Hilbert modular Shimura variety and the Ratner's ergodic theory on the p-adic Lie group, it would be reasonable to think that it is required to develop new way of extending the original Ferrero-Washington's argument. The possible approach to the case of cyclotomic modular L-values could be hinted from the observation of the different expressions for the special L-values.

Usually the special L-values with anti-cyclotomic twists are described as the evaluation of an automorphic form of GL(2) at the special divisors on the modular variety such as the divisor of CM points. This is the reason why Zariski density should be studied to get the non-vanishing property. On the other hand, one would find that the special L-values of a modular form with Dirichlet twist is expressible as a cup product pairing between a relative homology class on the modular curve and a cohomology class associated with the modular form. This topological point of view, called modular symbol approach, is not new and has been utilized successfully by several mathematicians since it is invented by Mazur and Swinnerton-Dyer ([18]). The toric or equivalently cylindrical analogue of the modular symbol approach, called abelian modular symbols, is explained well in [12] and as we describe in Chapter 1, there is a nice homological argument to deduce the non-vanishing. Therefore the natural step we could take next is the generalization of the discussion in Chapter 1 to the case of modular L-values.

In this chapter, we shall try to adapt the idea of homological independence to the situation of modular curves $X_0(N)$ or $X_1(N)$ instead of the cylinder X_N°. In order to achieve it, we first discuss about the basic properties of modular symbols. Some algorithmic feature shall be explained and the conjectures about homological independence in the modular curve setting which is the analogue of Proposition 2.6.2 shall be presented. Finally under the assumption of these conjectures the non-vanishing of complex L-values is discussed. To make things simple, we restrict ourselves to the situation of modular forms of weight 2. Every description in this chapter could be generalized to higher weight after introducing coefficient modules and we expect that similar results could be obtained. We also expect that the homological independence can be applied to determine the vanishing of μ-invariant of p-adic modular L-functions using the discussions [19] and [8].

3.1 Modular Symbols of weight 2

In this section, we describe modular symbols and some conjectures mentioned in the introduction. Main references are [5] and [30]. We consider the homology groups $H_1(X, \{cusps\}, \mathbb{Z})$ and $H_1(X, \mathbb{Z})$ where X is the standard modular curve $X_0(N)$ or $X_1(N)$ of level N. Observe that from the exact sequence

$$0 = H_1(\{cusps\}, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \to H_1(X, \{cusps\}, \mathbb{Z})$$

we can regard $H_1(X, \mathbb{Z})$ as a submodue of $H_1(X, \{cusps\}, \mathbb{Z})$. There is a relative homology class $\{\alpha, \beta\}_\Gamma$, which is called a *modular symbol* of weight 2 for the congruence group $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$ correspondingly as X and is corresponding to a geodesic connecting α and β in X. They enjoy the following properties for all α, β, $\delta \in \mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$:

1. $\{\alpha, \beta\} + \{\beta, \delta\} + \{\delta, \alpha\} = 0$
2. $\{\alpha, \alpha\} = 0$
3. $\{\alpha, \beta\} = -\{\beta, \alpha\}$.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the symbols canonically by the action on $\mathfrak{H}^*$ and the action of Γ is defined to be trivial. It is well known that there is a surjective homomorphism $\phi : \Gamma \to H_1(X, \mathbb{Z})$ defined by $\gamma \mapsto \{z, \gamma(z)\}_\Gamma$ for any $z \in \mathfrak{H}^*$ and one has one more relation of modular symbols.

4. $\{z, ghz\} = \{z, gz\} + \{z, hz\}$ for all $g, h \in \Gamma$.

Let $\mathcal{M}$ be the free abelian group generated by $\{\alpha, \beta\}$. For the congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Q})$, we define $\mathcal{M}_2(\Gamma, \mathbb{Z})$ to be a quotient space of $\mathcal{M}$ by the relations $\gamma \cdot U - U$ for all $g \in \Gamma$ and $U \in \mathcal{M}$. The $\mathbb{Z}$-module $\mathcal{M}_2(\Gamma, \mathbb{Z})$ is called the space of *modular symbols of weight* 2 *for* Γ. For a general ring R, we can define $\mathcal{M}_2(\Gamma, R)$ in a similar way. The complex vector space $\mathcal{M}_2(\Gamma, \mathbb{C})$ is canonically isomorphic to $\mathcal{M}_2(\Gamma, \mathbb{Z}) \otimes \mathbb{C}$. Let $\mathcal{B}$ be the free $\mathbb{Z}$-module $\langle \{\alpha\} \big| \alpha \in \mathbb{P}^1(\mathbb{Q}) \rangle$. The action of $\mathrm{SL}_2(\mathbb{Q})$ on $\mathcal{B}$ is similar as before and hence we define $\mathcal{B}_2(\Gamma, \mathbb{Z})$ in the same way. We consider the canonical boundary map $\partial : \mathcal{M} \to \mathcal{B}$ given by

$$\partial : \{\alpha, \beta\} \mapsto \{\beta\} - \{\alpha\}.$$

Let $\mathcal{S}_2(\Gamma, \mathbb{Z}) = \ker \partial$. It is called the space of *cuspidal modular symbols*. Then we have

Proposition 3.1.1 ([22],[30]). *We have the exact sequences*

$$0 \to \mathcal{S}_2(\Gamma, \mathbb{Z}) \to \mathcal{M}_2(\Gamma, \mathbb{Z}) \xrightarrow{\partial} \mathcal{B}_2(\Gamma, \mathbb{Z}) \xrightarrow{\theta} \mathbb{Z} \to 0,$$

where $\theta(\sum_i c_i\{\alpha_i\}) = \sum_i c_i$ *and we have the isomorphism*

$$\mathcal{S}_2(\Gamma, \mathbb{Z}) \simeq H_1(X, \mathbb{Z}).$$

We define a pairing between $S_2(\Gamma) \oplus \overline{S_2(\Gamma)}$ and $\mathcal{M}_2(\Gamma, \mathbb{C})$ as follows.

$$\langle f_1 + f_2, \{\alpha, \beta\}\rangle = \int_\alpha^\beta f_1(z)dz + \int_\alpha^\beta f_2(z)d\overline{z},$$

where $f_1 \in S_2(\Gamma)$, $f_2 \in \overline{S_2(\Gamma)}$. Generally speaking, the pairing is degenerate. The space of elements $E \in \mathcal{M}_2(\Gamma, \mathbb{C})$ such that $\langle f, E\rangle = 0$ for all f is the space of *Eisenstein elements.* But if we restrict ourselves to the space $\mathcal{S}_2(\Gamma, \mathbb{C})$, one has the non-degeneracy of the pairing as follows :

Theorem 3.1.2 ([22]). *The pairing* $\langle \cdot\, , \cdot\rangle$ *is non-degenerate when it is restricted to* $S_2(\Gamma) \oplus \overline{S_2(\Gamma)} \times \mathcal{S}_2(\Gamma)$.

Observe that $j^{-1}\Gamma j = \Gamma$ for $j = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Define an operator $*$ on $g \in \Gamma$ and $z \in \mathfrak{H}$ such that $g^* = j^{-1}gj \in \Gamma$ and $z^* = -\overline{z} \in \mathfrak{H}$. One obtains that $(gz)^* = g^*z^*$. Therefore the operator $*$ gives a well-defined involution on X and on the modular symbols. Similarly the involution $*$ has an action on $S_2(\Gamma) \oplus \overline{S_2(\Gamma)}$ such that $f^*(z) = \overline{f(z^*)}$ for $f \in S_2(\Gamma) \oplus \overline{S_2(\Gamma)}$. It satisfies the followings:

$$f^*|g = (f|g^*)^*, \ \langle f^*, \gamma^*\rangle = \overline{\langle f, \gamma\rangle} \text{ for } g \in \Gamma \text{ and } \gamma \in \mathcal{M}_2(\Gamma).$$

Therefore the involution $*$ interchanges forms between $S_2(\Gamma)$ and $\overline{S_2(\Gamma)}$. The effect of the involution on the q-expansion of $f \in S_2(\Gamma)$ is

$$f^*(q) = \sum_{n=1}^{\infty} \overline{a(n, f)}q^n.$$

3.2 Manin Symbols

In this section, we discuss a special class of modular symbols, so called Manin symbols. The basic references are [20] and [5]. A Manin symbol $[\gamma]$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is defined as

$$[\gamma] = \gamma \cdot \{0, \infty\} = \{\gamma(0), \gamma(\infty)\} \in H_1(X, \{cusps\}, \mathbb{Z}).$$

Each modular symbol is actually a rational homology class from the Manin-Drinfeld theorem. In particular, the Manin symbol is also in $H_1(X, \mathbb{Q})$.

Theorem 3.2.1 ([17]). *All modular symbols are $\mathbb{Q}$-linear combinations of homology classes. In other words, we have*

$$\mathcal{M}_2(\Gamma, \mathbb{Z}) \subseteq H_1(X, \mathbb{Q}).$$

The right action of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ on Manin symbols $[\gamma']$ is defined as $[\gamma'] \cdot \gamma = [\gamma'\gamma]$. With the matrices $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, we have the following relations between Manin symbols for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$:

$$[\gamma] + [\gamma]\sigma = 0 \text{ and } [\gamma] + [\gamma]\tau + [\gamma]\tau^2 = 0. \tag{3.2.1}$$

And these are all possible relations between Manin symbols (see [20]). Let $\{\gamma_i\}$ be a right coset representative of Γ in $\mathrm{SL}_2(\mathbb{Z})$. Since $[\gamma'\gamma] = [\gamma]$ for all $\gamma' \in \Gamma$, a Manin symbol is equal to a unique $[\gamma_i]$ for some i. With the computational viewpoint, Manin's symbol is very practical to implement. The advantage of making use of the Manin symbol is that every modular symbols are representable as a linear combination of Manin symbols as follows.

For an arbitrary modular symbol $\{\alpha, \beta\}$, we split them into two parts $\{\alpha, \beta\} = \{0, \beta\} - \{0, \alpha\}$. Hence we may consider the symbols of the form $\{0, \alpha\}$. Now let us consider the continued fraction expansion of α and $\left\{\frac{p_j}{q_j} \,\middle|\, -2 \leq j \leq k\right\}$ be the convergent of the expansion with the convention $p_{-2} = q_{-1} = 0$ and $p_{-1} = q_{-2} = 1$. The following identity is well-known:

$$p_j q_{j-1} - p_{j-1} q_j = (-1)^{j-1} \text{ for } -1 \leq j \leq k.$$

Now setting $\gamma_j = \begin{pmatrix} (-1)^{j-1} p_j & p_{j-1} \\ (-1)^{j-1} q_j & q_{j-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, one has

$$\{0, \alpha\} = \sum_{j=-1}^{k} \left\{\frac{p_{j-1}}{q_{j-1}}, \frac{p_j}{q_j}\right\} = \sum_j \{\gamma_j \cdot 0, \gamma_j \cdot \infty\} = \sum_j [\gamma_j].$$

Therefore any modular symbol in $H_1(X_0(N), \{cusps\}, \mathbb{Z})$ can be expressed in terms of Manin symbols. Since each homology class is expressible as a modular symbol and each Manin symbol is a rational homology class by the Manin-Drinfeld theorem, we have the complete description of $H_1(X, \mathbb{Q})$ using the the basic relation (3.2.1).

One is also able to give a description of $H_1(X, \mathbb{Z})$ using Manin symbols. The modular symbol space $\mathcal{M}_2(\Gamma, \mathbb{Z})$ is generated by the Manin symbols and the integral homology group $H_1(X, \mathbb{Z})$ is nothing but the kernel of $\partial : \mathcal{M}_2(\Gamma, \mathbb{Z}) \to \mathcal{B}_2(\Gamma, \mathbb{Z})$ by Proposition 3.1.1. Once the basis of $\mathcal{M}_2(\Gamma, \mathbb{Z})$ is chosen explicitly, the computation of null space of the boundary map ∂ is an elementary linear algebra. Now let us specialize the discussion to the case of $\Gamma = \Gamma_0(N)$.

From Proposition 1.42 in [25], we have the list of coset representatives for $\Gamma_0(N)$ as follows:

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, (c, d) = 1,\ d|N,\ 0 \leq c < \frac{N}{d} \text{ with } ad - bc = 1 \right\}.$$

We choose more specific elements such that

$$\left\{\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & d\end{pmatrix}, \begin{pmatrix}* & *\\ u & v\end{pmatrix} \,\middle|\, \begin{matrix}0 \le c < N,\ 0 \le d < N,\ (d,N) > 1\\ 2 < u < N,\ u \mid N,\ (u,v) = 1,\ (v,N) > 1\end{matrix}\right\}$$

where the third elements have the equivalence relation

$$\begin{pmatrix}* & *\\ u & v\end{pmatrix} \sim \begin{pmatrix}* & *\\ u' & v'\end{pmatrix} \text{ if } uv' \equiv u'v \bmod N.$$

In total, every Manin symbols are written as $[c,1]$, $[1,d]$, $[u,v]$ with the condition in the above coset representatives.

3.3 Homological Independence

Now we consider modular symbols $\left\{0, \frac{a}{\ell^n}\right\}$ for an integer a with $\gcd(a,\ell) = 1$ and $n \ge 1$. Since $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \in \Gamma$ and $\{u,v\} = 0$ for all integers u, v, it is of no harm to think of a as an element of $\mathbb{Z}_\ell^\times$. We consider the map

$$j_n : \frac{\ell^{-n}\mathbb{Z}}{\mathbb{Z}} \to H_1(X, \{cusps\}, \mathbb{Z})^{\frac{\ell-1}{2}}, \frac{a}{\ell^n} \mapsto \left(\left\{\frac{a\eta}{\ell^n}, \infty\right\}\right)_{\eta \in \mu_{\ell-1}/\{\pm 1\}}$$

and the submodule $M_n(\ell)$ of $H_1(X,\mathbb{Z})$ generated by the image of j_n. Then let us state the conjecture which is an analogue of the Proposition 2.2.

Conjecture 3.3.1. *For almost all prime p, the p-adic valuation*

$$v_p\left(\left[H_1(X,\mathbb{Z})^{\frac{\ell-1}{2}} : M_n(\ell)\right]\right)$$

is 0 for all odd prime ℓ for all sufficiently large n.

In order to obtain more concrete statement, we consider a slight different situation. Let $\Gamma = \Gamma_0(N)$. For an integer a with $\gcd(a,\ell) = 1$, there are integers

u, v such that $\gamma = \begin{pmatrix} u & a \\ Nv & \ell^n \end{pmatrix} \in \Gamma$ for each n. The image under ϕ of this element is $\{0, \gamma(0)\} = \{0, \frac{a}{\ell^n}\} \in H_1(X_0(N), \mathbb{Z})$. Let $M_n(\ell)$ be the submodule of $H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}}$ generated by the modular symbols

$$\left(\left\{\frac{a\eta}{\ell^n}, \frac{b\eta}{\ell^n}\right\}\right)_{\eta \in \mu_{\ell-1}/\{\pm 1\}}$$

with $a, b \in 1+\ell^m\mathbb{Z}_\ell$ for some fixed positive integer m. We set $\nu(N) = \prod_{q|N}(q-1)$, where q is a prime divisor of the integer N. Then we consider another conjecture related to previous one.

Conjecture 3.3.2. *For all prime $p \nmid \nu(N)$ and odd prime ℓ, we have*

$$v_p\left(\left[H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}}) : M_n(\ell)\right]\right) = 0$$

for all sufficiently large n.

Consider the case $N = 11$. The modular curve $X_0(11)$ is of genus 1 with cusps 0 and ∞. Hence the rank of $H_1(X_0(11), \mathbb{Z})$ is 2. The right coset representative of $\Gamma_0(11)$ in $\mathrm{SL}_2(\mathbb{Z})$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \cdots, \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the corresponding modular symbols are

$$\{0, \infty\}, \{0, 1\} = 0, \left\{0, \frac{1}{2}\right\}, \cdots, \left\{0, \frac{1}{10}\right\}, \{\infty, 0\}.$$

Since every elements in $H_1(X_0(11), \mathbb{Z})$ are linear combination of the above Manin symbols which are in $H_1(X_0(11), \mathbb{Q})$ except the first and last ones, one can conclude that $H_1(X_0(11), \mathbb{Z})$ is generated by them except $\{0, \infty\}$.

One can expand the rational number $\frac{a}{\ell^n}$ in continued fractions and hence represent each modular symbol $\{0, \frac{a}{\ell^n}\}$ as the linear combination of Manin symbols.

For example, we have $11/7^3 = [0; 31, 5, 2]$ and their convergents are $1/31$, $5/156$, $11/7^3$. From this, in $H_1(X_0(11), \mathbb{Z})$ one obtains

$$\left\{0, \frac{11}{7^3}\right\} = \left\{0, \frac{1}{31}\right\} + \left\{\frac{1}{31}, \frac{5}{156}\right\} + \left\{\frac{5}{156}, \frac{11}{7^3}\right\} = \left[\begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix}\right] + 0 + 0.$$

We are able to verify the following proposition on some special case of above conjectures.

Proposition 3.3.3. *The index* $[H_1(X_0(11), \mathbb{Z}) : M_k(3)]$ *is* 1 *or* 5 *for all sufficiently large* k.

Proof. The module $H_1(X_0(11), \mathbb{Z})$ is a free module with basis

$$[8] = \left\{0, \frac{1}{8}\right\}, \ [9] = \left\{0, \frac{1}{9}\right\}.$$

We consider a submodule of M_k generated by $\left\{\frac{1+a\cdot 3^{k-2}}{3^k}, \frac{1+b\cdot 3^{k-2}}{3^k}\right\}$ for $a, b = 1$, 2, 4, 5, 7, or 8. By considering the continued fraction expansion of $\frac{1+a3^{k-2}}{3^k}$, we have the following calculations for sufficiently large k :

$$\begin{aligned}
\left\{0, \frac{1+3^{k-2}}{3^k}\right\} &= [8] + [-8^{-1}9] + [9^{-1}(3^{k-2}-1)] + [-(3^{k-2}-1)^{-1}3^k]. \\
\left\{0, \frac{1+2\cdot 3^{k-2}}{3^k}\right\} &= [4] + [-4^{-1}9] + [9^{-1}(3^{k-2}-5)] \\
&+ [(-3^{k-2}+5)^{-1}(-1+2\cdot 3^{k-2})] + [(1-2\cdot 3^{k-2})3^k]. \\
\left\{0, \frac{1+4\cdot 3^{k-2}}{3^k}\right\} &= [2] + [-2^{-1}9] + [9^{-1}(3^{k-2}-7)] \\
&+ [(-3^{k-2}+7)^{-1}(2+3^{k-2})] + [(2+3^{k-2})(-1+4\cdot 3^{k-2}].
\end{aligned}$$

$$\left\{0, \frac{1+5\cdot 3^{k-2}}{3^k}\right\} = [-2]+[2^{-1}7]+[-7^{-1}9]+[9^{-1}(3^{k-2}-2)]$$
$$+ [(-3^{k-2}+2)^{-1}(1+4\cdot 3^{k-2})]+[(1+4\cdot 3^{k-2})^{-1}3^k].$$
$$\left\{0, \frac{1+7\cdot 3^{k-2}}{3^k}\right\} = [-4]+[4^{-1}5]+[-5^{-1}9]+[9^{-1}(3^{k-2}-4)]$$
$$+ [(-3^{k-2}+4)^{-1}(1+2\cdot 3^{k-2})]+[(1+2\cdot 3^{k-2})^{-1}3^k].$$
$$\left\{0, \frac{1+8\cdot 3^{k-2}}{3^k}\right\} = [-9]+[9^{-1}(3^{k-2}-8)]$$
$$+ [(-3^{k-2}+8)^{-1}(1+3^{k-2})]+[(1+3^{k-2})^{-1}3^k].$$

Here by $[m]$ we denote $\left[\begin{pmatrix}1 & 0\\ m & 1\end{pmatrix}\right]$ and the inverse in the argument is the inverse modulo 11. Since $[m]$ depends only on m modulo 11, we conclude that the index divides 5 from the following calculations:

When $k \equiv 0 \bmod 5$,

$$\left\{\frac{1+2\cdot 3^{k-2}}{3^k}, \frac{1+4\cdot 3^{k-2}}{3^k}\right\} = -5[9], \left\{\frac{1+4\cdot 3^{k-2}}{3^k}, \frac{1+5\cdot 3^{k-2}}{3^k}\right\} = [8]+2[9].$$

When $k \equiv 1 \bmod 5$,

$$\left\{\frac{1+3^{k-2}}{3^k}, \frac{1+2\cdot 3^{k-2}}{3^k}\right\} = -[8]+3[9], \left\{\frac{1+4\cdot 3^{k-2}}{3^k}, \frac{1+5\cdot 3^{k-2}}{3^k}\right\} = [8]+2\cdot[9].$$

When $k \equiv 2 \bmod 5$,

$$\left\{\frac{1+3^{k-2}}{3^k}, \frac{1+2\cdot 3^{k-2}}{3^k}\right\} = 2[8]-[9], \left\{\frac{1+4\cdot 3^{k-2}}{3^k}, \frac{1+5\cdot 3^{k-2}}{3^k}\right\} = [8]+2\cdot[9].$$

When $k \equiv 3 \bmod 5$,

$$\left\{\frac{1+2\cdot 3^{k-2}}{3^k}, \frac{1+4\cdot 3^{k-2}}{3^k}\right\} = -[8]+3[9], \left\{\frac{1+4\cdot 3^{k-2}}{3^k}, \frac{1+5\cdot 3^{k-2}}{3^k}\right\} = -2[8]+[9].$$

When $k \equiv 4 \bmod 5$,

$$\left\{\frac{1+2\cdot 3^{k-2}}{3^k}, \frac{1+4\cdot 3^{k-2}}{3^k}\right\} = -3[8]-[9], \left\{\frac{1+7\cdot 3^{k-2}}{3^k}, \frac{1+8\cdot 3^{k-2}}{3^k}\right\} = -[8]-2[9].$$

This finishes the proof. □

A similar argument can be used to calculate the index for higher level and larger prime powers. In the last section we discuss how to compute the index and check the meaning of the result of computations.

3.4 Modular L-function twisted by Dirichlet characters

Let f be a cusp form of type (k, N, ϕ). It has the q-expansion

$$f(z) = \sum_{n=1}^{\infty} a(n, f)q^n, \;\; q = \exp(2\pi i z).$$

To the cusp form f and a Dirichlet character χ with conductor D, an L-function twisted by χ is associated such that

$$L(s, f, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n, f)}{n^s}$$

for some right half region of $\mathbb{C}$. Setting $\Lambda(s, f, \chi) = (D^2N)^{s/2}(2\pi)^{-s}\Gamma(s)L(s, f, \chi)$ and $f_\chi(z) = \sum_{n=1}^{\infty} \chi(n)a(n, f)q^n$, one shows that it is the Mellin transform of f_χ which is of type $(k, N, \phi\chi^2)$:

$$\Lambda(s, f, \chi) = (D^2N)^{s/2} \int_0^{\infty} f_\chi(iy)y^s \frac{dy}{y}. \tag{3.4.1}$$

Furthermore $\Lambda(s, f, \chi)$, hence, $L(s, f, \chi)$ can be analytically continued to the entire $\mathbb{C}$ and satisfies the functional equation (see [25])

$$\Lambda(s, f, \chi) = i^k \phi(D)\chi(N)D^{-1}G(\chi)^2\Lambda(k - s, f|W_N, \overline{\chi})$$

where $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ and $G(\chi) = \sum_{r=0}^{D-1} \chi(r)\exp(2\pi i r/D)$.

Let χ be a Dirichlet character with ℓ-power conductor and f be a normalized eigen cusp form of level N with weight 2. The field K_f generated by the Fourier coefficient $a(n, f)$ is a finite extension of $\mathbb{Q}$. From the expression (3.4.1) one gets the integral representation of the special L-values $L(f,\chi,1)$ as follows:

$$L(f,\chi,1) \;=\; 2\pi i \int_0^{i\infty} f_\chi(z)\,dz. \tag{3.4.2}$$

It is well-known fact ([12]) that there exists a transcendental period $\Omega_f \in \mathbb{C}$ such that $\dfrac{L(f,\chi,1)}{2\pi i\Omega_f}$ is algebraic over $\mathbb{Q}$ and for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, one has

$$\left(\frac{L(f,\chi,1)}{2\pi i\Omega_f}\right)^\sigma = \frac{L(f^\sigma,\chi^\sigma,1)}{2\pi i\Omega_f}$$

where $f^\sigma(q) = \sum_{n=1}^\infty a(n,f)^\sigma q^n$ is also an eigen cusp form.

3.5 Non-vanishing of Cyclotomic L-values

In this section, we study the non-vanishing of complex L-values twisted by Dirichelt characters of prime power conductors assuming the conjectures. From now on we restrict ourselves to the case of modular form with weight two. The present discussion can be extended to higher weights by introducing the coefficient module and the extension of definition of modular symbols with coefficient module [22]. Following the discussions in [1], we hope to be able to get the mod p non-vanishing result.

Let ψ be the restriction of χ to $\mu_{\ell-1} \subseteq (\mathbb{Z}/\ell^n\mathbb{Z})^\times$. The integral representation of the L-value in (3.4.2) can be written as

$$\begin{aligned}
\frac{L(f,\chi,1)}{2\pi i\Omega_f} &= \frac{1}{\Omega_f}\int_0^{i\infty} f_\chi(z)\,dz \\
&= \frac{1}{G(\chi^{-1})}\sum_r \chi^{-1}(r)\frac{1}{\Omega_f}\int_0^{i\infty} f\left(z+\frac{r}{\ell^n}\right)\,dz \\
&= \frac{1}{G(\chi^{-1})}\sum_r \chi^{-1}(r)\frac{1}{\Omega_f}\int_{\frac{r}{\ell^n}}^{i\infty} f(z)\,dz \\
&= \frac{1}{G(\chi^{-1})}\sum_r \chi^{-1}(r)\frac{1}{\Omega_f}\left\langle f, \left\{\frac{r}{\ell^n},\infty\right\}\right\rangle.
\end{aligned} \tag{3.5.1}$$

Arguing with the inversion formula for χ, we are able to check that the pairing $\frac{1}{\Omega_f}\left\langle f,\left\{\frac{r}{\ell^n},\infty\right\}\right\rangle$ is algebraic and for $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have

$$\left(\frac{1}{\Omega_f}\left\langle f,\left\{\frac{r}{\ell^n},\infty\right\}\right\rangle\right)^{\sigma}=\frac{1}{\Omega_f}\left\langle f^{\sigma},\left\{\frac{r}{\ell^n},\infty\right\}\right\rangle.$$

In order to establish the non-vanishing of the complex L-values by deducing a contradiction, we assume that the complex values $L(f,\chi,1)$ are zero for infinitely many χ's with the fixed $\psi=\chi|_{\mu_{\ell-1}}$. By multiplying $\chi(v)$ for $v\in 1+\ell\mathbb{Z}_\ell$ and applying $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/K_f(\psi))$ to the formula (3.5.1), we have

$$\sum_{r\in(\mathbb{Z}/\ell^n\mathbb{Z})^\times}\chi^{-\sigma}(rv^{-1})\frac{1}{\Omega_f}\left\langle f,\left\{\frac{r}{\ell^n},\infty\right\}\right\rangle=0$$

for infinitely many χ's. Adding up the last equality for $\sigma\in\mathrm{Gal}(K_f(\chi)/K_f(\psi))$ and decomposing $(\mathbb{Z}/\ell^n)^\times=\mu_{\ell-1}\times\Gamma/\Gamma_n$ with $\Gamma_n=1+\ell^n\mathbb{Z}$ and $\Gamma=1+\ell\mathbb{Z}$, we have

$$\sum_{\eta\in\mu_{\ell-1}}\psi^{-1}(\eta)\sum_{r\in\Gamma/\Gamma_n}\mathrm{Tr}_{K_f(\chi)/K_f(\psi)}(\chi^{-1}(rv^{-1}))\frac{1}{\Omega_f}\left\langle f,\left\{\frac{r\eta}{\ell^n},\infty\right\}\right\rangle=0.$$

Let m be the largest integer such that $\mu_{\ell^m}\subset K_f(\psi)$. The easy calculation shows that for $r\in\Gamma$, we have

$$\mathrm{Tr}_{K_f(\chi)/K_f(\psi)}(\chi^{-1}(r))=\begin{cases}[K_f(\chi):K_f(\psi)]\chi^{-1}(r) & \text{if } r\in\Gamma_{n-m}\\ 0 & \text{otherwise}\end{cases}.$$

Therefore the last equality reduces to

$$\sum_{\eta\in\mu_{\ell-1}}\psi^{-1}(\eta)\sum_{r\in\Gamma_{n-m}/\Gamma_n}\chi^{-1}(r)\left\langle f,\left\{\frac{rv\eta}{\ell^n},\infty\right\}\right\rangle=0.$$

Writing down the representatives of the quotient Γ_{n-m}/Γ_n explicitly, we have

$$\sum_{\eta}\psi^{-1}(\eta)\sum_{c\in\mathbb{Z}/\ell^m\mathbb{Z}}\chi^{-1}(1+\ell^{n-m}c)\left\langle f,\left\{\frac{v\eta}{\ell^n}+\frac{cv\eta}{\ell^m},\infty\right\}\right\rangle=0.$$

Since there are infinitely many χ's which give the vanishing, we may assume that $n > 2m$ and, then, $\chi^{-1}(1 + c\ell^{n-m}) = \zeta_m^c$ for a primitive ℓ^m-th root ζ_m of unity. We arrange the last equality as

$$\sum_{\eta} \psi^{-1}(\eta) \sum_{c \in \mathbb{Z}/\ell^m\mathbb{Z}} \zeta_m^c \left\langle f \middle| \begin{pmatrix} 1 & \frac{cv_0\eta}{\ell^m} \\ 0 & 1 \end{pmatrix}, \left\{ \frac{v\eta}{\ell^n}, \infty \right\} \right\rangle = 0.$$

Here $v \in v_0(1 + \ell^m\mathbb{Z})$. Denote by $f_a(z)$ for $a \in \mathbb{Z}_\ell^\times$ the sum

$$f_a(z) = \sum_{c \in \mathbb{Z}/\ell^m\mathbb{Z}} \zeta_m^c f \middle| \begin{pmatrix} 1 & \frac{ac}{\ell^m} \\ 0 & 1 \end{pmatrix} (z) = \sum_{n \equiv a^{-1}(\ell^m)} a_n q^n.$$

As is shown in [12], one can show that if $f \in S_2(\Gamma_1(N))$, then $f_a \in S_2(\Gamma_1(N\ell^{2m}))$ as follows : Let $\gamma \in \Gamma_1(N\ell^{2m})$, say $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we have

$$\gamma' = \begin{pmatrix} 1 & \frac{v}{\ell^m} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \frac{-d^2v}{\ell^m} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + c\frac{v}{\ell^m} & b + \frac{dv}{\ell^m} - \frac{d^2v}{\ell^m}(a + \frac{cv}{\ell^m}) \\ c & d - \frac{cd^2v}{\ell^m} \end{pmatrix}.$$

From this calculations, one can show that

$$f_a|\gamma = \sum_{c \in \mathbb{Z}/\ell^{m_0}\mathbb{Z}} \zeta_m^c f \middle| \begin{pmatrix} 1 & \frac{ac}{\ell^m} \\ 0 & 1 \end{pmatrix} \gamma = \sum_{c \in \mathbb{Z}/\ell^m\mathbb{Z}} \zeta_m^c f \middle| \gamma' \begin{pmatrix} 1 & \frac{acd^2}{\ell^m} \\ 0 & 1 \end{pmatrix} = f_a.$$

Since $f \in S_2(\Gamma_1(N))$ and $d \equiv 1 \mod N\ell^{2m}$, we conclude that $f_a \in S_2(\Gamma_1(N\ell^{2m}))$. Considering the sum over $\mu_{\ell-1}/\{\pm 1\}$, from the last equality we have

$$\sum_{\eta \in \mu_{\ell-1}/\{\pm 1\}} \psi^{-1}(\eta) \left\langle f_{v_0\eta}, \left\{ \frac{v\eta}{\ell^n}, \infty \right\} \right\rangle + \psi^{-1}(-\eta) \left\langle f_{-v_0\eta}, \left\{ \frac{-v\eta}{\ell^n}, \infty \right\} \right\rangle = 0.$$

The second pairing, z being replaced with $z^* = -\overline{z}$, equals

$$\psi^{-1}(-\eta) \left\langle \overline{f^*_{-v_0\eta}}, \left\{ \frac{v\eta}{\ell^n}, \infty \right\} \right\rangle$$

and the last vanishing is equal to

$$\sum_{\eta \in \mu_{\ell-1}/\{\pm 1\}} \left\langle \psi^{-1}(\eta) f_{v_0\eta} + \psi^{-1}(-\eta) \overline{f^*_{-v_0\eta}}, \left\{ \frac{v\eta}{\ell^n}, \infty \right\} \right\rangle = 0. \qquad (3.5.2)$$

Now we are ready to verify the non-vanishing of the complex L-values.

Theorem 3.5.1. *Assuming the conjectures, we have*

$$L(1, f, \chi) \neq 0$$

for almost all Dirichlet characters χ of conductor ℓ-power with a fixed ψ.

Proof. From Theorem 3.1.2, one is able to find a unique element $\gamma \in \mathcal{S}_2(\Gamma_1(N)) = H_1(X_1(N\ell^{2m}), \mathbb{C})$ such that

$$\int_\gamma g(z)dz = a(1, g), \text{ and } \int_\gamma h(z)d\bar{z} = 0$$

for all $g \in S_2(\Gamma_1(N\ell^{2m}))$ and $h \in \overline{S_2(\Gamma_1(N\ell^{2m}))}$. We choose γ so that it corresponds to the linear functional $S_2(\Gamma_1(N\ell^{2m}), \mathbb{C}) \to \mathbb{C}$ given by $g \mapsto a(1, g)$. Similar choice of $\gamma' \in \mathcal{S}_2(\Gamma_1(N\ell^{2m}), \mathbb{C})$ can be made so that

$$\int_{\gamma'} g(z)dz = 0, \text{ and } \int_{\gamma'} h(z)d\bar{z} = \overline{a(1, \bar{h})}.$$

For an arbitrary Hecke operator T and its action on γ, the conjecture enables us to choose $v_j \in 1 + \ell^m\mathbb{Z}_\ell$ and $c_j \in \mathbb{C}$, $\eta_0 \in \mu_{\ell-1}/\{\pm 1\}$ and n such that

$$\sum_j \left(c_j \left\{ \frac{v_j \eta}{\ell^n}, \infty \right\} \right)_\eta = (0, 0, \cdots, [T\gamma], \cdots, 0) \in H_1(X_1(N\ell^{2m}), \mathbb{Z})^{\frac{\ell-1}{2}}$$

where the non-trivial position corresponds to η_0. Setting $T = T(r)$, the standard Hecke operator for a positive integer r, the vanishing (3.5.2) becomes

$$\begin{aligned}
0 =& \left\langle \psi^{-1}(\eta_0) f_{v_0\eta_0} + \psi^{-1}(-\eta_0)\, \overline{f^*_{-v_0\eta_0}} \,,\, T(r)\gamma \right\rangle \\
=& \left\langle \psi^{-1}(\eta_0) f_{v_0\eta_0} | T(r) + \psi^{-1}(-\eta_0)\, \overline{f^*_{-v_0\eta_0}} | T(r) \,,\, \gamma \right\rangle \\
=& \psi^{-1}(\eta_0) a(r, f_{v_0\eta_0})
\end{aligned}$$

Hence $a(r, f_{v_0\eta_0}) = 0$ and similar calculations for γ' give us

$$a(r, \overline{f^*_{-v_0\eta_0}}) = \overline{a(r, f_{-v_0\eta_0})} = 0.$$

Since $f = \sum_{v_0,\eta} f_{v_0\eta} + f_{-v_0\eta}$ and $a(r, f_{v_0\eta}) = a(r, f_{-v_0\eta}) = 0$ for all η, r and $v_0 \in (1 + \ell\mathbb{Z}_\ell)/(1 + \ell^m\mathbb{Z}_\ell)$, we have $f = 0$ which is the contradiction. Therefore we conclude the theorem. □

Using the Proposition 3.3.3, we obtain

Corollary 3.5.2. *For a mormalized eigen cusp form $f \in S_2(\Gamma_0(11))$, we have*

$$L(1, f \otimes \chi) \neq 0$$

for almost all Dirichlet characters χ of 3-power conductors with a fixed $\psi = \chi|_{\{\pm 1\}}$.

Proof. Recall that $f(z) = (\Delta(z)\Delta(11z))^{\frac{1}{12}}$ since $S_2(\Gamma_0(11))$ is one-dimensional and spanned by f. In the previous proof, we have $K_f(\psi) = \mathbb{Q}$. Therefore $m = 0$ and the formula (3.5.2) becomes

$$\left\langle f + \overline{f^*}, \left\{ \frac{a}{3^n}, \infty \right\} \right\rangle = 0$$

for all $a \in 1 + 3\mathbb{Z}_3$ and infinitely many n's. From the Proposition 3.3.3, we verify the corollary in the same way of the theorem. □

3.6 Numerical Computation

In this section, we describe how we verify the conjecture computationally. We have the boundary map

$$\delta : \mathcal{M}_2(\Gamma_0(N), \mathbb{Z}) \to \mathcal{D}_0, \{\alpha, \beta\} \mapsto \{\alpha\} - \{\beta\}$$

for the abelian group $\mathcal{D}_0 \subset \mathcal{B}_2(\Gamma, \mathbb{Z})$ of divisors with zero degree of $X_0(N)$ supported on $\mathbb{P}^1(\mathbb{Q})$. From Proposition 3.1.1, one shows that

$$\ker(\delta) = \mathcal{S}_2(\Gamma, \mathbb{Z}) = H_1(X_0(N), \mathbb{Z}).$$

From the coset representatives of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ described in Section 3.2, the explicit generators of $H_1(X_0(N), \{cusps\}, \mathbb{Z}) \subseteq H_1(X_0(N), \mathbb{Q})$ and hence the explicit generators of $\ker(\delta)$ can be obtained using continued fraction expansions. Using the basis of $H_1(X_0(N), \mathbb{Z})$ consisting of the linear combinations of Manin symbols and the lattice reduction algorithm for $M_n(\ell)$, we checked the index $[H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}} : M_n(\ell)]$. Recall that $M_n(\ell)$ is the submodule of $H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}}$ generated by the $\frac{\ell-1}{2}$-tuples $\left(\left\{\frac{v\eta}{\ell^n}, \frac{w\eta}{\ell^n}\right\}\right)_{\eta \in \mu_{\ell-1}/\{\pm 1\}}$ for v, w in a subset Z of $1 + \ell\mathbb{Z}$ with non-trivial measure.

Note that if one tries to obtain the index, the required number of elementary calculations increases exponentially as n grows. In order to avoid a huge amount of operations, instead of computing $M_n(\ell)$, we consider its submodule which is reasonably simpler than $M_n(\ell)$. In fact we have considered the submodule generated by

$$\left\{\frac{1 + a \cdot \ell^{k-3}}{\ell^k}, \frac{1 + (a+1) \cdot \ell^{k-3}}{\ell^k}\right\} \in H_1(X_0(N), \mathbb{Z})$$

for $1 \le a \le \ell^3 - 1$ and $k - 3 > v_\ell(N)$ under the condition that $\ell^3 > 2g$ where g is the genus of $X_0(N)$. Recall that the $\mathbb{Z}$-rank of $H_1(X_0(N), \mathbb{Z})$ is $2g$. For the larger g, we choose ℓ^4 or even larger ℓ-power. The above memberships come from the criterion for the equivalence among cusps of $X_0(N)$ which can be found in [5].

Proposition 3.6.1 ([5])**.** *Let α, $\beta \in \mathbb{Q}$ be two cusps of $X_0(N)$ with the reduced form $\alpha = \frac{a}{c}$ and $\beta = \frac{b}{d}$. Then the two cusps are equivalent if and only if there exists a u with $(u, N) = 1$ such that*

$$c \equiv ud \bmod N, \text{ and } ua \equiv b \bmod (c, N).$$

Although the new index should be divisible by the index

$$\left[H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}} : M_n(\ell)\right], \tag{3.6.1}$$

in many cases of small prime number ℓ, it turned out that they are same.

It seems to be worthwhile to remark that when the level N is a prime number, the prime divisors of the index is Eisenstein in the sense of [21]; i.e., the prime divisors of the numerator of $\frac{N-1}{12}$. In the following table, we list the possible prime factors p of the index (3.6.1) in the last column according to N and ℓ^k. The blanks in the table represent for the repetition of the previous numbers.

Table 3.1: Prime Divisors of the Index

N	genus	ℓ	k_0	p	N	genus	ℓ	k_0	p
11	1	5	100	5	$5 \cdot 7$	3	5	100	2, 3
		7	100				7	70	
		11	100				11	50	
		13	100		$3 \cdot 13$	3	5	100	2
		17	70				7	100	
$2 \cdot 7$	1	5	100	3	41	3	5	100	2, 5
		7	100				7	100	
		11	100		$2 \cdot 5 \cdot 7$	9	5	100	2, 3
		13	70				7	70	
$3 \cdot 5$	1	5	100	2			11	70	
		7	100		71	6	5	100	5, 7
		11	70				7	75	
17	1	5	100	2	$7 \cdot 11$	7	5	70	2, 3, 5
		7	100				7	60	
		11	100		103	8	5	100	17
$2 \cdot 11$	2	5	100	5			7	100	
		7	100		$3 \cdot 5 \cdot 7$	13	5	80	2, 3
		11	100				7	80	
23	2	5	100	11			11	50	
		7	100				13	75	
		11	100						
$3 \cdot 11$	3	5	100	2, 5	$5^2 \cdot 7$	15	5	100	2, 3
		7	100				7	100	
		11	70				11	75	
		13	100		$5 \cdot 7^2$	21	5	100	2, 3

CHAPTER 4

Anti-cyclotomic Modular L-values

The original equi-distribution argument by Ferrero and Washington is successfully generalized by Vatsal, who uses ergodic theory on p-adic Lie group to get the information on the μ-invariant of p-adic anti-cyclotomic L-functions. Using the Gross-Zagier formula, Vatsal relates the special L-values to a sum of character and values of modular form at Heegner points on the Gross curve. On the other hand, a geometric interpretation of Sinnott's argument is devised by Hida as the Zariski density property of CM points on a Hilbert modular Shimura variety and this is one of main ingredient in the result of non-vanishing of Hecke L-values with anti-cyclotomic twists of ℓ-power conductors.

In this chapter, we prove the Theorem 4.8.3, which concerns the non-vanishing mod p of the special values of L-functions associated with an eigencusp form on $\mathrm{GL}_2(\mathbb{A})$ with anti-cyclotomic twists of ℓ-power conductors. Instead of Gross-Zagier formula we have used Harris-Kudla formula to get an expression of special L-values. The advantage of the use of Harris-Kudla formula is that we are able to consider weights higher than 2. Let us give a brief explanation about how the Zariski density plays a role in proving the non-vanishing of L-values.

We consider an eigencusp newform f for the congruence group $\Gamma_0(N)$ and set Ξ as a set of Heegner (CM) points associated with K of ℓ-power conductors. The special value $L(1, f, \theta_\chi)$ which is a Rankin product of f and the theta series θ_χ of the ring class character χ has a nice analogue of the formula (2.4.1). Check the

Theorem 4.7.7.

Let Z be the inverse limit $\varprojlim_n \operatorname{Pic}(\mathcal{O}_{\ell^n})$ of ring class groups of an imaginary quadratic field K with ℓ-power conductors (see section 3.1), which has the decomposition $Z = \Delta \times \Gamma$ such that Δ is finite and Γ is torsion-free subgroup with a topological generator. There is an injection $Z \hookrightarrow X_0(N)_{/\overline{\mathbb{F}}_p}$, $\mathfrak{a} \mapsto [\mathfrak{a}]$, of which the image is a collection Ξ of special points, so-called CM or Heegner points on $X_0(N)$ of ℓ-power conductors. Then for a certain algebraically independent subset U of Δ and the U-copy, $X_0(N)^U$ of X we consider the twisted diagonal map

$$Z \to X_0(N)^U, \ \mathfrak{a} \mapsto [\,\eta \cdot \mathfrak{a}\,]_{\eta \in U}.$$

The Zariski density result due to Hida [10] is that the image of the previous map is Zariski dense in $X_0(N)^U$. Check the Theorem 4.3.3 for the precise statement.

If one assumes the contrary of the non-vanishing result, i.e. Theorem 4.8.3, then by following the standard treatment by Sinnott we deduce

$$\sum_{\eta \in U} f_\eta([\,\eta\mathfrak{a}\,]) \equiv 0 \ (\bmod \mathcal{B})$$

for each $\mathfrak{a} \in \Gamma$ and some functions f_η obtained from f. Therefore the Zariski-density result enable us to conclude that $f_\eta \equiv 0 \ (\bmod \mathcal{B})$ for each $\eta \in U$. From this one may deduce that $f \equiv 0 \,(\bmod \mathcal{B})$ which is not the case and, hence, one is able to conclude the non-vanishing result.

With the assumption that d_K is odd and f is an eigen cusp form of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$, we show in Section 4.8 that

Theorem 4.0.2 (Theorem 4.8.3)**.** *With the previous restrictions and split prime ℓ in K we have the followings. For almost all anti-cyclotomic χ of ℓ-power conductor with a fixed restriction to Δ, we have*

$$\frac{\pi^{k-1} L\big(\frac{1}{2}, \pi_f \otimes \pi(\chi')\big)}{\Omega^{2k}} \not\equiv 0 \mod \mathcal{B},$$

where $L(s, \pi_f \otimes \pi(\chi'))$ is a Rankin-Selberg L-function for two automorphic representations π_f and $\pi(\chi')$.

We expect to be able to extend the result to the case of non-trivial level N which is square-free and its prime divisors split in K. Furthermore, it would be natural that one expects to extend above discussion to the case of quaternionic Shimura curve. The main difficulty in such extension is that the corresponding modular form doesn't have q-expansion as the complex points of the Shimura curve is compact. In the first few sections, we review the Shimura variety obtained from a quaternion algebra over $\mathbb{Q}$ and the description of Heegner (CM) points on it and automorphic representation associated with an eigencusp form.

4.1 Shimura Curves

Let D be a quaternion algebra over $\mathbb{Q}$. In other words, D is a central simple algebra of rank 4 over $\mathbb{Q}$. For a place v of $\mathbb{Q}$, set $D_v = D \otimes \mathbb{Q}_v$. Note that for a prime p, D_p is either a division algebra over $\mathbb{Q}_p$ or a matrix algebra $\mathrm{M}_2(\mathbb{Q}_p)$. In this case, p is said to be ramified or p is said to split respectively. D is said to be indefinite if it splits at ∞ and to be definite if it is ramified at ∞. By the discriminant N^- of D, we denote the product of all ramified primes p. An embedding $h : K \hookrightarrow D$ exists if and only if $K \otimes \mathbb{Q}_v$ is a field for all $v \mid N^-$. In particular, all imaginary quadratic fields have an embedding into $\mathrm{M}_2(\mathbb{Q})$.

Let D be an indefinite quaternion algebra over $\mathbb{Q}$. For each split prime q, fix an isomorphism $\phi_q : D_q \to \mathrm{M}_2(\mathbb{Q}_q)$. The order $R_q = \phi_q^{-1}(\mathrm{M}_2(\mathbb{Z}_q))$ is a maximal order of D_q. There is a unique maximal order in D_q for ramified prime q. The isomorphism ϕ_q is chosen so that $R_0 = D \cap \prod_q R_q$ is not empty. Then R_0 is a maximal order of D. For a (square-free) integer N^+ prime to the discriminant

N^-, an Eichler order R of level N^+ is defined to be

$$R = \left\{ x \in R_0 \,\middle|\, \phi_q(x) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod q \text{ for all } q \mid N^+ \right\}.$$

From the strong approximation theorem, one can deduce

$$D(\mathbb{A})^\times = D^\times \cdot (\widehat{R}^\times \times (D_\infty^\times)^+).$$

We consider a homogeneous space $S = \mathbb{C} - \mathbb{R} = \mathfrak{H} \cup \overline{\mathfrak{H}}$ on which the group $D^\times$ acts. Here $\mathfrak{H}$ is the upper half plane of $\mathbb{C}$. Using the isomorphism $\phi_\infty : D_\infty \to \mathrm{M}_2(\mathbb{R})$, there is an action of the group $D^\times$ on S. We observe that S can be identified with the set of algebra homomorphisms $\mathrm{Hom}(\mathbb{C}, D_\infty)$ as follows : Any algebra homomorphism $s : \mathbb{C} \to D_\infty$ gives rise to a group action of $\mathbb{C}^\times$ on S. There are two common eigenvectors $(z,1)^t$, $(\overline{z},1)^t$ for all elements γ of $s(\mathbb{C}^\times)$. Its eigenvalues are $j(\gamma, z)$ or $j(\gamma, \overline{z})$ respectively. Since the eigenvalues of $s(i)$ are i and $-i$, we can choose the homomorphism s so that $j(s(i), z) = i$. So we have the identification $S \cong \mathrm{Hom}(\mathbb{C}, D_\infty)$, $z \mapsto s$. Note that for $z \in S$, the converse is given by

$$z \mapsto s : i \mapsto \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \; cz^2 - 2az - b = 0, \; a^2 + bc = -1.$$

We consider the open Shimura curve

$$Y(\mathbb{C}) = D^\times \backslash S \times D(\mathbb{A}^{(\infty)})^\times / \widehat{R}^\times.$$

For the group $\Gamma = \phi_\infty(R^{(1)})$, which is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, we have the identification

$$\Gamma \backslash \mathfrak{H} \cong \phi_\infty(R^\times) \backslash S \cong Y(\mathbb{C}), \;\; z \mapsto [z, 1].$$

If the discriminant N^- is equal to 1, in other words, $D = \mathrm{M}_2(\mathbb{Q})$, the curve $Y(\mathbb{C})$ is nothing but the classical open modular curve $Y_0(N)(\mathbb{C}) = \Gamma_0(N) \backslash \mathfrak{H}$ of level

N. It can be compactified by adjoining a set of cusps and its compactification is denoted by $X_0(N)$. If $N^- \neq 1$; i.e., D is a division algebra, then Y is already compact.

The curve $Y_0(N)(\mathbb{C})$ has a nice interpretation as a moduli space of elliptic curves equipped with the cyclic subgroup of order N. For each point $z \in Y_0(N)(\mathbb{C})$, it corresponds to the pair (E_z, C_z) where

$$E_z = \frac{\mathbb{C}}{\mathbb{Z} + z\mathbb{Z}},\ C_z = \frac{\frac{1}{N}\mathbb{Z} + z\mathbb{Z}}{\mathbb{Z} + z\mathbb{Z}}.$$

Since any pair (E, C) over $\mathbb{C}$ is isomorphic to (E_z, C_z) for some $z \in \mathfrak{H}$ and $(E_z, C_z) \simeq (E_{z'}, C_{z'})$ exactly when $z' = \gamma z$ for some $\gamma \in \Gamma_0(N)$, the complex points $Y_0(N)(\mathbb{C})$ classify the pair in the above way. This moduli problem can be used to define an integral structure $Y_0(N)$ by interpreting it as a coarse moduli scheme representing a functor $\mathcal{P}_0(N)$ between $\mathbb{Z}[1/N]$-schemes and sets (see [6]).

First of all, let $N > 3$. The functor $\mathcal{P}_0(N)$ is defined as follows: for a $\mathbb{Z}[1/N]$-scheme S,

$$\mathcal{P}_0(N)(S) = \{(E, C)_{/S}\}/\simeq$$

where E is an elliptic curve over S and C is a finite flat subgroup scheme of E so that the geometric fibers are cyclic of order N and " $\simeq$ " means the isomorphism between the pairs. Then there is a $\mathbb{Z}[1/N]$-scheme $\mathcal{Y}_0(N)$ which is a coarse moduli scheme for $\mathcal{P}_0(N)$. In other words, for each $\mathbb{Z}[1/N]$-scheme S, there is a map $\phi_S : \mathcal{P}_0(N)(S) \to \mathcal{Y}_0(N)(S)$ and for a field k, $\mathcal{Y}_0(N)(k)$ is identified with $\mathcal{P}_0(N)(k)$ by ϕ_k. There is a similar description for $N = 2, 3$ and $\mathcal{Y}_0(1)$ is nothing but Spec$[\mathbb{Z}[j]]$ for j-invariant.

4.2 CM points

One can show that if there is an element α in $D^\times$ that fixes a point in $\mathfrak{H}$, then $\mathbb{Q}(\alpha) \subset D$ is an imaginary quadratic extension of $\mathbb{Q}$. Conversely, if there is an embedding $h : K \hookrightarrow D$ for an imaginary quadratic field, then the subgroup $h(K^\times)$ has a unique fixed point in $\mathfrak{H} \cap K$. The fixed point is called a *CM point* associated to the field K. Hence there exists an one to one correspondence between CM points and the embeddings. The CM points can be described with another view point : Consider an imaginary quadratic field K with $(disc(K), N) = 1$. One can extend each $h : K \hookrightarrow D$ to $h : \mathbb{C} \hookrightarrow D_\infty$ by extension of scalars and, hence, obtains the inclusion $\mathrm{Hom}(K, D) \subset \mathrm{Hom}(\mathbb{C}, D_\infty)$, and setting

$$\mathfrak{h}_N(K) := D^\times \backslash (\mathrm{Hom}(K, D) \times D(\mathbb{A}^{(\infty)})^\times)/\widehat{R}^\times,$$

we have $\mathfrak{h}_N(K) \subset Y(\mathbb{C})$. A point $[h, \gamma] \in Y(\mathbb{C})$ is called a CM point associated to K if it is in $\mathfrak{h}_N(K)$; i.e., $h \in \mathrm{Hom}(K, D)$. Observe that because of the above normalization of the map $S \cong \mathrm{Hom}(\mathbb{C}, D_\infty)$, one can verify that for $x \in K^\times$,

$$j(\Phi_\infty(h(x)), z) = x,$$

when z is associated with h by above correspondence. This is because $cz + d$ is a root of irreducible polynomial of x over $\mathbb{Q}$. By abuse of language, we set $z = [h, \gamma]$. Let $\mathcal{O}_c$ be an order of K with conductor c; i.e., $\mathcal{O}_c = \mathbb{Z}[c\omega]$, where the maximal order $\mathcal{O}_K$ of K is $\mathbb{Z}[\omega]$. We say that $[h, \gamma]$ is a CM point of conductor c associated to K or is a CM point associated to $\mathcal{O}_c$, if we have

$$h(K) \cap \gamma \widehat{R} \gamma^{-1} = h(\mathcal{O}_c).$$

Due to Shimura, we know that the CM point $[h, \gamma] \in \mathfrak{h}_N(K)$ is defined over K^{ab}. We set $\mathfrak{h}_N(K, c) = \mathfrak{h}_N(\mathcal{O}_c)$ as the set of CM points associated to $\mathcal{O}_c$. The

CM points $\mathfrak{h}_N(\mathcal{O}_c)$ with $(N, c) = 1$ are dependent only on N rather than on the decomposition $N = N^+N^-$ because $\mathfrak{h}_N(K, c)$ is non-empty exactly when N satisfies the following Heegner condition [3]:

$$N^+ = \prod_{q \text{ splits}} q, \; N^- = \prod_{q \text{ inert}} q.$$

We call a CM point $[h, \gamma]$ as Heegner point if $h(\mathcal{O}_K) \subset R$ and this condition is equivalent to $\chi_K(v) = 1$ for all $v \mid N^+$ and $\chi_K(v) = -1$ for all $v \mid N^-$ for a quadratic character χ_K associated with K.

Let $h_{\mathbb{A}} : K_{\mathbb{A}} \to D(\mathbb{A})$ be the extension of h by tensoring $\mathbb{A}$; i.e., $h_{\mathbb{A}} = h \otimes 1_{\mathbb{A}}$ and set $h_{\mathbb{A}}^{\gamma} = \gamma^{-1} h_{\mathbb{A}} \gamma$ for $\gamma \in D(\mathbb{A})^{\times}$. The Picard group of the order $\mathcal{O}_c$; i.e., the ring class group of K with the conductor c is

$$\mathrm{Pic}(\mathcal{O}_c) = K^{\times}_{\mathbb{A}^{(\infty)}} / K^{\times} \widehat{\mathcal{O}}_c^{\times}$$

which is isomorphic to the Galois group $\mathrm{Gal}(H_c/K)$ of the ring class field H_c of K using the Artin symbol. Using the homomorphism

$$K^{\times}_{\mathbb{A}^{(\infty)}} / K^{\times} \widehat{\mathcal{O}}_c^{\times} \xrightarrow{h_{\mathbb{A}}} D^{\times} \backslash D(\mathbb{A}^{(\infty)})^{\times} / \widehat{R}^{\times},$$

one has the action of $[x] \in \mathrm{Pic}(\mathcal{O}_c)$ on a CM point $[h, \gamma] \in \mathfrak{h}_N(\mathcal{O}_c)$ of the conductor c so that

$$[x] \cdot [h, \gamma] = [h, h_{\mathbb{A}}(x)\gamma].$$

This action is well-defined because one has $h_{\mathbb{A}}(\widehat{\mathcal{O}}_c) \subset \gamma \widehat{R} \gamma^{-1}$ from the definition of the CM point $[h, \gamma]$.

4.3 Zariski density of CM points

From now on we consider $D = \mathrm{M}_2(\mathbb{Q})$; i.e, $N^- = 1$. Since $\mathcal{O}_{\ell^{n+1}} \subseteq \mathcal{O}_{\ell^n}$, there is a canonical surjection $\mathrm{Pic}(\mathcal{O}_{\ell^{n+1}}) \twoheadrightarrow \mathrm{Pic}(\mathcal{O}_{\ell^n})$. Define

$$\mathrm{Pic}(\mathcal{O}_{\ell^\infty}) := \varprojlim_n \mathrm{Pic}(\mathcal{O}_{\ell^n}).$$

Let $\mathcal{O}_K = \mathbb{Z}[\omega]$. We consider an explicit collection of CM points $[h_n, 1]$ of conductor ℓ^n for a prime ℓ. The homomorphism $h_n : K \hookrightarrow \mathrm{M}_2(\mathbb{Q})$ is defined by

$$h_n : \omega \mapsto \begin{pmatrix} a + \mathrm{Tr}(\omega) & \frac{1}{\ell^n} \\ \ell^n b N^+ & -a \end{pmatrix},$$

where $-\mathrm{N}_{K/\mathbb{Q}}(\omega) = a^2 + a\mathrm{Tr}_{K/\mathbb{Q}}(\omega) + bN^+$. Then we have $h_m = \begin{pmatrix} 1 & 0 \\ 0 & \ell^{m-n} \end{pmatrix} \cdot h_n$. In the case that $\omega = \sqrt{d}$ for $d < 0$, $[h_n, 1]$ corresponds to $z_n = \frac{a+sgn(b)\sqrt{d}}{\ell^n b N^+}$.

Along with the CM points we defined above, each element of $\mathrm{Pic}(\mathcal{O}_{\ell^n})$ can be regarded as CM points on the modular curve as follows:

$$\mathrm{Pic}(\mathcal{O}_{\ell^n}) = K^\times_{\mathbb{A}^{(\infty)}}/K^\times \mathcal{O}^\times_{\ell^n} \overset{h_{n,\mathbb{A}}}{\longrightarrow} D^\times \backslash \mathrm{Hom}(\mathbb{C}, D_\infty) \times D(\mathbb{A}^{(\infty)})^\times / \widehat{R}^\times.$$

$$[x] \mapsto [h_n, h_{n,\mathbb{A}}(x)].$$

Let us describe the relations between these CM points. Let $m > n$ and $x \in K^\times_{\mathbb{A}^{(\infty)}}$. Because of the decomposition

$$\widehat{\mathcal{O}}^\times_{\ell^n} = \bigcup_{u \bmod \ell^{m-n}} (1 + u\ell^n\omega)\widehat{\mathcal{O}}^\times_{\ell^m},$$

the following set

$$\left\{ [x(1 + u\ell^n\omega)] \in \mathrm{Pic}(\mathcal{O}_{\ell^m}) \,\middle|\, u \bmod \ell^{m-n} \right\}$$

is the complete list of all elements, which map to $[x]$ in $\mathrm{Pic}(\mathcal{O}_{\ell^n})$ by the surjection $p_{n,m} : \mathrm{Pic}(\mathcal{O}_{\ell^m}) \twoheadrightarrow \mathrm{Pic}(\mathcal{O}_{\ell^n})$. Here $1 + u\ell^n\omega$ is an element in $\mathcal{O}_{\ell^n} \otimes \mathbb{Z}_\ell$; i.e. it is the ℓ-component $(1+u\ell^n\omega)_\ell$ of the idele element $1+u\ell^n\omega$ in $K \subseteq K^\times_{\mathbb{A}^{(\infty)}}$. From the calculation

$$h_{m,\mathbb{A}}((1+u\ell^n\omega)_\ell) = \begin{pmatrix} 1+(a+\mathrm{Tr}(\omega))u\ell^n & \frac{u}{\ell^{m-n}} \\ \ell^{m+n}N^+bu & 1-au\ell^n \end{pmatrix}_\ell = \begin{pmatrix} 1 & -\frac{u}{\ell^{m-n}} \\ 0 & 1 \end{pmatrix}_\ell \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

where the last matrix is in $\mathrm{GL}_2(\mathbb{Z}_\ell)$, we deduce that

$$[h_m, h_{m,\mathbb{A}}((1+u\ell^n\omega)_\ell)] = \left[\begin{pmatrix} 1 & \frac{u}{\ell^{m-n}} \\ 0 & 1 \end{pmatrix} h_n, 1\right].$$

In general, for each $x \in K^\times_{\mathbb{A}^{(\infty)}}$ we also have

$$[h_m, h_{m,\mathbb{A}}(x(1+u\ell^n\omega)_\ell)] = (x, K)\left[\begin{pmatrix} 1 & \frac{u}{\ell^{m-n}} \\ 0 & 1 \end{pmatrix} h_n, 1\right].$$

By approximation theorem, we can choose x in $K^\times_{\mathbb{A}^{(\ell\infty)}}$; i.e $x_\ell = 1$. Since the strong approximation implies $D^\times(\mathbb{A}^{(\ell\infty)}) = D^\times \cdot \widehat{R}^{(\ell)\times}$, we can write $h_{n,\mathbb{A}}(x) = d \cdot r$ for $d \in D^\times$ and $r \in \widehat{R}^\times$ with $r_\ell = 1$. Then $[h_n, h_n(x)] = [d^{-1}h_n, 1]$ and setting $U = \begin{pmatrix} 1 & u \\ 0 & \ell^{m-n} \end{pmatrix}$, we have

$$x \cdot [Uh_n, 1] = [Uh_n, UdrU^{-1}] = [Ud^{-1}h_n, UrU^{-1}].$$

Since $U(\widehat{R}^{(\ell)\times})U^{-1} \subseteq \widehat{R}^{(\ell)\times}$, it is equal to

$$[Ud^{-1}h_n, 1] = U \cdot [h_n, h_n(x)].$$

In total, we have the following result

Lemma 4.3.1. *For the projection* $p_{n,m} : \mathrm{Pic}(\mathcal{O}_{\ell^n}) \to \mathrm{Pic}(\mathcal{O}_{\ell_m})$, *we have*

$$\Big\{ [h_m, h_m(y)] \,\Big|\, p_{n,m}([y]) = [x] \Big\} = \Big\{ \begin{pmatrix} 1 & \frac{u}{\ell^{m-n}} \\ 0 & 1 \end{pmatrix} \cdot [h_n, h_n(x)] \,\Big|\, 0 \leq u < \ell^{m-n} \Big\}.$$

We define a Hecke operator on the coarse moduli scheme $\mathcal{Y}_0(N)$ as follows. Let $N' = Nd'_K$. The modular curve $X_0(N')$ classifies a pair (E, C) for an elliptic curve E and a cyclic subgroup C of order N'. For each divisor $d \mid d'_K$, we define an operator $[d]$ on the the pair (E, C) such that

$$(E, C)|[d] = (E/C_d, C/C_d),$$

where C_d is the cyclic factor of order d of C. The operator is the degeneracy map $[d] : X_0(N') \to X_0(N'/d)$. The operator $[d]$ sends a modular form f in $M_k(\Gamma_0(N), R)$ to a modular form $f|[d]$ in $M_k(\Gamma_0(dN), R)$. The corresponding action on the complex points $X_0(N')(\mathbb{C})$ is $[d] \cdot z = dz$. Note that

$$[d]([h, 1]) = \Big[\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \cdot h, 1\Big] = \Big[h, \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\Big].$$

Let q be a ramified prime in K, say $\mathfrak{q}^2 = (q)$ for a prime ideal $\mathfrak{q}$ in K. The $\mathfrak{q}$ corresponds $\varpi_{\mathfrak{q}} \in K$, a uniformizer of $K_{\mathfrak{q}} \subset K_{\mathbb{A}}^{\times}$. With this identification, the idele $\varpi_{\mathfrak{q}}$ generate Δ_0 and their action on the CM points is described as follows. Note that the following proposition is same as Cornut's description in [4].

Lemma 4.3.2. *Let $z \in X_0(qN)$ be a CM point. Then we have*

$$[q] \cdot z = \varpi_q \cdot (\pi_q(z)).$$

Proof. Let $[h, 1]$ be a CM point in $X_0(N)(\mathbb{C})$, where $h : K \to \mathrm{M}_2(\mathbb{Q})$ for some imaginary quadratic field K. Since $\varpi_{\mathfrak{q}} = \omega$ for a prime $q \mid d_K$, we have

$$\varpi_{\mathfrak{q}}([h_n, 1]) = [h_n, h_{n,\mathbb{A}}(\omega)_q] = \left[h_n, \begin{pmatrix} a + \mathrm{Tr}(\omega) & \frac{1}{\ell^n} \\ \ell^n N b & -a \end{pmatrix}_q\right].$$

We have

$$\begin{pmatrix} a + \mathrm{Tr}(\omega) & \frac{1}{\ell^n} \\ \ell^n N b & -a \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\ell^n} \\ \ell^n & -a - \mathrm{Tr}(\omega) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a\ell^n & d_K \end{pmatrix}.$$

Since $(d_K, N) = 1$, we can find integers x, y such that $Ny = -a\ell^n + d_K x$. Then we have

$$\begin{pmatrix} 1 & 0 \\ -a\ell^n & d_K \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Ny & d_K \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Ny & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & d_K \end{pmatrix}.$$

From the action of $\widehat{R}^\times$ and $D^\times$, we deduce that

$$\varpi_{\mathfrak{q}}([h_n, 1]) = \left[\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -Ny & 1 \end{pmatrix} \cdot h_n, 1\right] = \iota_q\left(\left[h_n, \begin{pmatrix} 1 & 0 \\ Ny & 1 \end{pmatrix}\right]\right).$$

Clearly $\pi_q\left(\left[h_n, \begin{pmatrix} 1 & 0 \\ Ny & 1 \end{pmatrix}\right]\right) = [h_n, 1]$ as $\begin{pmatrix} 1 & 0 \\ Ny & 1 \end{pmatrix} \in R^\times$. $\square$

Let Δ be a set of representatives of $\mathrm{Pic}(\mathcal{O}_{\ell^\infty})/\Delta_0\Gamma$. Each element $\eta \in \Delta$ can be chosen so that $\eta = \varpi_{\mathfrak{q}}$ for a split prime $q = \mathfrak{q}\mathfrak{q}^c$ with $(q, N\ell p) = 1$ and the projections η_Γ to Γ are all distinct. By applying a suitable element $g_0 \in \mathrm{GL}_2(\mathbb{A}^{(\infty)\times})$, we may assume that $h_n(\eta)$ is diagonal corresponds to the decomposition $K_q = K_{\mathfrak{q}} \oplus K_{\mathfrak{q}^c}$. Since $q \neq \ell$, $h_n(\eta)$ is independent of n. The previous setting gives us the identification $h_n(\eta) = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ since $\varpi_{\mathfrak{q}}$ corresponds to $(q, 1)$ in $K_{\mathfrak{q}} \oplus K_{\mathfrak{q}^c}$. Its action on the CM points z is given as

$$\eta \cdot z = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \cdot z = qz.$$

Obviously there is a moduli interpretation of the action and hence for each modular form f, we can define the action $f|[\eta]$ as before.

Let Δ be the coset representatives of $\mathrm{Pic}(\mathcal{O}_{\ell^\infty})/\Delta_0\Gamma$ for an ℓ-profinite subgroup Γ and the genus subgroup $\Delta_0 = \left\langle Frob_{\mathfrak{q}} \,\middle|\, \mathfrak{q}^2 = (q),\, q \,|\, d_K \right\rangle \subset \Delta$ and $U = \Delta/\Delta_0$. With the Heegner points $h_n \in \mathrm{Hom}(K, D)$ given in the previous section, we have an embedding

$$\varrho_n : \mathrm{Pic}(\mathcal{O}_{\ell^n}) \hookrightarrow Y(\mathbb{C})$$

given by $\varrho_n([x]) = [h_n, h_{n,\mathbb{A}}(x)]$. Let $\Xi = \cup_{n\geq 1}\varrho_n(\text{ Pic}(\mathcal{O}_{\ell^n})) \subset \mathfrak{h}_N(K, \ell^\infty)$. We consider the map j defined as

$$j : \Xi \to X^U_{/\overline{\mathbb{F}}_p},\ \varrho_n([x]) \mapsto \big(\ \varrho_n([x\eta])\ \big)_{\eta\in U} \text{ for } x \in \text{ Pic}(\mathcal{O}_{\ell^n}).$$

Here $X^U_{\overline{\mathbb{F}}_p}$ is the U-copy, $\prod_U X_{\overline{\mathbb{F}}_p}$ of the modular curve X over $\overline{\mathbb{F}}_p$. In [10], the author proves that the image of j is Zariski dense. More precisely, the author proves that

Theorem 4.3.3. *For $n_1 < n_2 < n_3 < \cdots$ a sequence of integers and the projection $\pi_n :\ Pic(\mathcal{O}_{\ell^n}) \to\ Pic(\mathcal{O}_{\ell^m})$ with a fixed m, the image under j of*

$$\Xi' = \left\{ \varrho_{n_j}([x]) \in X(\overline{\mathbb{F}}_p) \,\middle|\, [x] \in\ Pic(\mathcal{O}_{\ell^{n_j}}) \text{ with } [x] \in\ Ker(\pi_{n_j}),\ j = 1, 2, 3, \cdots \right\}$$

is Zariski dense in $X^U_{/\overline{\mathbb{F}}_p}$.

Hence if modular forms f_η on $X_0(N)_{/\overline{\mathbb{F}}_p}$ for $\eta \in U$ satisfy the relation

$$\sum_{\eta\in U} f_\eta(\rho_n([x\eta])) = 0$$

for infinitely many n's, then we are able to conclude from the theorem that $f_\eta = 0$ for each $\eta \in U$.

4.4 Rankin-Selberg L-function

An automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ is a function F on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$ satisfying the following conditions:

1. For some grossencharacter ψ on $\mathbb{A}^\times$ and the center $Z(\mathbb{A})$ of $\mathrm{GL}_2(\mathbb{A})$,

$$F(zg) = \psi(z)F(g) \text{ for all } z \in Z(\mathbb{A}) \text{ and } g \in \mathrm{GL}_2(\mathbb{A})$$

2. F is right K-finite.

3. Regarded as a function on $\mathrm{GL}_2(\mathbb{R})$, F is smooth and $\mathfrak{z}$-finite, where $\mathfrak{z}$ is the center of the universal enveloping algebra of $\mathrm{GL}_2(\mathbb{R})$.

4. F is slowly increasing in the sense that for all sufficiently large $|a|$ and compact subset T of $\mathrm{GL}_2(\mathbb{A})$ there exists a positive constant N such that

$$F\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) = O(|a|^N) \text{ for all } g \in T.$$

If it satisfies the following additional condition

$$\int_{\mathbb{Q}\backslash\mathbb{A}} F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0 \text{ for almost all } g,$$

then F is called a cuspidal automrophic form. Let $L^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}}, \psi)$ be the space of L^2-functions satisfying the condition (1). By $L_0^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}}, \psi)$ denote the collection of all functions in $L^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}}, \psi)$ satisfying the cuspidal condition.

Let f be a modular form of weight k and ψ a (Dirichelt) character for the group $\Gamma_0(N)$. In other words, for each $\gamma \in \Gamma_0(N)$ the holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ satisfies that

$$f\big|[\gamma](z) := f(\gamma(z))j(\gamma, z)^{-k} = \psi(\gamma)f(z).$$

For a modular form f with Nebentypus ψ and weight k, we can define a function F on $\mathrm{GL}_2(\mathbb{A})$ by

$$F(g) = f(g_\infty(i))j(g_\infty, i)^{-k} \det(g_\infty)^{k/2}\widetilde{\psi}(k_0),$$

where $g = \gamma g_\infty k_0$ and $\widetilde{\psi}$ is the grossencharacter canonically defined by ψ. Then F is called an adelic lift of f and is an automorphics form on $\mathrm{GL}_2(\mathbb{A})$ satisfying

1. $F(gk_0) = F(g)\widetilde{\psi}(k_0)$ for all $k_0 \in \prod K_p^N$

2. $F(gr(\theta)) = \exp(-ik\theta)F(g)$ for $r(\theta) \in \mathbf{SO}_2(\mathbb{R})$.

3. F is a function on $\mathrm{GL}_2^+(\mathbb{R})$ satisfies

$$\Delta F = -\frac{k}{2}\left(\frac{k}{2} - 1\right) F.$$

The group $\mathrm{GL}_2(\mathbb{A})$ acts on $L_0^2(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}), \psi)$ by the right translation. For an eigencusp form f with nebentypus ψ for all $T(p)$, $p \nmid N$, we define a representation π_f of $\mathrm{GL}_2(\mathbb{A})$ as the restriction of the action of $\mathrm{GL}_2(\mathbb{A})$ to the subspace spanned by right translates of F under $\mathrm{GL}_2(\mathbb{A})$. Then one can show that π_f is irreducible.

To the anti-cyclotomic character $\chi' : K_{\mathbb{A}}^\times / K^\times \to \mathbb{C}^\times$, one can also associate another automorphic representation $\pi(\chi')$ comes from a two dimensional representation of the Weil group $W_{K/\mathbb{Q}}$ given by χ'. The Rankin-Selberg L-function $L(s, \pi_f \otimes \pi(\chi'))$ is defined such that if the q-local factors are

$$\pi_{f,q} = \pi(\mu_1, \mu_2),\ \pi(\chi')_q = \pi(\nu_1, \nu_2)$$

for almost all q, then the Euler q-factor of the Rankin-Selber L-series is given by

$$L_q(s, \pi_{f,q}, \pi(\chi')_q) = \prod_{i,j} L(s, \mu_i \nu_j).$$

Let $\pi_{f,K}$ be the base change of π_f to K. One can obtain the identity

$$L(s, \pi_f \otimes \pi(\chi')) = L(s, \pi_{f,K} \otimes \chi')).$$

The Rankin Selberg L-function $L(s, \pi_f \otimes \pi(\chi'))$ satisfies the following functional equation:

$$L(s, \pi_f \otimes \pi(\chi')) = \varepsilon(s, \pi_f \otimes \pi(\chi'))L(1 - s, \widehat{\pi}_f \otimes \widehat{\pi}(\chi'))$$

where $\widehat{\pi}_f$ and $\widehat{\pi}(\chi')$ are the contragredient representations of π_f and $\pi(\chi')$ respectively. Since the central character of π_f is trivial, $\widehat{\pi}_f$ is isomorphic to π_f.

Note that one has $\widehat{\pi}(\chi') \simeq \chi'^{-2}\pi(\chi') = \pi(\chi'^{-1})$ as the central character of $\pi(\chi')$ is χ'^2. Since χ' is anti-cyclotomic, we have $\chi'^{-1} = \chi' \circ c$ for the complex conjugation c. For any non-split prime q in $\mathbb{Q}$, the q-component χ'_q is equal to $\chi'_q \circ c$ as c induces the identity in $\mathrm{Gal}(K_q/\mathbb{Q}_q)$. On the other hand, for split $q = \mathfrak{q}\mathfrak{q}^c$ we have $\pi(\chi')_q = \pi(\chi'_{\mathfrak{q}}, \chi'_{\mathfrak{q}^c})$ and $\pi(\chi' \circ c)_q = \pi(\chi'_{\mathfrak{q}^c}, \chi'_{\mathfrak{q}}) \simeq \pi(\chi'_{\mathfrak{q}}, \chi'_{\mathfrak{q}^c})$. From the discussion, we have $\widehat{\pi}(\chi') \simeq \pi(\chi')$. Therefore we have the following functional equation for anti-cycltomic χ':

$$L(s, \pi_f \otimes \pi(\chi')) = \varepsilon(s, \pi_f \otimes \pi(\chi'))L(1-s, \pi_f \otimes \pi(\chi')).$$

Evaluating at $s = \frac{1}{2}$, the functional equation reduces to

$$L(1/2, \pi_f \otimes \pi(\chi')) = \varepsilon(1/2, \pi_f \otimes \pi(\chi'))L(1/2, \pi_f \otimes \pi(\chi')).$$

So the need for computing the ε-factor arises.

The explicit expression of the ε-factors of the Rankin-Selberg L-function in terms of representation theory can be found in [**?**, Ch.V]. Since f is a modular form for $\mathrm{SL}_2(\mathbb{Z})$, for all finite places q the q-component of π_f is the principal series representation $\pi_{f,q} = \pi(\mu_1, \mu_2)$ for unramified characters μ_1, μ_2 of $\mathbb{Q}_q^\times$. For each q, let λ_q be a constant introduced in [**?**, §1] so that their product over all places is trivial. For non-split prime q and $\mu'_i = \mu_i \circ \mathrm{N}_{K_q/\mathbb{Q}_q}$, one has

$$\varepsilon(s, \pi_{f,q} \otimes \pi(\chi')_q) = \lambda_q^2 \varepsilon(s, \mu'_1\chi'_q)\varepsilon(s, \mu'_2\chi'_q).$$

Since μ'_i and χ'_q are all unramified, each ε-factors in the right hand side are all trivial. Let us consider a split prime q. Then one has

$$\varepsilon(s, \pi_{f,q} \otimes \pi(\chi')_q) = \lambda_q^2 \varepsilon(s, \mu_1\chi'_{\mathfrak{q}})\varepsilon(s, \mu_1\chi'_{\mathfrak{q}^c})\varepsilon(s, \mu_2\chi'_{\mathfrak{q}})\varepsilon(s, \mu_2\chi'_{\mathfrak{q}^c}).$$

If q is different from ℓ, then all characters in the above expression are unramified and, therefore, all ε-factors are 1. On the other hand, the ε-factors in the last equality at the prime $\ell = \mathfrak{l}\mathfrak{l}^c$ are

$$\varepsilon(s, \mu_i \chi'_{\mathfrak{l}}) = \ell^{n(s-1/2)} g(\chi'_{\mathfrak{l}}),\ \varepsilon(s, \mu_i \chi'_{\mathfrak{l}^c}) = \ell^{n(s-1/2)} g(\chi'_{\mathfrak{l}^c})$$

where $g(\chi'_{\mathfrak{l}})$, $g(\chi'_{\mathfrak{l}^c})$ are the normalized Gauss sums. Since $\chi'_{\mathfrak{l}^c} = \chi'^{-1}_{\mathfrak{l}}$, one has

$$g(\chi'_{\mathfrak{l}}) g(\chi'_{\mathfrak{l}^c}) = \chi'_{\mathfrak{l}}(-1).$$

At the infinite place, we set $\mathrm{N}(z) = |z|^2$ for $z \in \mathbb{C}^\times$. Then we have

$$\varepsilon(s, \pi_{f,\infty} \otimes \pi(\chi')_\infty) = \lambda_\infty^2 \varepsilon(s, \mathrm{N}^{(k-1)/2} \chi'_\infty) \varepsilon(s, \mathrm{N}^{-(k-1)/2} \chi'_\infty)$$

and from the identity $\mathrm{N}(z)^{\pm(k-1)/2} \chi'_\infty(z) = |z|^{\pm(k-1)} z^k |z|^{-k} = z^0 \bar{z}^{-k} |z|^{k \pm (k-1)}$, we conclude that

$$\varepsilon(s, \mathrm{N}^{(k-1)/2} \chi'_\infty) = \varepsilon(s, \mathrm{N}^{-(k-1)/2} \chi'_\infty) = i^k.$$

Putting all the terms together we have

$$\varepsilon(1/2, \pi_f \otimes \pi(\chi')) = i^{2k} \chi'_{\mathfrak{l}}(-1)^2 = 1.$$

4.5 Special Values of L-functions

Let $\mathrm{Gal}(K/\mathbb{Q}) = \langle c \rangle$ and $\chi : K_{\mathbb{A}}^\times / K^\times \to \overline{\mathbb{Q}}^\times$ be an arithmetic Hecke character of infinity type $k + 0 \cdot c$. In other words, we have $\chi_\infty(v) = v^k \overline{v}^0 = v^k$ for $v \in K_\infty = \mathbb{C}$. And set $\chi' = \chi |\cdot|^k$. Let z be a CM point of conductor c, which corresponds to $[h, 1]$. Let F be an automorphic form on $\mathrm{GL}_2(\mathbb{A})$ which is the adelic lift of f. For $\tau \in \mathrm{GL}_2(\mathbb{A})$ with $\tau_\infty(i) = z$, we set

$$\Sigma(z, \chi, F) = \frac{1}{h_c} \sum_{x \in \mathrm{Pic}(\mathcal{O}_c)} F(h_{\mathbb{A}}(x)\tau) \chi'(x).$$

In fact, if $d^\times x$ is a Haar measure on $K^\times \backslash K^\times_{\mathbb{A}^{(\infty)}}$ of which total volume is 1, then one has

$$\Sigma(z, \chi, F) = \int_{K^\times K_\infty^\times \backslash K_{\mathbb{A}}^\times} F(h_{\mathbb{A}}(x)\tau) \chi'(x) dx$$

since F is right $\widehat{R}^\times$-invariant.

Express $D = K + K \cdot \mathrm{J}$ for some $\mathrm{J} \in N_K(D^\times) \setminus K^\times$ such that $\mathrm{J}^\iota = -\mathrm{J}$ and $x\mathrm{J}{=}\mathrm{J}\overline{x}$ for all $x \in K^\times$. (This is an orthogonal decomposition for the norm form on D). Let f be λ-adically integral and p is unramified in K, then by following the proof of Proposition 2.8 in [24], one obtains the integrality of $\Sigma(z, \chi, F)$ as follows:

Proposition 4.5.1. *The algebraic number*

$$(2j(J,z)Im(z))^k h_K^2 \frac{(2\pi)^{2k}\Sigma(z,\chi,F)^2}{\Omega^{2k}}$$

is a $\mathcal{B}$-adic integer for all anti-cyclotomic χ of ℓ-power conductors.

Let us explain briefly how the integral values are connected to the special values $L(1/2, \pi_f \otimes \pi(\chi'))$ of Rankin-Selberg L-functions. By the work of Waldspurger, Harris and Kudla(see [14]), one can transform $\Sigma(z, \chi, F)^2$ to an integration of a theta lift $\theta_\varphi(F)$ against the character value χ' over a torus. Using see-saw duality, we transform the previous integration to one of theta lifts $\theta_\varphi(\chi')$ and $\theta_\varphi(1)$ of the character χ' and the constant function 1 against the value of F over $Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$. By Siegel-Weil formula, the theta lift of the constant function 1 is nothing but an Eisenstein series and, therefore, one obtains the special values of Rankin-Selberg $L(s, \pi_f \otimes \pi(\chi'))$ with some ambiguous local factors. It is Prasana([24]) who computed the local factors explicitly when χ' is unramified. As the grossencharacter χ' is ramified at ℓ, it is required to compute the local ℓ-factor separately.

In the next few sections, we shall compute the local ℓ-factor and, hence, a slight extension of the result in [24].

4.6 Theta correspondence

Let W be a symplectic space over $\mathbb{Q}$ and V be an orthogonal space of dimension d over $\mathbb{Q}$. There is a natural symplectic structure for $W \otimes V$ and for an additive character $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}$, the Weil representation ω_ψ of $\mathrm{Sp}(W \otimes V)$ is realized on the Schwartz space $\mathcal{S}(W_1 \otimes V(\mathbb{A}))$, where W has the isotropic decomposition $W = W_1 \oplus W_2$. Two groups $(\mathrm{Sp}(W), \mathrm{O}(V))$ form a dual reductive pair. By restricting, one can regard ω_ψ is a representation of $\mathrm{Sp}(W) \times \mathrm{O}(V)$ and the action of $\mathrm{O}(V)$ is given by the regular representation on $\mathcal{S}(W_1 \otimes V(\mathbb{A}))$. For 2 dimensional space W, $\mathcal{S}(W_1 \otimes V(\mathbb{A})) = \mathcal{S}(V(\mathbb{A}))$ and $\mathrm{Sp}(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A})$, and with this identification, one has the explicit action of ω_ψ,

1. $\omega_\psi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \phi(x) = \chi_V(a)|a|^{d/2}\phi(ax)$ for $\phi \in \mathcal{S}(V(\mathbb{A}))$.
2. $\omega_\psi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \phi(x) = \psi\left(\frac{\langle bx, x\rangle_V}{2}\right)\phi(x)$.
3. $\omega_\psi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi(x) = \gamma\widehat{\phi}(x)$.

Here $\widehat{\phi}$ is the Fourier transform

$$\int_{V(\mathbb{A})} \phi(y)\psi(-\langle x, y\rangle_V)dy,$$

where the Haar measure dy is given as $\widehat{\widehat{\phi}}(x) = \phi(-x)$ and γ is an 8-th root of unity dependent on ψ and $\langle \cdot, \cdot \rangle_V$. Following [14], we consider the extension of the action of ω_ψ to $\mathrm{GO}(V)$ and $\mathrm{G}(\mathrm{SL}(2) \times \mathrm{O}(V))(\mathbb{A})$ given by

$$\mathrm{G}(\mathrm{SL}(2) \times \mathrm{O}(V))(\mathbb{A}) = \left\{(x, y) \in \mathrm{GL}_2(\mathbb{A}) \times \mathrm{GO}(V)(\mathbb{A}) \,\middle|\, \det(x) = \nu(y)\right\}$$

where ν is a similitude character of $\mathrm{GO}(V)$. The action is defined by

1. For $h \in \mathrm{GO}(V)(\mathbb{A})$ and $\phi \in \mathcal{S}(V(\mathbb{A}))$, $h \cdot \psi\phi(v) = |\nu(h)|^{d/4}\phi(h^{-1}v)$.

2. For $(x, y) \in \mathrm{G}(\mathrm{SL}(2) \times \mathrm{O}(V))(\mathbb{A})$, $\omega_\psi(x, y) = \omega_\psi(x^{(1)}) \cdot y$ with $x^{(1)} = x\begin{pmatrix} 1 & 0 \\ 0 & \det(x)^{-1} \end{pmatrix}$.

For $\varphi \in \mathcal{S}(V(\mathbb{A}))$, we define a theta kernel by $\Theta_\varphi(g, g') = \sum_{v \in V} \omega_\psi(g, g')\varphi(x)$ and the theta lift $\theta_\varphi(F)$ of an automorphic form F of $\mathrm{GL}_2(\mathbb{A})$ to $\mathrm{GO}(V)(\mathbb{A})$ by

$$\theta_\varphi(F)(g) = \int_{\mathrm{GL}_2(\mathbb{Q})^{(1)} \backslash \mathrm{GL}_2(\mathbb{A})^{(1)}} \Theta_\varphi(hh', g)F(hh')d^{(1)}h$$

where $h' \in \mathrm{GL}_2(\mathbb{A})$ with $\det(h') = \nu(g)$. Similarly the theta lift $\theta_\varphi(G)$ of a form G on $\mathrm{GO}(V)(\mathbb{A})$ to $\mathrm{GL}_2(\mathbb{A})^{(\nu)}$ is defined by

$$\theta_\varphi(G)(g) = \int_{\mathrm{GO}(V)(\mathbb{Q})^{(1)} \backslash \mathrm{GO}(V)(\mathbb{A})^{(1)}} \Theta_\varphi(g, hh')G(hh')d^{(1)}h.$$

The subgroup $\mathrm{GL}_2(\mathbb{A})^{(\nu)}$ of $\mathrm{GL}_2(\mathbb{A})$ is given by $\mathrm{GL}_2(\mathbb{A})^{(\nu)} = \det^{-1}\nu(\mathrm{GO}(V)(\mathbb{A}))$.

Now we set $V = D = \mathrm{M}_2(\mathbb{Q})$ with the quadratic form $\langle a, b \rangle = ab^\iota + a^\iota b$. Define $\rho : D^\times \times D^\times \to \mathrm{GO}(D)$ by $\rho(a, b)(d) = adb^{-1}$. The the image of ρ is $\mathrm{GO}(D)^\circ$, the identity component of $\mathrm{GO}(D)$. Let $\widetilde{\theta}_\varphi(F)$ be the pull-back of $\theta_\varphi(F)$ to $D^\times(\mathbb{A}) \times D^\times(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$. Now we need to extend the following formula verified in [24, 37] to include the case of a ramified prime. We have

Proposition 4.6.1.

$$\widetilde{\theta}_\varphi(F) = F \otimes F.$$

Proof. For a cusp form f of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$ and Schwarz-Bruhat function φ on $\mathrm{M}_2(\mathbb{A})$, we set $\Theta(\varphi) = \sum_{v \in \mathrm{M}_2(\mathbb{Q})} \varphi^{(\infty)}(v)\overline{\varphi}_\infty(v)$,

$$\Theta(\varphi)(\tau; z, w) = (\mathrm{Im}(z)\mathrm{Im}(w)\mathrm{Im}(\tau))^{-k}\Theta(\varphi)(\tau; z, w).$$

and

$$\Theta_\varphi(f) = \int_{\Gamma\backslash\mathfrak{H}} \Theta(\varphi)(\tau; z, w) f^*(\tau)\mathrm{Im}(\tau)^k d\mu(\tau)$$

for a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ fixing $f(\tau)\Theta(\varphi)(\tau)$. Here $d\mu(\tau) = \frac{dxdy}{y^2}$ for $\tau = x + iy$, and $f|_k\alpha(z) = \det(\alpha)^{k-1} f(\alpha w) j(\alpha, w)^{-k}$ for $\alpha \in \mathrm{M}_2(\mathbb{Q})$ with $\det(\alpha) > 0$. Recall that $j = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$ and $f^*(z) = \overline{f(z^*)}$. It is not hard to reduce the previous definition of $\widetilde{\theta}_\varphi(F)$ to this classical setting. From Theorem 3.2 in [11], we have

$$\Theta_\varphi(f) = (2i)^k \sum_{\substack{\alpha \in \Gamma\backslash \mathrm{M}_2(\mathbb{Q}) \\ \det(\alpha)>0}} \widetilde{\varphi^{(\infty)}}(j\alpha) \exp(2\pi i \det(\alpha) z) f|_k\alpha(w) \tag{4.6.1}$$

where $\widetilde{\varphi^{(\infty)}}$ is the partial Fourier transform defined by

$$\widetilde{\varphi^{(\infty)}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \int_{\mathbb{A}^2} \varphi^{(\infty)}\begin{pmatrix} x & y \\ c & d \end{pmatrix} \psi_{\mathbb{A}}(ay - bx) dxdy$$

for the additive character $\psi_{\mathbb{A}}$ with $\psi_\infty(z) = \exp(2\pi i z)$. The partial Fourier transform of the function φ in (4.7.2) is

$$\widetilde{\varphi^{(\infty)}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbb{I}_{1+\ell^n\widehat{\mathbb{Z}}}(a)\mathbb{I}_{\widehat{\mathbb{Z}}}(b)\mathbb{I}_{\ell^n(1+\ell^n\widehat{\mathbb{Z}})}(c)\mathbb{I}_{\widehat{\mathbb{Z}}}(d)$$

which is supported on

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{c} b, d \in \widehat{\mathbb{Z}},\ a \in 1 + \ell^n\widehat{\mathbb{Z}}, \\ c \in \ell^n(1 + \ell^n\widehat{\mathbb{Z}}) \end{array} \right\} = L_n \cdot \widehat{\Delta}$$

where

$$\widehat{\Delta} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{c} a \in 1 + \ell^n\widehat{\mathbb{Z}}, \\ b, d \in \widehat{\mathbb{Z}}, c \in \ell^{2n}\widehat{\mathbb{Z}} \end{array} \right\},\ L_n = \begin{pmatrix} 1 & 0 \\ \ell^n & 1 \end{pmatrix}.$$

Now we set $\Gamma = \widehat{\Delta} \cap \mathrm{SL}_2(\mathbb{Q})$ and $\Delta = \{\alpha \in \mathrm{M}_2(\mathbb{Q}) \mid \det(\alpha) > 0\} \cap \widehat{\Delta}$. Then the formula (4.6.1) becomes

$$\begin{aligned}\Theta_\varphi(f) &= (2i)^k \sum_{\alpha \in \Gamma\backslash\Delta} \exp(2\pi i \det(\alpha) z) f|_k L_n \alpha(w) \\ &= (2i)^k \sum_{m=1}^{\infty} \exp(2\pi i m z) f|T(m)(w) \\ &= (2i)^k \sum_{m=1}^{\infty} a(m,f) \exp(2\pi i m z) f(w) = (2i)^k f(z) f(w).\end{aligned}$$

This proves the proposition. □

4.7 Waldpurger's computation of $\Sigma(z,\chi,F)^2$

Let χ be an anti-cyclotomic Hecke character on $K_{\mathbb{A}}^\times$ of infinite type $(k,0)$ that is unramified at all places except ℓ. Let $j : K \hookrightarrow D$ be an embedding associated with a Heegner point. Using this, we regard $K_{\mathbb{A}}^\times$ as a subgroup of $D^\times(\mathbb{A})$. For the orthogonal decomposition $D = D_1 \oplus D_2 = K \oplus \mathrm{J} \cdot K$, we have $\mathrm{GO}(D_1)^\circ = \mathrm{GO}(D_2)^\circ = K^\times$ and, hence, $\mathrm{G}(\mathrm{O}(D_1) \times \mathrm{O}(D_2))^\circ = \mathrm{G}(K^\times \times K^\times)$. Then we have

$$\begin{aligned}\Theta_\varphi(g,(x\tau,y\tau)) &= \sum_{d \in D} \omega_\psi(g) L(x\tau,y\tau)\varphi(d) \\ &= \sum_{d_1,d_2} \omega_\psi(g)\varphi(\tau^{-1}x^{-1}(d_1+d_2)y\alpha) \\ &= \sum_{d_1,d_2} \omega_\psi(g)\varphi(\tau^{-1}(x^{-1}d_1 y + x^{-1}d_2 y)\tau) \\ &= \sum_{d_1,d_2} \omega_\psi(g)\varphi(\tau^{-1}(x^{-1}y d_1 + (x^{-1}\overline{y} d_2)\tau) \\ &= \Theta_{\varphi'}(g,(xy^{-1}, x\overline{y}^{\,-1}))\end{aligned}$$

where $\varphi'(d) = \varphi(\tau^{-1} d\tau)$. Hence The restriction of ρ to $K^\times \times K^\times$ is $\rho(x,y) = (xy^{-1}, x\overline{y}^{-1})$. Since χ is anti-cyclotomic, we have $\chi'(y\overline{y}) = 1$ for all $y \in K_{\mathbb{A}}^\times$ and

$$\chi'(xy) = \chi'(x\overline{y}^{-1})\chi'(x\overline{y}) = \chi'(x\overline{y}^{-1}).$$

Setting $\mathrm{T} = \mathrm{G}(\mathrm{O}(D_1)^\circ \times \mathrm{O}(D_2)^\circ)$, we obtain

$$\begin{aligned}\Sigma(z,\chi,F)^2 &= \int\limits_{(K^\times K_\infty^\times \backslash K_{\mathbb{A}}^\times)^2} F(x\tau)F(y\tau)\chi'(xy)d^\times x d^\times y \\ &= \int\limits_{(K^\times K_\infty^\times \backslash K_{\mathbb{A}}^\times)^2} \widetilde{\theta}_\varphi(F)(a\tau, b\tau)\chi'(ab)d^\times a d^\times b \\ &= \int\limits_{\mathrm{T}(\mathbb{Q})\ \mathrm{T}(\mathbb{R})\backslash\ \mathrm{T}(\mathbb{A})} \theta_{\varphi'}(F)\big|_{\ \mathrm{T}(\mathbb{A})}(a,b)\chi'(b)d^\times a d^\times b\end{aligned}$$

The theta lift $\theta_\varphi(1\otimes\chi')$ lives on $\mathrm{GL}_2(\mathbb{A})^{(\mathrm{N})} = \det^{-1}\mathrm{N}(K_{\mathbb{A}}^\times)$ for $\mathrm{N} = \mathrm{N}_{K/\mathbb{Q}}$. It can be extended to $\mathrm{GL}_2(\mathbb{A})$ by making it left invariant under $\mathrm{GL}_2(\mathbb{Q})$ and setting it zero outside $\mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2(\mathbb{A})^{(\mathrm{N})}$. Note also that if $\varphi = \sum_\varphi \varphi_1\otimes\varphi_2$ for $\varphi\in \mathcal{S}(D(\mathbb{A})) = \mathcal{S}(D_1(\mathbb{A}))\otimes\mathcal{S}(D_2(\mathbb{A}))$ one obtains

$$\theta_\varphi(1\otimes\chi') = \sum_\varphi \theta_{\varphi_1}(1)\theta_{\varphi_2}(\chi').$$

From the see-saw duality (see [14]), we have

$$\begin{aligned}\Sigma(z,\chi,F)^2 &= \int\limits_{\mathrm{Z}(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})} F(g)\theta_\varphi(1\otimes\chi')\big|_{\mathrm{GL}_2(\mathbb{A})}(g)d^\times g \\ &= \sum_\varphi \int\limits_{\mathrm{Z}(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})} F(g)\theta_{\varphi_1}(1)(g)\theta_{\varphi_2}(\chi')(g)d^\times g\end{aligned}$$

First, let us consider $\theta_{\varphi_1}(1)$. Setting $\Psi_{\varphi_1}(g) = \omega_\psi(g,g')\varphi_1(0)$ for $g\in\mathrm{GL}_2(\mathbb{A})^{(\mathrm{N})}$ and $g'\in K_{\mathbb{A}}^\times$ with $\mathrm{N}(g') = \det(g)$, one obtains a function $\Psi_{\varphi_1}:\ \mathrm{B}(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})\to\mathbb{C}$ such that

$$\Psi_{\varphi_1}\left(\begin{pmatrix} ad & b \\ 0 & d\end{pmatrix} g\right) = \eta_K(d)|a|^{1/2}\Psi_{\varphi_1}(g) \text{ for } g\in\mathrm{GL}_2(\mathbb{A}).$$

From the formula of Siegel-Weil (see [14, Theorem 13.3]) with $\varphi_{1,\infty} = e^{-\pi\mathrm{N}_{K/\mathbb{Q}}(x)}$, we have

$$\theta_{\varphi_1}(1) = E(0,\Psi_{\varphi_1},-)|_{\mathrm{GL}_2(\mathbb{A})^{(\mathrm{N})}}, \tag{4.7.1}$$

where $E(s,\Psi_{\varphi_1},g)$ is the Eisenstein series attached to Ψ_{φ_1}.

Secondly, as computed in [24, Section 3.3] the function

$$\theta_{\varphi_2}(\chi')(g) = \int\limits_{K^{\times(1)}\backslash K_{\mathbb{A}}^{\times(1)}} \theta_{\varphi_2}(g,hh')\chi'(hh')d_1^{\times}h$$

with $\det(g)=\mathrm{N}(h')$ has the Fourier expansion

$$\theta_{\varphi_2}(\chi')(g) = \sum_{\substack{\xi\in\mathbb{Q}^{\times} \\ \mathrm{N}(\mathrm{J})^{-1}\xi\in\mathrm{N}(K^{\times})}} W_{\varphi_2}\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}g\right),$$

where

$$W_{\varphi_2}(g) = \int_{K_{\mathbb{A}}^{\times(1)}} \omega_{\psi}\left(\begin{pmatrix}\mathrm{N}(\mathrm{J})^{-1} & 0\\ 0 & 1\end{pmatrix}g, hh''\right)\varphi_2(\mathrm{J})\chi'(hh'')d_1^{\times}h.$$

Here h'' satisfies $\mathrm{N}(h'')=\mathrm{N}(\mathrm{J})^{-1}\det(g)$. Let $U_q^{(1)}$ be the group of norm 1 elements in $(\mathcal{O}_K\otimes\mathbb{Z}_q)^{\times}$ for finite prime q and $U_{\infty}^{(1)}=K_{\infty}^{\times(1)}$. Then let $d^{\times}d_q$ be the measure on $K_q^{(1)}$ of which volume of $U_q^{(1)}$ is 1 for unramified q or ∞ and 2 for the others. Set $d^{\times}h=\prod_q d^{\times}d_q$. Then one has $d_1^{\times}=\frac{1}{h_K}d^{\times}h$. For $\varphi_2=\otimes_q\varphi_{2,q}$, we have $h_K W_{\varphi_2}=\otimes_q W_{\varphi_{2,q}}$ where

$$W_{\varphi_{2,q}}(g) = \int_{K_q^{(1)}} \omega_{\psi_q}\left(\begin{pmatrix}\mathrm{N}(\mathrm{J})^{-1} & 0\\ 0 & 1\end{pmatrix}g, h_qh'\right)\varphi_2(\mathrm{J})\chi_q'(h_qh')d^{\times}h_q$$

with $\mathrm{N}(\mathrm{J})^{-1}\det(g)=\mathrm{N}(h')$.

In total, unfolding the Eisenstein series we have

$$\begin{aligned}\Sigma(z,\chi,F)^2 &= \sum_{\varphi}\int\limits_{\mathrm{Z}(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})}\sum_{\xi}F(g)E(0,\Psi_{\varphi_1},g)W_{\varphi_2}\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}g\right)d^{\times}g\\ &= \sum_{\varphi}\int\limits_{\mathrm{Z}(\mathbb{A})\,\mathrm{N}(\mathbb{A})\backslash\mathrm{GL}_2(\mathbb{A})} W_F(g)\Psi_{\varphi_1}(g,\tfrac{1}{2})W_{\varphi_2}\left(\begin{pmatrix}-1 & 0\\ 1 & 0\end{pmatrix}g\right)d^{\times}g,\end{aligned}$$

where

$$\Psi_{\varphi_1}(g,s) = \left|\frac{a_1}{a_2}\right|^{s-1/2} \Psi_{\varphi_1}(g)$$

for the decomposition $g = n\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} k \in NAK$. So $\Sigma(z,\chi,F)^2$ is the evaluation at $s = 1/2$ of the analytic continuation of

$$\sum_{\varphi} \int_{\mathrm{Z}(\mathbb{A})\mathrm{N}(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} W_F(g)\Psi_{\varphi_1}(g,s)W_{\varphi_2}\left(\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} g\right) d^\times g$$

$$= \sum_{\varphi} \int_{\mathrm{K}(\mathbb{A})} \int_{\mathbb{A}^\times} W_F\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) \Psi_{\varphi_1}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right)$$

$$\times W_{\varphi_2}\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} k\right) |a|^{-1} d^\times a dk.$$

Here $\mathrm{K}(\mathbb{A}) = \prod_q \mathrm{K}_q$ with $\mathrm{K} = \mathrm{SO}_2(\mathbb{R})$, $\mathrm{K}_q = \mathrm{GL}_2(\mathbb{Z}_q)$ and $dk = \prod_q dk_q$ such that dk_∞ is the usual measure on $\mathrm{SO}_2(\mathbb{R})$ and dk_q is the restriction of $\zeta_q(2)d^\times g$ on K_q. One obtains $vol(\mathrm{SO}_2(\mathbb{R})) = 2\pi$ and $vol(\mathrm{K}_q) = 1$.

From now on, we consider the Schwarz-Bruhat function φ in $\mathcal{S}(\mathrm{M}_2(\mathbb{A}))$ such that $\varphi_q = \varphi_{1,q} \otimes \varphi_{2,q}$ for all places q except the ramified ones, where $\varphi_{1,q} \otimes \varphi_{2,q}$ are in $\mathcal{S}(K_q) \otimes \mathcal{S}(K_q \cdot \mathrm{J}) = \mathcal{S}(\mathrm{M}_2(\mathbb{Q}_q))$. The last integral is now the product of the local factors L_q which is given by

$$\int_{\mathrm{K}_q} \int_{\mathbb{Q}_q^\times} W_{F,q}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) W_{\varphi_{2,q}}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) \Phi_{\varphi_{1,q}}\left(s, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) |a|^{-1} d^\times a dk.$$

Basic calculations of these local factors are done in [24] except the ramified place ℓ.

To compute the local ℓ-factor, it is required to choose the Schwarz-Bruhat functions φ_1, φ_2 in $\mathcal{S}(\mathrm{M}_2(\mathbb{Q}_\ell))$ appropriately. By choosing suitable element $g_0 \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and apply it for the embedding of K, we may assume that $\mathrm{M}_2(\mathbb{Q}_\ell) =$

$K_\ell \oplus K_\ell$ J. Here K_ℓ is diagonal and $K_\ell \cdot$ J is anti-diagonal part. Note that now we assume $\mathrm{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We choose $\varphi_\ell \in \mathcal{S}(\mathrm{M}_2(\mathbb{Q}_\ell))$ as

$$\varphi_\ell \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbb{I}_{\mathbb{Z}_\ell}(a)\mathbb{I}_{\mathbb{Z}_\ell}(d)\psi(b)\mathbb{I}_{\ell^{-n}\mathbb{Z}_\ell}(b)\mathbb{I}_{\ell^n(1+\ell^n)\mathbb{Z}_\ell}(c), \tag{4.7.2}$$

which is supported on $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, d \in \mathbb{Z}_\ell,\ b \in \ell^{-n}\mathbb{Z}_\ell, c \in \ell^n(1 + \ell^n\mathbb{Z}_\ell) \right\}$. We have a decomposition $\varphi = \varphi_1 \otimes \varphi_2$ such that

$$\varphi_1(a, d) = \mathbb{I}_{\mathbb{Z}_\ell}(a)\mathbb{I}_{\mathbb{Z}_\ell}(d),\ \varphi_2(c, d) = \psi(b)\mathbb{I}_{\ell^{-n}\mathbb{Z}_\ell}(b)\mathbb{I}_{\ell^n(1+\ell^n\mathbb{Z}_\ell)}(c).$$

From the straightforward calculations, we obtain the following Fourier transforms of φ_1 and φ_2:

Lemma 4.7.1. $\widehat{\varphi}_1 = \varphi_1,\ \widehat{\varphi}_2(x, y) = \mathbb{I}_{\ell^{-2n}\mathbb{Z}_\ell}(x)\psi(\ell^n x)\mathbb{I}_{1+\ell^n\mathbb{Z}_\ell}(y),$

Proof. Let us consider the second case. By definition, we have

$$\begin{aligned}
\widehat{\varphi}_2(x, y) &= \int_{D_2} \varphi_2(u, v)\psi(-\langle (u, v), (x, y)\rangle)dudv \\
&= \int_{\mathbb{Q}_\ell}\int_{\mathbb{Q}_\ell} \psi(u)\mathbb{I}_{\ell^{-n}\mathbb{Z}_\ell}(u)\mathbb{I}_{\ell^n(1+\ell^n\mathbb{Z}_\ell)}(v)\psi(uy + vx)dudv \\
&= \int_{\mathbb{Q}_\ell} \mathbb{I}_{\ell^n(1+\ell^n\mathbb{Z}_\ell)}(v)\psi(vx)dv \int_{\mathbb{Q}_\ell} \psi(u)\mathbb{I}_{\ell^{-n}\mathbb{Z}_\ell}(u)\psi(uy)du \\
&= \mathbb{I}_{\ell^{-2n}\mathbb{Z}_\ell}(x)\psi(\ell^n x)\mathbb{I}_{1+\ell^n\mathbb{Z}_\ell}(y)
\end{aligned}$$

This finishes the proof. □

We have the following evaluation of Φ_{φ_1}.

Lemma 4.7.2. *For all $k \in \mathrm{GL}_2(\mathbb{Z}_\ell)$, we have*

$$\Phi_{\varphi_1}\left(s, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) = |a|^{s-1}.$$

Proof. The group $\mathrm{GL}_2(\mathbb{Z}_\ell)$ is generated by $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $u, v \in \mathbb{Z}_\ell^\times$ and $w \in \mathbb{Z}_\ell$. The actions of the generators are as follows:

1. $\omega_\psi \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, 1 \right) \varphi_1(t) = \psi(u\mathrm{N}(t))\varphi_1(t) = \varphi_1(t).$

2. For $c = (v, u) \in K_\ell^\times = \mathbb{Q}_\ell^\times \times \mathbb{Q}_\ell^\times$ and $t = (x, y)$, we have $\mathrm{N}(c) = uv$ and

$$\omega_\psi \left(\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, c \right) \varphi_1(t) = \varphi_1((v^{-1}, u^{-1})(ux, uy)) = \varphi_1(v^{-1}ux, y) = \varphi_1(t)$$

since $v^{-1}u$ is a unit.

3. We have the action of of $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ as follows.

$$\omega_\psi \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \varphi_1(t) = \widehat{\varphi}_1(t) = \varphi_1(t).$$

In total, we have

$$\Phi_{\varphi_1} \left(s, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \right) = |a|^{s-1/2} \omega_\psi \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a^{-1}\tilde{h} \right) \varphi_1(0) = |a|^{s-1}\varphi_1(0) = |a|^{s-1}.$$

This finishes the proof. $\square$

We set $\Gamma_1(\ell^{2n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_\ell) \,\middle|\, a, d \in 1 + \ell^{2n}\mathbb{Z}_\ell,\ c \in \ell^{2n}\mathbb{Z}_\ell \right\}$. Observe that φ_2 is invariant under the action of $\Gamma_1(\ell^{2n})$ as follows. The group $\Gamma_1(\ell^{2n})$ is generated by

$$\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$$

for $w \in \ell^{2n}\mathbb{Z}$, $v \in \mathbb{Z}_\ell$ and $u \in 1 + \ell^{2n}\mathbb{Z}_\ell$. For the first generator, we have

$$\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$\omega_\psi \left(\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, 1 \right) \varphi_2(t) = \omega_\psi \left(-\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \psi(-wxy)\widehat{\varphi}_2(x, y).$$

Since $x \in \ell^{-2n}\mathbb{Z}_\ell$ and $y \in 1 + \ell^n\mathbb{Z}_\ell$, we have $wxy \in \mathbb{Z}_\ell$ and $\psi(wxy) = 1$. Hence we verify that

$$\omega_\psi \left(\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, 1 \right) \varphi_2(t) = \omega_\psi \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1 \right) \widehat{\widehat{\varphi}}_2(t) = \varphi_2(t).$$

For the second generator, we have

$$\omega_\psi \left(\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, 1 \right) \varphi_2(t) = \varphi_2(ut) = \varphi_2(t)$$

since $\psi_n(ux) = \psi_n(x)$ for $x \in \ell^{-n}\mathbb{Z}_\ell$. The action of the last generator is same as one for φ_1.

Now we decompose dk as $dk = d^\times z dm$ which corresponds to the decomposition

$$\mathrm{GL}_2(\mathbb{Z}_\ell) = \bigcup_{z \in \mathbb{Z}_\ell^\times} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \mathrm{SL}_2(\mathbb{Z}_\ell).$$

such that $vol(\mathbb{Z}_\ell^\times) = 1$ and $vol(\mathrm{SL}_2(\mathbb{Z}_\ell)) = 1$. Note that in [24], dk is chosen so that $vol(\mathrm{GL}_2(\mathbb{Z}_\ell)) = 1$. Observe that there is a coset decomposition

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}_\ell) = & \bigcup_{\substack{u \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times \\ w \bmod \ell^n, \ell \mid w}} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \Gamma_1(\ell^n) \\ & \cup \bigcup_{\substack{u \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times \\ w \bmod \ell^n}} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} w & 1 \\ -1 & 0 \end{pmatrix} \Gamma_1(\ell^n). \end{aligned}$$

We arrange the coset representatives and get

$$\begin{aligned}\mathrm{GL}_2(\mathbb{Z}_\ell) = & \bigcup_{\substack{z,\in\mathbb{Z}_\ell^\times, u\in(\mathbb{Z}/\ell^{2n}\mathbb{Z})^\times \\ w \bmod \ell^{2n}, \ell | w}} \begin{pmatrix} u & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}\Gamma_1(\ell^{2n}) \\ \cup & \bigcup_{\substack{z\in\mathbb{Z}_\ell^\times, u\in(\mathbb{Z}/\ell^{2n}\mathbb{Z})^\times \\ w \bmod \ell^{2n}}} \begin{pmatrix} u & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} w & 1 \\ -1 & 0 \end{pmatrix}\Gamma_1(\ell^{2n}).\end{aligned}$$

The local ℓ-factor

$$L_\ell = \int_{\mathbb{Q}_\ell}\int_{\mathrm{GL}_2(\mathbb{Z}_\ell)} W_{F,\ell}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) W_{\varphi_2}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}k\right)|a|^{s-1}dkda$$

is then divided into the two parts:

$$C\int_{\mathbb{Q}_\ell} W_{F,\ell}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\left[\sum_{u,w}\int_{\mathbb{Z}_\ell^\times} W_{\varphi_2}\left(\begin{pmatrix} au & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}\right)d^\times z\right]|a|^{s-1}da \tag{4.7.3}$$

$$+\,C\int_{\mathbb{Q}_\ell} W_{F,\ell}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\left[\sum_{u,w}\int_{\mathbb{Z}_\ell^\times} W_{\varphi_2}\left(\begin{pmatrix} au & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} w & 1 \\ -1 & 0 \end{pmatrix}\right)d^\times z\right]|a|^{s-1}da \tag{4.7.4}$$

where $C = \int_{\Gamma_1(\ell^{2n})} dm = \dfrac{1}{\ell^{2n}+\ell^{2n-1}}$. We need to check each term by considering 4 cases separately as follows:

1. (4.7.3) $w = 0$ case.
2. (4.7.3) $w \neq 0$ and $v_\ell(w) \leq n$.
3. (4.7.3) $w \neq 0$ and $n < v_\ell(w) < 2n$.
4. (4.7.4) for all u, w.

We will see that the integrals in every cases produce zero except the first case. Since we have the formula for $\pi_f = \pi(\mu_1, \mu_2)$ (see [24])

$$W_{F,\ell}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}_{\mathbb{Z}_\ell}(a)\frac{\mu_1(a\ell) - \mu_2(a\ell)}{\mu_1(\ell) - \mu_2(\ell)},$$

we may assume that $a \in \mathbb{Z}_\ell$.

We consider the first case in the above list. We compute the integrals

$$\sum_u \int_{\mathbb{Z}_\ell^\times} W_{\varphi_2}\left(\begin{pmatrix} au & 0 \\ 0 & z \end{pmatrix}\right) d^\times z.$$

Recall that we have

$$W_{\varphi_2}\begin{pmatrix} au & 0 \\ 0 & z \end{pmatrix} = \int_{K_\ell^{(1)}} \omega_\psi\left(\begin{pmatrix} au & 0 \\ 0 & z \end{pmatrix}, h\tilde{h}\right)\varphi_2(1,1)\chi'(h\tilde{h})d^\times h,$$

where $\tilde{h} = (z, au)$ with respect to the identification $K_\ell = \mathbb{Q}_\ell \times \mathbb{Q}_\ell$. Now we set $\lambda(t) = \chi'(t, 1)$. Then we also have $\chi'(1, t) = \lambda(t)^{-1}$ since χ' is anti-cyclotomic i.e $\chi'(t, t) = 1$. With this setting and the identification $K_\ell^{(1)} = \{(t, t^{-1}) \mid t \in \mathbb{Q}_\ell^\times\}$, $h = (t, t^{-1})$, the last integral equals

$$\begin{aligned} &\int_{\mathbb{Q}_\ell^\times} \varphi_2(aut^{-1}z^{-1}, t)\lambda(z)\lambda(au)^{-1}\lambda(t)^2 d^\times t \\ = &\int_{\mathbb{Q}_\ell^\times} \psi(aut^{-1}z^{-1})\mathbb{I}_{\ell^{-n}\mathbb{Z}_\ell}(at^{-1})\mathbb{I}_{\ell^n(1+\ell^n\mathbb{Z}_\ell)}(t)\lambda(z)\lambda(au)^{-1}\lambda(t)^2 d^\times t \end{aligned}$$

Now summing up the last integral over $u \in (\mathbb{Z}/\ell^{2n}\mathbb{Z})^\times$, we have

$$\int_{\substack{at^{-1} \in \ell^{-n}\mathbb{Z}_\ell \\ t \in \ell^n(1+\ell^n\mathbb{Z}_\ell)}} \sum_u \psi(at^{-1}z^{-1}u)\lambda^{-1}(u)\lambda(z)\lambda(a)^{-1}\lambda(t)^2 d^\times t.$$

We state an easy lemma.

Lemma 4.7.3. *For $x \in \mathbb{Q}_\ell^\times$, we have*

$$\sum_{u \in (\mathbb{Z}/\ell^{2n}\mathbb{Z})^\times} \lambda(u)\psi(xu) = \begin{cases} \ell^n G(\lambda, \psi)\lambda^{-1}(x\ell^n) & \text{if } v_\ell(x) = -n \\ 0 & \text{otherwise} \end{cases}.$$

Proof. We decompose the index set as

$$(\mathbb{Z}/\ell^{2n})^\times = \frac{1+\ell^n\mathbb{Z}}{1+\ell^{2n}\mathbb{Z}} \times (\mathbb{Z}/\ell^n\mathbb{Z})^\times.$$

Then for $(v,u) \in (\mathbb{Z}/\ell^{2n})^\times$ we have

$$\sum_{u\in(\mathbb{Z}/\ell^{2n}\mathbb{Z})^\times} \lambda(u)\psi(xu) = \sum_{v,u} \lambda(u)\psi(xuv)$$

Using usual Gauss sum with respect to $\psi_n(x) = \psi(\ell^{-n}x)$, the last sum equals

$$\sum_v G(\lambda,\psi)\lambda(xv\ell^n)^{-1} = \ell^n G(\lambda,\psi)\lambda(x\ell^n)^{-1}$$

only when $v_\ell(xv) = v_\ell(x) = -n$. □

Hence with the condition $v_\ell(a) - v_\ell(t) = -n$, the last integral equals

$$\begin{aligned}
&\int_{\substack{at^{-1}\in\ell^{-n}\mathbb{Z}_\ell \\ t\in\ell^n(1+\ell^n\mathbb{Z}_\ell)}} \ell^n G(\lambda^{-1},\psi)\lambda(azt^{-1})\lambda^{-1}(az)\lambda(t)^{2n} d^\times t \\
&= \mathbb{I}_{\mathbb{Z}_\ell^\times}(a) G(\lambda^{-1},\psi)\lambda(\ell^n)\ell^n \int_{t\in\ell^n(1+\ell^n\mathbb{Z}_\ell)} d^\times t \\
&= \mathbb{I}_{\mathbb{Z}_\ell^\times}(a) G(\lambda^{-1},\psi)\lambda(\ell^n).
\end{aligned}$$

Note that $t\in\ell^n(1+\ell^n\mathbb{Z}_\ell)$ implies $a\in\mathbb{Z}_\ell^\times$. So we conclude the first case in the list.

To consider the second and the third case in the list, we need the following computation:

Lemma 4.7.4. *For $w \in \ell\mathbb{Z}/\ell^{2n}\mathbb{Z}$ with $w \neq 0$, let $\phi_w(x,y) = \psi(wxy)\widehat{\varphi}_2(x,y)$. Then the Fourier Transform of ϕ_w is*

$$\widehat{\phi}_w(x,y) = \begin{cases} |w|_\ell^{-1}\mathbb{I}_{w\ell^{-2n}\mathbb{Z}_\ell}(x)\psi(-w^{-1}x(y+\ell^n))\mathbb{I}_{w+\ell^n+w\ell^n\mathbb{Z}_\ell}(-y) & \text{if } v_\ell(w)\le n \\ \psi(x)\mathbb{I}_{w+\ell^n+\ell^{2n}\mathbb{Z}_\ell}(-y) & \text{if } n < v_\ell(w) < 2n. \end{cases}$$

Proof. By the definition of Fourier Transform, we have

$$
\begin{aligned}
\widehat{\phi}_w(x,y) &= \int_{\mathbb{Q}_\ell}\int_{\mathbb{Q}_\ell} \mathbb{I}_{\ell^{-2n}\mathbb{Z}_\ell}(v)\psi(\ell^n u)\mathbb{I}_{1+\ell^n\mathbb{Z}_\ell}(v)\psi(wuv)\psi(uy+vx)dudv \\
&= \int_{\ell^{-2n}\mathbb{Z}_\ell} \psi(\ell^n u)\psi(uy)\int_{1+\ell^n\mathbb{Z}_\ell}\psi(vx)\psi(wuv)dvdu \\
&\overset{v\mapsto 1+v}{=} \psi(x)\int_{\ell^{-2n}\mathbb{Z}_\ell}\psi(wu+\ell^n u+yu)\int_{\ell^n\mathbb{Z}_\ell}\psi((x+wu)v)dv \\
&= \psi(x)\int_S \psi((w+\ell^n+y)u)du
\end{aligned}
$$

where $S=\ell^{-2n}\mathbb{Z}_\ell \cap w^{-1}(-x+\ell^{-n}\mathbb{Z}_\ell)$. Fisrt, if $v_\ell(w)\le n$, then we have $S\neq\emptyset \Leftrightarrow \mathbb{Z}_\ell\cap -w^{-1}x\ell^{2n}+w^{-1}\ell^n\mathbb{Z}_\ell \neq \emptyset \Leftrightarrow w^{-1}x\ell^{2n}\in\mathbb{Z}_\ell$ i.e. $x\in w\ell^{-2n}\mathbb{Z}_\ell$. In this case, we have $S=w^{-1}(-x+\ell^{-n}\mathbb{Z}_\ell)$. Hence the last line of the previous computation equals

$$
\begin{aligned}
& \psi(x)\mathbb{I}_{w\ell^{-2n}\mathbb{Z}_\ell}(x)\int_{w^{-1}(-x+\ell^{-n}\mathbb{Z}_\ell)}\psi((w+\ell^n+y)u)du \\
=\ & \psi(x)\mathbb{I}_{w\ell^{-2n}\mathbb{Z}_\ell}(x)\int_{\ell^{-n}\mathbb{Z}_\ell}\psi((w+\ell^n+y)(-w^{-1}x)+(w+\ell^n+y)w^{-1}u)|w|_\ell^{-1}du \\
=\ & |w|_\ell^{-1}\mathbb{I}_{w\ell^{-2n}\mathbb{Z}_\ell}(x)\psi(-w^{-1}x\ell^n-w^{-1}xy)\int_{\ell^{-n}\mathbb{Z}_\ell}\psi((1+w^{-1}\ell^n+w^{-1}y)u)du \\
=\ & |w|_\ell^{-1}\mathbb{I}_{w\ell^{-2n}\mathbb{Z}_\ell}(x)\psi(-w^{-1}x(y+\ell^n))\mathbb{I}_{\ell^n\mathbb{Z}_\ell}(1+w^{-1}w\ell^n+w^{-1}y).
\end{aligned}
$$

Now let us assume that $n<v_\ell(w)<2n$. Then $S\neq\emptyset\Leftrightarrow \mathbb{Z}_\ell\cap -w^{-1}x\ell^{2n}+w^{-1}\ell^n\mathbb{Z}_\ell\neq\emptyset \Leftrightarrow -w^{-1}x\ell^{2n}\in w^{-1}\ell^n\mathbb{Z}_\ell$ i.e. $x\in\ell^{-n}\mathbb{Z}_\ell$. In this case we have $S=\ell^{-2n}\mathbb{Z}_\ell$. So the last line of previous computation equals

$$
\begin{aligned}
\psi(x) & \int_{\ell^{-2n}\mathbb{Z}_\ell}\psi((w+\ell^n+y)u)du \\
= & \ \psi(x)\mathbb{I}_{\ell^{2n}\mathbb{Z}_\ell}(w+\ell^n+y) \\
= & \ \psi(x)\mathbb{I}_{-w-\ell^n+\ell^{2n}\mathbb{Z}_\ell}(y).
\end{aligned}
$$

Hence we obtain the result. □

Now we consider the second case in the list. We compute the term

$$\sum_{\substack{u,w\\ v_\ell(w)\le n}} \int_{\mathbb{Z}_\ell^\times} W_{\varphi_2}\left(\begin{pmatrix} au & 0\\ 0 & z\end{pmatrix}\begin{pmatrix} 1 & 0\\ w & 1\end{pmatrix}\right) d^\times z.$$

Now from the lemma and the expression

$$\begin{pmatrix} 1 & 0\\ w & 1\end{pmatrix} = \begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix} 1 & -w\\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},$$

we have

$$\begin{aligned}
&\omega_\psi\left(\begin{pmatrix} au & 0\\ 0 & z\end{pmatrix}\begin{pmatrix} 1 & 0\\ w & 1\end{pmatrix}, h(z, au)\right)\varphi_2(x,y)\\
&= \omega_\psi\left(\begin{pmatrix} au & 0\\ 0 & z\end{pmatrix}, h(z, au)\right)\widehat{\phi}_w(-x,-y)\\
&= \widehat{\phi}_w(-aut^{-1}z^{-1}x, -ty).
\end{aligned}$$

Here we choose $\tilde{h} = (z, au)$ and $h = (t, t^{-1})$. The term W_{φ_2} in the last integral equals

$$\begin{aligned}
&\sum_w |w|_\ell^{-1} \int_{\substack{at^{-1}\in w\ell^{-2n}\mathbb{Z}_\ell\\ t\in w+\ell^n+w\ell^n\mathbb{Z}_\ell}} \sum_u \psi(w^{-1}az^{-1}(\ell^n t^{-1}-1)u)\lambda^{-1}(u)\lambda(z)\lambda^{-1}(a)\lambda(t)^2 d^\times t\\
&= |w|_\ell^{-1}\ell^n G(\lambda^{-1},\psi) \int_{\substack{at^{-1}\in w\ell^{-2n}\mathbb{Z}_\ell\\ t\in w+\ell^n+w\ell^n\mathbb{Z}_\ell\\ v_\ell(w^{-1}a(\ell^n t^{-1}-1))=-n}} \lambda(w^{-1}az^{-1}(\ell^n t^{-1}-1)\lambda(z)\lambda(a)^{-1}\lambda(t)^2 d^\times t\\
&= |w|_\ell^{-1}\ell^n G(\lambda^{-1},\psi)\lambda(w^{-1}) \int_{\substack{at^{-1}\in w\ell^{-2n}\mathbb{Z}_\ell\\ t\in w+\ell^n+w\ell^n\mathbb{Z}_\ell\\ v_\ell(w^{-1}a(\ell^n t^{-1}-1))=-n}} \lambda(\ell^n t^{-1}-1)\lambda(t)^2 d^\times t
\end{aligned}$$

Now note that $\lambda(\ell^n t^{-1}-1)\lambda(t) = \lambda(\ell^n - t)$. Since $t \in w + \ell^n + w\ell^n\mathbb{Z}_\ell$, we have $t - \ell^n \in w(1+\ell^n\mathbb{Z}_\ell)$. Hence $\lambda(\ell^n - t) = \lambda(-w)$ and the last integral equals

$$|w|_\ell^{-1}\ell^n G(\lambda^{-1},\psi)\lambda(-1) \int_{w+\ell^n+w\ell^n\mathbb{Z}_\ell} \lambda(t)d^\times t \tag{4.7.5}$$

with the condition $v(a) - v(t) = -n$. If we have $w + \ell^n \in \ell^n w\mathbb{Z}_\ell$, then by multiplying $\lambda(u) \neq 1$ for a unit u to the integral we are able to conclude

$$\int_{w+\ell^n+w\ell^n\mathbb{Z}_\ell} \lambda(t)d^\times t = \int_{w\ell^n\mathbb{Z}_\ell} \lambda(t)d^\times t = 0.$$

Therefore we assume that $w + \ell^n \notin w\ell^n\mathbb{Z}_\ell$. Let $m = v_\ell(w) + n - v_\ell(w + \ell^n)$. Then if $m < n$, then we have

$$\int_{w+\ell^n+w\ell^n\mathbb{Z}_\ell} \lambda(t)d^\times t = \lambda(w + \ell^n) \int_{1+\ell^m\mathbb{Z}_\ell} \lambda(t)d^\times t = 0$$

as the non-trivial character sum over a group is trivial. Hence we may assume $m = n$. This implies $v_\ell(w) = v_\ell(w + \ell^n)$. In addition, if $v_\ell(n) < n$ then $v_\ell(t) = v_\ell(w + \ell^n) = v_\ell(w) < n$ and $v_\ell(a) = v_\ell(t) - n < 0$ which is not the case since $a \in \mathbb{Z}_\ell$.

In total, we assume that $m = n$ i.e. $v_\ell(w) = v_\ell(w + \ell^n)$ and $v_\ell(w) = n$. Set $w = \ell^n u$ for a unit u. The previous assumption implies that $u + 1$ is also a unit. Let $H = \{u \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times \mid u + 1 \in \mathbb{Z}_\ell^\times\}$. We have the following lemma.

Lemma 4.7.5. *Let $v \in 1 + \ell\mathbb{Z}_\ell$. Then the map on H given by*

$$v : H \to H,\ u \mapsto (vu + v - 1) \bmod \ell^n$$

is one to one correspondence.

Proof. First of all, $vu+v-1$ is a unit since $v-1$ is divisible by ℓ and $vu+v-1+1 = v(u + 1) \in \mathbb{Z}_\ell^\times$. Therefore the map is well-defined. Clearly the map is injective. Since H is a finite set, it is surjective. □

Summing up the integral in the formula (4.7.5) over H, we have

$$\sum_{u\in H} \int_{\ell^n(u+1)+\ell^{2n}\mathbb{Z}_\ell} \lambda(t)d^\times t = \lambda(\ell^n) \sum_{u\in H} \lambda(u + 1) \int_{1+\ell^n\mathbb{Z}_\ell} d^\times t$$

We choose $v \in 1+\ell\mathbb{Z}_\ell$ such that $\lambda(v) \neq 1$. Then from the lemma we have

$$\lambda(v)\sum_{u\in H}\lambda(u+1) = \sum_{u\in H}\lambda((uv+v-1)+1) = \sum_{u\in H}\lambda(u+1).$$

Hence it is equal to zero and we conclude the second case in the list.

For the third case, we compute the terms in a similar way but easier than before. We need to compute the term

$$\sum_{\substack{u,w \\ n<v_\ell(w)<2n}} \int_{\mathbb{Z}_\ell^\times} W_{\varphi_2}\left(\begin{pmatrix} au & 0 \\ 0 & z\end{pmatrix}\begin{pmatrix} 1 & 0 \\ w & 1\end{pmatrix}\right) d^\times z.$$

Same as before, using the Lemma 4.7.4 we compute

$$\begin{aligned}
& \sum_{u,w}\int_{\mathbb{Q}_\ell^\times} \widehat{\phi}_w(-aut^{-1}z^{-1}x, -ty)\lambda(z)\lambda(au)^{-1}\lambda(t)^2 d^\times t \\
= & \sum_{w} \int_{w+\ell^n+w\ell^n\mathbb{Z}_\ell} \sum_u \psi(-at^{-1}z^{-1}u)\lambda^{-1}(u)\lambda(z)\lambda(u)^{-1}\lambda(t)^2 d^\times t \\
= & \sum_{w} G(\lambda^{-1},\psi)\ell^n\lambda(\ell^n) \int_{w+\ell^n+w\ell^n\mathbb{Z}_\ell} \lambda(-at^{-1}z^{-1})\lambda(z)\lambda(a)^{-1}\lambda(t)^2 d^\times t \\
= & \sum_{w} G(\lambda^{-1},\psi)\ell^n\lambda(-\ell^n) \int_{w+\ell^n+w\ell^n\mathbb{Z}_\ell} \lambda(t) d^\times t \\
= & \sum_{w} G(\lambda^{-1},\psi)\ell^n\lambda(-\ell^n)\lambda(w+\ell^n) \int_{1+\ell^m\mathbb{Z}_\ell} \lambda(t) d^\times = 0
\end{aligned}$$

since $0 < m = 2n - v_\ell(w+\ell^n) = 2n - v_\ell(w) < n$. Therefore we conclude the third case in the list.

Finally, let us consider the last case in the list. From the expression

$$\begin{pmatrix} w & 1 \\ -1 & 0\end{pmatrix} = \begin{pmatrix} 1 & -w \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix},$$

we compute

$$\begin{aligned}
& \omega_\psi\left(\begin{pmatrix} au & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} w & 1 \\ -1 & 0 \end{pmatrix}, h(z, au)\right)\varphi_2(x,y) \\
= \quad & \omega_\psi\left(\begin{pmatrix} au & 0 \\ 0 & z \end{pmatrix}, h(z, au)\right)\psi(wxy)\widehat{\varphi}_2(x,y) \\
= \quad & \psi(wauz^{-1}xy)\mathbb{I}_{\ell^{-2n}\mathbb{Z}_\ell}(aut^{-1}z^{-1}tx)\psi(\ell^n aut^{-1}z^{-1}x)\mathbb{I}_{1+\ell^n\mathbb{Z}_\ell}(ty).
\end{aligned}$$

The inner term in the formula (4.7.4) equals

$$\sum_w \int_{\mathbb{Q}_\ell^\times} \sum_u \psi(\ell^n at^{-1}z^{-1}u)\lambda^{-1}(u)\lambda(z)\lambda^{-1}(au)d^\times t$$

But it is equal to zero since $\psi(\ell^n at^{-1}z^{-1}u) = 1$ for $t \in 1+\ell^n\mathbb{Z}_\ell$ and $\sum_u \lambda^{-1}(u) = 0$. We conclude the last case and obtain

Proposition 4.7.6.

$$L_\ell = \frac{(1-\ell^{-2s})^{-1}}{\ell^{2n-1}(\ell+1)} G(\lambda^{-1},\psi)\lambda(\ell^n)L(2s,\eta_K)^{-1}L(s,\pi_f\otimes\pi(\chi')).$$

Proof. We know that $\pi_{f,\ell} = \pi(\mu_1,\mu_2)$ is a principal series representation with unramified quasi-characters μ_1 and μ_2 and the Whittaker coefficient is given by

$$W_{F,\ell}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = |a|^{1/2}\mathbb{I}_{\mathbb{Z}_\ell}(a)\frac{\mu_1(a\ell)-\mu_2(a\ell)}{\mu_1(\ell)-\mu_2(\ell)}.$$

So we have

$$\begin{aligned}
L_\ell \quad = \quad & \frac{G(\lambda^{-1},\psi)\lambda(\ell^n)}{\ell^{2n-1}(\ell+1)}\int_{\mathbb{Q}_\ell^\times}\frac{\mu_1(a\ell)-\mu_2(a\ell)}{\mu_1(\ell)-\mu_2(\ell)}\mathbb{I}_{\mathbb{Z}_\ell^\times}(a)|a|^s d^\times a \\
= \quad & \frac{G(\lambda^{-1},\psi)\lambda(\ell^n)}{\ell^{2n-1}(\ell+1)}\int_{\mathbb{Z}_\ell^\times} d^\times a \\
= \quad & \frac{G(\lambda^{-1},\psi)\lambda(\ell^n)}{\ell^{2n-1}(\ell+1)}.
\end{aligned}$$

Recall that since $\chi'_{\mathfrak{l}_1}$, $\chi'_{\mathfrak{l}_2}$ are ramified at ℓ, so are $\mu_i\chi'_{\mathfrak{l}_j}$. Hence we have

$$L(s, \pi_f \otimes \pi(\chi')) = \prod_{i,j} L(s, \mu_i \chi'_{\mathfrak{l}_j}) = 1$$

and $L(2s, \eta_K)^{-1}L(s, \pi_f \otimes \pi(\chi')) = (1 - \ell^{-2s})$. So we have the trivial relation

$$L_\ell = \frac{(1-\ell^{-2s})^{-1}}{\ell^{2n-1}(\ell+1)} G(\lambda^{-1}, \psi)\lambda(\ell^n) L_\ell(2s, \eta_K)^{-1} L_\ell(s, \pi_f \otimes \pi(\chi')).$$

This finishes the proof. □

Evaluating at $s = 1/2$ and considering the prime $\mathcal{B}$ in $\overline{\mathbb{Q}}$ over p, and combining the calculations done in [24], we verify that

Theorem 4.7.7. *Let f be a modular form for $\mathrm{SL}_2(\mathbb{Z})$. Up to $\mathcal{B}$-adic units, we have the following equality:*

$$j(J, z)^k \; Im(z)\Sigma(\chi', z, F)^2 = \frac{2k!}{\pi h_K} \frac{1}{\ell^2 - 1} L(1, \eta_K)^{-1} \frac{L(1/2, \pi_f \otimes \pi(\chi'))}{\Omega_f^{2k}}.$$

4.8 Non-vanishing result: Sinnott's argument

In this section we prove the Theorem mentioned in the introduction following the Sinnott's argument. We restrict ourselves to the case that d_K is even and f is an eigen cusp form of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$. Any arithmetic Hecke character of type $k + 0 \cdot c$ of ℓ-power conductor has the decomposition $\chi\psi$ for χ a character on Γ and ψ the restriction to Δ. We fixed ψ and now assume that the special L-values of $\pi_f \otimes \chi'$ vanish modulo $\mathcal{B}$ for infinitely many such χ's. From the Theorem 4.7.7, this is equivalent to the vanishing of the value:

$$\Sigma(\chi', z, F) = \frac{1}{h_n} \sum_{x \in G_n} F(x\tau)\chi'(x) \equiv 0 \bmod \mathcal{B}. \qquad (4.8.1)$$

Let D_p be the decomposition group of $\mathcal{B}$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Applying an automorphism $\sigma \in D_p$, we have

$$\sigma\Sigma(z, \chi', F) = \frac{1}{h_n} \sum_{x \in G_n} \sigma F(x\tau)\sigma\chi'(x) \equiv 0 \bmod \mathcal{B}.$$

By Shimura's reciprocity law([26] 26.8), for each idele s in $K_{\mathbb{A}}^{(\ell\infty)\times}$ and $\sigma = (s, K)$, we obtain $\sigma F(z) = F(s^{-1}z)$ and

$$\sum_{x \in G_n} \sigma F(x\tau)\sigma\chi'(x) = \chi^{\sigma}(s) \sum_{x \in G_n} F(x\tau)\chi'^{\sigma}(x)$$

by the change of variable $s^{-1}x \mapsto x$. So we also conclude from (4.8.1) that

$$\sum_{x \in G_n} F(x\tau)\chi'^{\sigma}(x) \equiv 0 \bmod \mathcal{B}.$$

Let us denote $\overline{\chi}$ for $\chi \bmod \mathcal{B}$. Let $K(\psi)$ be the number field obtained by adjoining the values of ψ to K. Let $k_n := \mathbb{F}_p(\mu_{\ell^n})$ be obtained by adjoining the ℓ^n-th roots of unity to $\mathbb{F}_p$ and m be the smallest integer such that k_m contains the residue field of $\mathcal{B} \cap K(\psi)$. If ℓ^n is the conductor of χ, then observe that we have

$$\mathrm{Tr}_{k_n/k_m}(\overline{\chi}(x)) = \begin{cases} [k_n : k_m]\overline{\chi}(x) & \text{if } \overline{\chi}(x) \in k_m, \\ 0 & \text{otherwise,} \end{cases}$$

Applying $\sigma \in D_p$ to the formula (4.8.1) and summing up over σ's, we obtain the following :

$$\sum_{\overline{\chi}(x) \in k_m} \chi(x)F(x\tau) \equiv 0 \mod \mathcal{B}.$$

Since Γ has $\mathbb{Z}_\ell$-rank 1, the kernel $\ker\chi$ is $\Gamma_n := \Gamma^{\ell^n}$ and

$$\overline{\chi}(x) \in k_m \text{ if and only if } x \in \Gamma_{n-m}.$$

Using the decomposition $G_\infty = \Delta \times \Gamma$, we have

$$\sum_{x \in \Delta} \psi(x) \sum_{y \in \frac{\Gamma_{n-m}}{\Gamma_n}} \chi(y) f(xy \cdot z_n) j(xy, z_n)^{-k} \equiv 0 \bmod \mathcal{B}.$$

For the genus subgroup Δ_0 of Δ, we have

$$\sum_{\eta\in\Delta/\Delta_0}\psi(\eta)\sum_{x\in\Delta_0}\psi(x)\sum_{y}\chi(y\eta_\Gamma)f_\alpha(\eta xy\cdot z_n)j(x\eta y,z_n)^{-k}\equiv 0 \bmod \mathcal{B}. \tag{4.8.2}$$

Here by abuse of notation, we write $h_{n,\infty}(\eta xy)\cdot z_n$ as $\eta xy\cdot z_n$ and $j(h_{n,\infty}(xy\eta),z_n)$ as $j(xy\eta,z_n)$. Since $h_{n,\infty}(y)$ is upper-triangular for each $y\in\Gamma$, we have

$$j(xy\eta,z_n)=j(y,x\eta z_n)j(x\eta,z_n)=j(x\eta,z_n).$$

The representatives of Γ_{n-m}/Γ_n is given by $\left\{1+u\ell^{n-m}\omega \,\middle|\, u \bmod \ell^m\right\}$ and, hence, the collection $\{y\cdot z_n\,|\,y\in\Gamma_{n-m}/\Gamma_n\}$ is equal to

$$\left\{\begin{pmatrix}1 & \frac{u}{\ell^m}\\ 0 & 1\end{pmatrix}\cdot z_{n-m}\,\middle|\, u \bmod \ell^m\right\}.$$

Note also that from the Proposition 4.3.2, we obtain that

$$\{x\cdot z_n\,\big|\,x\in\Delta_0\}=\{\,dz_n\,\big|\,d\text{ is a divisor of }d'_K\},$$

where d'_K is the product of all prime divisors of d_K. Set $\varpi_d=\prod_{q|d}\varpi_q\in\Delta_0$. The left hand side of the above congruence (4.8.2) can be written as

$$\begin{aligned}
&\sum_{\eta}\psi(\eta)\sum_{x}\psi(x)\sum_{y}\chi(y\eta_\Gamma)j(\eta x,z_n)^{-k}f_\alpha(\eta xy\cdot z_n)\\
=&\sum_{\eta}\psi(\eta)\sum_{x}\psi(x)\sum_{y}\chi(y)f_\alpha(\eta xy\eta_\Gamma^{-1}\cdot z_n)\\
=&\sum_{\eta}\psi(\eta)\sum_{d|d'_K}\psi(\varpi_d)\sum_{u}\zeta^u f\left|\begin{pmatrix}1 & \frac{u}{\ell^m}\\ 0 & 1\end{pmatrix}\right|[d]\big|[\eta](\eta_\Gamma^{-1}z_{n-m}).
\end{aligned}\tag{4.8.3}$$

Here $j(\varpi_d\eta,z_n)=1$ and $\zeta=\chi(1+\ell^{n-m}\omega)$ is the primitive ℓ^m-th root of unity. In the third line (4.8.3), the condition that $n>2m$ implies that $\chi(1+u\ell^{n-m}\omega)=\zeta^u$. Now for each $\eta\in\Delta/\Delta_0$, we set

$$f_\eta=\sum_{d|d'_K}\psi(\varpi_d)\sum_{u}\zeta^u f\left|\begin{pmatrix}1 & \frac{u}{\ell^m}\\ 0 & 1\end{pmatrix}\right|[d]\big|[\eta].$$

Recall that η_Γ is the Γ-part of η which are all distinct. Then for infinitely many n's we obtain the following vanishing

$$\sum_{\eta\in\Delta/\Delta_0} \psi(\eta) f_\eta(\eta_\Gamma^{-1} z_{n-m}) \equiv 0 \bmod \mathcal{B}.$$

From the Theorem 4.3.3 which is the Zariski density result, we are able to conclude

$$f_\eta \equiv 0 \bmod \mathcal{B}.$$

We shall show that this is impossible for non-trivial modular form f.

Let us calculate the q-expansion of f_η to get the non-triviality. Its Fourier coefficient $a(n, f_\eta)$ of $\exp(2\pi i n z)$ is given by

$$a(n, f_\eta) = \begin{cases} \ell^{m-n} \sum_{d|(d'_K, \frac{n}{q})} \psi(\varpi_d) a\left(\frac{n}{qd}, f\right) & \text{if } n \equiv -1 \bmod \ell^{m-n},\ q \mid n \\ 0 & \text{otherwise} \end{cases}.$$

Now we consider a coefficient $a(rq, f_\eta)$ for a prime number r which is unramified in K i.e. $r \nmid d_K$ and $\eta = \varpi_q$. From the previous formula for q-expansion we have

$$a(rq, f_\eta) = a(r, f)$$

and it is required to chooose $r \equiv -q' \mod \ell^{m-n}$ where q' is the multiplicative inverse of q modulo ℓ^{m-n}. At this moment, let us quote a theorem about the Galois representation attached to the eigencusp forms for later use.

Proposition 4.8.1. *Let $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_f)$ be the Galois representation attached to f with nebentypus ψ. Then for all prime $p \nmid N$, ρ_f is unramified at p and for a Frobenius element F_p, one has*

$$\mathrm{Tr}\rho(F_p) = a(p, f), \ \textit{and} \ \det \rho_f(F_p) = \psi(p) p^{k-1}.$$

There are many references for the proof of this well-known fact. For example, see [6] and [13].

Proposition 4.8.2. *For each $\eta \in W$, we can find a prime number r such that*

$$r \equiv -q' \bmod \ell^m \text{ and } a(r,f) \not\equiv 0 \bmod \mathcal{B}.$$

Therefore f_η is not zero modulo $\mathcal{B}$.

Proof. Let ρ_f be the Galois representation given by f, i.e. $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_f)$, where $\mathcal{O}_f$ is the integer ring of the number field generated by the eigenvalues of f over $\mathbb{Q}$. And set $\overline{\rho}_f = \rho_f \bmod \mathcal{B}$ and $K = \overline{\mathbb{Q}}^{\mathrm{Ker}\,\overline{\rho}_f}$. Since ρ_f is unramified at ℓ, we know that ℓ is unramified in $K/\mathbb{Q}$. Since ℓ is totally ramified in $\mathbb{Q}(\mu_{\ell^m})/\mathbb{Q}$, we can conclude that K and $\mathbb{Q}(\mu_{\ell^m})$ is linearly disjoint. From this we have

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cong \mathrm{Gal}(K/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^m})/\mathbb{Q}).$$

Now choose an element c in $\mathrm{Gal}(K/\mathbb{Q}) = \mathrm{Im}(\overline{\rho}_f)$ such that $\mathrm{Tr}(c) \not\equiv 0 \mod \mathcal{B}$. There exists an element $\sigma \in \mathrm{Gal}(K(\mu_{\ell^m})/\mathbb{Q})$ such that

$$\sigma|_K = c, \text{ and } \sigma|_{\mathbb{Q}(\mu_{\ell^m})} = -q'.$$

The Cebotarev density theorem allows us to choose an F_r such that $F_q = \sigma$. Then we have $a(r,f) = \mathrm{Tr}(F_q) = \mathrm{Tr}(c) \neq 0$ and $r \equiv -q' \bmod \ell^m$. □

Now from the previous proposition, we are able to conclude that

$$f_\eta \not\equiv 0 \pmod{\mathcal{B}},$$

which is a contradiction to above conclusion from the assumption that there exist infinitely many characters which give the vanishing of L-values. By previous arguments and the Proposition 4.8.2, now we are able to conclude that

Theorem 4.8.3. *Let d_K be odd, ℓ split in K and f be an eigen cusp form of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$. Then for almost all anti-cyclotomic χ of ℓ-power* *conductor with a fixed* $\psi = \chi|_{\Delta}$, *we have*

$$\frac{\pi^{k-1} L\big(\frac{1}{2}, \pi_f \otimes \pi(\chi')\big)}{\Omega^{2k}} \not\equiv 0 \mod \mathcal{B}.$$

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