Homological Interpretation of a Theorem of L. Washington

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Abstract

We give a homological interpretation of Washington's theorem about non-vanishing mod p of special L-values twisted by Dirichlet characters with ℓ-power conductors. Then we make some conjectures to generalize the argument to the case of elliptic modular L-values using modular symbols. The conjecture is checked using numerical calculations.

Key words: homology, modular symbols, modular curve, special L-values, Washington's theorem.

1 Introduction

In this paper, we discuss a homological interpretation of Washington's classical result about special value of Dirichlet L-functions. To state the result, let us consider two different odd primes, p and ℓ, and two Dirichlet characters $\lambda : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}$ of conductor N, and $\chi : (\mathbb{Z}/\ell^n\mathbb{Z})^\times \to \mu_{\ell^\infty}$ of ℓ-power conductor respectively. Here N is relatively prime to both p and ℓ and $\mu_{\ell^\infty} = \cup_{n\geq 1}\mu_{\ell^n}$ with the group μ_{ℓ^n} of ℓ^n-th roots of unity. It is well-known that the special value of Dirichlet L-function $L(0, \lambda\chi)$, after excluding its Euler ℓ-factor, is integral. Let $\mathcal{B}$ be a prime in $\overline{\mathbb{Q}}_p$ over p. Washington proved in [14] that

Theorem 1 *Let λ be an odd Dirichlet character of conductor N. Then for only finitely many Dirichlet characters $\chi : (\mathbb{Z}/\ell^n\mathbb{Z})^\times \to \mu_{\ell^\infty}$, we have $L(0, \lambda\chi) \equiv 0 \pmod{\mathcal{B}}$.*

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Note that this implies the finiteness ofl exponents of the prime divisor ℓ in the number h_n^-, which is the relative class number of each subfield K_n of cyclotomic $\mathbb{Z}_\ell$-extension K_∞ of $\mathbb{Q}$. Using Kummer type argument studied in [15], we also get the finiteness of h_n, the class number of K_n. The group $\mathbb{Z}_\ell^\times = \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q})$ has the decomposition $\mathbb{Z}_\ell^\times = \mu_{\ell-1} \times \Gamma$ and the profinite group, $\Gamma \cong \mathbb{Z}_\ell$ is the Galois group $\mathrm{Gal}(K_\infty/\mathbb{Q})$. A factor $\mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$ is the $(\ell-1)$-th roots of unity in $\mathbb{Z}_\ell^\times$. One of main ingredients in [14] is that the maximal subset of $\mu_{\ell-1}$ which are linearly independent over $\mathbb{Q}$ generates a subset of $(\ell^{-\infty}\mathbb{Z}/\mathbb{Z})^{\frac{\ell-1}{2}}$ so that it is uniformly distributed on $(0,1)^{\frac{\ell-1}{2}}$. This will be discussed briefly in Section 2.

In [11], Sinnott invented a totally different approach to the proof of this problem. He studied a suitable measure on $\mathbb{Z}_\ell^\times$, which are defined by a rational function twisted by the elements in $\mu_{\ell-1}$, called formal functions. Instead of uniform distribution property, he made use of the algebraic independence of such formal functions. It was Hida who realized that Sinnott's method can be written in an algebro-geometric language in [4] and this (modular) version of Sinnott's proof is explained in the appendix of the present paper. Let U be a maximally linearly independent subset of $\mu_{\ell-1}$ and consider the map

$$j : \mu_{\ell^\infty} \to \mathbb{G}_m^U,\ \zeta \mapsto (\zeta^\eta)_{\eta\in U}.$$

Here $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t,t^{-1}])$. Then the algebraic independence result can be reformulated as follows.

Proposition 2 *The image of j is Zariski dense in $\mathbb{G}_m^U$.*

He generalize this to the situation of Hecke L-values of anti-cyclotomic characters of prime power conductors. In this context, the $\mathbb{G}_m$ is replaced by a Hilbert modular Shimura variety and μ_{ℓ^∞} corresponds to various CM points of prime power conductor on it. Like the case of a rational function on $\mathbb{G}_m$, Eisenstein series is made use of to get the special L-values. For details, see [5].

By an integral representation of L-functions, we can think of special L-values as a (cup product) pairing of a homology class and a cohomology class defined by a power series which is actually a rational function whose poles are roots of unity. This is an abelian version of the Mazur's modular symbol method, which are discussed in great details in [6]. The crucial point of this paper, we call it homological independence, is a mixture of Washington's original uniform distribution result and Hida's Zariski density argument. Let us explain the situation briefly. We consider a punctured cylinder $\mathbb{C}/\mathbb{Z} - S_N \cong \mathbb{G}_m(\mathbb{C}) - \mu_N(\mathbb{C})$, where $S_N = \{j/N | 0 \le j < N\}$ for an integer N. We add two points $\{\pm i\infty\}$ to the cylinder in order to get T_N^0. Let $v(r)$ be the vertical line $r + it$, $-\infty \le t \le \infty$ for $0 < r < 1$, $r \notin S_N$. Hence $v(r)$ are in the relative homology

$H_1(T_N^0, \{\pm i\infty\}, \mathbb{Z})$. We consider a map

$$j : \ell^{-\infty}\mathbb{Z}/\mathbb{Z} \to H_1(T_N^0, \{\pm i\infty\}, \mathbb{Z})^{\frac{\ell-1}{2}}.$$

We will prove an independence result related to above map j in the Section 2 from this homological setting and we shall give a new proof of Theorem 1 in Section 3, interpreting Washington's argument in a homological way. Although the proof itself is not much new, it seems to be simple enough to be extended to the case of elliptic modular L-values

There have been several results concerned about non-vanishing property for prime modulus of special L-values. In addition to the results [14] and [5], Vatsal considered the modular L-values defined over the imaginary quadratic field twisted by anti-cyclotomic characters (see [12,13]). Vatsal has used the Gross' formula for special L-values of a modular form defined on the definite quaternion algebra over $\mathbb{Q}$([12]) and the uniform distribution of Heegner points on the modular curve associated with the quaternion algebra([13]). Using a modular symbol method, we hope to be able to get a generalization of our method to modular L-values twisted by Dirichlet characters of prime power conductor in near future. Having in mind this hope, in Section 4 we review briefly the modular symbols and Manin symbols and discuss some conjecture related to the homology group of modular curve and the analog of the above map j. In other words, we consider a map

$$j : \ell^{-\infty}\mathbb{Z}/\mathbb{Z} \to H_1(X_0(N), \{cusps\}, \mathbb{Z})^{\frac{\ell-1}{2}}$$

defined by modular symbols (see Conjecture 6). Even though the map j is defined geometrically, we obtain many numerical evidences which suggest that verification of the homological independence in the elliptic modular setting would be algebraic. In the appendix, using algorithmic result in [1] we collect some numerical evidences that the behavior of j in this setting is similar to the Proposition 2.

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2 Homological Independence

As mentioned in the introduction, we consider T_N^0, which is the cylinder $\mathbb{C}/\mathbb{Z}$ punctured at points S_N and with two points $\{\pm i\infty\}$ added. Each vertical line

$v(r)$ is in the relative homology $H_1(T_N^0, \{\pm i\infty\}, \mathbb{Z})$. Let $c(k/N) \in H_1(T_N^0, \mathbb{Z})$ be the closed path on T_0^N starting from a fixed base point $z_0 \in \mathbb{C}$, then turning around k/N counterclockwise and returning to z_0. The homology $H_1(T_0^N, \mathbb{Z})$ is generated by such $c(\frac{k}{N})$. For any $r \in (\frac{k-1}{N}, \frac{k}{N})$, and $s \in (\frac{k}{N}, \frac{k+1}{N})$, we have $c(\frac{k}{N}) = v(r) - v(s)$. The main idea is that given any partition of $(0,1)$, say $(0, \frac{1}{N}) \cup \cdots \cup (\frac{N-2}{N}, \frac{N-1}{N})$, we can find two vertical lines from $\{v(r) | r \in \ell^{-\infty}\mathbb{Z}/\mathbb{Z}\}$, which are on the two consecutive partitions $(\frac{k-1}{N}, \frac{k}{N})$ and $(\frac{k}{N}, \frac{k+1}{N})$ to get the homology class $c(k/N)$. To make this idea working, the uniform distribution property should be brought into the present picture.

We consider a $\mathbb{Q}$-linearly independent maximal subset U of $\mu_{\ell-1}$. It is a well-known fact([3]) that the collection of points $\{\, (\frac{a\xi}{\ell^n})_{\xi \in U} \,|\, n \geq 1 \,\}$ is uniformly distributed on $[0,1)^U$ for almost all $a \in \mathbb{Z}_\ell^\times$; i.e., for all a outside of a measure zero subset of $\mathbb{Z}_\ell^\times$. Here the ℓ-adic numbers $\frac{a\xi}{\ell^n}$ are thought of as the element $\frac{s_n(a\xi)}{\ell^n}$ in T_N^0 for n-th partial sum s_n of ℓ-adic integers. More specifically, when a subset $Z \subseteq \mathbb{Z}_\ell^\times$ with non-zero measure and a point $x_\xi \in (0,1)$ and $\epsilon_\xi > 0$ are given for each $\xi \in U$, then for all sufficiently large n, we can find $a \in Z$ so that $|x_\xi - \frac{a\xi}{\ell^n}| < \epsilon_\xi$ for each $\xi \in U$. Roughly speaking, for a given vector $(x_\xi)_{\xi \in U}$, we can find a point $(\frac{a\xi}{\ell^n})_{\xi \in U}$ which is close enough to $(x_\xi)_{\xi \in U}$.

Set $U = \{\eta_1 = \eta, \eta_2, \cdots, \eta_t\}$ and $\mu_{\ell-1}/\{\pm 1\} = \{\eta_1, \cdots, \eta_t, \tau_1, \cdots, \tau_s\}$. We have an integral $s \times t$ matrix A such that

$$(\tau_1, \cdots, \tau_s) = (\eta_1, \cdots, \eta_t)A.$$

Since $\mu_{\ell-1}/\{\pm 1\}$ is $\mathbb{Z}$-multiplicatively independent i.e. there are no integers m, n such that $m\eta \neq n\eta'$ for distinct η, $\eta' \in \mu_{\ell-1}/\{\pm 1\}$, all the columns of A have at least two non-zero entries. Now let us state a lemma.

Lemma 3 *Let $\alpha_1, \cdots, \alpha_t$ be any real numbers and set*

$$P(\alpha_1, \cdots, \alpha_t) := (\alpha_1, \cdots, \alpha_t)\,(I|A)$$

for a $t \times t$ identity matrix I. For each $k \in \mathbb{Z}$, we can find numbers α_1', α_1'', and α_2^o, $\cdots$, α_t^o such that (1) $\alpha_1' \in (\frac{k-1}{N}, \frac{k}{N})$, $\alpha_1'' \in (\frac{k}{N}, \frac{k+1}{N})$, and (2) $P(\alpha_1', \alpha_2^o, \cdots, \alpha_t^o)$, $P(\alpha_1'', \alpha_2^o, \cdots, \alpha_t^o)$ are away from $\left(\frac{1}{N}\mathbb{Z}\right)^{\frac{\ell-1}{2}}$. In other words, no coordinate of them are in $\frac{1}{N}\mathbb{Z}$.

PROOF. Consider the function $P(\frac{k}{N}, \alpha_2, \cdots, \alpha_t)$ of $\alpha_2, \cdots, \alpha_t$. Since A is an integral matrix, $P(\frac{k}{N}, 0, \cdots, 0)$ is in $\left(\frac{1}{N}\mathbb{Z}\right)^{\frac{\ell-1}{2}}$. The $\mathbb{Z}$-multiplicative independence of $\mu_{\ell-1}/\{\pm 1\}$ implies that no coordinate in $P(\alpha_1, \cdots, \alpha_t)$ is $\mathbb{Z}$-multiple of α_1 except itself. Hence, $P(\frac{k}{N}, \alpha_2, \cdots, \alpha_t)$ is an open map; i.e., no coordinates are constant except the first one and we can find $(\alpha_2^o, \cdots, \alpha_t^o)$ around $(0, \cdots, 0)$ such that each coordinate of $P(\frac{k}{N}, \alpha_2^o, \cdots, \alpha_t^o)$ is not in $\frac{1}{N}\mathbb{Z}$ except the first one that is k/N. Since $P(\alpha_1, \alpha_2^o, \cdots, \alpha_t^o)$ is continuous in the variable α_1, we can find $\alpha_1' \in (\frac{k-1}{N}, \frac{k}{N})$, $\alpha_1'' \in (\frac{k}{N}, \frac{k+1}{N})$ satisfying the condition (2). $\square$

We define $j : \ell^{-\infty}\mathbb{Z}/\mathbb{Z} \to H_1(T_N^0, \{\pm i\infty\}, \mathbb{Z})^{\frac{\ell-1}{2}}$ for $\ell^{-\infty}\mathbb{Z} = \cup_n \ell^{-n}\mathbb{Z}$ by

$$j\left(\frac{a}{\ell^n}\right) = \left(v\left(\frac{a\xi}{\ell^n}\right)\right)_{\xi\in U}.$$

The following proposition is a homological variant of Proposition 2.

Proposition 4 *$H_1(T_N^0, \mathbb{Z})^{\frac{\ell-1}{2}}$ $(\subseteq H_1(T_N^0, \{\pm i\infty\}, \mathbb{Z})^{\frac{\ell-1}{2}})$ is generated by the image (under j) of*

$$\left\{\frac{a}{\ell^{n_j}}\,\Big|\, j \geq 0,\, a \in Z\right\}$$

for any increasing sequence $\{n_j\}$ of positive integers and any subset Z of $\mathbb{Z}_\ell^\times$ with non-zero measure. In particular, $H_1(T_N^0, \mathbb{Z})^{\frac{\ell-1}{2}}$ is generated by the image of j.

PROOF. As discussed above, the subset $\{\, (\frac{a\xi}{\ell^n})_{\xi\in U} \,|\, n \geq 1, a \in Z\,\}$ are uniformly distributed on $(0,1)^U$. Hence it is possible to find suitable n, m in $\{n_j\}$ and a, b in Z such that the vectors $(\frac{a\xi}{\ell^n})_{\xi\in U}$, $(\frac{b\xi}{\ell^m})_{\xi\in U}$ are close enough to $(\alpha_1', \alpha_2^o, \cdots, \alpha_t^o)$, $(\alpha_1'', \alpha_2^o, \cdots, \alpha_t^o)$ in the lemma respectively and, therefore, all coordinates of $P((\frac{a\xi}{\ell^n})_{\xi\in U})$, $P((\frac{b\xi}{\ell^m})_{\xi\in U})$ are inside of the same partitions except the first ones that are in consecutive partitions $(\frac{k-1}{N}, \frac{k}{N})$, $(\frac{k}{N}, \frac{k+1}{N})$ respectively. Due to this result, we have

$$j(\frac{a}{\ell^n}) - j(\frac{b}{\ell^m}) = (\,[0], \cdots, [0], c(k/N), [0], \cdots, [0]\,) \in H_1(T_N^0, \mathbb{Z})^{\frac{\ell-1}{2}},$$

where the position of the non-zero homology class corresponds to η. Since η and k are chosen arbitrarily, we prove the proposition. $\square$

3 Special L-values and a theorem of L. Washington

For a Dirichlet character ψ, we define

$$f_\psi(z) = \begin{cases} \sum_{n\geq 1} \psi(n) q^n & Im(z) > 0, \\ -\sum_{n\geq 1} \psi(-n) q^{-n} & Im(z) < 0 \end{cases}$$

for $q = \exp(2\pi i z)$. As shown in [6, §4.2], the differential $f_\psi(z)dz$ gives a cohomology class $\omega(f_\psi)$ in $H^1_c(T^0_N, \mathbb{C})$, and we have

$$\int_{-i\infty}^{i\infty} f_{\lambda\chi}(z)dz = -G(\chi)\frac{L(1,\lambda\chi)}{2\pi i} = -L(0,\lambda^{-1}\chi^{-1}).$$

Here $G(\chi)$ is the Gauss sum defined by χ. We define a pairing between v in $H_1(T^0_N, \{\pm\infty\}, \mathbb{Z})$ and ω in $H^1_c(T^0_N, \mathbb{C})$ as follows:

$$\langle v, \omega\rangle = \frac{1}{2\pi i}\int_v \omega.$$

There are operators $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ acting on T^0_N for each $r \in \ell^{-\infty}\mathbb{Z}/\mathbb{Z}$ defined by

$$z \mapsto \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \cdot z := z + r.$$

For any $\omega \in H^1_c(T^0_N, \mathbb{C})$, we define $\omega\Big| \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ by the natural action of $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ on $H^1_c(T^0_N, \mathbb{C})$ induced from the one on T^0_N.

From the decomposition

$$f_{\lambda\chi}(z) = \frac{1}{G(\chi)} \sum_{a\in(\mathbb{Z}/\ell^n\mathbb{Z})^\times} \chi(a) f_\lambda(z + \frac{a}{\ell^n}),$$

we have

$$\begin{aligned} -L(0,\lambda^{-1}\chi^{-1}) &= \frac{1}{G(\chi)} \sum_{a\in(\mathbb{Z}/\ell^n\mathbb{Z})^\times} \chi(a) \int_{-i\infty}^{i\infty} f_\lambda(z + \frac{a}{\ell^n})dz \\ &= \frac{1}{G(\chi)} \sum_a \chi(a) \int_{v(\frac{a}{\ell^n})} f_\lambda(z)dz \\ &= \frac{1}{G(\chi)} \sum_a \chi(a) \left\langle v\Big(\frac{a}{\ell^n}\Big), \omega(f_\lambda)\right\rangle. \end{aligned}$$

By inversion formula we obtain

$$\left\langle v\Big(\frac{a}{\ell^n}\Big), \omega(f_\lambda)\right\rangle = \frac{-1}{\ell^{n-1}(\ell-1)} \sum_{\psi \,(\bmod\, \ell^n)} \psi^{-1}(a) L(0, \lambda^{-1}\psi^{-1}),$$

and hence it is algebraic. The pairing is an element of $\mathbb{Q}(\lambda)$ because it is fixed under the $\mathrm{Gal}(\mathbb{Q}(\lambda,\psi)/\mathbb{Q}(\lambda))$.

Let k be the finite field over $\mathbb{F}_p$ generated by λ and the ℓ-th root of unity. There is an integer m such that $\mu_{\ell^\infty} \cap k = \mu_{\ell^m}$. Setting $k_j = k(\mu_{\ell^{m+j}})$ and $k_\infty = \bigcup_{j\geq 1} k_j$, we have $\mathrm{Gal}(k_\infty/k) = \Gamma_m$ with the action on μ_{ℓ^∞} given by $\zeta \mapsto \zeta^t$ for $t \in \Gamma_m$.

Now assume that $L(0, \lambda^{-1}\chi^{-1}) \equiv 0 \,(\bmod\, \mathcal{B})$ for a Dirichelt character χ of conductor ℓ^n and $n \geq 2m$. Multiplying $\chi(a)$ for $a \in \Gamma_m$ and taking modulus $\mathcal{B}$ and $\mathrm{Tr}_{k_n/k}$, and using the fact

$$\mathrm{Tr}_{k_n/k}(\chi(a)) = \begin{cases} [k_n : k]\chi(a) & \text{if } \chi(a) \in \mu_{\ell^m}, \\ 0 & \text{otherwise} \end{cases}$$

the previous formula is reduced to

$$[k_n : k]\chi(a) \sum_{\eta\in\mu_{\ell-1}} \sum_{b\in a\frac{\Gamma_{n-m}}{\Gamma_n}} \chi^{-1}(b) \left\langle v(\frac{b\eta}{\ell^n}), \omega(f_\lambda) \right\rangle = 0$$

for all $a \in \Gamma_m$. Furthermore, writing down the representatives of the quotient $a\Gamma_{n-m}/\Gamma_n$ explicitly, we get

$$\sum_\eta \sum_{c\in\mathbb{Z}/\ell^m\mathbb{Z}} \chi^{-1}(1 + \ell^{n-m}c) \left\langle v\left(\frac{a\eta}{\ell^n} + \frac{\eta c}{\ell^m}\right), \omega(f_\lambda) \right\rangle = 0.$$

Since $2(n-m) \geq n$ and $(1+\ell^{n-m})^c \equiv 1+\ell^{n-m}c \,(\bmod\, \ell^n)$, setting $\chi(1+\ell^{n-m}) = \zeta_m$, a primitive ℓ^m-th root of unity, we have

$$\sum_\eta \left\langle v\left(\frac{a\eta}{\ell^n}\right), \sum_{c\in\mathbb{Z}/\ell^m\mathbb{Z}} \zeta_m^c \omega(f_\lambda)\Big| \begin{pmatrix} 1 & \frac{\eta c}{\ell^m} \\ 0 & 1 \end{pmatrix} \right\rangle = 0.$$

Define for any $b \in \mathbb{Z}_\ell^\times$

$$f_{\lambda,b}(z) = \frac{\sum\limits_{r=0, r\equiv b(\ell^m)}^{\ell^m N-1} \lambda(r)q^r}{q^{\ell^m N} - 1}.$$

Above equality is written as

$$\sum_\eta \left\langle v\left(\frac{a\eta}{\ell^n}\right), \omega(f_{\lambda,\eta^{-1}}) \right\rangle = 0$$

for all $a \in \Gamma_m$ because

$$f_{\lambda,b}(z) = \sum_{c\in\mathbb{Z}/\ell^m\mathbb{Z}} \zeta_m^c f_\lambda \Big| \begin{pmatrix} 1 & \frac{bc}{\ell^m} \\ 0 & 1 \end{pmatrix} (z) = \sum_{\substack{n=1 \\ n\equiv b^{-1}(\ell^m)}}^{\infty} \lambda(n)q^n.$$

The function $f_{\lambda,b}(z)$ has poles only at $z = \frac{k}{\ell^m N}$ for $0 \le k < \ell^m N$ and $(k,N) = 1$ with residues

$$\frac{\lambda(k)G(\lambda^{-1})(\zeta_m^{b+1}\eta_N)^k}{N\ell^m}.$$

Here η_N is a primitive N-th root of unity. Note that these are algebraic and p-integral. There is no pole at $q = \infty$ for function $f_{\lambda,b}(q)$. In other words, there is no holomorphic term in $f_{\lambda,b}(q)$. Hence, we have the expression

$$f_{\lambda,b}(q) = \sum_{(k,N)=1} \frac{\lambda(k)G(\lambda^{-1})(\zeta_m^{b+1}\eta_N)^k}{N\ell^m(q-(\eta_N\zeta_m)^k)}.$$

Since the pairing is a $\mathbb{Z}$-linear combination of residues of $f_{\lambda,b}$, above fractional expansion of $f_{\lambda,b}(q)$ guarantees p-integrality of the pairing $\langle v, \omega(f_{\lambda,b})\rangle$ for each $v \in H_1(T^0_{N'}, \mathbb{Z})$, and $N' = N\ell^m$, and thus $\omega(f_{\lambda,b}) \in H^1(T^0_{N'}, \mathbb{Z}_{(p)})$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p.

Note also that vanishing of the pairing on whole homology group

$$\Big\langle H_1(T^0_{N'}, \mathbb{Z}), \omega(f_{\lambda,b})\Big\rangle \equiv 0 \pmod{\mathcal{B}}$$

implies that all residues are congruent to $0 \pmod{\mathcal{B}}$. Therefore non-existence of holomorphic part in $f_{\lambda,b}(q)$ forces us to conclude that $f_{\lambda,b} \equiv 0 \pmod{\mathcal{B}}$.

Now we prove Theorem 1. Let us assume the contrary that $L(0,\lambda\chi) \equiv 0 \pmod{\mathcal{B}}$ for infinitely many χ's. It implies that for infinitely many integers n and all $a \in \Gamma_m$ we have

$$\sum_{\eta\in\mu_{\ell-1}} \left\langle v\left(\frac{a\eta}{\ell^n}\right), \omega(f_{\lambda,\eta^{-1}})\right\rangle \equiv 0 \pmod{\mathcal{B}}.$$

From the identity $f_{\lambda,-b}(q) = f_{\lambda,b}(q^{-1})$; i.e., $f_{\lambda,-b}(z) = f_{\lambda,b}(-z)$, we have

$$\left\langle v\left(\frac{a\eta}{\ell^n}\right), \omega(f_{\lambda,\eta^{-1}})\right\rangle = \left\langle v\left(\frac{-a\eta}{\ell^n}\right), \omega\left(f_{\lambda,(-\eta)^{-1}}\right)\right\rangle,$$

and

$$\sum_{\xi\in\mu_{\ell-1}/\{\pm1\}} \left\langle v\Big(\frac{a\xi}{\ell^n}\Big), \omega(f_{\lambda,\xi^{-1}})\right\rangle \equiv 0 \pmod{\mathcal{B}}.$$

By Proposition 4, we know that the linear map $H_1(T^0_{N'}, \mathbb{Z})^{\frac{\ell-1}{2}} \to \overline{\mathbb{F}}_p$, which is given by

$$(v_\xi)_\xi \mapsto \sum_{\xi\in\mu_{\ell-1}/\{\pm1\}} \langle v_\xi, \omega(f_{\lambda,\xi^{-1}})\rangle \pmod{\mathcal{B}}$$

is zero. For a fixed η we can find $(v_\xi)_\xi \in H_1(T^0_{N'}, \mathbb{Z})^{\frac{\ell-1}{2}}$ such that $v_\xi = [0]$ if $\xi \ne \eta$ and v_η is arbitrary. Then we have vanishing $\langle H_1(T^0_{N'}, \mathbb{Z}), \omega(f_{\lambda,\eta})\rangle \equiv 0 \pmod{\mathcal{B}}$ and, hence, vanishing $\pmod{\mathcal{B}}$ of $f_{\lambda,\eta}$ for each $\eta \in \mu_{\ell-1}$, which is contradiction. So we have proved the theorem. □

Remark 5 *Actually, in [14] Washington chose a_1, $a_2 \in \Gamma_m$ such that*

$$\mathrm{Tr}_{k_n/k}(\chi(a_1)L(0,\lambda^{-1}\chi^{-1})) \neq \mathrm{Tr}_{k_n/k}(\chi(a_2)L(0,\lambda^{-1}\chi^{-1})).$$

As expressed before, this can be rewritten as

$$\sum_{\eta}\left\langle v\left(\frac{a_1\eta}{\ell^n}\right), \omega(f_{\lambda,\eta^{-1}})\right\rangle - \sum_{\eta}\left\langle v\left(\frac{a_2\eta}{\ell^n}\right), \omega(f_{\lambda,\eta^{-1}})\right\rangle \neq 0.$$

Two numbers a_1, a_2 are chosen specially so that above difference of the sums over $\mu_{\ell-1}$ is equal to

$$\left\langle v\left(\frac{a_1\eta_0}{\ell^n}\right), \omega(f_{\lambda,\eta_0^{-1}})\right\rangle - \left\langle v\left(\frac{a_2\eta_0}{\ell^n}\right), \omega(f_{\lambda,\eta_0^{-1}})\right\rangle$$

for a fixed $\eta_0 \in \mu_{\ell-1}$ and the last difference is nothing but a residue of $f_{\lambda,\eta_0^{-1}}$. It is not a coincidence that the proof of Lemma 3 is similar to the proof of the Proposition 2 in [14] in choosing suitable a_1, a_2.

4 Modular Symbols of weight 2

In this section, we describe modular symbols and some conjectures mentioned in the introduction. Main references are [1] and [16]. We consider the homology groups $H_1(X, \{cusps\}, \mathbb{Z})$ and $H_1(X, \mathbb{Z})$ where X is the standard modular curve $X_0(N)$ or $X_1(N)$ of level N. There is a homology class $\{\alpha, \beta\}_\Gamma$, which is called a modular symbol of weight 2 for the congruence group $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$ correspondingly as X and is corresponding to a geodesic connecting α and β in X. They enjoy the following three properties for all α, β, $\delta \in \mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$:

(1) $\{\alpha, \beta\} + \{\beta, \delta\} + \{\delta, \alpha\} = 0$
(2) $\{\alpha, \alpha\} = 0$
(3) $\{\alpha, \beta\} = -\{\beta, \alpha\}$.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the symbol canonically by the action on $\mathfrak{H}^*$ and the action of Γ is defined to be trivial. It is well known that there is a surjective homomorphism $\phi : \Gamma \to H_1(X, \mathbb{Z})$ defined by $\gamma \mapsto \{z, \gamma(z)\}_\Gamma$ for any $z \in \mathfrak{H}^*$.

For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma = \Gamma_0(N)$, a Manin symbol $[\gamma]$ is defined as

$$[\gamma] = \gamma \cdot \{0, \infty\} = \{\gamma(0), \gamma(\infty)\} \in H_1(X_0(N), \{cusps\}, \mathbb{Z}).$$

The right action of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ on Manin symbols $[\gamma']$ is defined as $[\gamma'] \cdot \gamma =$

$[\gamma'\gamma]$. With the matrices $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, we have the following relations between Manin symbols for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$:

$$[\gamma] + [\gamma]\sigma = 0 \text{ and } [\gamma] + [\gamma]\tau + [\gamma]\tau^2 = 0.$$

And these are all possible relations between Manin symbols. Let $\{\gamma_i\}$ be a right coset representative of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$. A Manin symbol is equal to a unique $[\gamma_i]$ for some i. Using the continued fraction expansion of rational numbers and their consecutive convergents, any modular symbol in $H_1(X_0(N), \{cusps\}, \mathbb{Z})$ can be expressed in terms of Manin symbols [1,8]. Hence explicit generators and their rules are given for the $\mathbb{Z}$-free module $H_1(X_0(N), \{cusps\}, \mathbb{Z})$. By the Drinfeld-Manin theorem(see [7]) , we have $H_1(X_0(N), \{cusps\}, \mathbb{Z}) \subset H_1(X_0(N), \mathbb{Q})$. Hence $H_1(X_0(N), \mathbb{Q}) = H_1(X_0(N), \{cusps\}, \mathbb{Z}) \otimes \mathbb{Q}$ is generated by the Manin symbols with complete description of the relations.

As $\mathrm{GL}_2(\mathbb{Q})$ acts on $H_1(X, \{cusps\}, \mathbb{Z})$, the action of Hecke operators is also defined on it. Let $\mathcal{S}_2(\Gamma)$ be the space of cusp forms of weight 2 for Γ and $\mathbb{T}$ be the subring of $\mathrm{End}(\mathcal{S}_2(\Gamma))$ generated by the Hecke operators T_m for all m. Let f be an eigencusp form of weight 2 with respect to the congruence group Γ. We set K_f as a finite extension of $\mathbb{Q}_p$ containing all eigenvaules of f and $\mathcal{O}$ the integer ring of K_f with maximal ideal λ. One can obtain an algebra homomorphism $\phi_f : \mathbb{T} \to \mathcal{O}$. Let $\mathfrak{m}$ be the preimage of λ. One can show that $\mathcal{S}_2(\Gamma_1(N), \mathbb{Z}_p)_\mathfrak{m}$ is free over $\mathbb{T}_\mathfrak{m}$ and $H_1(X_1(N), \mathbb{Z})_\mathfrak{m}$ is free of rank two over $\mathbb{T}_\mathfrak{m}$ when N is not divisible by p (see [2, §12.5]).

Now we consider modular symbols $\{0, \frac{a}{\ell^n}\}$ for some a with $\gcd(a, \ell) = 1$ and $n \geq 1$. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$ and $\{u, v\} = 0$ for all integers u and v, it is of no harm to think of a as an element of $\mathbb{Z}_\ell^\times$. We consider the map

$$j : \frac{\ell^{-\infty}\mathbb{Z}}{\mathbb{Z}} \to H_1(X, \{cusps\}, \mathbb{Z})^{\frac{\ell-1}{2}}, \frac{a}{\ell^n} \mapsto \left(\left\{0, \frac{a\eta}{\ell^n}\right\}\right)_{\eta \in \mu_{\ell-1}/\{\pm 1\}}$$

and the submodule $M(\ell)$ of $H_1(X, \mathbb{Z})$ generated by the image of j. Then let us state the conjectures which are analogs of Proposition 4.

Conjecture 6 *For almost all prime p, the p-adic valuation*

$$v_p\left(\left[H_1(X, \mathbb{Z})^{\frac{\ell-1}{2}} : M(\ell)\right]\right)$$

is 0 *for all odd prime ℓ.*

With the previous arguments and the $\mathbb{T}_\mathfrak{m}$-module $M(\ell)_\mathfrak{m} := M(\ell) \otimes_\mathbb{Z} \mathbb{T}_\mathfrak{m}$, we also state :

Conjecture 7 *For almost all prime p, we have*

$$H_1(X_1(N), \mathbb{Z}_p)_{\mathfrak{m}}^{\frac{\ell-1}{2}} = M(\ell)_{\mathfrak{m}}$$

as $\mathbb{T}_{\mathfrak{m}}$-modules for almost all odd prime ℓ.

Concerned about the conjecture, we consider a slight different situation. Let $\Gamma = \Gamma_0(N)$. For an integer a with $\gcd(a, \ell) = 1$, there are integers u, v such that $\gamma = \begin{pmatrix} u & a \\ Nv & \ell^n \end{pmatrix} \in \Gamma$ for each n. The image under ϕ of this element is $\{0, \gamma(0)\} = \{0, \frac{a}{\ell^n}\} \in H_1(X_0(N), \mathbb{Z})$. We consider the map

$$j_n : \ell^{-n}\mathbb{Z}/\mathbb{Z} \to H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}}, \frac{a}{\ell^n} \mapsto \left(\left\{0, \frac{a\eta}{\ell^n}\right\}\right)_{\eta \in \mu_{\ell-1}/\{\pm 1\}}.$$

Let $M_n(\ell)$ be the submodule of $H_1(X_0(N), \mathbb{Z})$ generated by the image of the map j_n and $\nu(n) = \prod_{q|n}(q-1)$, where q is a prime divisor of the integer n. Then we consider another conjecture related to previous one.

Conjecture 8 *For all prime $p \nmid \nu(N)$ and odd prime ℓ, we have*

$$v_p\left(\left[H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}}) : M_n(\ell)\right]\right) = 0$$

for all sufficiently large n.

Example 9 *Consider the case $N = 11$. The modular curve $X_0(11)$ is of genus 1 with cusps 0 and ∞. Hence the rank of $H_1(X_0(11), \mathbb{Z})$ is 2. The right coset representative of $\Gamma_0(11)$ in $\mathrm{SL}_2(\mathbb{Z})$ is*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \cdots, \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the corresponding modular symbols are

$$\{0, \infty\}, \{0, 1\} = 0, \left\{0, \frac{1}{2}\right\}, \cdots, \left\{0, \frac{1}{10}\right\}, \{\infty, 0\}.$$

Since every elements in $H_1(X_0(11), \mathbb{Z})$ are linear combination of the above Manin symbols which is in $H_1(X_0(11), \mathbb{Q})$ except the first and last ones, one can conclude that $H_1(X_0(11), \mathbb{Z})$ is generated by them excluding $\{0, \infty\}$.

One can expand the rational number $\frac{a}{\ell^n}$ in continued fractions and hence represent each modular symbol $\{0, \frac{a}{\ell^n}\}$ as the linear combination of Manin symbols. For example, we have $11/7^3 = [0; 31, 5, 2]$ and their convergents are $1/31$, 5/156, $11/7^3$. *From this, one have in* $H_1(X_0(11),\mathbb{Z})$

$$\left\{0,\frac{11}{7^3}\right\}=\left\{0,\frac{1}{31}\right\}+\left\{\frac{1}{31},\frac{5}{156}\right\}+\left\{\frac{5}{156},\frac{11}{7^3}\right\}=\left[\begin{pmatrix}1&0\\9&1\end{pmatrix}\right]+0+0.$$

We can verify that the index $[H_1(X_0(11),\mathbb{Z}):M_k(3)]$ *is* 1 *or* 5 *for all sufficiently large* k. *The module* $H_1(X_0(11),\mathbb{Z})$ *is a free module with basis*

$$[8]=\left\{0,\frac{1}{8}\right\},\ [9]=\left\{0,\frac{1}{9}\right\}.$$

We consider a submodule of M_k *generated by* $\left\{0,\frac{1+a\cdot 3^{k-2}}{3^k}\right\}$ *for* $a=1,\ 2,\ 4,\ 5,\ 7$, *or* 8. *By considering the continued fraction expansion of* $\frac{1+a3^{k-2}}{3^k}$, *we have the following calculations for sufficiently large* k *:*

$$\begin{aligned}
\left\{0,\frac{1+3^{k-2}}{3^k}\right\}&=[8]+[-8'9]+[9'(3^{k-2}-1)]+[-(3^{k-2}-1)'3^k].\\
\left\{0,\frac{1+2\cdot 3^{k-2}}{3^k}\right\}&=[4]+[-4'9]+[9'(3^{k-2}-5)]\\
&\quad+[(-3^{k-2}+5)'(-1+2\cdot 3^{k-2})]+[(1-2\cdot 3^{k-2})3^k].\\
\left\{0,\frac{1+4\cdot 3^{k-2}}{3^k}\right\}&=[2]+[-2'9]+[9'(3^{k-2}-7)]\\
&\quad+[(-3^{k-2}+7)'(2+3^{k-2})]+[(2+3^{k-2})(-1+4\cdot 3^{k-2}].\\
\left\{0,\frac{1+5\cdot 3^{k-2}}{3^k}\right\}&=[-2]+[2'7]+[-7'9]+[9'(3^{k-2}-2)]\\
&\quad+[(-3^{k-2}+2)'(1+4\cdot 3^{k-2})]+[(1+4\cdot 3^{k-2})'3^k].\\
\left\{0,\frac{1+7\cdot 3^{k-2}}{3^k}\right\}&=[-4]+[4'5]+[-5'9]+[9'(3^{k-2}-4)]\\
&\quad+[(-3^{k-2}+4)'(1+2\cdot 3^{k-2})]+[(1+2\cdot 3^{k-2})'3^k].\\
\left\{0,\frac{1+8\cdot 3^{k-2}}{3^k}\right\}&=[-9]+[9'(3^{k-2}-8)]\\
&\quad+[(-3^{k-2}+8)'(1+3^{k-2})]+[(1+3^{k-2})'3^k].
\end{aligned}$$

Here by $[m]$ *we denote* $\left[\begin{pmatrix}1&0\\m&1\end{pmatrix}\right]$ *and* m' *is the inverse of* m *modulo* 11. *Since* $[m]$ *depends only on* m *modulo* 11, *we conclude that the index is* 1 *or* 5 *as follows:*

(1) When $k\equiv 0\,(\mathrm{mod}\ 5)$, *we have* $\left\{0,\frac{1+3^{k-2}}{3^k}\right\}=[8]+2\cdot[9]$, $\left\{0,\frac{1+4\cdot 3^{k-2}}{3^k}\right\}=-[8]+3\cdot[9]$. *The index is a divisor of* 5.

(2) *When* $k \equiv 1 \pmod 5$, *we have* $\left\{0, \frac{1+3^{k-2}}{3^k}\right\} = -[8] + [9]$, $\left\{0, \frac{1+5\cdot 3^{k-2}}{3^k}\right\} = -[8]$. *The index is* 1.

(3) *When* $k \equiv 2 \pmod 5$, *we have* $\left\{0, \frac{1+3^{k-2}}{3^k}\right\} = [9]$, $\left\{0, \frac{1+7\cdot 3^{k-2}}{3^k}\right\} = -[8] - [9]$. *The index is* 1.

(4) *When* $k \equiv 3 \pmod 5$, *we have* $\left\{0, \frac{1+3^{k-2}}{3^k}\right\} = -[8] + 2[9]$, $\left\{0, \frac{1+2\cdot 3^{k-2}}{3^k}\right\} = -[9]$ *The index is* 1.

(5) *When* $k \equiv 4 \pmod 5$, *we have* $\left\{0, \frac{1+3^{k-2}}{3^k}\right\} = [8] - [9]$, $\left\{0, \frac{1+4\cdot 3^{k-2}}{3^k}\right\} = 2\cdot[9]$, $\left\{0, \frac{1+7\cdot 3^{k-2}}{3^k}\right\} = -3\cdot[9]$. *The index is* 1. □

We have the boundary map

$$\delta : H_1(X_0(N), \{cusps\}, \mathbb{Z}) \to \mathcal{D}_0, \{\alpha, \beta\} \mapsto \{\alpha\} - \{\beta\}$$

for the group $\mathcal{D}_0$ of divisors of $X_0(N)$ supported on $\mathbb{P}^1(\mathbb{Q})$. One can show(see [8,16]) that

$$H_1(X_0(N), \mathbb{Z}) = \ker(\delta).$$

From the coset representatives of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$, the explicit generators of $H_1(X_0(N), \{cusps\}, \mathbb{Z}) = H_1(X_0(N), \mathbb{Q})$ and hence the explicit generators of $\ker(\delta)$ can be obtained using Manin symbols method [1]. Using the basis of $H_1(X_0(N), \mathbb{Z})$ consisting of Manin symbols and the lattice reduction algorithm for $M_n(\ell)$, we checked the index $[H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}} : M_n(\ell)]$. In the appendix II, we have calculated the list of the possible prime divisors of the index of the submodule generated by

$$\left\{0, \frac{1 + a\cdot \ell^{k-3}}{\ell^k}\right\} \in H_1(X_0(N), \mathbb{Z})$$

for $1 \le a \le \ell^3 - 1$ when $\ell \not| N$. In the case $\ell \mid N$, we consider the module generated by

$$\left\{0, \frac{1 + a\cdot \ell^{k-3}}{\ell^k}\right\} - \left\{0, \frac{1 + (a+1)\cdot \ell^{k-3}}{\ell^k}\right\}$$

for $1 \le a \le \ell^3 - 1$. Using the criterion for the equivalence among cusps of $X_0(N)$ in [1, Proposition 2.2.3], it can be verified that they are in $H_1(X_0(N), \mathbb{Z})$ for $k > 3$. Although the new index defined above should be divided by the number $\left[H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}} : M_k(\ell)\right]$, in many cases of small prime number ℓ, it turned out that both indices are same.

It seems to be worthwhile to remark that when the level N is a prime number, the prime divisors of the index is Eisenstein in the sense of [10]; i.e., the prime divisors of the numerator of $\frac{N-1}{12}$.

5 Appendix I. Modular version of Sinnott's proof

In this appendix, we review the modification of Sinnott's proof [11] of Theorem 1 due to Hida [4].

For a commutative ring R, let $\lambda : (\mathbb{Z}/N\mathbb{Z})^\times \to R^\times$ be an odd Dirichlet character, $\chi : \mathbb{Z}_\ell^\times \to \mu_{\ell^\infty}(R)$ and

$$\Phi_\lambda(t) = \sum_{n=1}^{\infty} \lambda(n)t^n = \frac{\sum_{a=1}^{N} \lambda(a)t^a}{1 - t^N}.$$

Considering the group scheme $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$ and the stalk $\mathcal{O}_{\mathbb{G}_m,1}$ at 1 of the sheaf $\mathcal{O}_{\mathbb{G}_m}$ of regular functions on $\mathbb{G}_m$, we have $\Phi_\lambda(t) \in \mathcal{O}_{\mathbb{G}_m,1/R}$. Note that from the well-known result of Euler, we obtain

$$\Phi_\lambda(1) = L(0, \lambda).$$

Define an R-valued measure $d\mu_\lambda$ on $\mathbb{Z}_\ell$ by

$$\int_{\mathbb{Z}_\ell} f d\mu_\lambda = (\ell\lambda(\ell))^{-n} \sum_{x \in \mathbb{Z}/\ell^n\mathbb{Z}} f(x)\Phi(\zeta_n^x),$$

where $f : \mathbb{Z}/\ell^n\mathbb{Z} \to R$; i.e., f is a locally constant function on $\mathbb{Z}_\ell$ and ζ_n is the primitive ℓ^n-th root of unity . Note that this definition is independent of $\mathbb{Z}/\ell^n\mathbb{Z}$ where f is defined. For a primitive Dirichlet character $\chi : (\mathbb{Z}/\ell^n\mathbb{Z})^\times \to R^\times$, we have

$$\begin{aligned}
\int \chi d\mu_\lambda(t) &= (\ell\lambda(\ell))^{-n} \sum_x \sum_m \chi(x)\lambda(m)(\zeta_n^x t)^m \\
&= (\ell\lambda(\ell))^{-n} \sum_m \left(\sum_{x \in \mathbb{Z}/\ell^n\mathbb{Z}} \chi(x)\zeta_n^{mx} \right) \lambda(m)t^m \\
&= \frac{G(\chi)}{(\ell\lambda(\ell))^n} \sum_m \chi^{-1}(m)\lambda(m)t^m
\end{aligned}$$

Evaluating this sum at t=1, we have

$$\int \chi d\mu_\lambda = \frac{G(\chi)}{(\ell\lambda(\ell))^n} L(0, \chi^{-1}\lambda).$$

Since $\chi : \mathbb{Z}_\ell \to \mu_{\ell^\infty}$ is trivial on $\mu_{\ell-1}$, we have

$$\int_{\mathbb{Z}_\ell} \chi d\mu_\lambda = \int_{\mathbb{Z}_\ell^\times} \chi d\mu_\lambda = \int_\Gamma \chi d\mu_\Psi,$$

where $\Gamma = \mathbb{Z}_\ell{}^\times/\mu_{\ell-1}$, and $d\mu_\Psi$ is a measure on Γ associated with the formal function Ψ on μ_{ℓ^∞} such that

$$\Psi(t) = \sum_{\epsilon \in \mu_{\ell-1}} \Phi(t^\epsilon) = \sum_{\epsilon \in \mu_{\ell-1}/\{\pm 1\}} (\Phi(t^\epsilon) + \Phi(t^{-\epsilon})).$$

In other words,

$$\int_\Gamma \chi d\mu_\Psi = \sum_{x \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times/\mu_{\ell-1}} \chi(x)\Psi(\zeta_n^x).$$

Now we consider the case that R is the algebraic closure $\overline{\mathbb{F}}_p$ of the finite field of characteristic p. Since Γ has a topological generator g, we have an 1-1 correspondence between the set of Dirichlet characters $\{\ \chi : \Gamma \to \mu_{\ell^\infty}\}$ and μ_{ℓ^∞}. More explicitly, the correspondence is $\chi \mapsto \chi(g)$ and if the conductor of χ is ℓ^n; i.e., $\ker(\chi) = 1 + \ell^n\mathbb{Z}_\ell =: \Gamma_n$ in Γ, then after fixing the primitive root of unity in μ_{ℓ^n} for each n, the character χ corresponds to a unique ζ_χ, which is an ℓ^n-th root of unity and $\chi(g)$=ζ_χ^ℓ. We have isomorphisms

$$1 + \ell\mathbb{Z}_\ell \xrightarrow{\log} \ell\mathbb{Z}_\ell,\ 1 + \ell\mathbb{Z}_\ell \xleftarrow{\exp} \ell\mathbb{Z}_\ell$$

and $g = \exp(\ell)$. As before, let m be the integer such that $k \cap \mu_{\ell^\infty} = \mu_{\ell^m}$. If $n \geq 2m$ and $x = \exp(\ell^{n-m}z) \in \Gamma_{n-m}$, then setting $v = n - m$ we have $2v \geq n$ and

$$\chi(x) = \chi(\exp(\ell))^{\ell^{v-1}z} = \zeta_\chi^{\ell^v z} = \zeta_\chi^{\log(x)} = \zeta_\chi^{x-1}$$

since $\log(x) = (x-1)$+$O((x-1)^2)$ and $\ell^{2v} \mid (x-1)^2$. Here we define

$$\Phi_a(t) = \frac{\sum_{r \equiv a^{-1}(\ell^m)} \lambda(r)t^r}{1 - t^{N\ell^m}},\ \text{where } 0\ \leq r\ < \ell^m N$$

for $a \in \mathbb{Z}_\ell{}^\times$. Then we have the following proposition.

Proposition 10 *Let n be the conductor of χ and $n \geq 2m$. Then we have*

$$\int_\Gamma \chi d\mu_\Psi = 0 \textit{ if and only if } \sum_{\epsilon \in \mu_{\ell-1}} \Phi_{\epsilon y}(\zeta_\chi^{\epsilon y}) = 0 \textit{ for all } y \in \Gamma.$$

The proof is similar with the argument in Section 3 in which the use of trace $\mathrm{Tr}_{k_n/k}$ results in the introduction of $f_{\lambda,b}$.

Let A be a free $\mathbb{Z}$-module of finite rank and $A^* = \mathrm{Hom}(A, \mathbb{Z})$. Then we have a multiplicative group t^{A^*} and a group ring $\mathbb{Z}[t^{A^*}]$. There is a torus

$$\mathbb{G}_m \otimes A = \mathrm{Spec}(\mathbb{Z}[t^{A^*}]).$$

Given the pairing $\langle \cdot\,,\,\cdot \rangle : A \times A^* \to \mathbb{Z}$ defined by $\langle a, f \rangle = f(a)$ and a basis $\{w_i\}$ $\subseteq A$, $\{w_i^*\} \subseteq A^*$ the dual basis, we have the isomorphism, $\mathbb{G}_m \otimes A \cong \mathbb{G}_m^{\mathrm{rank}\,A}$.

More explicitly, the isomorphism is induced from

$$\mathbb{Z}[t^{A^*}] \to \mathbb{Z}[t_1, \cdots, t_n],\ t^{w_i^*} \mapsto t_i,\ n = \operatorname{rank} A$$

and for a commutative ring R, we have an isomorphism $R^\times \otimes A \cong (R^\times)^{\operatorname{rank} A}$ defined by $x \otimes \alpha \to (x^{\langle \alpha, w_i^* \rangle})_{i=1}^n$, and $x \otimes w_i \leftarrow (1, \cdots, x, \cdots, 1)$ with x in i-th position. We have a commutative diagrams in which all the maps are injective :

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{x \mapsto x \otimes 1} & (\mathbb{G}_m \otimes \mathbb{Z}[\mu_{\ell-1}])^{\mu_{\ell-1}} \\ \Big\uparrow & & \Big\uparrow {\scriptstyle i : x \mapsto (x \otimes \epsilon)_\epsilon} \\ \mu_{\ell^\infty} & \xrightarrow{j} & \mathbb{G}_m^{\mu_{\ell-1}} \end{array} .$$

This diagram is commutative since $\zeta^\epsilon \otimes 1 = \zeta \otimes \epsilon$ for $\zeta \in \mu_{\ell^\infty}$ and $\epsilon \in \mu_{\ell-1}$. Let us make a remark that j is not an algebraic map. But after introducing i and having in mind the commutativity of above diagram, we may regard j as a restriction of an algebraic map.

If V is the subscheme of $(\mathbb{G}_m \otimes \mathbb{Z}[\mu_{\ell-1}])^{\mu_{\ell-1}}$ defined by the equations

$$\Big\{ \prod_\epsilon t_\epsilon^{n_\epsilon w} - 1 \ \Big|\ \sum n_\epsilon \epsilon = 0, w \in \mathbb{Z}[\mu_{\ell-1}]^* \Big\},$$

then each row is factorized through V and $V \cap \mathbb{G}_m^{\mu_{\ell-1}}$; i.e., the image is contained in V and we have the diagram

$$\begin{array}{ccccc} \mathbb{G}_m & \longrightarrow & V & \longrightarrow & (\mathbb{G}_m \otimes \mathbb{Z}[\mu_{\ell-1}])^{\mu_{\ell-1}} \\ \Big\uparrow & & \Big\uparrow & & {\scriptstyle i}\Big\uparrow \\ \mu_{\ell^\infty} & \xrightarrow{j} & V \cap \mathbb{G}_m^{\mu_{\ell-1}} & \longrightarrow & \mathbb{G}_m^{\mu_{\ell-1}} \end{array} .$$

Here, $V \cap \mathbb{G}_m^{\mu_{\ell-1}}$ is the subscheme of $\mathbb{G}_m^{\mu_{\ell-1}}$ defined by the equation

$$S = \Big\{ \prod_\epsilon t_\epsilon^{n_\epsilon} - 1 \ \Big|\ \sum n_\epsilon \epsilon = 0 \Big\}.$$

Let $\{\alpha_1 = 1, \alpha_2, \cdots, \alpha_n\}$ be a maximal subset of $\mu_{\ell-1}$ which is linearly independent over $\mathbb{Q}$. Then $V \cap \mathbb{G}_m^{\mu_{\ell-1}}$ is isomorphic to $\mathbb{G}_m^n$ by the isomorphism

$$\mathbb{Z}[\{t_\epsilon, t_\epsilon^{-1}\}]/\langle S \rangle \to \mathbb{Z}[t_1, t_1^{-1} \cdots, t_n, t_n^{-1}],\ t_\epsilon \mapsto \prod_i t_i^{c_i(\epsilon)},$$

where $\epsilon = \sum_i c_i(\epsilon) \alpha_i$, i.e. $c_i(\epsilon) = \langle \epsilon, \alpha_i^* \rangle$. Then we restate Proposition 2 :

Theorem 11 *The image of* $j : \mu_{\ell^\infty} \to V \cap \mathbb{G}_m^{\mu_{\ell-1}} \cong \mathbb{G}_m^n$, $\zeta \mapsto (\zeta^{\alpha_i})_{i=1}^n$ *is Zariski dense*

As mentioned in the introduction, the proof is an immediate consequence of the Theorem 2.2 in [11]. The proof also can be established using the Dirichlet's Theorem on the linear independence of characters by considering each formal monomial t^α, $\alpha \in \mathbb{Z}_\ell$ as a character on μ_{ℓ^∞}.

Now we consider the regular function

$$F(\{t_\epsilon\}) = \sum_\epsilon \Phi_{\lambda,\epsilon}(t_\epsilon^{\alpha_1^*})$$

on $(\mathbb{G}_m \otimes \mathbb{Z}[\mu_{\ell-1}])^{\mu_{\ell-1}}$ and

$$i^*F(\{t_\epsilon\}) = \sum_\epsilon \Phi_{\lambda,\epsilon}(t_\epsilon)$$

on $\mathbb{G}_m^{\mu_{\ell-1}}$. Suppose that $\mathbb{G}_m \subseteq V(F)$; i.e., F is identically zero on $\mathbb{G}_m$. Then i^*F is identically zero on μ_{ℓ^∞}. Since $j(\mu_{\ell^\infty})$ is Zariski dense in $V \cap \mathbb{G}_m^{\mu_{\ell-1}}$, i^*F is identically zero on it. Since i^*F is of the form in $\mathbb{G}_m^n$

$$\sum_{\epsilon \in \mu_{\ell-1}/\{\pm 1\}} \Phi_{\lambda,\epsilon}\left(\prod_i t_i^{c_i(\epsilon)}\right) + \Phi_{\lambda,-\epsilon}\left(\prod_i t_i^{-c_i(\epsilon)}\right),$$

and $\prod_i t_i^{c_i(\epsilon)}$ are pairwise $\mathbb{Z}$-mutiplicatively independent, they cannot be identically zero on $\mathbb{G}_m^n$. Hence, $\mathbb{G}_m \nsubseteq V(F)$. We know that the equality $L(0, \chi\lambda) = 0 \pmod{\mathfrak{B}}$ implies that $\sum_\epsilon \Phi_{\lambda,\epsilon}(\zeta_\chi^\epsilon) = 0$. Since these ζ_χ's are in the Zariski closed set $\mathbb{G}_m \cap V(F)$, we can conclude that there are finite number of such ζ_χ's. □

6 Appendix II. Numerical Calculations

In the table 1, we list the possible prime factors p of the index

$$[H_1(X_0(N), \mathbb{Z})^{\frac{\ell-1}{2}} : M_k(\ell)]$$

in the last column according to N and ℓ for all $k \leq k_0$. The blanks in the table represent for the repetition of the previous numbers.

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Table 1
Prime divisors of the index

N	genus	ℓ	k_0	p	N	genus	ℓ	k_0	p
11	1	5	100	5	$5 \cdot 7$	3	5	100	2, 3
		7	100				7	70	
		11	100				11	50	
		13	100		$3 \cdot 13$	3	5	100	2
		17	70				7	100	
$2 \cdot 7$	1	5	100	3	41	3	5	100	2, 5
		7	100				7	100	
		11	100		$2 \cdot 5 \cdot 7$	9	5	100	2, 3
		13	70				7	70	
$3 \cdot 5$	1	5	100	2			11	70	
		7	100		71	6	5	100	5, 7
		11	70				7	75	
17	1	5	100	2	$7 \cdot 11$	7	5	70	2, 3, 5
		7	100				7	60	
		11	100		103	8	5	100	17
$2 \cdot 11$	2	5	100	5			7	100	
		7	100		$3 \cdot 5 \cdot 7$	13	5	80	2, 3
		11	100				7	80	
23	2	5	100	11			11	50	
		7	100				13	75	
$3 \cdot 11$	3	5	100	2, 5	$5^2 \cdot 7$	15	5	100	2, 3
		7	100				7	100	
		11	70				11	75	

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