Derivative of power series attached to Γ-transform of p-adic measures

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Abstract

We extend the result of Anglès [1], namely $f'(T;\theta) \not\equiv 0 \,(\bmod\, p)$ for the Iwasawa power series $f(T;\theta) \in \overline{\mathbb{Z}}_p[[T-1]]$. For the derivative $D = T\frac{d}{dT}$, a numerical polynomial Q on $\mathbb{Z}_p$, and a prime π in $\overline{\mathbb{Z}}_p$ over p, we show that $Q(D)f(T;\theta) \equiv 0 \,(\bmod\, \pi)$ if and only if $Q \equiv 0 \,(\bmod\, \pi)$ i.e. $Q(x) \equiv 0 \,(\bmod\, \pi)$ for all $x \in \mathbb{Z}_p$. This result comes from a similar assertion for the power series attached to the Γ-transform of a p-adic measure which is related to a certain rational function in $\overline{\mathbb{Z}}_p[[T-1]]$.

Key words: p-adic L-function, Γ-transform, Iwasawa power series, p-adic measure, equi-distribution, Ferrero-Washington's theorem.

1 Introduction

Let p be a prime number. Let A be a complete $\mathbb{Z}_p$-algebra included in $\overline{\mathbb{Z}}_p$ and $\mathfrak{m}(\mathbb{Z}_p, A)$ be an A-module of A-valued measures on $\mathbb{Z}_p$. The A-module $\mathfrak{m}(\mathbb{Z}_p, A)$ is isomorphic to the power series ring $A[[T-1]]$.

Let $W \subset \mathbb{Z}_p^\times$ be the set of torsion elements in $\mathbb{Z}_p^\times$ and γ be a topological generator of $1 + 2p\mathbb{Z}_p$. In other words, one has $W = \mu_{p-1}$, the set of $(p-1)$-th roots of unity in $\mathbb{Z}_p^\times$ if $p > 2$ and $W = \{\pm 1\}$ if $p = 2$. Let V be a subset of W, which is a set of representatives of $W/\{\pm 1\}$. Let γ^y be the isomorphism

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$\mathbb{Z}_p \simeq 1 + 2p\mathbb{Z}_p$ defined by $y \mapsto \gamma^y$. By abuse of notation, Γ-*transform* is an A-linear map from $\mathfrak{m}(\mathbb{Z}_p, A)$ to itself defined by

$$\Gamma : \mathfrak{m}(\mathbb{Z}_p, A) \to \mathfrak{m}(\mathbb{Z}_p, A), \tau \mapsto \widetilde{\tau}(\gamma^y),$$

where

$$\widetilde{\tau} := \sum_{\eta \in W} \tau \circ \eta.$$

Since the p-adic Dirichlet L-function is represented by a Γ-transform of a rational function measure, the study on the map Γ gives much information on the p-adic L-functions.

For example, let K be an abelian number field and θ an even Dirichlet character associated to $K(\zeta_p)$ for a primitive p-th root ζ_p of unity. Let k be a finite extension of $\mathbb{Q}_p$ containing θ and π be a uniformizer in k. Let $\mathfrak{o}$ be the ring of integers of k. Iwasawa ([4]) has established a power series $f(T;\theta) \in \mathfrak{o}[[T-1]]$ such that

$$2f(\chi(\gamma)\gamma^{1-n};\theta) = -(1 - \chi\omega^{-n}(p)p^{n-1})L(1-n, \chi\omega^{-n})$$

for even Dirichlet characters χ associated to the cyclotomic $\mathbb{Z}_p$-extension K_∞/K. In fact, when p is an odd prime number we can define a measure $\tau_\theta \in \mathfrak{m}(\mathbb{Z}_p, \mathfrak{o})$ such that

$$2f(T;\theta) = \frac{G_{\tau_\theta}(T)}{1 - \alpha^{-1}T^{-p}},$$

where G_{τ_θ} is a power series attached to $\Gamma(\tau_\theta)$ and α is an integer (For a more precise formulation, see the formula (4.7) and Proposition 7).

The celebrated theorem of Ferrero and Washington ([2]) has shown that

$$G_{\tau_\theta}(T) \not\equiv 0 \pmod{\pi}$$

or equivalently

$$f(T;\theta) \not\equiv 0 \pmod{\pi}.$$

Sinnott ([5]) also has proved the congruence $G_\sigma(T) \not\equiv 0 \pmod{\pi}$ for a certain rational function measure σ i.e. a measure of which power series is a rational function in $\mathfrak{o}[[T-1]]$ such that

$$\sigma|_{\mathbb{Z}_p^\times} + \sigma|_{\mathbb{Z}_p^\times} \circ (-1) \not\equiv 0 \pmod{\pi}.$$

A natural question one can raise next would be whether the derivative of $f(T;\theta)$ or G_{τ_θ} is non-zero modulo p or not. In fact if $K = \mathbb{Q}(\zeta_p)$, Anglès ([1]) has shown that

$$f'(T;\theta) \not\equiv 0 \pmod{p}.$$

Let $D = T\frac{d}{dT}$ and $p \geq 2$. The purpose of present paper is to show that for a numerical polynomial Q on $\mathbb{Z}_p$ such that $Q \not\equiv 0 \pmod{\pi}$ i.e., $Q(x) \not\equiv 0 \pmod{\pi}$ for some $x \in \mathbb{Z}_p$, we have

$$Q(D)G_{\tau_\theta}(T) \not\equiv 0 \pmod{\pi} \tag{1.1}$$

and

$$Q(D)f(T;\theta) \not\equiv 0 \pmod{\pi}.$$

Main ingredients of the proof are to define the measure τ_θ in a homological way as discussed in [3] and to reduce the congruence (1.1) to a statement about a congruence which the measure τ_θ satisfies:

$$\widetilde{\tau_\theta}|_U \equiv 0 \pmod{\pi}$$

for some open subset U of $\mathbb{Z}_p^\times$, which is turned out to be impossible using the homological argument that is also used in [6] in order to get a homological proof for a theorem of Washington.

Throughout the paper, for $x \in \mathbb{Z}_p^\times$ we define $\omega(x)$ and $\langle x \rangle$ as the projections of x into W and $1 + 2p\mathbb{Z}_p$ respectively. ω is the Teichmüller character.

2 Γ-transform of p-adic measures

Let k be a finite extension of $\mathbb{Q}_p$ and $\mathfrak{o}$ be the ring of integers with a uniformizer π. Let $\mathcal{C}(\mathbb{Z}_p, \mathfrak{o})$ be the $\mathfrak{o}$-algebra of continuous functions from $\mathbb{Z}_p$ to $\mathfrak{o}$. Let τ be an $\mathfrak{o}$-valued p-adic measure on $\mathbb{Z}_p$. A power series $P_\tau \in \mathfrak{o}[[T-1]]$ is associated to the measure τ in the following way:

$$P_\tau(T) = P(\tau; T) := \int_{\mathbb{Z}_p} T^x d\tau(x)$$

where T^x is the power series in $T-1$ defined by

$$T^x = \sum_{n\geq 0} \binom{x}{n} (T-1)^n, \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} \in \mathbb{Z}_p.$$

From Mahler's theorem, the statement that

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\tau(x) \equiv 0 \pmod{\pi} \text{ for all } n \geq 0$$

is equivalent to

$$\int_{\mathbb{Z}_p} f(x) d\tau(x) \equiv 0 \pmod{\pi} \text{ for all } f \in \mathcal{C}(\mathbb{Z}_p, \mathfrak{o}).$$

In other words, $P_\tau(T) \equiv 0 \pmod{\pi}$ if and only if $\tau \equiv 0 \pmod{\pi}$ in $\mathfrak{m}(\mathbb{Z}_p, \mathfrak{o})$.

From the identification $\mathfrak{m}(\mathbb{Z}_p, \mathfrak{o}) \simeq \mathrm{Hom}(\mathcal{C}(\mathbb{Z}_p, \mathfrak{o}), \mathfrak{o})$, there is an action of $\mathcal{C}(\mathbb{Z}_p, \mathfrak{o})$ on $\mathfrak{m}(\mathbb{Z}_p, \mathfrak{o})$ defined by

$$d(f \cdot \tau)(x) = f(x) d\tau(x) \text{ for } f \in \mathcal{C}(\mathbb{Z}_p, \mathfrak{o}), \tau \in \mathfrak{m}(\mathbb{Z}_p, \mathfrak{o}).$$

Since we have the isomorphism

$$\mathfrak{m}(\mathbb{Z}_p, \mathfrak{o}) \simeq \mathfrak{o}[[T-1]], \tau \mapsto P_\tau,$$

there is also an action of $\mathcal{C}(\mathbb{Z}_p, \mathfrak{o})$ on $\mathfrak{o}[[T-1]]$. Clearly we have

$$f \cdot P_\tau = P_{f \cdot \tau}.$$

For a polynomial $Q \in k[T]$, Q is called *numerical* if $Q(x) \in \mathfrak{o}$ for all $x \in \mathbb{Z}_p$. In other words, $Q \in \mathcal{C}(\mathbb{Z}_p, \mathfrak{o})$. For the differential operator $D = T\frac{d}{dT}$ on $\mathfrak{o}[[T-1]]$ one can show that $D(T^x) = xT^x$ for $x \in \mathbb{Z}_p$, and hence

$$D \int_{\mathbb{Z}_p} T^x d\tau(x) = \int_{\mathbb{Z}_p} T^x x d\tau(x),$$

and for a numerical polynomial $Q(T) \in k[T]$, we obtain that

$$Q(D) P_\tau(T) = P(Q(x) d\tau(x); T) = P_{Q \cdot \tau}(T).$$

From this observation, we establish an elementary but crucial statement.

Proposition 1 *For $f \in \mathcal{C}(\mathbb{Z}_p, \mathfrak{o})$ and $\tau \in \mathfrak{m}(\mathbb{Z}_p, \mathfrak{o})$, we have*

$$f \cdot P_\tau(T) \equiv 0 \pmod{\pi} \text{ if and only if } \tau|_{f^{-1}(\mathfrak{o}^\times)} \equiv 0 \pmod{\pi}.$$

Therefore $Q(D)P_\tau(T) \equiv 0 \pmod{\pi}$ if and only if $\tau|_{Q^{-1}(\mathfrak{o}^\times)} \equiv 0 \pmod{\pi}$.

PROOF. Let $P_{f \cdot \tau}(T) \equiv 0 \pmod{\pi}$. Then we have $f(x) d\tau(x) \equiv 0 \pmod{\pi}$. Let $g(x)$ be an $\mathfrak{o}$-valued continuous function on $f^{-1}(\mathfrak{o}^\times)$. We define a function on $\mathbb{Z}_p$ such that

$$h(x) = \begin{cases} \frac{g(x)}{f(x)} & \text{if } x \in f^{-1}(\mathfrak{o}^\times) \\ 0 & \text{otherwise .} \end{cases}$$

Since h is an $\mathfrak{o}$-valued continuous function, we obtain that

$$\int_{f^{-1}(\mathfrak{o}^\times)} g(x) d\tau(x) = \int_{\mathbb{Z}_p} h(x) f(x) d\tau(x) \equiv 0 \pmod{\pi}.$$

Since g is arbitrary, we conclude that $\tau|_{f^{-1}(\mathfrak{o}^\times)} \equiv 0 \pmod{\pi}$.

The converse is immediate since we have $f(x) d\tau(x)|_{f^{-1}(\pi\mathfrak{o})} \equiv 0 \pmod{\pi}$ and $\mathbb{Z}_p = f^{-1}(\mathfrak{o}^\times) \cup f^{-1}(\pi\mathfrak{o})$. We conclude the proposition. $\square$

The Γ-*transform* or *p-adic Mellin transform* of a measure τ on $\mathbb{Z}_p$ is a p-adic analytic function $\Gamma(s,\tau)$ on $\mathbb{Z}_p$ defined by

$$\Gamma(s,\tau) = \int_{\mathbb{Z}_p^\times} \langle x\rangle^{-s} d\tau(x). \tag{2.1}$$

If γ is a topological generator of $1+2p\mathbb{Z}_p$, then there exists a power series $G_\tau \in \mathfrak{o}[[T-1]]$ such that

$$G_\tau(\gamma^{-s}) = \Gamma(s,\tau).$$

In fact, since $\gamma^y : \mathbb{Z}_p \simeq 1+2p\mathbb{Z}_p$ is an isomorphism the definition (2.1) can be written as

$$\Gamma(s,\tau) = \sum_{\eta\in W} \int_{1+2p\mathbb{Z}_p} x^{-s} d\tau\circ\eta(x) = \int_{\mathbb{Z}_p} \gamma^{-sy} d\left(\sum_\eta \tau\circ\eta(\gamma^y)\right).$$

If we set

$$\tilde{\tau} = \sum_\eta \tau\circ\eta,$$

then we have

$$G_\tau(T) = P(\tilde{\tau}(\gamma^y); T).$$

We also set

$$\overline{\tau} = \sum_{\eta\in V} \tau\circ\eta$$

and

$$G_\tau^\circ(T) := P\left(\overline{\tau}(\gamma^y); T\right).$$

Note that if $\tau = \tau\circ -1$, then we have $G_\tau = 2G_\tau^\circ$.

To a periodic function $\lambda : \mathbb{Z} \to \mathfrak{o}$ of a period N, $p \nmid N$, we associate a rational function $R_\lambda(T)$ defined by

$$R_\lambda(T) = \frac{\sum_{r=1}^N \lambda(r)T^r}{1-T^N}.$$

Let us assume that $\sum_{r=1}^N \lambda(r) = 0$. Since $\frac{1-T^N}{1-T} \in \mathfrak{o}[[T-1]]^\times$, we know that $R_\lambda(T) \in \mathfrak{o}[[T-1]]$. Let σ_λ be the measure corresponds to R_λ. Sinnott ([5]) has shown that $G_{\sigma_\lambda}(T) \equiv 0 \pmod{\pi}$ if and only if

$$\sigma_\lambda^* + \sigma_\lambda^* \circ -1 \equiv 0 \pmod{\pi},$$

where σ_λ^* is a measure obtained by restricting σ_λ to $\mathbb{Z}_p^\times$. The latter congruence is equivalent to

$$R_\lambda^*(T) + R_\lambda^*(T^{-1}) \equiv 0 \pmod{\pi}$$

where $R_\lambda^*(T) = R(T) - \frac{1}{p}\sum_{\zeta^p=1} R_\lambda(\zeta T)$. In order to study the congruence of the type

$$f\cdot G_\tau(T) \equiv 0 \pmod{\pi}$$

with $f \in \mathcal{C}(\mathbb{Z}_p, \mathfrak{o})$ and $\tau \in \mathfrak{m}(\mathbb{Z}_p, \mathfrak{o})$, we need to define a special set of measures as follows.

Definition 2 *A set of measures* $\{\sigma_1, \sigma_2, \cdots, \sigma_n\} \subset \mathfrak{m}(\mathbb{Z}_p, \mathfrak{o})$ *is called independent modulo* π *if we have*

$$\sum_i \sigma_i|_U \not\equiv 0 \,(\bmod\ \pi)$$

for all open subsets U *of* $\mathbb{Z}_p$*.*

In the next section, the set of measures $\{\sigma_\lambda \circ \eta | \eta \in V\}$ is shown to be independent modulo π if λ is a Dirichlet character of conductor N with $p \nmid N$ or $\lambda = \lambda_0$ (See Section 4 for the definition of λ_0). Using this we deduce that for $f \in \mathcal{C}(\mathbb{Z}_p, \mathfrak{o})$ and odd prime number p, the following congruence

$$f \cdot G_{\tau_\theta}(T) \equiv 0 \,(\bmod\ \pi)$$

is impossible unless $f \equiv 0 \,(\bmod\ \pi)$. In particular, $Q(D)G_{\tau_\theta}(T) \equiv 0 \,(\bmod\ \pi)$ is true only for $Q \equiv 0 \,(\bmod\ \pi)$ (See Theorem 9). For $p = 2$ case, we modify the statement using $G^\circ_{\tau_\theta}$.

3 Homological Equi-distribution

In this section we discuss the homology of a punctured cylinder. For a more complete discussion of the topic, see Hida ([3]).

Let T_N, $p \nmid N$ be a punctured cylinder given by

$$T_N = \mathbb{C}/\mathbb{Z} - \left\{ \frac{r}{N} \,\middle|\, 1 \le r < N \right\}.$$

We compactify T_N by adding boundaries to the holes i.e. by taking out small open disks around the points $S = \{\frac{r}{N}\} \cup \{\pm i\infty\}$. We denote it by T_N^S. For any subset $S' \subset S$, we do the same procedure and denote it by $T_N^{S'}$. Let $S_0 = \{\pm i\infty\}$ and A be a commutative ring and $H_c^1(T_N^{S-S_0}, A)$ be the cohomology group on $T_N^{S-S_0}$ with compact support. We have the identification

$$H_c^1(T_N^{S-S_0}, A) \simeq H^1(T_N^{S_0}, \partial T_N^{S_0}, A) = \mathrm{Hom}(H_1(T_N^{S_0}, \partial T_N^{S_0}, \mathbb{Z}), A). \quad (3.1)$$

For each $0 \le r < N$ or $\pm i\infty$, let c_r be the homology class associated to a path on $T_N^{S-S_0}$ that is starting from a fixed base point and winding the hole around the point $\frac{r}{N}$ counterclockwise. For each $x \in \mathbb{R}/\mathbb{Z}$, let

$$v(x) = x + i\mathbb{R}$$

be a vertical line passing through x from $-i\infty$ to $i\infty$. For $x \in N^{-1}\mathbb{Z}/\mathbb{Z}$, we modify $v(x)$ so that $v(x) = v_{-\rho} \cup c_\rho \cup v_\rho$ where

(1) $v_{-\rho} = \{x + it | -\infty \leq t \leq \rho\}$,
(2) $c_\rho = \{x + \rho e^{i\theta} | -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$,
(3) $v_\rho = \{x + it | \rho \leq t \leq \infty\}$

for a sufficiently small $\rho > 0$. Note that we have

$$H_1(T_N^S, \partial T_N^{S_0}, \mathbb{Z}) = \mathbb{Z}v(0) \oplus \bigoplus_{0 \leq r < N-1} \mathbb{Z}c_r.$$

Furthermore we have the relations $c_0 + \cdots + c_{N-1} = 0$ and $v(x) = v(0) + c_1 + \cdots + c_r$ for $r = \lfloor Nx \rfloor$. In particular for $\frac{r-1}{N} < x < \frac{r}{N} < y < \frac{r+1}{N}$, we have

$$c_r = v(y) - v(x).$$

We also obtain

$$H_1(T_N^S, \partial T_N^{S_0}, \mathbb{Z}) = \bigoplus_{0 \leq r < N} \mathbb{Z}v(x_r) \text{ for } \frac{r}{N} < x_r < \frac{r+1}{N}.$$

One can also verify that

$$H_1(T_N^{S-S_0}, \mathbb{Z}) = \mathbb{Z}c_\infty \oplus \bigoplus_{0 \leq r < N} \mathbb{Z}c_r.$$

Note that we have the relation $c_\infty + \sum_{r=0}^{N-1} c_r + c_{-\infty} = 0$ in $H_1(T_N^{S-S_0}, \mathbb{Z})$.

Let $\overline{T}_N$ be the punctured sphere defined by adding S_0 to $T_N^{S-S_0}$ i.e.

$$\overline{T}_N = T_N^{S-S_0} \cup S_0.$$

Observe that we have the isomorphism

$$H_1(\overline{T}_N, \{\pm i\infty\}, \mathbb{Z}) \simeq H_1(T_N^S, \partial T_N^{S_0}, \mathbb{Z}).$$

Since we have the inclusion

$$H_1(\overline{T}_N, \mathbb{Z}) \subset H_1(\overline{T}_N, \{\pm i\infty\}, \mathbb{Z}),$$

we regard $H_1(\overline{T}_N, \mathbb{Z})$ as a subgroup of $H_1(T_N^S, \partial T_N^{S_0}, \mathbb{Z})$. Also observe that

$$H_1(\overline{T}, \mathbb{Z}) = \bigoplus_{1 \leq r < N} \mathbb{Z}c_r.$$

For a periodic function λ with a period N and $\lambda(0) = 0$, we consider a cohomology class $\omega(R_\lambda)$ in $H_c^1(T_N^{S-S_0}, \mathbb{C})$ given by

$$\omega(R_\lambda) = R_\lambda(e^{2\pi iz})dz.$$

We are also able to say that $\omega(R_\lambda) \in H^1(T_N^{S-S_0}, \mathbb{C})$ or $H^1(\overline{T}_N, \mathbb{C})$. Define the Fourier transform $\widehat{\lambda}$ of λ as follows:

$$\widehat{\lambda}(r) = \sum_{s=0}^{N-1} \lambda(s)\zeta_N^{rs}.$$

Observe that we have

$$\widehat{\widehat{\lambda}}(r) = N\lambda(-r).$$

Since we have a following formula

$$\int_{c_r} R_\lambda(e^{2\pi i z})dz = \frac{1}{N}\widehat{\lambda}(r)$$

and

$$\lim_{q\to 0} R_\lambda(q) = 0$$

i.e. $\omega(R_\lambda)(c_\infty) = 0$, we are able to conclude that $\omega(R_\lambda) \in H^1(T_N^{S-S_0}, \mathbb{Z}_p[\widehat{\lambda}])$ and $\omega(R_\lambda) \in H^1(\overline{T}_N, \mathbb{Z}_p[\widehat{\lambda}])$. Here $\mathbb{Z}_p[\widehat{\lambda}]$ is the $\mathbb{Z}_p$-algebra generated by the values of $\widehat{\lambda}$. Furthermore, since we have

$$\int_{v(0)} \omega(R_\lambda) = \sum_{r=1}^{N} \widehat{\lambda}(r) B_1\left(\frac{r}{N}\right) \in \mathbb{Z}_p[\widehat{\lambda}],$$

we also conclude that

$$\omega(R_\lambda) \in H_c^1(T_N^{S-S_0}, \mathbb{Z}_p[\widehat{\lambda}]).$$

Recall that we have the decomposition $\mathbb{Z}_p^\times = W \times (1 + 2p\mathbb{Z}_p)$ for $W = \mu_{p-1}$ if $p > 2$ and $W = \{\pm 1\}$ if $p = 2$. We set U be a maximal linearly independent subset of W over $\mathbb{Q}$. For $x \in \mathbb{Z}_p$, let $(x)_n$ be the $(n-1)$-th partial sum of the p-adic expansion of x and we also set

$$\mathbf{x}_n(\alpha) := \left(\frac{(\alpha\eta)_n}{p^n}\right)_{\eta \in U} \in [0,1)^U.$$

In the next proposition, we consider a property of equi-distribution in a slightly different way from [2].

Proposition 3 *Let $a + p^m\mathbb{Z}_p \subset \mathbb{Z}_p$. Let $(a_\eta) \in [0,1)^U$ and $\epsilon > 0$ be given. Then for all sufficiently large n, there exists $\alpha \in a + p^m\mathbb{Z}_p$ such that*

$$\left|\frac{(\alpha\eta)_n}{p^n} - a_\eta\right| < \epsilon \text{ for all } \eta \in U.$$

PROOF. Let $\mathbf{x} = (x_\eta) \in [0,1)^U$. Define a function f on $[0,1)^U$ such that

$$f(\mathbf{x}) = \begin{cases} \prod_\eta \sin(2\pi\epsilon^{-1}(x_\eta - a_\eta)) & \text{if } |x_\eta - a_\eta| \le \epsilon \text{ for all } \eta \in U \\ 0 & \text{otherwise} \end{cases}.$$

Let $c_{\mathbf{n}}$ be the coefficient of the Fourier expansion

$$f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^U} c_{\mathbf{n}} e^{2\pi i \mathbf{x}\cdot\mathbf{n}}.$$

Since we have the evaluation of the integration

$$\int_{\alpha-\epsilon}^{\alpha+\epsilon} \sin\left(2\pi\epsilon^{-1}(x-\alpha)\right) e^{2\pi n i x} dx = \frac{\epsilon(e^{2n\pi i(\alpha+\epsilon)} - e^{2n\pi i(\alpha-\epsilon)})}{2\pi(n^2\epsilon^2 - 1)},$$

we obtain that

$$c_{\mathbf{n}} = O\left(\frac{1}{|\mathbf{n}|^{2|U|}}\right).$$

Let $Z = a + p^m\mathbb{Z}_p$, $Z_n := \{x \pmod{p^n} | x \in Z\}$. In order to verify the proposition, it suffices to show that

$$\frac{1}{|Z_n|}\sum_{\alpha \in Z_n} |f(\mathbf{x}_n(\alpha))|^2 > 0 \text{ for } n \gg 1.$$

For $\mathbf{n} = (n_\eta) \in \mathbb{Z}^U$, we set

$$\sigma_{\mathbf{n}} = \sum_{\eta \in U} n_\eta \eta.$$

From the formula

$$\sum_{\alpha \in Z_n} e^{\frac{2\pi i \alpha\beta}{p^n}} = \begin{cases} e^{\frac{2\pi i a\beta}{p^n}} |Z_n| & \text{if } p^{n-m}|\beta \\ 0 & \text{otherwise} \end{cases},$$

we have

$$\frac{1}{|Z_n|}\sum_{\alpha \in Z_n} |f(\mathbf{x}_n(\alpha))|^2 = \sum_{\mathbf{n}} |c_{\mathbf{n}}|^2 + \lim_{M\to\infty} \delta_{n,M},$$

where

$$\delta_{n,M} = \sum_{\substack{|\mathbf{n}|,|\mathbf{m}|<M, \\ \mathbf{n}\neq\mathbf{m} \\ \sigma_{\mathbf{n}} \equiv \sigma_{\mathbf{m}} (p^{n-m})}} e^{\frac{2\pi i(\sigma_{\mathbf{n}} - \sigma_{\mathbf{m}})}{p^n}} c_{\mathbf{n}} \overline{c}_{\mathbf{m}}.$$

Observe that $\lim_{n\to\infty} \delta_{n,M} = 0$ for all $M > 0$. Since the sum $\sum_{\mathbf{n}} c_{\mathbf{n}}$ converges absolutely, we conclude the proposition. $\square$

For $x \in \mathbb{Z}_p$, we set $v\left(\frac{x}{p^n}\right) = v\left(\frac{(x)_n}{p^n}\right)$ and for $Z \subset \mathbb{Z}_p^\times$ and $n \ge 1$ let $M_n(Z)$ be a subgroup generated by

$$\left\{ \left(v\left(\frac{\alpha\eta}{p^n}\right)\right)_{\eta \in V} \middle| \alpha \in Z \right\} \subset H_1(T_N^S, \partial T_N^{S_0}, \mathbb{Z})^{\frac{p-1}{2}}.$$

The following homological equi-distribution statement is a stronger version than one in [6, Proposition 4] which has been used to obtain a homological proof of Washington's theorem on non-vanishing mod p of special Dirichlet L-values.

Proposition 4 *Let Z be an open subset of $\mathbb{Z}_p^\times$. For all sufficiently large n, we have*

$$M_n(Z) \supseteq H_1(\overline{T}_N, \mathbb{Z})^{\frac{p-1}{2}}. \tag{3.2}$$

PROOF. Let $\eta \in V$ be chosen. Set $U = \{\eta_1 = \eta, \eta_2, \cdots, \eta_t\}$ and $W/\{\pm 1\} = \{\eta_1, \cdots, \eta_t, \xi_1, \cdots, \xi_s\}$. We have an integral $t \times s$ matrix A such that

$$(\xi_1, \cdots, \xi_s) = (\eta_1, \cdots, \eta_t)A.$$

Let $\alpha_1, \cdots, \alpha_t$ be any real numbers and set

$$P(\alpha_1, \cdots, \alpha_t) := (\alpha_1, \cdots, \alpha_t)\,(I|A)$$

for a $t \times t$ identity matrix I and a $t \times \frac{p-1}{2}$ block matrix $(I|A)$. Note that we have

$$P(\eta_1, \cdots, \eta_t) = (\eta_1, \cdots, \eta_t, \xi_1, \cdots, \xi_s).$$

For each integer i with $1 \le i < \frac{p-1}{2}$ we set $P(\alpha_1, \cdots, \alpha_t)_{(i)}$ be the i-th coordinate of $P(\alpha_1, \cdots, \alpha_t)$. For an integer r with $1 \le r < N$, consider a function $P(\frac{r}{N}, \alpha_2, \cdots, \alpha_t)$ of real variables $\alpha_2, \cdots, \alpha_t$. Since A is an integral matrix, we have

$$P\Big(\frac{r}{N}, 0, \cdots, 0\Big)_{(i)} \in \frac{1}{N}\mathbb{Z} \text{ for each } i.$$

Since V is $\mathbb{Z}$-multiplicatively independent i.e. $\eta/\eta' \notin \mathbb{Q}$ for $\eta \neq \eta' \in V$, $P(\alpha_1, \cdots, \alpha_t)_{(i)}$ is not a $\mathbb{Z}$-multiple of α_1 for $i = 2, \cdots, \frac{p-1}{2}$. Hence, the function $P(\frac{r}{N}, \alpha_2, \cdots, \alpha_t)_{(i)}$ is not constant for each $i = 2, \cdots, \frac{p-1}{2}$ and we can find a real vector $(\alpha_2^\circ, \cdots, \alpha_t^\circ)$ near $(0, \cdots, 0)$ such that

$$P\Big(\frac{r}{N}, \alpha_2^\circ, \cdots, \alpha_t^\circ\Big)_{(i)} \notin \frac{1}{N}\mathbb{Z}$$

for $i = 2, \cdots, \frac{p-1}{2}$ and $P(\frac{r}{N}, \alpha_2^\circ, \cdots, \alpha_t^\circ)_{(1)} = \frac{r}{N}$. Since $P(\alpha_1, \alpha_2^\circ, \cdots, \alpha_t^\circ)$ is continuous in the variable α_1, there exist two numbers α_1', α_1'' such that

(1) $P(\alpha_1', \alpha_2^\circ, \cdots, \alpha_t^\circ)_{(1)} = \alpha_1' \in (\frac{r-1}{N}, \frac{r}{N})$,
(2) $P(\alpha_1'', \alpha_2^\circ, \cdots, \alpha_t^\circ)_{(1)} = \alpha_1'' \in (\frac{r}{N}, \frac{r+1}{N})$
(3) $P(\alpha_1', \alpha_2^\circ, \cdots, \alpha_t^\circ)_{(i)}, P(\alpha_1'', \alpha_2^\circ, \cdots, \alpha_t^\circ)_{(i)} \in \Big(\frac{l_i-1}{N}, \frac{l_i}{N}\Big)$ for all $i = 2, \cdots, \frac{p-1}{2}$ and an integer $1 \le l_i < N$.

Now it suffices to show the proposition with $Z = a + p^m\mathbb{Z}_p$ for an $a \in \mathbb{Z}_p^\times$ and $m \ge 1$. By Proposition 3, for sufficiently large n it is possible to find suitable

α, β in Z such that the vectors $(\frac{\alpha\eta_1}{p^n}, \cdots, \frac{\alpha\eta_t}{p^n})$, $(\frac{\beta\eta_1}{p^n}, \cdots, \frac{\beta\eta_t}{p^n})\ (\bmod\ 1)$ are close enough to vectors $(\alpha_1', \alpha_2^\circ, \cdots, \alpha_t^\circ)$, $(\alpha_1'', \alpha_2^\circ, \cdots, \alpha_t^\circ)$ respectively. The vectors $P(\frac{\alpha\eta_1}{p^n}, \cdots, \frac{\alpha\eta_t}{p^n})$, $P(\frac{\beta\eta_1}{p^n}, \cdots, \frac{\beta\eta_t}{p^n})$ satisfy the above conditions (1), (2), and (3). Since we have

$$v\left(\frac{\alpha\eta_i}{p^n}\right) - v\left(\frac{\beta\eta_i}{p^n}\right) = [0]$$

for each $i = 2, \cdots, \frac{p-1}{2}$ and

$$v\left(\frac{\alpha\eta_1}{p^n}\right) - v\left(\frac{\beta\eta_1}{p^n}\right) = -c_r \text{ in } H_1(T_N^S, \partial T_N^{S_0}, \mathbb{Z}),$$

we obtain

$$\left(v\left(\frac{\alpha\eta}{p^n}\right)\right)_\eta - \left(v\left(\frac{\beta\eta}{p^n}\right)\right)_\eta = (\,[0], \cdots, [0], -c_r, [0], \cdots, [0]\,) \in M_n(Z),$$

where the position of the non-zero homology class corresponds to η. Since η and r are chosen arbitrarily and $H_1(\overline{T}_N, \mathbb{Z})$ is generated by c_r for $r = 1, \cdots, N-1$, we prove the proposition. $\square$

For $1 \le r < N$ and $\omega \in H_c^1(T_N^{S-S_0}, \mathbb{C})$ we set

$$\mathrm{Res}_{z=\frac{r}{N}}\omega := \int_{c_r}\omega.$$

Note that when $\omega = \omega(R_\lambda)$, then we have

$$\mathrm{Res}_{z=\frac{r}{N}}\omega = \mathrm{Res}_{z=\frac{r}{N}}R_\lambda(e^{2\pi i z}) = \frac{1}{N}\widehat{\lambda}(r).$$

An immediate consequence of the proposition is the following corollary, which is a cohomological analogue of Sinnott's algebraic independence result (See [5, Proposition 3.2]).

Corollary 5 *Let Z be an open set and n be a sufficiently large integer. If we have*

$$\sum_{\eta\in V}\int_{v\left(\frac{a\eta}{p^n}\right)}\omega_\eta \equiv 0\ (\bmod\ \pi)$$

for $\omega_\eta \in H_c^1(T_N^{S-S_0}, \overline{\mathbb{Z}}_p)$ and all $a \in Z$, then

$$\mathrm{Res}_{z=\frac{r}{N}}\omega_\eta \equiv 0\ (\bmod\ \pi)$$

for each $\eta \in W/\{\pm 1\}$ and $r = 1, \cdots, N-1$.

4 Power series attached to p-adic Dirichlet L-functions

Let $\alpha > 0$ be an integer relatively prime to p and λ_0 be a periodic function with a period α defined by

$$\lambda_0(r) = \begin{cases} 1 & \text{if } \alpha \nmid r \\ 1-\alpha & \text{if } \alpha \mid r \end{cases}.$$

Observe that

$$\widehat{\lambda_0}(r) = \begin{cases} -\alpha & \text{if } \alpha \nmid r \\ 0 & \text{if } \alpha \mid r \end{cases}.$$

Let λ be a Dirichlet character of conductor N if $N \neq 1$ and $\lambda = \lambda_0$ if $N = 1$. We define a $\mathbb{Z}_p[\widehat{\lambda}]$-valued function σ_λ on the set of basic open subsets of $\mathbb{Z}_p$ as follows:

$$\sigma_\lambda(a + p^m\mathbb{Z}_p) = \frac{1}{\widehat{\lambda}(p^m)} \int_{v(a/p^m)} \omega(R_{\widehat{\lambda}}).$$

For each case $N > 1$ or $N = 1$, σ_λ is a distribution on $\mathbb{Z}_p$. In fact, σ_λ becomes a p-adic measure (see [3]). When $N > 1$, we normalize σ_λ to obtain another measure $\sigma_{\lambda,\alpha}$ such that

$$\sigma_{\lambda,\alpha} = \sigma_\lambda - \alpha^{-1}\sigma_\lambda \circ \alpha.$$

Since we have

$$\begin{aligned} \sigma_{\lambda,\alpha}(a + p^m\mathbb{Z}_p) &= \frac{1}{\widehat{\lambda}(p^m)} \int_{v(a/p^m)} R_{\widehat{\lambda}}(q)dz - \frac{\alpha^{-1}}{\widehat{\lambda}(p^m)} \int_{v(\alpha a/p^m)} R_{\widehat{\lambda}}(q)dz \\ &= \frac{1}{\widehat{\lambda}(p^m)} \int_{v(a/p^m)} R_{\widehat{\lambda}}(q) - R_{\widehat{\lambda}}(q^\alpha)dz, \end{aligned}$$

the measure $\sigma_{\lambda,\alpha}$ is associated to a cohomology class defined by the following rational function:

$$R_{\lambda_\alpha}(q) = R_{\widehat{\lambda}}(q) - R_{\widehat{\lambda}}(q^\alpha)$$

of the cylinder $T_{\alpha N}^{S-S_0}$ where a periodic function λ_α of a period αN is defined by

$$\lambda_\alpha(r) = \begin{cases} \widehat{\lambda}(r) & \text{if } \alpha \nmid r \\ \widehat{\lambda}(r) - \widehat{\lambda}(r/\alpha) & \text{if } \alpha \mid r \end{cases}. \tag{4.1}$$

We also have the following interpolation formula.

Proposition 6 *For a Dirichlet character $\chi \neq 1$ on $\mathbb{Z}_p^\times$, we have*

$$\int_{\mathbb{Z}_p} \chi(x)x^j d\sigma_{\lambda,\alpha}(x) = -\chi(N)^{-1}N^{-j}(1 - \chi(\alpha)\alpha^{j-1})L(-j, \chi\lambda) \text{ if } N > 1 \tag{4.2}$$

$$\int_{\mathbb{Z}_p} \chi(x)x^j d\sigma_{\lambda_0}(x) = -\chi(N)^{-1}N^{-j}(1 - \chi(\alpha)\alpha^{j-1})L(-j, \chi) \text{ if } N = 1. \tag{4.3}$$

PROOF. From [3, Theorem 4.4.1], we obtain the formula (4.3) and for $N > 1$ we also have the interpolation

$$\int_{\mathbb{Z}_p} \chi(x)x^j d\sigma_\lambda(x) = -\chi(N)^{-1}N^{-j}L(-j,\chi\lambda).$$

Hence by the definition of $\sigma_{\lambda,\alpha}$ we have

$$\begin{aligned}\int_{\mathbb{Z}_p} \chi(x)x^j d\sigma_{\lambda,\alpha}(x) &= \int_{\mathbb{Z}_p} \chi(x)x^j d\sigma_\lambda(x) - \alpha^{-1}\int_{\mathbb{Z}_p} \chi(x)x^j d\sigma_\lambda(\alpha x) \\ &= (1-\chi^{-1}(\alpha)\alpha^{-j-1})\int_{\mathbb{Z}_p} \chi(x)x^j d\sigma_\lambda(x).\end{aligned}$$

This concludes the proof of the proposition. □

The relation between the integration and the special values suggests us to define a p-adic analytic function

$$L_p(s,\chi\lambda_1) = \frac{1}{1-\alpha^{-1}\chi^{-1}\omega(\alpha)\langle\alpha\rangle^{-s}}\int_{\mathbb{Z}_p^\times} \chi\omega^{-1}(x)\langle x\rangle^{-s}d\sigma(N^{-1}x), \qquad (4.4)$$

where $\lambda_1 = \lambda$, $\sigma = \sigma_{\lambda,\alpha}$ if $N > 1$ and $\lambda_1 = 1$, $\sigma = \sigma_{\lambda_0}$ if $N = 1$. From Proposition 6 and from the observation that

$$\sigma \circ p = \lambda_1(p)\sigma,$$

we have the interpolation

$$L_p(-j,\chi\lambda_1) = -(1-\chi\lambda_1\omega^{-j-1}(p)p^j)L(-j,\chi\lambda_1\omega^{-j-1}). \qquad (4.5)$$

From now on, we set

$$\sigma = \sigma_{\lambda,\alpha} \text{ or } \sigma_{\lambda_0}.$$

We set $\chi_n = \chi\omega^{-n}$ for $n \geq 0$. Let θ be the restriction of $\chi\lambda_1$ to $W \times (\mathbb{Z}/N\mathbb{Z})^\times$ and ξ be the restriction of χ to W. Hence we have $\theta = \xi\lambda_1$. Define a measure τ_θ as

$$\tau_\theta := \xi\omega^{-1}\cdot(\sigma\circ N^{-1}).$$

Since $\sigma \circ -1 = \lambda_1(-1)\sigma$ and $\theta(-1) = 1$, we obtain that

$$\tau_\theta \circ -1 = \theta(-1)\tau_\theta = \tau_\theta \text{ and } G_{\tau_\theta} = 2G^\circ_{\tau_\theta}.$$

Observe that the integration $\int_{\mathbb{Z}_p^\times} \chi\omega^{-1}(x)\langle x\rangle^{-s}d\sigma(N^{-1}x)$ can be obtained from the Γ-transform of τ_θ. Actually we have

$$\int_{\mathbb{Z}_p^\times} \chi\omega^{-1}(x)\langle x\rangle^{-s}d\sigma(N^{-1}x) = G_{\tau_\theta}(\chi(\gamma)\gamma^{-s}), \qquad (4.6)$$

where $G_{\tau_\theta}(T)$ is the power series given by

$$G_{\tau_\theta}(T) = \int_{\mathbb{Z}_p} T^y d\widetilde{\tau}_\theta(\gamma^y).$$

From the fact that the set of negative integers is dense in $\mathbb{Z}_p$, we conclude that the definition (4.4) is independent of choice of α.

Now we choose γ and α such that $\gamma = 1 + 2p$ and $\alpha = \gamma^p$, and set

$$F(T) = -\frac{G^\circ_{\tau_\theta}(T)}{1 - \alpha^{-1}T^p}. \tag{4.7}$$

From (4.5) and (4.6), we obtain that

$$2F(\chi(\gamma)\gamma^{1-n}) = (1 - \chi_n\lambda_1(p)p^{n-1})L(1-n, \chi_n\lambda_1).$$

Note that θ is the first factor of $\chi\lambda_1$ in the sense of Iwasawa ([4]). Then there exists a power series $f(T;\theta) \in \mathfrak{o}[[T-1]]$ such that

$$2f(\chi(\gamma)\gamma^{1-n};\theta) = -(1 - \chi_n(p)p^{n-1})\frac{B_{n,\chi_n\lambda_1}}{n}$$

Since we have the following equality for all Dirichlet character ψ with $\psi(-1) = (-1)^n$ and integers $n \geq 1$

$$L(1-n, \psi) = -\frac{B_{n,\psi}}{n},$$

we conclude that

Proposition 7 *We have $F(T) = f(T;\theta)$. In other words,*

$$(1 - \alpha^{-1}T^p)f(T;\theta) = -G^\circ_{\tau_\theta}(T).$$

Now observe that $D(1 - \alpha^{-1}T^p) \equiv 0 \,(\bmod\ \pi)$. Therefore we obtain that

$$(1 - \alpha^{-1}T^p)Q(D)f(T;\theta) \equiv -Q(D)G^\circ_{\tau_\theta}(T) \,(\bmod\ \pi).$$

Now we show that the measures obtained by twisting τ_θ by $\eta \in V$ are independent modulo π. Recall that $\sigma = \sigma_{\lambda_0}$ if $N = 1$ and $\sigma = \sigma_{\lambda,\alpha}$ if $N > 1$.

Proposition 8 *(1) Let $g \in \mathcal{C}(\mathbb{Z}_p, \mathfrak{o})$ and $\tau \in \mathfrak{m}(\mathbb{Z}_p, \mathfrak{o})$ be a measure such that the measures*

$$\{\tau \circ \eta \,|\, \eta \in V\}$$

are independent modulo π. Then we have

$$g \cdot G^\circ_\tau(T) \equiv 0 \,(\bmod\ \pi) \text{ if and only if } g \equiv 0 \,(\bmod\ \pi).$$

In particular, for a numerical polynomial $Q \in k[T]$ *we have*

$$Q(D)G_\tau^\circ(T) \not\equiv 0 \,(\bmod\ \pi) \text{ if and only if } Q \equiv 0 \,(\bmod\ \pi).$$

(2) The set of measures

$$\{\tau_\theta \circ \eta \,|\, \eta \in V\}$$

is independent modulo π

PROOF. Obviously $g \equiv 0 \,(\bmod\ \pi)$ implies $g \cdot G_\tau^\circ \equiv 0 \,(\bmod\ \pi)$.

For the reverse direction, in order to deduce a contradiction, we assume the contrary that

$$g \not\equiv 0 \,(\bmod\ \pi).$$

Since we have

$$g \cdot G_\tau^\circ(T) = P\left(g(y)d\overline{\tau}(\gamma^y); T\right),$$

from Proposition 1 we obtain that

$$\overline{\tau}(\gamma^y)|_{g^{-1}(\mathfrak{o}^\times)} \equiv 0 \,(\bmod\ \pi).$$

From the assumption, we know that $g^{-1}(\mathfrak{o}^\times)$ is not empty, and by the hypothesis for τ we conclude the proposition (1).

To prove (2), first let $N = 1$ and U be an open subset of $\mathbb{Z}_p^\times$ such that

$$\sum_{\eta \in V} \tau_\theta \circ \eta|_U \equiv 0 \,(\bmod\ \pi). \tag{4.8}$$

There is a basic open subset $a + p^m\mathbb{Z}_p$ of U such that

$$\sum_\eta \xi(a\eta)\sigma_{\lambda_0}|_{\eta a + p^m\mathbb{Z}_p} \equiv 0 \,(\bmod\ \pi).$$

From the definition of σ_{λ_0} we have

$$\sum_{\eta \in V} \int_{v(\alpha\eta/p^n)} \xi(a\eta) R_{\widehat{\lambda_0}}(q)dz \equiv 0 \,(\bmod\ \pi)$$

for all sufficiently large n and $\alpha \in a + p^m\mathbb{Z}_p$. From Corollary 5, we obtain that

$$\mathrm{Res}_{z=\frac{r}{\alpha}} R_{\widehat{\lambda_0}}(q) = \frac{\alpha\lambda_0(-r)}{\alpha} \equiv 0 \,(\bmod\ \pi) \text{ for each } r = 1, \cdots, \alpha - 1.$$

Since we have

$$\lambda_0(r) = 1 \text{ for } r = 1, \cdots, \alpha - 1,$$

this is a contradiction.

For $N > 1$ case, we do a similar calculation as above. The congruence (4.8) enables us to deduce

$$\mathrm{Res}_{z=\frac{r}{\alpha N}} R_{\lambda_\alpha}(q) = \frac{\widehat{\lambda_\alpha}(r)}{\alpha N} \equiv 0 \,(\bmod\ \pi) \text{ for each } r = 1, \cdots, \alpha N - 1.$$

In fact from (4.1) we have

$$\begin{aligned}\widehat{\lambda_\alpha}(r) &= \sum_{s=1}^{\alpha N} \widehat{\lambda}(s)\zeta_{\alpha N}^{rs} - \sum_{s=1}^{N} \widehat{\lambda}(\alpha s)\zeta_N^{rs} \\ &= \begin{cases} -N\lambda(-r\overline{\alpha}) & \text{if } \alpha \nmid r \\ N\,(\alpha-1)\,\lambda(-r/\alpha) & \text{if } \alpha \mid r \end{cases}.\end{aligned}$$

This concludes the proof of the proposition. □

Putting the propositions together, we obtain

Theorem 9 *Let $g \in \mathcal{C}(\mathbb{Z}_p, \mathfrak{o})$. Then we have*

$$g \cdot G^\circ_{\tau_\theta}(T) \equiv 0 \,(\bmod\ \pi) \textit{ if and only if } g \equiv 0 \,(\bmod\ \pi).$$

In particular, we have

$$Q(D)f(T;\theta) \equiv 0 \,(\bmod\ \pi) \textit{ if and only if } Q \equiv 0 \,(\bmod\ \pi).$$

Observe that for $\tau \in \mathfrak{m}(\mathbb{Z}_p, \mathfrak{o})$

$$T^m \left(\frac{d}{dT}\right)^m P(\tau;T) = m! \int_{\mathbb{Z}_p} T^x \binom{x}{m} d\tau(x) = P\left(m! \binom{x}{m} d\tau(x); T\right).$$

In other words we obtain that $T^m(\frac{d}{dT})^m = m!\binom{D}{m}$. Since $m! \equiv 0 \,(\bmod\ \pi)$ for all $m \geq p$, we obtain that $m!\binom{x}{m} d\tau(x) \equiv 0 \,(\bmod\ \pi)$ and

$$\left(\frac{d}{dT}\right)^m P_\tau(T) \equiv 0 \,(\bmod\ \pi) \text{ for all } m \geq p.$$

Since the numerical polynomial $Q(x) = m!\binom{x}{m}$ is not identically zero modulo π if $0 \leq m < p$, we conclude that

Corollary 10 *For $0 \leq m < p$, we have*

$$\left(\frac{d}{dT}\right)^m G^\circ_{\tau_\theta}(T) \not\equiv 0 \,(\bmod\ \pi)$$

and

$$\left(\frac{d}{dT}\right)^m f(T;\theta) \not\equiv 0 \,(\bmod\ \pi).$$

Of course the case of $m = 0$ corresponds to the theorem of Ferrero-Washington ([2]), namely the vanishing of μ-invariant of an abelian number field.

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