STATEMENT OF PAST RESEARCH AND FUTURE PLANS

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Abstract. In this statement, we discuss the results from the author's paper and dissertation, as well as plans for future research. The main results are a new proof of Washington's theorem using our homological method, and a proof of non-vanishing mod p of cuspidal modular L-values with anti-cyclotomic twists. An outline of extending the homological method to the case of cyclotomic twists is given. Future research will focus on the generalization of the first result to Hecke L-values with cyclotomic twists and of the second result to broader cases. We will work towards a proof of the conjectures dealing with an extension of the homological method introduced in the statement.

1. Past Research

Since L-functions were first introduced by Dirichlet, there have been deep interactions between the behavior of L-values and equidistribution (or density result). For example, non-vanishing of Dirichlet L-values is a key ingredient in the proof of equidistribution of prime numbers in each class of $\mathbb{Z}/N\mathbb{Z}$ for an integer N. Another example is a dynamical L-function, which is an Euler-like product over prime closed geodesics on a compact smooth Riemannian manifold with negative curvature, such as modular curve. One can study this L-function to get the equidistribution property of closed geodesics. More precisely, one can study asymptotic behavior of the counting function of homology classes corresponding to geodesics on the manifolds with given length as in [11].

An example in the reverse direction is a theorem of Washington. Consider two distinct odd primes p and ℓ. Let λ be a Dirichlet character with conductor N prime to $p\,\ell$. It is well-known that the special value $L(0, \lambda\chi)$, after excluding its Euler ℓ-factor, is integral for all Dirichlet characters χ of ℓ-power conductors. Let $\mathcal{B}$ be the prime in $\overline{\mathbb{Q}}_p$ over p. Washington proved the following theorem in [27]: for a fixed λ, the special value $L(0, \lambda\chi)$ vanishes modulo $\mathcal{B}$ only for finitely many χ's. In the original proof, Washington verified Weyl's criterion and therefore the equidistribution of a certain sequence in $[0, 1)^d$ originating from ℓ-adic numbers, so-called *normal numbers*. Later, an algebraic approach was developed by Sinnott ([19]), whose argument can be transformed into a geometric one that a certain subset of $\mu_{\ell^\infty}^d$ is Zariski dense in the torus $\mathbb{G}_m^d$, where μ_{ℓ^∞} is the collection of all ℓ-power roots of unity. For the detailed description of Washington's theorem in this context, see [22].

The two main topics I have been studying in my thesis are L-values with anti-cyclotomic twists and cyclotomic twists. In [10], Ferrero and Washington proved Iwasawa's conjecture on the vanishing of the μ-invariant and Washington later developed the proof of his own theorem. At first glance the proofs look so subtle and complicated that the long and elementary calculation appears to be understood only by authors' ingenious intuition. In my paper [22], a more conceptual homological equidistribution argument (*homological independence*) has been developed to give a new proof of Washington's theorem. I will give an outline of a generalization to *cuspidal* modular L-values. So far nobody has been able to prove an analogue of the theorem for cuspidal modular L-values twisted by Dirichlet character χ. This we refer as the *cyclotomic case*. The cyclotomic case has its origin in arithmetic problems to determine the exponent of ℓ-power factors of class numbers in the cyclotomic $\mathbb{Z}_\ell$-extensions of $\mathbb{Q}$. Iwasawa had proved that if $K_1 \subset K_2 \subset \cdots \subset K_\infty$ is a $\mathbb{Z}_\ell$-tower of a number field K_1, the ℓ-exponents of class numbers $h(K_n)$ are of the form $\mu\ell^n + \lambda n + \nu$ for some μ, λ, $\nu \in \mathbb{Z}_{\geq 0}$ and all sufficiently large n. He also predicted that the μ-invariant is zero, and this conjecture was established by Ferrero and Washington([10]) when $K_1 = \mathbb{Q}$. The exponent of the p-power factor of $h(K_n)$ is believed to be bounded, and this conjecture for $K_1 = \mathbb{Q}$ is verified by Washington ([27]) as an application of his theorem. This example shows the proof of non-vanishing modulo p of L-values is not just a technical challenge, but has deep applications in arithmetic of number fields.

Twists by anti-cyclotomic characters χ (of imaginary quadratic fields) have been dealt with quite differently from the cyclotomic case, and some generalizations have been given by Hida([7]), Vatsal([24],[25]), and other researchers. This *anti-cyclotomic case* involves more sophisticated tools, but is philosophically more well-understood than the cyclotomic case. It is Vatsal([25]) who successfully transferred the original equidistribution idea of [10] to the case of modular L-values for a modular form of weight 2 under the twists by ring class characters. As an application of non-vanishing, a conjecture of Mazur on the special values of elliptic L-function is established. He has used the equidistribution of Heegner points on Gross curve associated with definite quaternion algebra and Ratner's ergodic theory on p-adic Lie group. On the other hand, even though Sinnott's algebraic proof of the theorem was elegant and simple, a generalization is not made until Hida interpreted the proof as a geometric one. He extended it to abelian Hecke L-series using the Zariski density of CM (Heegner) points on a Hilbert modular variety. We plan to apply Hida's method to cuspidal L-values to obtain more general results than in the earlier works. In the next few paragraphs, we sketch these two topics in detail:

Geometric proof (Originated by Sinnott). Let $X_{/\overline{\mathbb{F}}_p}$ be an algebraic variety over $\overline{\mathbb{F}}_p$, K a number field, Ξ a collection of special points in X, and f a section of a line bundle on X. Let Z be a profinite group which is the projective limit of (ring or standard) class groups of K with ℓ-power conductors. It has the decomposition $Z = \Delta \times \Gamma$, where Δ is finite and Γ is torsion-free subgroup. Suppose we have an action of Z on Ξ. The (square root of) special L-values are expressible as a finite sum of the characters against evaluations of f over Z. In these examples, the value of f at $x \in \Xi$ is a partial L-value. Assuming the vanishing of twisted L-values for infinitely many characters of Γ, we derive

$$(*) \quad \sum_{\eta \in \Delta} f(\eta \cdot \mathfrak{a}) \equiv 0 \mod \mathcal{B} \text{ for each } \mathfrak{a} \in \Xi.$$

Then the Zariski-density of the image of the map $j : \Xi \to X^{\Delta}$, $\mathfrak{a} \mapsto (\eta \cdot \mathfrak{a})_{\eta \in \Delta}$ often enables us to conclude that $f \equiv 0 \mod \mathcal{B}$, which is a contradiction. There are two examples: (1) Dirichlet L-values: $K = \mathbb{Q}$, $Z \cong \mathbb{Z}_\ell^\times$, $X = \mathbb{G}_m$, $f =$ a suitable rational function and $\Xi = \mu_{\ell^\infty}$. (2) The anti-cyclotomic case: $K = \mathbb{Q}(\sqrt{-d})$, $d > 0$, $Z = \varprojlim_n \mathrm{Pic}(\mathcal{O}_{\ell^n})$ (see section 3), $X =$ a modular curve $X_0(N)$, $f =$ an eigencusp newform of level N and $\Xi =$ Heegner (CM) points associated to K of ℓ-power conductors. This case deals with the modular L-value $L(1, f, \chi)$, which is a Rankin product of f and the theta series of χ.

Homological proof (originated by Washington and Ferrero). We adopt a different approach, namely the homological analogue of the anti-cyclotomic case. We consider a Riemann surface X, a finite subset $\mathfrak{b}$ of X, a subset Ξ of $H_1(X, \mathfrak{b}, \mathbb{Z})$ with an action of $\mathbb{Z}_\ell^\times$, and a differential form ω_f on X. Set $\Delta = \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$. There is an integral representation of L-values which is interpretable as a sum of (cup-product) pairings between homology classes in Ξ and a cohomology class on X. This is the modular symbol approach. After the same process as before, we get vanishing similar to $(*)$:

$$(**) \quad \sum_{\eta \in \Delta} \langle \eta \cdot v, \omega_f \rangle \equiv 0 \mod \mathcal{B} \text{ for each } v \in \Xi.$$

Hence, if the image of the map $j : \Xi \to H_1(X, \mathfrak{b}, \mathbb{Z})^{\Delta}$, $v \mapsto (\eta \cdot v)_\eta$ generates whole homology group $H_1(X, \mathbb{Z})^{\Delta}$, or in other words, if the image has an equidistribution property, then we expect to get a contradiction similar to the anti-cyclotomic case. There are two examples: (1) Dirichlet L-values: $X = (\mathbb{C}/\mathbb{Z} - S_N) \bigcup \{\pm i\infty\}$ for $S_N = \left\{ \frac{j}{N} \,\middle|\, 0 \le j < N \right\}$, $\mathfrak{b} = \{\pm i\infty\}$, and $\Xi = \{v(r) = r + i\mathbb{R} \,|\, r \in \ell^{-\infty}\mathbb{Z}/\mathbb{Z}\}$. (2) The cyclotomic case: $X = X_0(N)$, $\mathfrak{b} = \{cusps\}$, and $\Xi = \{v(r) = r + i\mathbb{R}_+ \,|\, r \in \ell^{-\infty}\mathbb{Z}/\mathbb{Z}\}$. This case deals with modular L-values $L(m, f, \chi)$, which are the cyclotomic twists $L(m, f \otimes \chi)$ by the characters χ.

2. **Dirichlet L-values : Homological version of Washington's proof**

Let T_N be a cylinder from which N points are removed and to which two points at infinity are added. In other words, we set $T_N = (\mathbb{C}/\mathbb{Z} - S_N) \bigcup \{\pm i\infty\}$ for $S_N = \{j/N | 0 \le j < N\}$ and an integer N. Let $v(r) = r + i\mathbb{R}$ be the vertical line on the punctured cylinder, which is an element of the relative homology $H_1(T_N, \{\pm i\infty\}, \mathbb{Z})$ and Ξ a collection of vertical lines $\{v(r) \,|\, r \in \ell^{-\infty}\mathbb{Z}/\mathbb{Z}\}$. The following map is a key ingredient in the homological version of Washington's proof:

$$j : \Xi \to H_1(T_N, \{\pm i\infty\}, \mathbb{Z})^{\frac{\ell-1}{2}}, \ v(a) \mapsto (v(a\eta))_{\eta \in U}.$$

In our paper [22], we give a concise proof of the following homological version of equidistribution:

Proposition. *For each sequence of integers $n_1 < n_2 < \cdots$ and a fixed m, the image of $\left\{ v\left(\frac{a}{\ell^{n_j}}\right) \,\middle|\, j = 1, 2, \cdots \right\}$ for all $a \in \Gamma_m = 1 + \ell^m\mathbb{Z}_\ell$ under j generates $H_1(T_N, \mathbb{Z})^{\frac{\ell-1}{2}}$ $\left(\subseteq H_1(T_N, \{\pm i\infty\}, \mathbb{Z})^{\frac{\ell-1}{2}}\right)$. Hence $H_1(T_N, \mathbb{Z})^{\frac{\ell-1}{2}}$ is* *generated by the image of j.*

For a Dirichlet character ψ, we define a rational function $f_\psi(z) = \sum_{n\geq 1} \psi(n)q^n$ on $\mathbb{G}_m$ ($q = \exp(2\pi iz)$). Special Dirichlet L-values $L(0,\psi)$ can be described as the integral along the vertical line $v(0)$ against the cohomology class $f_\psi(z)dz$ in $H^1_c(T_N, \mathbb{C})$. Using this, we deduce $(**)$. This gives a short and more conceptual proof of Washington's theorem (see [22]).

3. Modular Cuspidal L-values with Anti-Cyclotomic Twists

Zariski density of Heegner points in the indefinite case. Let $D_{/\mathbb{Q}}$ be an indefinite quaternion algebra of discriminant N^- with $\phi_\infty : D_\infty \cong M_2(\mathbb{R})$, ι be the involution inducing the reduced norm of D and R be an Eichler order of D, of level N^+, which is square free and relatively prime to N^-. We consider a homogeneous space $S = \mathfrak{H} \cup \overline{\mathfrak{H}}$ on which the group $D^\times$ acts and the Shimura curve (as a complex manifold)

$$Y(\mathbb{C}) = D^\times \backslash S \times D(\mathbb{A}^{(\infty)})^\times / \widehat{R}^\times.$$

The classical open modular curve $\Gamma\backslash\mathfrak{H}$ is recognized as $Y(\mathbb{C})$ through the identification $Y(\mathbb{C}) \cong \phi_\infty(R)\backslash S \cong \Gamma\backslash\mathfrak{H}$, $[z,1] \mapsto z$ for Γ, the image under ϕ_∞ of reduced norm 1 elements of R. The curve $Y(\mathbb{C})$ has a suitable compactification $X(\mathbb{C})$, which is $Y(\mathbb{C})$ itself if the discriminant N^- is non-trivial (i.e. D is a division algebra). Due to Shimura, $X(\mathbb{C})$ is known to be complex points of an algebraic curve X canonically defined over $\mathbb{Q}$.

Each imaginary quadratic field K (rather its order $\mathcal{O}_c$ of conductor c) gives rise to the set $\mathfrak{h}_N(K,c)$ of *Heegner points* of conductor c. By the reciprocity law of Shimura, a Heegner point $z \in \mathfrak{h}_N(K,c)$ is rational over the ring class field of $\mathcal{O}_c$; therefore the *Picard group* of the order $\mathcal{O}_c$, i.e. the ring class group of K with the conductor c, has an action on z via the Artin map $\sigma(x)$. The projective limit of Picard groups $\mathrm{Pic}(\mathcal{O}_{\ell^n})$ decomposes into a finite subgroup Δ and an ℓ-profinite subgroup Γ. There is a subset U of Δ, which is in a sense algebraically independent and in fact is equal to $U = \Delta/\Delta_0$ for the genus subgroup Δ_0 of Δ. Let Ξ be a collection of Heegner points with ℓ-power conductors. We consider the map j defined by

$$j : \Xi \to X^U_{/\overline{\mathbb{F}}_p},\ z \mapsto \left(z^{\sigma(\eta)} \right)_{\eta\in U}.$$

In [7], Zariski density of the image of j is established by Hida.

Special L-values: Waldspurger, Harris and Kudla formula. Let f be a normalized eigencusp newform for $\Gamma_0(N)$ with weight k and π_f be the corresponding cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Let χ be a ring class character on $\mathrm{Pic}(\mathcal{O}_{\ell^n})$ of conductor ℓ^n. By the Jacquet-Langlands correspondence, there is a normalized eigencusp newform g on Γ with the same Hecke eigenvalues as those of f, and its adelic lift Ψ. Then, for a Heegner point $z = [h,1]$ of conductor ℓ^n, we set

$$\Sigma(z,\chi,\Psi) = \frac{1}{h_K} \sum_{x\in \mathrm{Pic}(\mathcal{O}_{\ell^n})} g(z^{\sigma(x)})\chi(x).$$

By the formula of Waldspurger, Harris and Kudla, and explicit computation of local zeta integrals at all places in [18], there exists a scalar multiple $\Psi' = \pm\Psi$ of Ψ such that

$$\frac{2\cdot k!}{\pi^k h_K} \prod_{q|N^-} \frac{q-1}{q+1} \frac{L\left(\frac{1}{2}, \pi_f \otimes \chi|\cdot|^k_{\mathbb{A}}\right)}{L(1,\eta_K)} = (4\Im(z) j(\kappa,z))^k \Sigma(z,\chi,\Psi')^2,$$

where η_K is the quadratic character of K and we have $K + \kappa K = M_2(\mathbb{Q})$ so that $\kappa^\iota = -\kappa$ and $x\kappa = \kappa\overline{x}$ for all $x \in K$.

Non-vanishing of L-values. The advantage of the use of indefinite quaternion algebras over the use of definite ones comes from the Harris-Kudla formula, which is valid for modular forms with weights ≥ 2, while Gross' formula could only handle the weight 2 case. If N^- is square-free and has an odd number of prime divisors, and χ is anti-cyclotomic, then the L-function has a functional equation with sign -1, and therefore its central critical value is equal to zero. Therefore we only consider the case that the number of prime divisors of square free N^- is even.

In the author's thesis [23], we consider the case that the discriminant N^- of D is 1. In other words, D is the matrix algebra $M_2(\mathbb{Q})$ over $\mathbb{Q}$. Hence for each prime divisor q of N, q splits in K by the Heegner condition([2]). Let f be a normalized $\mathcal{B}$-integral eigencusp newform for $\Gamma_0(N)$ with the previous condition on N. There exists a canonical CM period $\Omega \in \mathbb{C}$ which is defined up to a p-adic unit. It is shown in [18] that $\pi^{k-1}L\big(\frac{1}{2}, \pi_f \otimes \chi|\cdot|_{\mathbb{A}}^k\big)/\Omega^{2k}$ is a $\mathcal{B}$-adic integer. For a prime ℓ that splits in K we have

Theorem ([23])**.** *For almost all anti-cyclotomic χ of ℓ-power conductor, we have* $\dfrac{\pi^{k-1}L\big(\frac{1}{2}, \pi_f \otimes \chi|\cdot|_{\mathbb{A}}^k\big)}{\Omega^{2k}} \not\equiv 0 \mod \mathcal{B}$.

By Hida's density result, we reach a version of (∗∗), and it is crucial to use the q-expansion principle and the Chebotarev's density theorem in concluding the absurdity: $f \equiv 0 \mod \mathcal{B}$ assuming the contrary of Theorem.

4. Modular Cuspidal L-values with Cyclotomic Twists

Modular Symbols and L-values. We consider the relative homology group $H_1(X, \{cusps\}, \mathbb{Z})$, and the homology group $H_1(X, \mathbb{Z})$ for a modular curve X of level N, which is associated with a congruence group Γ. For example, X is $X_0(N)$ or $X_1(N)$. There is a (relative) homology class $\{\alpha, \beta\}_\Gamma \in H_1(X, \{cusps\}, \mathbb{Z})$, that is called a *modular symbol* of weight 2 for Γ, corresponding to a geodesic connecting α and β on X. To each eigen cusp form $f \in \mathcal{S}_2(\Gamma)$ of weight 2, we relate a differential form $\omega_f = f(z)dz$. Let χ be a Dirichlet character with conductor ℓ^n and set $V_\chi = \sum_{r=0}^{\ell^n-1} \overline{\chi}(r) \left\{\frac{r}{\ell^n}, i\infty\right\}_\Gamma$. One obtains the fundamental identity:

$$L(f, \chi, 1) = \frac{2\pi i}{G(\overline{\chi})} \int_{V_\chi} \omega_f.$$

Homological Independence. For $r \in \mathbb{Q}$, let $v(r) = \{r, i\infty\}_\Gamma \in H_1(X, \{cusps\}, \mathbb{Z})$ and $\Xi = \{v(r) \,|\, r \in \ell^{-\infty}\mathbb{Z}/\mathbb{Z}\}$. In light of the homological proof of Washington's theorem, we consider the map

$$j : \Xi \to H_1(X, \{cusps\}, \mathbb{Z})^{\frac{\ell-1}{2}}, \; v(a) \mapsto (v(a\eta))_\eta.$$

Let $M(\ell)$ be a submodule of $H_1(X, \mathbb{Z})$ generated by the image of j. In some easy cases and also by numerical calculations, we verify the following conjectures:

Conjecture 1. *For almost all prime p, p-adic valuation $v_p\left(\left[H_1(X, \mathbb{Z})^{\frac{\ell-1}{2}} : M(\ell)\right]\right)$ is 0 for all odd primes ℓ.*

Let $\mathbb{T}$ be the subring of $\mathrm{End}(\mathcal{S}_2(\Gamma))$ generated by the Hecke operators T_m for all m. Let K_f be the finite extension of $\mathbb{Q}_p$ containing all eigenvaules of f and $\mathcal{O}$ the integer ring of K_f with maximal ideal λ. One can obtain an algebra homomorphism $\phi_f : \mathbb{T} \to \mathcal{O}$. Let $\mathfrak{m}$ be the preimage of λ. One can show that $\mathcal{S}_2(\Gamma_1(N), \mathbb{Z}_p)_\mathfrak{m}$ is free over $\mathbb{T}_\mathfrak{m}$, and $H_1(X_1(N), \mathbb{Z})_\mathfrak{m}$ is free of rank two over $\mathbb{T}_\mathfrak{m}$, when N is not divisible by p (see [9, §12.5]). With the $\mathbb{T}_\mathfrak{m}$-module $M(\ell)_\mathfrak{m} = M(\ell) \otimes_\mathbb{Z} \mathbb{T}_\mathfrak{m}$, we also state

Conjecture 2. *For almost all primes p, we have $H_1(X_1(N), \mathbb{Z}_p)_\mathfrak{m}^{\frac{\ell-1}{2}} = M(\ell)_\mathfrak{m}$ as $\mathbb{T}_\mathfrak{m}$-modules for all odd primes ℓ.*

Let $X = X_0(N)$, $\Xi_n = \{v(r) | r \in \ell^{-n}\mathbb{Z}/\mathbb{Z}\}$, and $\nu(N) = \prod_{q|N}(q-1)$. Let $M_n(\ell)$ be the submodule of $M(\ell)$ generated by the image of Ξ_n. A more explicit version of above conjectures is

Conjecture 3. *For all primes $p \nmid \nu(N)$ and odd primes ℓ, we have $v_p\left(\left[H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}}) : M_n(\ell)\right]\right) = 0$ for all sufficiently large n.*

Numerical Computation. With the algorithms and descriptions of modular symbols in [3] and [21], one is able to verify Conjecture 3 computationally. In our paper [22], we construct a submodule of $M_n(\ell)$ using *Manin symbols* and calculate the list of possible prime numbers dividing the index of the submodule and the whole homology group $H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}}$, which is divisible by the index of $M_n(\ell)$ and the homology group. Numerical computations seem to suggest that if N is prime, then the prime divisors of the index $\left[H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}} : M_n(\ell)\right]$ are *Eisenstein* in the sense of [13]. In other words, they are divisors of the numerator of reduced form of $\frac{N-1}{12}$.

5. **Future Plans**

Anti-cyclotomic case. In the thesis [23], special values of elliptic modular L-functions are studied, which correspond to the case $N^- = 1$. Apparently, the next step would be the treatment of the case where D is a division algebra over $\mathbb{Q}$ (so $N^- > 1$). The main difficulty of extending the argument to the case of non-trivial discriminant is that there is no q-expansion of g as the corresponding modular curve is compact. One may consider the Galois representation ρ_g associated with g and try to establish representation theoretic analogue of descriptions of elliptic modular case and extend it to the division algebra case. The eigenvalues of g will provide much information, as they are traces of Frobenius elements under ρ_g. We expect to deduce the theorem after proving an Ihara's lemma type result such as discussed in [4].

Anticipated proof of the conjectures on homological independence. It seems that there are two possible approaches to attack the conjectures. Dealing with eigencusp form f, it suffices to verify Conjecture 2 to get a non-vanishing result. In [16], Merel shows that if N is a sufficiently large prime and $\mathbf{e}$ is the cycle from 0 to $i\infty$, then $T_1\mathbf{e}$, $T_2\mathbf{e}, \cdots,$ $T_D\mathbf{e}$ are linearly independent in $H_1(X_0(N)(\mathbb{C}), \mathbb{Q})$ for Hecke operators T_n and $\max(D^8, 400D^4) < N/(\log N)^4$. This result suggests that $M_n(\ell) \otimes \mathbb{T}_\mathfrak{m}$ may contain enough linearly independent elements. One hopes to analyze and modify his proof to get an idea of how to verify Conjecture 2. In another direction, Duke in [5] studied the distribution of closed geodesics on $\Gamma_0(1)\backslash\mathfrak{H}$ using suitable "Weyl sums", which are treated as Fourier coefficients of a certain (non) holomorphic modular form of weight 1/2. I would like to develop a multi-dimensional version of "Weyl sum" to establish Conjecture 3.

The vanishing of μ_p-invariant of p-adic L-functions. I have shown in the brief note which is included in the application that the homological independence can be used to get the vanishing of μ_p-invariant of p-adic Dirichlet L-functions. Main ingredient is the definition of a suitable measure using modular symbols similar to one defined in [14]. It would be interesting to consider the cyclotomic μ-invariant of p-adic modular L-functions. The calculation in [22] suggests that we need to exclude the case that p is the Eisenstein prime.

Cyclotomic Hecke L-values over a totally real field. Obviously, a study of non-vanishing of Hecke L-values over a totally real number field F would be a natural extension of Washington's theorem. Following the geometric setting in the Introduction, one considers a multi-dimensional torus $\mathbb{G}_m^d$ with $d = [F : \mathbb{Q}]$ and a rational function on it. More precisely, one considers $X = \mathbb{G}_m \otimes \mathcal{O}_F \cong \mathbb{G}_m^d$ for the ring $\mathcal{O}_F$ of integers, and a rational function $\Phi(\mathfrak{S}; t)$ on X defined by a *Shintani (cone) decomposition* $\mathfrak{S}$ of $F \otimes \mathbb{R}$. Let us assume that ℓ is inert in F for simplicity. Even though we do verify the Zariski density property as in the corresponding situation for $X = \mathbb{G}_m$ ([8]), the existence of the non-trivial unit group $\mathcal{O}_F^\times$ gives rise to the following problem. As the unit group is contained in the kernel of a Hecke character χ, the quotient $V_n = \mathrm{Ker}(\chi)/\mathrm{Ker}(\chi)^{\ell^n}$ is not trivial. Roughly speaking, the vanishing of $L(0, \chi)$ is reduced to the vanishing of sum $\sum_{u \in V_n} \Phi(\mathfrak{S}; t^u)$, which is dependent on χ, and therefore is not a rational function. This obstacle prevents us from using a Zariski density argument.

The above observation also brings our attention to the following speculation. The characters that give rise to the vanishing of L-values correspond to the zeros of $\Phi(\mathfrak{S}_n; t)$. Having this fact in mind one studies the degree of zero divisors of $\Phi(\mathfrak{S}_n; t)$. Using a certain naive compactification $\mathbb{P}^d(\mathfrak{S}_n)$ of X that comes from $\mathfrak{S}_n$, we verify in [8] that $\Phi(\mathfrak{S}_n; t)$ does not have any poles at prime divisors outside X. Therefore it suffices to compute the degree of polar divisor inside X whose support consists of zeros of simple monomials. However, it turns out that the estimate for the degree is too large for practical application. I would like to study this topic in this direction with toroidal compactification of X, which is dealt with in [1].

Homological Independence for Hecke L-values. Extending the homological independence argument, one also expects to avoid the above obstacle as follows: Following the homological setting in the Introduction, one considers a multi-dimensional cylinder that is analogous to one introduced in Section 2. In other words, we set

$$T_\mathfrak{S} = (\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{C}/\mathcal{O}_F - \{\text{poles of } \Phi(\mathfrak{S}; t)\}) \cup \{\mathcal{O}_F \otimes i\infty\}.$$

One also considers a (relative) homology class $v(\alpha) = \alpha + C(\mathfrak{S})$ that is a translation of the Shintani cone $C(\mathfrak{S})$ in $\mathcal{O}_F \otimes i\mathbb{R}$ by $\alpha \in \mathcal{O}_F \otimes \mathbb{R}/\mathcal{O}_F$. Let U be a certain finite subset of the unit group $\mathcal{O}_{F,\ell}^\times$ of the ℓ-adic completion of $\mathcal{O}_F$. Let Ξ be the submodule generated by $v(\alpha)$ for $\alpha \in \ell^{-\infty}\mathcal{O}_F/\mathcal{O}_F$. I would like to study the image of Ξ

under a map

$$j : \Xi \to H_d(T_{\mathfrak{S}}, \{\mathcal{O}_F \otimes i\infty\}, \mathbb{Z})^U,\ v(\alpha) \mapsto \big(v(\alpha\eta)\big)_{\eta \in U}.$$

and in particular check if the images of $\sum_{u \in V_n} v(u\alpha)$ generate $H_d(T_{\mathfrak{S}}, \mathbb{Z})^U$ for all sufficiently large n.

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