PERIODIC ZETA FUNCTIONS AND ABELIAN MODULAR SYMBOLS

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Abstract. We develop a homological method to compute the μ-invariant of Γ-transform of a measure on $\mathbb{Z}_p^\times$ defined by an abelian modular symbol on a punctured cylinder. As a consequence we compute the μ-invariant of a p-adic periodic zeta function and deduce the theorem of Ferrero-Washington. We also study the non-vanishing mod p property of special L-values of abelian modular symbols twisted by Dirichlet characters of ℓ-power conductors.

1. Introduction

Ever since Ferrero and Washington proved the celebrated conjecture of Iwasawa on vanishing of μ-invariant in the abelian case, major progress in the research have been achieved when one succeeded in translating the methods for the abelian case into general situations.

For example, in [17, 18] Vatsal has proved the equi-distribution property of Heegner points on the modular curve, which is an extension of the equi-distribution property ([1]) of normal numbers and has calculated the μ-invariant of anti-cyclotomic p-adic L-function of elliptic curves. The elegant method by Sinnott ([14, 15]), namely algebraic independence of rational functions twisted by irrational p-adic integers, has also been generalized by several mathematicians([2, 3, 4]). The geometric version of the algebraic independence, namely Zariski density of CM points on the Hilbert Shimura modular variety, has been used by Hida ([7, 8]) in order to obtain information on μ-invariant of p-adic Hecke L-functions and non-p-part of special values of Hecke L-functions for anti-cyclotomic $\mathbb{Z}_p$-extension of CM fields.

The goal of present paper is to introduce a homological way in describing the abelian case having in mind its extension to the case of cyclotomic μ-invariant of elliptic curves. In [16], a numerical evidence is presented. In this paper, we study an abelian modular symbol which is nothing but an element in cohomology group with compact support of a punctured cylinder. This is an abelian analogue of modular symbol construction ([12]), which is well explained in [6].

The approach using abelian modular symbols in the paper has flavor of homological counterpart for Sinnott's algebraic treatment using rational function measures. Let us briefly review Sinnott's description. In [14], Sinnott considers a measure σ_λ associated to a rational function $R_\lambda(T) = \frac{\sum_{r=1}^N \lambda(r)T^r}{1-T^N}$ for a periodic function λ. Usually one can associate μ-invariant to a power series, equivalently to a measure (See Section 4). Sinnott shows how to calculate μ-invariant of Γ-transform or Mellin transform $\Gamma(\sigma_\lambda)$ of the measure σ_λ. More precisely, he shows that

$$\mu(\Gamma(\sigma_\lambda)) = \mu(\sigma_\lambda|_{\mathbb{Z}_p^\times} + \sigma_\lambda|_{\mathbb{Z}_p^\times} \circ -1). \tag{1.1}$$

In the paper, we consider a set $\mathcal{R}$ of measures obtained by tensoring continuous functions on $\mathbb{Z}_p$ and measures associated to a class of modular symbols. For $\sigma \in \mathcal{R}$, we also obtain the formula (1.1). Note that $\mathcal{R}$ includes measures which are not rational. For example, a measure $\sigma \in \mathcal{R}$ defined by $d\sigma(x) = \left(\frac{1}{x} + \frac{1}{x^2}\right) d\sigma_\lambda|_{\mathbb{Z}_p^\times}(x)$ is not rational and satisfies the formula (1.1) (See Theorem 4.5). The main ingredient of Sinnott's proof is the algebraic independence ([14, Proposition 3.2]) of rational functions twisted by $\eta \in \mu_{p-1} \subset \mathbb{Z}_p^\times$. In our case, Corollary 3.4 plays a similar role to deduce the formula (1.1) for $\sigma \in \mathcal{R}$.

The paper consists of three parts. In Section 2, we collect classical results on Bernoulli distributions with a view point of continuous parametrization. The p-adic periodic zeta function is defined by an integral with respect to the measure on $\mathbb{Z}_p^\times$ with continuous parametrization. In Section 3 and Section 4, we discuss a homological interpretation of the theorem of Ferrero-Washington ([1]) using abelian modular symbols after discussing properties of the punctured cylinder. In Section 5, we study p-divisibility of special L-values of modular symbols twisted by Dirichlet characters of ℓ-power conductors for a prime number $\ell \nmid 2p$. Of course, this section also can be regarded as the homological analogue of [15]. Especially Theorem 5.1 is the cohomological analogue of [15, Theorem 3.2].

In the paper, we fix two morphisms $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Let v_p be the valuation on $\mathbb{C}_p$ which is normalized so that $v_p(p) = 1$. For an integer β, $\overline{\beta}$ is the inverse of β with respect to a suitable modulus. We denote i by $\sqrt{-1}$ and ζ_M by a primitive M-th root of unity. For $a \in \mathbb{Z}_p$, $(a)_m$ is the m-th partial sum of p-adic expansion of a. In other words, $(a)_m$ is the integer such that $0 \le (a)_m < p^m$ and $a \equiv (a)_m \pmod{p^m}$.

All periodic functions in the paper are assumed to have values in $\overline{\mathbb{Z}}$, the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$. Let λ be a periodic function with a period N and z be an M-th root of unity. Then λz is a periodic function of period MN defined by $(\lambda z)(r) = \lambda(r) z^r$. For an integer β, we set $\beta^*\lambda(r) = \lambda(\beta r)$. Note that $\beta^*\lambda$ depends on the congruence class of β modulo N.

2. Continuous Bernoulli distribution and p-adic periodic zeta function

The modified Bernoulli numbers $Q^{k-1}\omega^{-k}(a)B_k(a/Q)$, $k \ge 0$ for $p \mid Q$, $1 \le a < Q$ can be interpolated by an analytic function on $\mathbb{Z}_p$. Indeed the analytic function $B(s, a, Q)$ in $s \in \mathbb{Z}_p$ defined by

$$B(s, a, Q) := \frac{\langle a \rangle^s}{Q} \sum_{n=0}^{\infty} \binom{s}{n} \left(\frac{Q}{a}\right)^n B_n. \tag{2.1}$$

interpolates the numbers

$$B(k, a, Q) = \frac{\langle a \rangle^k}{Q} \sum_{n=0}^{k} \binom{k}{n} \left(\frac{Q}{a}\right)^n B_n = Q^{k-1}\omega(a)^{-k} B_k\left(\frac{a}{Q}\right).$$

Definition 2.1. Regarding λ as a periodic function with the period Np^m, for $s \in \mathbb{Z}_p$ we define a function $\mu_{s,\lambda}$ on basic open subsets of $\mathbb{Z}_p^\times$ such that

$$\mu_{s,\lambda}(a + p^m\mathbb{Z}_p) = \sum_{\substack{r \equiv a (p^m) \\ 1 \le r < Np^m}} \lambda(r) B(s, r, Np^m). \tag{2.2}$$

When $N = 1$, $s = k \geq 0$, and $\lambda = \omega^k$, it is the Bernoulli distribution. Furthermore when $\lambda = \xi$ for an N-th root of unity, one can easily show that

$$\mu_{k,\omega^k\xi}(a + p^m\mathbb{Z}_p) = \left(T\frac{d}{dT}\right)^{k-1} \frac{T^{(a)_m}}{T^{p^m} - 1}\bigg|_{T=\xi}.$$

In particular, $\mu_{1,\omega\xi}(a + p^m\mathbb{Z}) = \frac{\xi^{(a)_m}}{\xi^{p^m}-1}$ is a measure on $\mathbb{Z}_p$, which is considered in [10] to deduce several properties of p-adic Dirichlet L-function. For general λ, we have

Proposition 2.1. *The function $\mu_{s,\lambda}$ becomes a distribution on $\mathbb{Z}_p^\times$.*

Proof. Let us consider the sum

$$\sum_{q=0}^{p-1} \mu_{s,\lambda}(a + p^m q + p^{m+1}\mathbb{Z}_p) = \sum_q {\sum_r}' \lambda(r)B(s, r, Np^{m+1}), \tag{2.3}$$

where $\sum'$ represents the sum over $1 \leq r < Np^{m+1}$ with the condition $r \equiv N(a + p^m q) \,(\bmod\, p^{m+1})$. Setting $r = Np^m\ell + t$ with $0 \leq \ell < p$ and $0 \leq t < Np^m$ and letting $\sum''_{t,\ell}$ be the sums over t and ℓ satisfying $t \equiv a \,(\bmod\, p^m)$ and $q \equiv \ell - \frac{t-a}{p^m} \,(\bmod\, p)$, the expression (2.3) is equal to

$$\sum_t \lambda(t) \sum_{\ell=0}^{p-1} B(s, Np^m\ell + t, Np^{m+1}) \tag{2.4}$$

Using the well-known distribution formula of the Bernoulli polynomial

$$M^{k-1}\sum_{\ell=0}^{M-1} B_k\left(\frac{\ell}{M} + t\right) = B_k(Mt)$$

for a positive integer M and $k \geq 1$, we obtain that

$$\sum_{\ell=0}^{q-1} B(s, a + M\ell, Mq) = B(s, a, M) \text{ with } p \mid M,\ s \in \mathbb{Z}_p \tag{2.5}$$

and the formula (2.4) is equal to

$$\sum_{t\equiv a(p^m)} \lambda(t)B(s, t, Np^m) = \mu_{s,\lambda}(a + Np^m).$$

Hence $\mu_{s,\lambda}$ is a distribution on $\mathbb{Z}_p^\times$. □

Remark 2.1. Let $s = k \geq 0$ be an integer. It can be easily checked that the following is a distribution on $\mathbb{Z}_p$: For $a + p^m\mathbb{Z}_p \subseteq \mathbb{Z}_p$,

$$\mu'_{k,\lambda}(a + p^m\mathbb{Z}_p) = (Np^m)^{k-1} \sum_{\substack{r\equiv a(p^m)\\ 1\leq r<Np^m}} \lambda(r)B_k\left(\frac{r}{Np^m}\right). \tag{2.6}$$

Furthermore we also have $\mu_{k,\lambda} = \mu'_{k,\lambda\omega^{-k}}$. Hence we can extend $\mu_{k,\lambda}$ to $\mathbb{Z}_p$ and from now on we regard $\mu_{k,\lambda}$ as a distribution on $\mathbb{Z}_p$.

Following Mazur's treatment, one can normalize the distribution in order to get a measure on $\mathbb{Z}_p^\times$.

Definition 2.2. Let α be an integer that is relatively prime to N, $\alpha \equiv 1 \,(\bmod p)$, and κ be another periodic function of a period M. For an open subset K of $\mathbb{Z}_p^\times$, we define a distribution $\mu_{s,\lambda,\kappa,\alpha}$ such that

$$\mu_{s,\lambda,\kappa,\alpha}(K) = \mu_{s,\lambda}(K) - \alpha^{-s}\mu_{s,\kappa}(\alpha K). \tag{2.7}$$

We set $\mu_{s,\lambda,\alpha} := \mu_{s,\lambda,\lambda,\alpha}$. We have the following proposition which is a continuous version of [13, Lemma 7.3].

Proposition 2.2. *(1) $\mu_{s,\lambda,\kappa,\alpha}$ is a measure on $\mathbb{Z}_p^\times$ if and only if*

$$\frac{1}{N}\sum_{r=1}^{N}\lambda(r) = \frac{1}{M}\sum_{r=1}^{M}\kappa(r). \tag{2.8}$$

In particular, $\mu_{s,\lambda}$ is a measure on $\mathbb{Z}_p^\times$ if and only if $\sum_r \lambda(r) = 0$.
(2) Assuming the condition (2.8), we obtain that for $s \in \mathbb{Z}_p - \{0\}$

$$d\mu_{s,\lambda,\kappa,\alpha}(x) = s\langle x\rangle^{s-1} d\mu_{1,\lambda,\kappa,\alpha}(x)$$

and

$$\left(\lim_{s\to 0} s^{-1} d\mu_{s,\lambda,\kappa,\alpha}\right)(x) = \langle x\rangle^{-1} d\mu_{1,\lambda,\kappa,\alpha}(x).$$

Proof. From the definition (2.2) we have

$$\begin{aligned}
\mu_{s,\lambda}(a+p^m\mathbb{Z}) &= \sum_r \lambda(r)\frac{\langle r\rangle^s}{Np^m}\sum_{n\geq 0}\binom{s}{n}\left(\frac{Np^m}{r}\right)^n B_n && (2.9)\\
&= \sum_{r\equiv a(p^m)} \lambda(r)\frac{\langle r\rangle^s}{Np^m}B_0 + O(1)\\
&= \frac{\langle a\rangle^s B_0}{Np^m}\sum_{r=1}^{N}\lambda(p^m r + a) + O(1), && (2.10)
\end{aligned}$$

where by $c_n = O(p^{-n\epsilon})$, $\epsilon \geq 0$ we mean that $p^{n\epsilon}c_n$ is p-adically bounded. Let us set $A = N^{-1}\sum_r \lambda(r)$ and $B = M^{-1}\sum_r \kappa(r)$. From the formula (2.10) we have the expression

$$\mu_{s,\lambda,\kappa,\alpha}(a+p^m\mathbb{Z}_p) = \frac{A\langle a\rangle^s - B\left\langle \alpha^{-1}(\alpha a)_m\right\rangle^s}{p^m}B_0 + O(1),$$

and therefore $\mu_{s,\lambda,\kappa,\alpha}$ is a measure on $\mathbb{Z}_p^\times$ if and only if $A = B$ since $\langle a\rangle^s \equiv \left\langle \alpha^{-1}(\alpha a)_m\right\rangle^s (\bmod p^m)$.

From the formula (2.9), we have

$$\mu_{s,\lambda}(a+p^m\mathbb{Z}_p) = \sum_{r\equiv a(p^m)} \lambda(r)\frac{\langle r\rangle^s}{Np^m} + \lambda(r)s\langle r\rangle^s r^{-1} + O\left(\frac{1}{p^{m-1}}\right). \tag{2.11}$$

We set $r = p^m q + a$ with $0 \leq q < N$. Since $\langle r\rangle^s = \langle a\rangle^s(1 + sp^m q a^{-1}) + O(p^{-2m})$, the formula (2.11) is equal to

$$\frac{\langle a\rangle^s}{Np^m}\sum_{q=1}^{N}\lambda(q) + s\langle a\rangle^s a^{-1}\sum_{q=1}^{N}\lambda(p^m q + a)\left(\frac{q}{N} - 1\right) + O\left(\frac{1}{p^{m-1}}\right).$$

If we set

$$\tau(s,\lambda,a) := s\langle a\rangle^s a^{-1}\sum_{q=1}^{N}\lambda(p^m q+a)\left(\frac{q}{N}-1\right).$$

then we have

$$\tau(s,\lambda,a) = s\langle a\rangle^{s-1}\tau(1,\lambda,a).$$

Note that $(\alpha a)_m = \alpha a - p^m g$ where $g = \left\lfloor \frac{\alpha a}{p^m}\right\rfloor$ and we have $\langle(\alpha a)_m\rangle^s = \langle \alpha a\rangle^s(1 - sp^m g(\alpha a)^{-1}) + O(p^{-2m})$. From this, we have

$$\begin{aligned}\mu_{s,\kappa}((\alpha a)_m + p^m\mathbb{Z}_p) &= \frac{\langle\alpha a\rangle^s}{Mp^m}\sum_{q=1}^{M}\kappa(q)\\ &+ s\langle \alpha a\rangle^s(\alpha a)^{-1}\sum_{q=1}^{M}\kappa(p^m q+(\alpha a)_m)\left(\frac{q-g}{M}-1\right) + O\left(\frac{1}{p^{m-1}}\right).\end{aligned}$$

Similarly we also set

$$\tau'(s,\kappa,a) := s\langle a\rangle^s a^{-1}\sum_{q=1}^{M}\kappa(p^m q+(\alpha a)_m)\left(\frac{q-g}{M}-1\right).$$

and we have

$$\tau'(s,\kappa,a) = s\langle a\rangle^{s-1}\tau'(1,\kappa,a).$$

In total, we have

$$\begin{aligned}\mu_{s,\lambda,\kappa,\alpha}(a+p^m\mathbb{Z}_p) &= \mu_{s,\lambda}(a+p^m\mathbb{Z}_p) - \langle\alpha\rangle^{-s}\mu_{s,\kappa}((\alpha a)_m + p^m\mathbb{Z}_p)\\ &= \tau(s,\lambda,a) - \alpha^{-1}\tau'(s,\kappa,(\alpha a)_m) + O\left(\frac{1}{p^{m-1}}\right)\\ &= s\langle a\rangle^{s-1}(\tau(1,\lambda,a) - \alpha^{-1}\tau'(1,\kappa,a)) + O\left(\frac{1}{p^{m-1}}\right)\\ &= s\langle a\rangle^{s-1}\mu_{1,\lambda,\kappa,\alpha}(a+p^m\mathbb{Z}_p) + O\left(\frac{1}{p^{m-1}}\right).\end{aligned}$$

To finish the proof, let us consider $|\langle x\rangle^{s-1} - \langle x\rangle^{-1}|_p$ for $x \in \mathbb{Z}_p^\times$. We have

$$|\langle x\rangle^{s-1} - \langle x\rangle^{-1}|_p = |\langle x\rangle^s - 1|_p \le |s|_p|\langle x\rangle - 1|_p.$$

Hence for all continuous function f on $\mathbb{Z}_p^\times$ with $||f||_p \le 1$, we have

$$\begin{aligned}&\Big|\int_{\mathbb{Z}_p^\times} f(x)\left(\frac{d\mu_{s,\lambda,\kappa,\alpha}(x)}{s} - x^{-1}d\mu_{1,\lambda,\kappa,\alpha}(x)\right)\Big|_p\\ =&\Big|\int_{\mathbb{Z}_p^\times} f(x)\left(\langle x\rangle^{s-1} - \langle x\rangle^{-1}\right)d\mu_{1,\lambda,\kappa,\alpha}(x)\Big|_p\\ \le&|s|_p\Big|\int_{\mathbb{Z}_p^\times} d\mu_{1,\lambda,\kappa,\alpha}\Big|_p = |s|_p\Big|\mu_{1,\lambda,\kappa,\alpha}(\mathbb{Z}_p^\times)\Big|_p\end{aligned}$$

Since the above inequality is independent of f, we obtain the result. □

Let A be an $\mathbb{Z}_p$-algebra. We define $\mathfrak{L}(\mathbb{Z}/N\mathbb{Z}, A)$ as the collection of A-valued periodic functions with a period N and $\mathfrak{L}(\mathbb{Z}_p, A)$ as the collection of A-valued locally constant functions on $\mathbb{Z}_p$. Let $\Psi \in \mathfrak{L}(\mathbb{Z}/cN\mathbb{Z}, A)$, where c is a p-power and $p \nmid N$.

For an integer $\alpha \equiv 1 \pmod{cp}$, we define a measure $\sigma_{\Psi,\alpha} \in \mathfrak{m}(\mathbb{Z}_p, A)$ as follows: We have a morphism $\mathfrak{L}(\mathbb{Z}/N\mathbb{Z}, A) \to \mathfrak{m}(\mathbb{Z}_p, A)$ by $\lambda \mapsto \mu_{1,\lambda,\alpha}$. We have

$$\mathfrak{L}(\mathbb{Z}/cN\mathbb{Z}, A) \subset \mathfrak{L}(\mathbb{Z}_p, A) \otimes \mathfrak{L}(\mathbb{Z}/N\mathbb{Z}, A) \to \mathfrak{m}(\mathbb{Z}_p, A),$$

where $\psi \otimes \lambda \mapsto \psi(x) d\mu_{1,\lambda,\alpha}$ for $\psi \in \mathfrak{L}(\mathbb{Z}_p, A)$ and $\lambda \in \mathfrak{L}(\mathbb{Z}/N\mathbb{Z}, A)$. We choose $\sigma_{\Psi,\alpha}$ as the image of Ψ under the map.

The following integrals are crucial to compute a residue of a p-adic meromorphic function on $\mathbb{Z}_p$ which is turned out to be a p-adic periodic zeta function.

Corollary 2.3. *Let $\alpha \equiv 1 \pmod{cp}$. Then we have*

$$\int_{\mathbb{Z}_p^\times} \langle x \rangle^{-1} d\sigma_{\Psi,\alpha}(x) = \frac{\log_p(\alpha)}{cN} \sum_{\substack{r=1 \\ p \nmid r}}^{cN} \Psi(r).$$

Proof. First let us consider the case that $\Psi = \psi\lambda$ for $\psi \in \mathcal{L}(\mathbb{Z}_p, \overline{\mathbb{Z}}_p)$ and $\lambda \in \mathcal{L}(\mathbb{Z}/N\mathbb{Z}, \overline{\mathbb{Z}}_p)$. Let c be a period of ψ. Then we have

$$d\sigma_{\Psi,\alpha}(x) = \psi(x) d\mu_{1,\lambda,\alpha}(x).$$

From Proposition 2.2, we obtain

$$\begin{aligned}
&\int_{\mathbb{Z}_p^\times} \langle x \rangle^{-1} d\sigma_{\Psi,\alpha}(x) = \lim_{s \to 0} \int_{\mathbb{Z}_p^\times} s^{-1} \psi(x) d\mu_{s,\lambda,\alpha}(x) \\
&= \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \psi(a) \lim_{s \to 0} s^{-1} (\mu_{s,\lambda}(a + c\mathbb{Z}_p) - \alpha^{-s} \mu_{s,\lambda}(\alpha a + c\mathbb{Z}_p)) \\
&= \lim_{s \to 0} s^{-1} (1 - \alpha^{-s}) \sum_a \psi(a) \mu_{0,\lambda}(a + p^m \mathbb{Z}_p) \\
&= \log_p(\alpha) \sum_a \psi(a) \sum_{r \equiv a (c)} \lambda(r) B(0, r, cN) \\
&= \frac{\log_p(\alpha)}{cN} \sum_{\substack{r=1 \\ p \nmid r}}^{cN} \psi\lambda(r).
\end{aligned}$$

By the linearity, we extend above calculations to a general Ψ. The corollary is verified. □

Definition 2.3. The *periodic zeta function* $L(s, \lambda)$ for a periodic function λ with a period N is defined by

$$L(s, \lambda) = \sum_{n \geq 1} \frac{\lambda(n)}{n^s} \text{ for } \Re(s) > 1.$$

The function $L(s, \lambda)$ has the meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ with the residue $\frac{1}{N} \sum_{r=1}^{N} \lambda(r)$. In fact, $L(s, \lambda)$ can be represented by the integral

$$\Gamma(s) L(s, \lambda) = \int_0^\infty R_\lambda(e^{-y}) y^{s-1} dy \text{ for } \Re(s) > 1 \tag{2.12}$$

and

$$(e^{2\pi i s} - 1) \Gamma(s) L(s, \lambda) = \int_{P(\rho)} R_\lambda(e^{-y}) y^{s-1} dy \text{ for } s \in \mathbb{C},$$

where the branch of log, $0 \le \log(y) < 2\pi$ is used for y^s, $P(\rho)$ is a positively oriented contour $\{\rho e^{i\theta} | 0 \le \theta < 2\pi\} \cup \{te^{i\theta} | \rho < t < \infty, \theta = 0, 2\pi\}$ with sufficiently small $\rho > 0$, and

$$R_\lambda(q) := \frac{\sum_{r=1}^N \lambda(r) q^r}{1 - q^N}.$$

We also have the expression

$$L(s, \lambda) = N^{-s} \sum_r \lambda(r) \zeta\left(s, \frac{r}{N}\right) \tag{2.13}$$

which enables us to get the functional equation (See [11]) of $L(s, \lambda)$ as follows:

$$(e^{2\pi i s} - 1)\Gamma(s) L(s, \lambda) = \left(\frac{2\pi i}{N}\right)^s \left(L(1-s, \widehat{\lambda}) - (-1)^{s-1} L(1-s, (-1)^* \widehat{\lambda})\right),$$

where $\widehat{\lambda}$ is the Fourier transform of λ defined by

$$\widehat{\lambda}(r) = \sum_{t=0}^{N-1} \lambda(t) \zeta_N^{-rt}.$$

Note that we have $\widehat{\widehat{\lambda}} = N(-1)^* \lambda$, $\widehat{\alpha^* \lambda} = \overline{\alpha}^* \widehat{\lambda}$. From (2.13), we obtain

$$L(1-k, \lambda) = -\frac{N^{k-1}}{k} \sum_r \lambda(r) B_k\left(\frac{r}{N}\right). \tag{2.14}$$

Let Ψ be a periodic function with a period cN for a p-power c and a positive integer N with $p \nmid N$. We define $\Psi^{(p)}$ as $\Psi^{(p)}(r) = \Psi(r)$ if $p \nmid r$ and $\Psi^{(p)}(r) = 0$ if $p \mid r$.

Theorem 2.4. *There exists a p-adic periodic zeta function $L_p(s, \Psi)$ which is a p-adic meromorphic function on $\mathbb{Z}_p$ with a simple pole at $s = 1$ and the residue*

$$\mathrm{Res}_{s=1} L_p(s, \Psi) = \frac{1}{cN} \sum_{\substack{r=1 \\ p \nmid r}}^{cN} \Psi(r). \tag{2.15}$$

It has the interpolation property as follows:

$$L_p(1-k, \Psi) = L(1-k, \Psi_k^{(p)}) \text{ for all } k \ge 1,$$

where $\Psi_k = \Psi \omega^{-k}$.

Proof. Choose an integer α so that $\alpha \equiv 1 \pmod{cp}$. Define the p-adic periodic zeta function of Ψ by

$$L_p(s, \Psi) = \frac{1}{\alpha^{s-1} - 1} \int_{\mathbb{Z}_p^\times} \langle x \rangle^{-s} d\sigma_{\Psi,\alpha}(x).$$

Note that it is a p-adic meromorphic function on $\mathbb{Z}_p$ with the pole at $s = 1$ and the residue

$$\mathrm{Res}_{s=1} L_p(s, \Psi) = \frac{1}{\log_p \alpha} \int_{\mathbb{Z}_p^\times} \langle x \rangle^{-1} d\sigma_{\Psi,\alpha} = \frac{1}{cN} \sum_{\substack{r=1 \\ p \nmid N}}^{cN} \Psi(r).$$

Suppose first that $\Psi = \psi\lambda$. We have the calculations

$$\int_{\mathbb{Z}_p^\times} \langle x \rangle^{k-1} d\sigma_{\Psi,\alpha} = \frac{1}{k} \int_{\mathbb{Z}_p^\times} \psi(x) d\mu_{k,\lambda,\alpha} = \frac{1}{k} \sum_{\substack{x=1 \\ p \nmid x}}^{c} \psi(x) \mu_{k,\lambda,\alpha}(x + c\mathbb{Z}_p).$$

Note that from (2.2) and (2.14) we have

$$\begin{aligned}\sum_{p\nmid x}\psi(x)\mu_{k,\lambda}(x+c\mathbb{Z}_p) &= (cN)^{k-1}\sum_{p\nmid x}\psi(x)\sum_{r\equiv x(c)}\lambda(r)\omega^{-k}(r)B_k\left(\frac{r}{cN}\right)\\ &= -kL(1-k,(\psi\lambda\omega^{-k})^{(p)}).\end{aligned}$$

Hence we have the interpolation.

$$\int_{\mathbb{Z}_p^\times}\langle x\rangle^{k-1}d\sigma_{\Psi,\alpha} = (\alpha^{-k}-1)L(1-k,(\psi\lambda\omega^{-k})^{(p)}). \tag{2.16}$$

By extending (2.16) linearly to a general Ψ, we conclude the theorem. □

The following construction of p-adic Dirichlet L-function is well-known:

Corollary 2.5. *Let $\chi : (\mathbb{Z}/cN\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a Dirichlet character with the conductor cN where c is a p-power and $(p,N)=1$. There exists a p-adic analytic function $L_p(s,\chi)$ unless $\chi=1$ so that*

$$L_p(1-k,\chi) = L(1-k,\chi_k^{(p)}) = (1-\chi\omega^{-k}(p)p^{k-1})L(1-k,\chi\omega^{-k}).$$

If $\chi=1$, then $L_p(s,1)$ is a p-adic meromorphic function on $\mathbb{Z}_p$ with the simple pole at $s=1$ and its residue is $1-p^{-1}$.

Proof. The residue of $L_p(s,\chi)$ at $s=1$ is

$$\frac{1}{cN}\sum_{\substack{r=1\\(Np,r)=1}}^{cN-1}\chi(r) = \begin{cases}0 & \text{if } \chi\neq 1\\ \frac{p-1}{p} & \text{otherwise.}\end{cases}$$

This concludes the proof. □

Remark 2.2. Observe that if $\chi(-1)=-1$ then $L_p(s,\chi)$ is identically zero. In fact, if we set $\chi=\chi_p\chi_N$ such that the conductors of χ_p and χ_N are a p-power and N respectively, then we have the calculations

$$\mu_{1,\chi_N}(-x+p^m\mathbb{Z}_p) = \sum_{\substack{1\le r<cN\\ r\equiv x(c)}}\chi_N\omega(-r)B_1\left(1-\frac{r}{cN}\right).$$

Since we have $B_1(1-x)=-B_1(x)$, we have

$$d\mu_{1,\chi_N,\alpha}\circ -1 = -\chi_N\omega(-1)d\mu_{1,\chi_N,\alpha}.$$

In conclusion, we obtain

$$\int_{\mathbb{Z}_p^\times}\chi_p(x)\langle x\rangle^{-s}d\mu_{1,\chi_N,\alpha}(x) = \chi(-1)\int_{\mathbb{Z}_p^\times}\chi_p(x)\langle x\rangle^{-s}d\mu_{1,\chi_N,\alpha}(x).$$

Therefore the p-adic L-function is identically zero if $\chi(-1)=-1$.

3. Modular symbols and homological equi-distribution

From now on, let us consider N with $p \nmid N$. Let T_N be a punctured cylinder given by $T_N = \mathbb{C}/\mathbb{Z} - \{\frac{r}{N} \,|\, 0 \le r < N\}$. We compactify T_N by adding boundaries to the holes i.e. by taking out small open disks around the points $S = \{\frac{r}{N}\} \cup \{\pm i\infty\}$. We denote it by T_N^S. For any subset $S' \subset S$, we do the same procedure and denote it by $T_N^{S'}$. Let A be a commutative ring and $H_c^1(T_N^{S-S_0}, A)$ be the cohomology group of $T_N^{S-S_0}$ with compact support. We have the identification

$$H_c^1(T_N^{S-S_0}, A) \simeq H^1(T_N^S, \partial T_N^{S_0}, A). \tag{3.1}$$

From the isomorphism (3.1), we also regard an abelian modular symbol as an element in $\mathrm{Hom}(H_1(T_N^{S_0}, \partial T_N^{S_0}, \mathbb{Z}), A)$. Note that we have the exact sequence

$$0 \to H^0(T_N^{S-S_0}, A) \to H^0(\partial T_N^{S_0}, A) \to H_c^1(T_N^{S-S_0}, A) \to H^1(T_N^{S-S_0}, A).$$

Definition 3.1. An A-valued *abelian modular symbol* on $T_N^{S-S_0}$ is an element in $H_c^1(T_N^{S-S_0}, A)$. An element in the image of $H^0(\partial T_N^{S_0}, A)$ is called an abelian *boundary symbol*. When A is a ring of integers for a finite extension of $\mathbb{Q}_p$ and π is a uniformizer, we call an A-valued modular symbol ω on $T_N^{S-S_0}$ a *boundary symbol modulo* π^e if ω is congruent to a boundary symbol modulo π^e for a positive integer e.

Let $N > 1$. For each $0 \le r < N$ or $\pm i\infty$, let c_r be the homology class of a path on $T_N^{S-S_0}$ that is starting from a fixed base point and winding the hole around the point $\frac{r}{N}$ counterclockwise. For each $x \in \mathbb{R}/\mathbb{Z}$, let $v(x) = x + i\mathbb{R}$ be the vertical line passing through x from $-i\infty$ to $i\infty$. For $x \in N^{-1}\mathbb{Z}/\mathbb{Z}$, we modify $v(x)$ so that $v(x) = v_{-\rho} \cup o_\rho \cup v_\rho$ where $v_{-\rho} = \{x + it | -\infty \le t \le \rho\}$, $o_\rho = \{x + \rho e^{i\theta} | -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$, and $v_\rho = \{x + it | \rho \le t \le \infty\}$ for a sufficiently small $\rho > 0$. Note that we have

$$H_1(T_N^S, \partial T_N^{S_0}, \mathbb{Z}) = \mathbb{Z}v(0) \oplus \bigoplus_{0 \le r < N-1} \mathbb{Z}c_r.$$

Furthermore we have the relations $c_0 + \cdots + c_{N-1} = 0$ and $v(x) = v(0) + c_1 + \cdots + c_r$ for $r = \lfloor Nx \rfloor$ and $x \notin N^{-1}\mathbb{Z}/\mathbb{Z}$. We also obtain

$$H_1(T_N^S, \partial T_N^{S_0}, \mathbb{Z}) = \bigoplus_{0 \le r < N} \mathbb{Z}v(x_r) \text{ for } \frac{r}{N} < x_r < \frac{r+1}{N}.$$

One can also verify that

$$H_1(T_N^{S-S_0}, \mathbb{Z}) = \mathbb{Z}c_\infty \oplus \bigoplus_{0 \le r < N} \mathbb{Z}c_r.$$

Note that we have the relation $c_\infty + \sum_{r=0}^{N-1} c_r + c_{-\infty} = 0$ in $H_1(T_N^{S-S_0}, \mathbb{Z})$.

Let $\overline{T}_N$ be a punctured sphere given by adding S_0 to $T_N^{S-S_0}$ i.e.

$$\overline{T}_N = T_N^{S-S_0} \cup S_0.$$

Since we have the isomorphism

$$H_1(\overline{T}_N, \{\pm i\infty\}, \mathbb{Z}) \simeq H_1(T_N^S, \partial T_N^{S_0}, \mathbb{Z})$$

and the inclusion

$$H_1(\overline{T}_N, \mathbb{Z}) \subset H_1(\overline{T}_N, \{\pm i\infty\}, \mathbb{Z}),$$

we regard $H_1(\overline{T}_N,\mathbb{Z})$ as a subgroup of $H_1(T_N^S,\partial T_N^{S_0},\mathbb{Z})$. Observe that we have

$$H_1(\overline{T}_N,\mathbb{Z}) = \bigoplus_{0\le r<N} \mathbb{Z}c_r.$$

Proposition 3.1. *ω is a boundary symbol if and only if $\omega(H_1(\overline{T}_N,\mathbb{Z}))=0$ and ω is a boundary symbol modulo π^e if and only if $\omega(H_1(\overline{T}_N,\mathbb{Z}))\equiv 0\,(\bmod\ \pi^e)$.*

Proof. The first statement follows from the fact that

$$\ker(H_1(T_N^{S-S_0},\partial T_N^{S_0},\mathbb{Z})\to H_0(\partial T_N^{S_0},\mathbb{Z})) = \oplus_r \mathbb{Z}c_r. \tag{3.2}$$

For the second statement, we consider a diagram

$$\begin{array}{ccc} H^0(\partial T_N^{S_0},A) & \longrightarrow & H_c^1(T_N^{S-S_0},A) \\ \downarrow & & \downarrow \\ H^0(\partial T_N^{S_0},A/\pi^e) & \longrightarrow & H_c^1(T_N^{S-S_0},A/\pi^e) \end{array}.$$

Since it is commutative, we conclude the proposition. □

For a periodic function λ with a period N, we consider a cohomology class $\omega(R_\lambda)=R_\lambda(e^{2\pi iz})dz$ in $H^1(T_N^{S-S_0},\mathbb{C})$. Since we have $\int_{c_r}R_\lambda(e^{2\pi iz})dz=\frac{1}{N}\widehat{\lambda}(r)$ and $\lim_{q\to 0}R_\lambda(q)=0$ i.e. $\omega(R_\lambda)(c_\infty)=0$, we conclude that $\omega(R_\lambda)\in H^1(T_N^{S-S_0},\mathbb{Z}_p[\widehat{\lambda}])$. When $\omega=\omega(R_\lambda)$ with $\lambda(0)=0$, it also can be regarded as a $\mathbb{C}$-valued abelian modular symbol of $T_N^{S-S_0}$.

The action of -1 on T_N^S induces an action on $H_1(T_N^S,\partial T_N^{S_0},\mathbb{Z})$. In particular, for $x\in\mathbb{R}/\mathbb{Z}$ and $0\le r<N$ we have

$$(-1)\cdot v(x) = -v(1-x), \text{ and } (-1)\cdot c_r = -c_{-r}$$

For a modular symbol ω on $T_N^{S-S_0}$, we define $(-1)^*\omega$ as $(-1)^*\omega(u)=\omega((-1)\cdot u)$ for each $u\in H_1(T_N^S,\partial T_N^{S_0},\mathbb{Z})$. Observe that if we set $R'(q)=R(q^{-1})$, then we have

$$(-1)^*\omega(R) = \omega(R').$$

For each $x\in M^{-1}\mathbb{Z}/\mathbb{Z}$ with $N\mid M$, we regard ω as a modular symbol on $T_M^{S-S_0}$ and define $\omega|\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ by the natural action of $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ on $T_M^{S-S_0}$. Observe that $\omega(R)|\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}=\omega(R'')$ where $R''(q)=R(e^{2\pi ix}q)$. The action of the Hecke operator is endowed to both (relative) homology group and (relative) cohomology group of the cylinder $T_N^{S-S_0}$. For a positive integer n with $\gcd(n,N)=1$ and a modular symbol ω, we have

$$\omega|T(n) = \sum_{r=1}^{n} \omega\left|\begin{pmatrix}1 & t\\ 0 & n\end{pmatrix}\right..$$

For details, check [6]. We have the explicit actions

$$c_r|T(n) = c_{\overline{n}r},\, v(0)|T(n) = \sum_t v(\frac{t}{n}), \text{ and } \omega(R_\lambda)|T(n) = \omega(R_{n^*\lambda}). \tag{3.3}$$

We consider the decomposition

$$\mathbb{Z}_p^\times = V\times(1+2p\mathbb{Z}_p)$$

for $V = \mu_{p-1}$ if $p > 2$ and $V = \mu_4$ if $p = 2$. We set U be a maximal linearly independent subset of V over $\mathbb{Q}$. We quote a proposition due to Ferrero and Washington. In [20], the reader can find a proof using compactness of the set $[0,1]^r$.

Proposition 3.2 ([19])**.** *Let $(x_\eta) \in [0,1)^U$ and $Z \subset \mathbb{Z}_p^\times$ be an open subset, and $\epsilon > 0$ be given. Then for all sufficiently large n, there exists $\alpha \in Z$ such that*

$$\left| \frac{(\alpha\eta)_n}{p^n} - x_\eta \right| < \epsilon \text{ for all } \eta \in U.$$

For $Z \subset \mathbb{Z}_p^\times$, let $M_n(Z)$ be a subgroup generated by

$$\left\{ \left(v\left(\frac{a\eta}{p^n}\right)\right)_{\eta \in \frac{V}{\{\pm 1\}}} \middle| \, a \in Z \right\} \subset H_1(T_N^S, \partial T_N^{S_0}, \mathbb{Z})^{\frac{p-1}{2}}.$$

The following homological equi-distribution statement is a modified version of [16, Proposition 4] which has been used to obtain a homological proof of Washington's theorem on the non-vanishing of special Dirichlet L-values.

Proposition 3.3. *Let Z be an open subset of $\mathbb{Z}_p^\times$. For all sufficiently large n, we have*

$$M_n(Z) \supseteq H_1(\overline{T}_N, \mathbb{Z})^{\frac{p-1}{2}}. \tag{3.4}$$

Proof. For an $\eta \in \mathbb{Z}_p^\times$, we set

$$U = \{\eta_1 = \eta, \eta_2, \cdots, \eta_t\} \text{ and } V/\{\pm 1\} = \{\eta_1, \cdots, \eta_t, \tau_1, \cdots, \tau_s\}.$$

For an integral $s \times t$ matrix A, we have $(\tau_1, \cdots, \tau_s) = (\eta_1, \cdots, \eta_t)A$. We set $P(\alpha_1, \cdots, \alpha_t) := (\alpha_1, \cdots, \alpha_t)\,(I|A)$ for a $t \times t$ identity matrix I and $\alpha_i \in \mathbb{R}$. Due to [16, Lemma 3], for each $r \in \mathbb{Z}$, we can find numbers α_1', α_1'', and α_2°, $\cdots$, α_t° such that $\alpha_1' \in (\frac{r-1}{N}, \frac{r}{N})$, $\alpha_1'' \in (\frac{r}{N}, \frac{r+1}{N})$, and no coordinate of $P(\alpha_1', \alpha_2^\circ, \cdots, \alpha_t^\circ)$, $P(\alpha_1'', \alpha_2^\circ, \cdots, \alpha_t^\circ)$ is in $\frac{1}{N}\mathbb{Z}$.

By Proposition 3.2, for sufficiently large n it is possible to find suitable α, β in Z such that the vectors $(\frac{\alpha\eta}{\ell^n})_{\eta \in U}$, $(\frac{\beta\eta}{\ell^n})_{\eta \in U}$ are close enough to vectors $(\alpha_1', \alpha_2^\circ, \cdots, \alpha_t^\circ)$, $(\alpha_1'', \alpha_2^\circ, \cdots, \alpha_t^\circ)$ given above respectively. Then all coordinates of $P((\frac{a\eta}{\ell^n})_{\eta \in U})$, $P((\frac{b\eta}{\ell^n})_{\eta \in U})$ are inside of the same intervals except the first ones that are in the consecutive intervals $(\frac{r-1}{N}, \frac{r}{N})$, $(\frac{r}{N}, \frac{r+1}{N})$ respectively. Due to this result, we have

$$\left(v\left(\frac{\beta\eta}{p^n}\right)\right)_\eta - \left(v\left(\frac{\alpha\eta}{p^n}\right)\right)_\eta = (\,0, \cdots, 0, c_r, 0, \cdots, 0\,) \in H_1(\overline{T}_N, \mathbb{Z})^{\frac{p-1}{2}},$$

where the position of non-zero homology class c_r corresponds to η. Since η and r are chosen arbitrarily, we prove the proposition. □

The following homological analogue of Sinnott's algebraic independence result (See [14, Proposition 3.2]) is an immediate consequence of Proposition 3.3.

Corollary 3.4. *Let n be a sufficiently large integer. If we have*

$$\sum_{\eta \in V/\{\pm 1\}} \omega_\eta \left(v\left(\frac{\alpha\eta}{p^n}\right)\right) \equiv 0 \,(\bmod\ \pi^e)$$

for modular symbols ω_η and all $\alpha \in Z$, then ω_η is a boundary symbol modulo π^e for each $\eta \in V/\{\pm 1\}$.

Remark 3.1. By projecting the inclusion (3.4) down to the position corresponding to $1 \in U$, we are able to deduce that for sufficiently large n, the set $\left\{v\left(\frac{a}{p^n}\right) \mid a \in Z\right\}$ generates $H_1(\overline{T}_N, \mathbb{Z})$. This leads us to conclude that if ω is a $\overline{\mathbb{Z}}_p$-valued modular symbol and $\omega\left(v\left(\frac{a}{p^n}\right)\right) \equiv 0 \,(\bmod\, \pi^e)$ for all $a \in Z$ and for a sufficiently large n, then ω is a boundary symbol modulo π^e.

4. p-adic measure attached to abelian modular symbols

Let A be a ring of integers for a finite extension over $\mathbb{Q}_p$ and σ be an A-valued measure on $\mathbb{Z}_p$. A power series $G(\sigma; T) \in A[[T-1]]$ is associated to the measure σ. The correspondence is that

$$\sigma \mapsto G(\sigma; T) = \int_{\mathbb{Z}_p} T^x d\sigma(x) := \sum_{n \geq 0} \int_{\mathbb{Z}_p} \binom{x}{n} d\sigma(x)(T-1)^n.$$

For example, for an integer $k \geq 0$, one can show that

$$G(x^k d\mu_{1,\lambda}(x); T) = \left(T\frac{d}{dT}\right)^k R_\lambda(T).$$

Furthermore, one can easily obtain that $G(x^{-k} d\mu_{1,\lambda}|_{\mathbb{Z}_p^\times}(x); T) = G_k(T)$ where $G_k(T)$ is a power series in $\overline{\mathbb{Z}}_p[[T-1]]$ such that

$$\left(T\frac{d}{dT}\right)^k G_k(T) = R_\lambda(T) - \frac{1}{p} \sum_{\zeta^p = 1} R_\lambda(\zeta T).$$

We define the *μ-invariant* $\mu(\sigma)$ of the measure σ as $\mu(G(\sigma; T))$, the minimum of the p-adic valuations of all coefficients of $G(\sigma; T)$. From this definition, we can easily deduce that $\mu(\sigma) = v_p(\pi)e$ if and only if e is the largest integer t satisfying $\sigma \equiv 0 \,(\bmod\, \pi^t)$ on $\mathbb{Z}_p$.

Let $\Lambda = 1 + 2p\mathbb{Z}_p$ and γ be a topological generator of Λ. Recall that $\mathbb{Z}_p^\times = V \times \Lambda$ and $V = \mu_{p-1}$ if $p > 2$ and $V = \mu_4$ if $p = 2$. As done in [14], we define a gamma transform (or Mellin transform) of σ such that

$$\Gamma(s, \sigma) = \int_{\mathbb{Z}_p^\times} \langle x \rangle^{-s} d\sigma(x) = \int_\Lambda x^{-s} d\widetilde{\sigma}(x),$$

where $\widetilde{\sigma} = \sum_{\eta \in V} \sigma \circ \eta$. We have $\Gamma(s, \sigma) = G(\widetilde{\sigma} \circ \gamma^x; \gamma^{-s})$ where $\gamma^x : \Lambda \xrightarrow{\sim} \mathbb{Z}_p$. Define the μ-invariant of the Gamma transform as

$$\mu(\Gamma(s, \sigma)) := \mu(G(\widetilde{\sigma} \circ \gamma^x; T)).$$

The discussion in the proof of Theorem 2.4 shows that

Proposition 4.1. $\Gamma(s, \sigma_{\Psi,\alpha}) = (\alpha^{s-1} - 1)L_p(s, \Psi)$ *for* $\alpha \equiv 1 \,(\bmod\, cp)$.

For an A-valued modular symbol ω on $T_N^{S-S_0}$ and a p-power P with $P \equiv 1 \,(\bmod\, N)$, we define an A-valued function σ_ω on the open subsets of $\mathbb{Z}_p$ by

$$\sigma_\omega(a + P^m\mathbb{Z}_p) := \omega\left(v\left(\frac{a}{P^m}\right)\right).$$

Observe that we have

$$\sigma_\omega \circ -1 = -\sigma_{(-1)^*\omega}.$$

For each $\mathbf{x} = (x_r) \in A^{N-1}$, we set

$$\mathcal{M}_N(\mathbf{x}, A) = \left\{ \sigma_\omega \,\middle|\, \omega \in H_c^1(T_N^{S-S_0}, A),\, \omega(v(0)) = \sum_{r=1}^{N-1} \omega(c_r) x_r \right\}.$$

Then we can show that

Proposition 4.2. *For each* $\mathbf{x} \in A^{N-1}$*,* $\mathcal{M}_N(\mathbf{x}, A) \subset \mathfrak{m}(\mathbb{Z}_p, A)$*.*

Proof. Let P be a p-power with $P \equiv 1 \pmod N$. Since $\mathbb{Z}_p = \varprojlim \mathbb{Z}_p/P^m\mathbb{Z}_p$, we check the following distribution relation for $\sigma_\omega \in \mathcal{M}_N(\mathbf{x}, A)$:

$$\sigma_\omega(a + P^m\mathbb{Z}_p) = \sum_{t=0}^{P-1} \sigma_\omega(a + P^m t + P^{m+1}\mathbb{Z}_p).$$

The right hand side is equal to

$$\sum_{t=0}^{P-1} \omega\left(v\left(\frac{a/P^m + t}{P} \right) \right) = \omega | T(P) \left(v\left(\frac{a}{P^m} \right) \right).$$

From the action of the Hecke operator $T(P)$ given in (3.3), we know that

$$\omega | T(P) = \omega.$$

Hence we conclude that σ_ω is a distribution on $\mathbb{Z}_p$. Observe that the number of homology classes $[v(r)]$, $r \in \mathbb{R}/\mathbb{Z}$ is finite and therefore $|\mu(a + P^m\mathbb{Z}_p)|_p$ is bounded. In conclusion, we show that σ_ω is a measure on $\mathbb{Z}_p$. □

For the convenience of calculations in the next proposition, we set

$$R_{k,\lambda}(q) := R_\lambda(q) - (-1)^k R_{(-1)^*\lambda}(q).$$

Proposition 4.3. *Let P be a p-power with $P \equiv 1 \pmod N$. Then we obtain*

$$L(1-k, \lambda) = -i^k N^{k-1} \int_0^\infty R_{k,\widehat{\lambda}}(e^{-2\pi y}) y^{k-1} dy. \tag{4.1}$$

$$\mu_{k,\lambda}(a + P^m\mathbb{Z}_p) = k i^k (NP^m)^{k-1} \int_0^\infty R_{k,\widehat{\lambda}\zeta_{P^m}^{-Na}}(e^{-2\pi y}) y^{k-1} dy. \tag{4.2}$$

Proof. From the functional equation of $L(s, \lambda)$, we have

$$L(1-s, \lambda) = \frac{1}{(e^{-2\pi i s} - 1)\Gamma(1-s)} \left(\frac{2\pi i}{N} \right)^{1-s} \times \left(L(s, \widehat{\lambda}) - (-1)^{-s} L(s, (-1)^*\widehat{\lambda}) \right).$$

Since $\lim_{s \to 1-k} (e^{2\pi i s} - 1)\Gamma(s) = \frac{(-1)^{k-1} 2\pi i}{(k-1)!}$, we have the formula (4.1) from the integral representation (2.12) of $L(s, \lambda)$.

Now we verify the formula (4.2). Regarding $\widehat{\lambda}$ as a periodic function with a period NP^m, we have

$$R_{\widehat{\lambda}}(\zeta_{P^m}^a q) = \frac{\sum_{r=0}^{NP^m - 1} \widehat{\lambda}(r) \zeta_{P^m}^{ar} q^r}{1 - q^{NP^m}}.$$

Since we have the expressions $R_{\widehat{\lambda}}(\zeta_{P^m}^a q) - R_{(-1)^*\widehat{\lambda}}(\zeta_{P^m}^a q) = R_{k,\lambda'}(q)$ for the periodic function $\lambda' = \widehat{\lambda}\zeta_{P^m}^a$ with a period NP^m and $\widehat{\widehat{\lambda'}} = NP^m(-1)^*\lambda'$, the formulae (2.14) and (4.1) imply that

$$i^k(NP^m)^k \int_0^\infty R_{k,(-1)^*\lambda'}(q)y^{k-1}dy = \frac{(NP^m)^{k-1}}{k} \sum_{r=0}^{NP^m-1} \widehat{\lambda'}(r) B_k\left(\frac{r}{NP^m}\right),$$

where $\lambda'(r) = \widehat{\lambda}(r)\zeta_{P^m}^{ar}$. Observe that

$$\begin{aligned}\widehat{\lambda'}(r) &= \sum_{s=0}^{N-1}\sum_{q=0}^{P^m-1} \widehat{\lambda}(Nq+s)\zeta_{P^m}^{-a(Nq+s)}\zeta_{P^m}^{qr}\zeta_{NP^m}^{sr} \\ &= \begin{cases} P^m \sum_{s=0}^{N-1} \widehat{\lambda}(s)\zeta_N^{\frac{r-Na}{P^m}s} = P^m\widehat{\widehat{\lambda}}\left(\frac{r-Na}{P^m}\right) & \text{if } r \equiv Na \,(\bmod P^m) \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

Since $\widehat{\widehat{\lambda}}\left(\frac{r-Na}{P^m}\right) = \widehat{\widehat{\lambda}}(r) = N(-1)^*\lambda(r)$, we obtain

$$-i^k \int_0^\infty R_{k,(-1)^*\lambda'}(q)y^{k-1}dy = -\frac{1}{k}\sum_{r\equiv Na(P^m)} (-1)^*\lambda(r)B_k\left(\frac{r}{NP^m}\right)$$

and the desired formula (4.2). □

Corollary 4.4. *Let $q = e^{2\pi i z}$ for $z \in \mathbb{C}$ and $\widehat{\lambda}(0) = 0$. We have*

$$L(1-k,\lambda) = N^{k-1}\int_{v(0)} R_{\widehat{\lambda}}(q)z^{k-1}dz. \tag{4.3}$$

$$\mu_{1,\lambda}(a + P^m\mathbb{Z}_p) = \omega(R_{\widehat{\lambda}})\left(v\left(\frac{\overline{N}a}{P^m}\right)\right) = \int_{v\left(\frac{\overline{N}a}{P^m}\right)} R_{\widehat{\lambda}}(q)dz. \tag{4.4}$$

We set $\mathcal{B}_N(A) = \{\mu_{1,\lambda} | \, \lambda \,$:A-valued and periodic$\}$. Observe that we have $\mathcal{B}_N(A) \simeq \mathfrak{L}(\mathbb{Z}/N\mathbb{Z}, A)$. We also set

$$\mathcal{B}_N^0(A) = \left\{\mu_{1,\lambda} \,\middle|\, \widehat{\lambda}(0) = 0\right\} \subset \mathfrak{m}(\mathbb{Z}_p, A).$$

From (2.14) and (4.3), we have

$$\mathcal{B}_N^0(A) = \mathcal{M}_N(\mathbf{x}_0, A),$$

where $\mathbf{x}_0 = (\frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N})$. Let $\mathcal{C}(\mathbb{Z}_p, A)$ be the space of A-valued continuous functions on $\mathbb{Z}_p$. We define

$$\mathcal{R}_N(\mathbf{x}, A) = \mathcal{C}(\mathbb{Z}_p, A) \otimes \mathcal{M}_N(\mathbf{x}, A).$$

We have an analogue of the Sinnott's result ([14, Theorem 1]).

Theorem 4.5. *For $\sigma \in \mathcal{R}_N(\mathbf{x}, A)$, we have $\mu(\Gamma(s,\sigma)) = \mu(\sigma + \sigma \circ -1|_{\mathbb{Z}_p^\times})$.*

Proof. We can represent σ as $d\sigma(x) = \sum_j f_j(x)d\sigma_{\omega_j}(x)$ for $f_j \in \mathcal{C}(\mathbb{Z}_p, A)$ and $\sigma_{\omega_j} \in \mathcal{M}_N(\mathbf{x}, A)$. Observe that $d\sigma(-x) = -\sum_j f_j(-x)d\sigma_{(-1)^*\omega_j}(x)$. From the definition $\Gamma(s,\sigma) = \int_{1+p\mathbb{Z}_p} x^{-s}d\widetilde{\sigma}$, the formula $\mu(\Gamma(s,\sigma)) = v_p(\pi)e$ implies that $\widetilde{\sigma} \equiv 0 \,(\bmod \pi^e)$ on $1 + p\mathbb{Z}_p$. We choose a locally constant $f_{j,e}$ such that $f_j(x) \equiv$

$f_{j,e}(x) \pmod{\pi^e}$ for all $x \in \mathbb{Z}_p^\times$. Let K be a sufficiently small open subset of $1+p\mathbb{Z}_p$ such that $f_{j,e}(\eta K) = \{c_{\eta,j}\}$ for some $c_{\eta,j} \in A$. For $a + P^m\mathbb{Z}_p \subseteq K$, we have

$$\widetilde{\sigma}(a + P^m\mathbb{Z}_p) = \sum_\eta \sum_j c_{\eta,j}\omega_j \left(v\left(\frac{a\eta}{P^m}\right)\right) \equiv 0 \pmod{\pi^e}.$$

Since $\omega_j \in \mathcal{R}_N(\mathbf{x}, A)$, by Corollary 3.4 we conclude that

$$\omega_\eta := \sum_j c_{\eta,j}\omega_j - c_{-\eta,j}(-1)^*\omega_j \equiv 0 \pmod{\pi^e}$$

for each $\eta \in V/\{\pm 1\}$. Observe that σ_{ω_η} is a measure on $\mathbb{Z}_p$ and that $\sigma + \sigma \circ -1|_{\eta K} \equiv \sigma_{\omega_\eta}|_{\eta K} \equiv 0 \pmod{\pi^e}$. Since η and K are arbitrary, we obtain that

$$\sigma + \sigma \circ -1|_{\mathbb{Z}_p^\times} \equiv 0 \pmod{\pi^e}.$$

Therefore we verify that

$$\mu(\Gamma(s,\sigma)) \leq \mu(\sigma + \sigma \circ -1|_{\mathbb{Z}_p^\times}).$$

Conversely, suppose that we have $\mu(\sigma + \sigma \circ -1|_{\mathbb{Z}_p^\times}) = v_p(\pi)e$ i.e. $\sigma + \sigma \circ -1 \equiv 0 \pmod{\pi^e}$ on $\mathbb{Z}_p^\times$. Clearly this implies $\widetilde{\sigma} \equiv 0 \pmod{\pi^e}$ and hence we have $\mu(\Gamma(s,\sigma)) \geq \mu(\sigma + \sigma \circ -1|_{\mathbb{Z}_p^\times})$. We conclude the theorem. ☐

The rational function $R_{\widehat{\lambda},\alpha}(q) := R_{\widehat{\lambda}}(q) - R_{\widehat{\lambda}}(q^\alpha)$ gives us a cohomology class

$$\omega_{\widehat{\lambda},\alpha} = \omega(R_{\widehat{\lambda},\alpha}) = R_{\widehat{\lambda},\alpha}(e^{2\pi i z})dz.$$

Observe that $R_{\lambda,\alpha}(q) = R_{\lambda_\alpha}(q)$ where λ_α is a periodic function with period αN defined by $\lambda_\alpha(r) = \lambda(r)$ if $\alpha \nmid r$ and $\lambda_\alpha(r) = 0$ otherwise. We also have the expression

$$\mu_{1,\lambda,\alpha}(a + P^m\mathbb{Z}_p) = \int_{v\left(\frac{a}{P^m}\right)} \omega_{\widehat{\lambda},\alpha} = \sigma_{\omega_{\widehat{\lambda},\alpha}}(a + P^m\mathbb{Z}_p).$$

In addition we are also able to show that $\mu_{1,\lambda,\alpha} \in \mathcal{B}^0_{\alpha N}(\mathbb{Z}_p[\lambda])$ as follows.

Proposition 4.6. *$\omega_{\widehat{\lambda},\alpha}$ is a $\mathbb{Z}_p[\lambda]$-valued modular symbol on $T_{\alpha N}^{S-S_0}$. Let $\alpha > N$. We have $\omega_{\widehat{\lambda},\alpha} \equiv 0 \pmod{\pi^e}$ if and only if $\lambda(r) \equiv 0 \pmod{\pi^e}$. Hence, for any open subset K of $\mathbb{Z}_p$ we have $\sigma_{\omega_{\widehat{\lambda},\alpha}}|_K \equiv 0 \pmod{\pi^e}$ if and only if $\lambda(r) \equiv 0 \pmod{\pi^e}$.*

Proof. Note that we have

$$\omega(R_{\widehat{\lambda},\alpha})(c_r) = \frac{1}{\alpha N}\sum_{s=1}^{\alpha N} \widehat{\lambda}_\alpha(s)\zeta_{\alpha N}^{rs} = \frac{1}{\alpha N}\sum_{s=1}^{\alpha N} \widehat{\lambda}(s)\zeta_{\alpha N}^{rs} - \frac{1}{\alpha N}\sum_{s=1}^{N} \alpha^*\widehat{\lambda}(s)\zeta_{N}^{rs}$$

$$(4.5) \qquad = \begin{cases} -\alpha^{-1}\overline{\alpha}^*\lambda(r) & \text{if } \alpha \nmid r \\ \left(1-\alpha^{-1}\right)\overline{\alpha}^*\lambda(r) & \text{if } \alpha \mid r \end{cases}.$$

In both cases, the values of the integration are in $\mathbb{Z}_p[\lambda]$. Observe that we have

$$\int_{v(0)} R_{\widehat{\lambda},\alpha}(q)dz = \int_{v(0)} R_{\widehat{\lambda}_\alpha}(q)dz = L(0, \widehat{\widehat{\lambda}_\alpha}) = -\sum_{r=0}^{\alpha N-1} \widehat{\widehat{\lambda}_\alpha}(r) B_1\left(\frac{r}{\alpha N}\right).$$

Since $\widehat{\widehat{\lambda}_\alpha}(r)$ is given by (4.5) and $\sum_r \widehat{\widehat{\lambda}_\alpha}(r) = 0$, we obtain $L(0, \widehat{\widehat{\lambda}_\alpha}) \in \mathbb{Z}_p[\lambda]$ and verify that $\omega_{\widehat{\lambda},\alpha}$ is a $\mathbb{Z}_p[\lambda]$-modular symbol. Note that the previous calculations show that $\omega_{\widehat{\lambda},\alpha} \equiv 0 \pmod{\pi^e}$ if and only if $\lambda(r) \equiv 0 \pmod{\pi^e}$ for each $0 \le r < N$ since $\alpha > N$.

Now assume that $\sigma_{\omega_{\widehat{\lambda},\alpha}}|_K \equiv 0 \pmod{\pi^e}$. Observe that from Remark 3.1, $\omega_{\widehat{\lambda},\alpha}$ becomes a boundary symbol modulo π^e. Again by (4.5), we conclude $\lambda \equiv 0 \pmod{\pi^e}$. Conversely, the condition $\lambda \equiv 0 \pmod{\pi^e}$ implies that $\sigma_{\omega_{\widehat{\lambda},\alpha}} \equiv 0 \pmod{\pi^e}$ and therefore we conclude the proposition. □

Let Ψ be a periodic function with a period cN, where c is a p-power and $p \nmid N$. Recall that the map $\mathfrak{L}(\mathbb{Z}/N\mathbb{Z}, A) \to \mathcal{M}_{\alpha N}(\mathbf{x}_0, A)$ is given by $\lambda \mapsto \mu_{1,\lambda,\alpha}$ and $\sigma_{\Psi,\alpha}$ is the image of Ψ under the morphism

$$\mathfrak{L}(\mathbb{Z}/cN\mathbb{Z}, A) \subset \mathcal{C}(\mathbb{Z}_p, A) \otimes \mathfrak{L}(\mathbb{Z}/N\mathbb{Z}, A) \to \mathcal{R}_{\alpha N}(\mathbf{x}_0, A).$$

Theorem 4.7. *Suppose that $\alpha > N$, and $\alpha \equiv 1 \pmod{cp}$. Then we have*

$$\mu(\Gamma(s, \sigma_{\Psi,\alpha})) = \min_{p \nmid t} v_p(\Psi(t) + \Psi(-t)).$$

Proof. Set $\mu = \mu(\Gamma(s, \sigma_{\Psi,\alpha}))$ and $\mu' = \min_{p\nmid t} v_p(\Psi(t) + \Psi(-t))$. We will show that $\mu \le \mu'$ and $\mu \ge \mu'$.

Let ψ_u be the characteristic function of $u + c\mathbb{Z}_p$ for $u \in \mathbb{Z}_p/c\mathbb{Z}_p$ and Ψ_u be a function of period N defined by

$$\Psi_u(r) = \Psi(rc\overline{c} + uN\overline{N}),$$

where $\overline{c}$, $\overline{N}$ are the inverses of c, N with respect to the moduli N, c, respectively. We have the relation $\Psi(x) = \sum_u \psi_u(x \bmod c)\Psi_u(x \bmod N)$. And that we also have

$$\sigma_{\Psi,\alpha} = \sum_u \psi_u \otimes \mu_{1,\Psi_u,\alpha} \in \mathcal{R}_{\alpha N}(\mathbf{x}_0, A). \tag{4.6}$$

Let $\widehat{\Psi}_u$ be the Fourier transform of Ψ_u. Setting $\omega = \omega(R_{\widehat{\Psi}_u,\alpha})$, we have

$$\sigma_{\Psi,\alpha}|_{u+c\mathbb{Z}_p} = \mu_{1,\Psi_u,\alpha}|_{u+c\mathbb{Z}_p} = \sigma_\omega|_{u+c\mathbb{Z}_p}.$$

Since we have

$$\sigma_{\Psi,\alpha} \circ -1 = \sum_u \psi_{-u} \otimes \mu_{1,\Psi_u,\alpha} \circ -1 = \sum_u \psi_u \otimes \mu_{1,\Psi_{-u},\alpha} \circ -1,$$

we obtain $\sigma_{\Psi,\alpha} \circ -1|_{u+c\mathbb{Z}_p} = \sigma_{\omega'}|_{u+c\mathbb{Z}_p}$ for $\omega' = \omega(R_{(-1)^*\widehat{\Psi}_{-u},\alpha})$.

By Theorem 4.5, we have $\mu(\Gamma(s, \sigma_{\Psi,\alpha})) = \mu(\sigma_{\Psi,\alpha} + \sigma_{\Psi,\alpha} \circ -1|_{\mathbb{Z}_p^\times})$. Assume that $\sigma_{\Psi,\alpha} + \sigma_{\Psi,\alpha} \circ -1 \equiv 0 \pmod{\pi^e}$ on $\mathbb{Z}_p^\times$. From this we get

$$\sigma_{\Psi,\alpha} + \sigma_{\Psi,\alpha} \circ -1|_{u+c\mathbb{Z}_p} = \sigma_\omega + \sigma_{\omega'}|_{u+c\mathbb{Z}_p} = \sigma_{\omega''}|_{u+c\mathbb{Z}_p} \equiv 0 \pmod{\pi^e}$$

for $\omega'' = \omega(R_{\widehat{\Psi}_u + (-1)^*\widehat{\Psi}_{-u},\alpha})$ for each $u \in \mathbb{Z}_p^\times$. From Proposition 4.6, this is equivalent to the congruence

$$\widehat{\Psi}_u + (-1)^*\widehat{\Psi}_{-u} \equiv 0 \pmod{\pi^e}.$$

In sum, by taking the Fourier transform to the both side of above congruence we conclude that $\Psi(rc\overline{c} + uN\overline{N}) + \Psi(-rc\overline{c} - uN\overline{N}) \equiv 0 \pmod{\pi^e}$ for each r, u and we verify that $\mu \le \mu'$.

Conversely, suppose that $dv_p(\pi) = \mu'$ i.e. $\Psi(t) + \Psi(-t) \equiv 0 \pmod{\pi^d}$ for all $p \nmid t$. Tracing back the above procedure, we obtain that $\sigma_{\Psi,\alpha} + \sigma_{\Psi,\alpha} \circ -1 \equiv 0 \pmod{\pi^d}$ on $\mathbb{Z}_p^\times$. We conclude that $\mu' \le \mu$. This verifies the theorem. □

Now let A be the ring of integers of $\mathbb{Q}_p(\Psi)$. There is a power series $H_{\Psi,\alpha} \in A[[T-1]]$ so that

$$\Gamma(s, \sigma_{\Psi,\alpha}) = H_{\Psi,\alpha}(\gamma^{-s}).$$

From Proposition 4.1, we obtain that

$$H_{\Psi,\alpha}(\gamma^{-1}) = \log_p(\alpha)\mathrm{Res}_{s=1}L_p(s,\Psi).$$

We set $\alpha = \gamma^d$ for some positive integer d and $K_\alpha(T) = \alpha^{-1}T^d - 1$. From Proposition 4.1, we have

$$L_p(s,\Psi) = \frac{H_{\Psi,\alpha}(\gamma^{-s})}{K_\alpha(\gamma^{-s})}.$$

The natural question would be when $H_{\Psi,\alpha}(T)K_\alpha(T)^{-1}$ becomes a power series in $A[[T-1]]$.

Proposition 4.8. *$H_{\Psi,\alpha}(T)K_\alpha(T)^{-1}$ is a power series in $A[[T-1]]$ if and only if $L_p(s,\Psi\varpi)$ is p-adically analytic for all $\varpi : (\mathbb{Z}/c\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ with $\varpi(V) = \{1\}$.*

Proof. We may assume that $K_\alpha(T) = \alpha^{-1}T^c - 1$. Then $H_{\Psi,\alpha}(T)K_\alpha(T)^{-1} \in A[[T-1]]$ if and only if $H_{\Psi,\alpha}(\gamma^{-1}\zeta_c^q) = 0$ for $0 \le q < c-1$. From (4.6), we have the representation

$$H_{\Psi,\alpha}(T) = \sum_{u \in (\mathbb{Z}/c\mathbb{Z})^\times} T^{\frac{\log_p(\langle u \rangle)}{\log_p(\gamma)}} P_u(T),$$

where

$$P_u(T) = \int_{\mathbb{Z}_p} T^{cx} d\mu_{1,\Psi_u,\alpha}(\alpha^x u).$$

Observe that from Corollary 2.3 we have

$$P_u(\gamma^{-1}\zeta_c^q) = P_u(\gamma^{-1}) = \frac{\langle u \rangle \log_p(\alpha)}{cN} \sum_{s \equiv u(p)} \Psi(u).$$

Therefore we have

$$H_{\Psi,\alpha}(\gamma^{-1}\zeta_c^q) = \frac{\log_p(\alpha)}{cN} \sum_{\substack{s=1 \\ p \nmid s}}^{cN} \Psi\varpi_0^q(s),$$

where $\varpi_0(u) = \zeta_c^{\frac{\log_p(\langle u \rangle)}{\log_p(\gamma)}}$. This calculation enables us to verify the proposition. □

From Remark 2.2 we need to assume that $\chi(-1) = 1$. For all primitive Dirichlet character χ with conductor cN and $\chi(-1) = 1$, we have $\chi\varpi(-1) = 1$ and $L_p(s,\chi\varpi)$ is p-adically analytic for all ϖ by Corollary 2.5. Hence we denote the μ-invariant of $L_p(s,\chi)$ by the μ-invariant of the power series $H_{\chi,\alpha}(T)K_\alpha(T)^{-1}$.

Corollary 4.9 ([1])**.** *The Iwasawa μ-invariant of an abelian number field vanishes.*

Proof. Let χ be a Dirichlet character with $\chi(-1) = 1$ and θ be the first factor of χ in the sense of [9]. Set $\gamma = 1 + p$ if $p > 2$ and $\gamma = 5$ if $p = 2$, and $\zeta = \chi(\gamma)$. There exists a power series $f(T;\theta) \in \mathfrak{o}_\theta[[T-1]]$ such that

$$2f(\zeta\gamma^{1-n};\theta) = -L_p(1-n,\chi),$$

where $\mathfrak{o}_\theta$ is the ring of integers of $\mathbb{Q}(\theta)$. The corollary follows from Theorem 4.7 since we have

$$\mu(2f(T;\theta)) = \mu(H_{\chi,\alpha}K_\alpha^{-1}) = \mu(H_{\chi,\alpha}) = \min_{p \nmid t}(v_p(2\chi(t))).$$

Hence $\mu(f(T;\theta)) = 0$ and we conclude the corollary. $\square$

5. Non-vanishing of periodic zeta functions

In this section, we generalize the discussion in [16] in terms of abelian modular symbols. We choose another prime ℓ that is different from 2 and p. For a Dirichlet character χ of conductor C, let A be a $\mathbb{Z}[\chi]$-algebra. We define a special L-value of an A-valued modular symbol ω by

$$L(\omega) = \omega(v(0))$$

and a special L-value twisted by χ by

$$L(\omega, \chi) = \omega\left(\sum_{r=0}^{C-1} \chi(r) v\left(\frac{r}{C}\right)\right) = \sum_r \chi(r)\omega\left(v\left(\frac{r}{C}\right)\right).$$

Observe that if $\omega = \omega(R_\lambda)$, from the formula (4.1) we have

$$L(\omega) = L(0, \widehat{\lambda}) \in \mathbb{Z}_p[\widehat{\lambda}]$$

and

$$L(\omega, \chi) = L(\omega(R_{\lambda\chi^{-1}})) = G(\chi)L(0, \widehat{\lambda}\chi) \in \mathbb{Z}_p[\chi, \widehat{\lambda}].$$

For a fixed positive integer m, $a \in \mathbb{Z}_p^\times$ and a modular symbol ω, we define a modular symbol $\omega_{a,m}$ on $T^{S-S_0}_{\ell^m N}$ such that

$$\omega_{a,m} := \sum_{t=0}^{\ell^m-1} \zeta_m^t \omega \Big| \begin{pmatrix} 1 & \frac{at}{\ell^m} \\ 0 & 1 \end{pmatrix}.$$

Here we regard ω as a modular symbol on $T^{S-S_0}_{\ell^m N}$. For each $b \in (\mathbb{Z}/\ell\mathbb{Z})^\times$, we set

$$\omega_b = \omega_{\overline{b},1} = \sum_{t=0}^{\ell-1} \zeta_\ell^{bt} \omega \Big| \begin{pmatrix} 1 & \frac{t}{\ell} \\ 0 & 1 \end{pmatrix}.$$

Note that we have $\omega(R_\lambda)_b = \omega\left(\sum_t \zeta_\ell^{bt} R_\lambda(\zeta_\ell^t q)\right)$.

Let $\mathcal{B}$ be a prime over p in $\mathbb{Q}_p(\mu_{\ell^\infty})$. For a character $\kappa : \mu_{\ell-1} \to \overline{\mathbb{Q}}^\times$, we define $\Xi_\kappa = \left\{\chi : \mathbb{Z}_\ell^\times \to \overline{\mathbb{Q}}^\times \,\middle|\, \chi|_{\mu_{\ell-1}} = \kappa\right\}$ and $\epsilon = \chi(-1)$. We have an analogue of Sinnott's result([15]).

Theorem 5.1. *Let ω be a $\overline{\mathbb{Z}}_p$-valued abelian modular symbol and $e > 0$ be an integer. Then the symbol $\omega_b - \epsilon(-1)^*\omega_{-b}$ is a boundary symbol modulo $\mathcal{B}^e$ for each $b \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ if and only if $L(\omega, \chi) \equiv 0 \pmod{\mathcal{B}^e}$ for infinitely many $\chi \in \Xi_\kappa$.*

Proof. First let χ be a Dirichlet character with the conductor ℓ^n, $n \geq 2$ and $\chi(-1) = \epsilon$. From the definition, we have

$$2\ell L(\omega, \chi) = \ell \sum_r \chi(r)\omega\left(v\left(\frac{r}{\ell^n}\right)\right) + \ell \sum_r \chi(r)\chi(-1)(-1)^*\omega\left(-v\left(\frac{r}{\ell^n}\right)\right). \tag{5.1}$$

Set $\zeta_\ell^t = \chi(1 + t\ell^{n-1})$. The first term can be written as

$$\begin{aligned} \ell \sum_r \chi(r)\omega\left(v\left(\frac{r}{\ell^n}\right)\right) &= \sum_{t=0}^{\ell-1}\sum_r \chi(r + t\ell^{n-1})\omega\left(v\left(\frac{r}{\ell^n} + \frac{t}{\ell}\right)\right) \\ &= \sum_r \chi(r) \sum_t \zeta_\ell^{\overline{r}t}\omega\left(v\left(\frac{r}{\ell^n} + \frac{t}{\ell}\right)\right) = \sum_r \chi(r)\omega_{\overline{r}}\left(v\left(\frac{r}{\ell^n}\right)\right). \end{aligned}$$

Recall that $v(\frac{r}{\ell^n}) = v(0) + w_r$ for some $w_r \in H_1(\overline{T}_N, \mathbb{Z})$. Arranging the last term, the first term in (5.1) is equal to

$$\sum_r \chi(r)\omega_{\overline{r}}(v(0)) + \sum_r \chi(r)\omega_{\overline{r}}(w_r).$$

Observe that since $\omega_{\overline{r}}$ only depends on $r \pmod{\ell}$, we have

$$\sum_r \chi(r)\omega_{\overline{r}}(v(0)) = 0.$$

Combined with similar calculations for the second term in (5.1), we obtain the formula

$$2\ell L(\omega, \chi) = \sum_r \chi(r)(\omega_{\overline{r}} - \epsilon(-1)^* \omega_{-\overline{r}})(w_r),$$

which is congruent to 0 modulo $\mathcal{B}^e$ by the hypothesis.

Conversely, assume that $L(\omega, \chi) \equiv 0 \pmod{\mathcal{B}^e}$ for infinitely many $\chi \in \Xi_\kappa$. Let k_0 be the finite extension of $\mathbb{Q}_p$ adjoining the values $\omega(u)$ for $u \in H_1(T_N^S, \partial T_N^{S_0}, \mathbb{Z})$, μ_ℓ, and $\kappa(\mu_{\ell-1})$. Set $k_\infty = k_0(\mu_{\ell^\infty})$ and $k_n = k(\mu_{\ell^n})$. The extension k_∞/k_0 is unramified at $\mathcal{B} \cap k_0$. Let H be the decomposition group of $\mathcal{B}$ and $k = k_\infty^H$. Then for all sufficiently large n, say for all $n > m_0$, we have $k_{n+1} \neq k_n$ and $\mathcal{B}$ is inert in k_∞/k_n.

Choose a Dirichlet character χ with the conductor ℓ^n, $n > 2m_0$. Note that we have $\tau L(\omega, \chi) = L(\omega, \chi^\tau)$. Taking trace $\mathrm{Tr} = \mathrm{Tr}_{k_n/k_{n-m}}$ after multiplying $\chi(a^{-1})$ for an $a \in 1 + \ell\mathbb{Z}_\ell$, we have

$$\mathrm{Tr}(\chi(a^{-1})L(\omega, \chi)) \equiv 0 \pmod{\mathcal{B}^e}.$$

Observe that $\mathrm{Tr}(\chi(x)) = 0$ if $x \notin 1 + \ell^{n-m}\mathbb{Z}_\ell$ and $\mathrm{Tr}(\chi(x)) = [k_n : k_{n-m}]\chi(x)$ otherwise. From the definition, we have

$$\sum_{\eta \in \mu_{\ell-1}} \chi(\eta) \sum_{r \in \frac{1+\ell^{n-m}\mathbb{Z}}{1+\ell^n\mathbb{Z}}} \chi(r)\omega\left(v\left(\frac{ra\eta}{\ell^n}\right)\right) \equiv 0 \pmod{\mathcal{B}^e}.$$

Using the representative $\{1 + \ell^{n-m}t \,|\, 0 \le t < \ell^m - 1\}$ of $1 + \ell^{n-m}\mathbb{Z}/1 + \ell^n\mathbb{Z}$, we have

$$\sum_\eta \chi(\eta) \sum_t \zeta_{\ell^m}^t \omega\Big|\begin{pmatrix} 1 & \frac{ta\eta}{\ell^m} \\ 0 & 1 \end{pmatrix}\left(v\left(\frac{a\eta}{\ell^n}\right)\right) \equiv 0 \pmod{\mathcal{B}^e}$$

where $\chi(1 + \ell^{n-m}t) = \chi(1 + \ell^{n-m})^t = \zeta_{\ell^m}^t$. Arranging the terms, we have

$$\sum_{\eta \in \frac{\mu_{\ell-1}}{\{\pm 1\}}} \chi(\eta)\left(\omega_{a\eta,m} - \epsilon(-1)^*\omega_{-a\eta,m}\right)\left(v\left(\frac{a\eta}{\ell^n}\right)\right) \equiv 0 \pmod{\mathcal{B}^e}.$$

By Corollary 3.4, we conclude that $\omega_{a\eta,m} - \epsilon(-1)^*\omega_{-a\eta,m}$ is a boundary symbol modulo $\mathcal{B}^r$ for each $a \in 1 + \ell\mathbb{Z}_\ell$ and η. The symbol $\omega_{a\eta,m}$ can be rewritten as

$$\omega_{a\eta,m} = \sum_t \zeta_{\ell^m}^{\overline{a}t}\omega\Big|\begin{pmatrix} 1 & \frac{t\eta}{\ell^m} \\ 0 & 1 \end{pmatrix}.$$

This implies that for $b \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ and $s \in \mathbb{Z}_\ell$ the following symbol

$$\frac{1}{\ell^{m-1}} \sum_{\substack{a=1 \\ a \equiv 1 (\ell)}}^{\ell^m} \zeta_{\ell^m}^{-\overline{a}s}(\omega_{ab\eta,m} - \epsilon(-1)^*\omega_{-ab\eta,m}) = \omega_b\Big|\begin{pmatrix} 1 & \frac{s\eta}{\ell^m} \\ 0 & 1 \end{pmatrix} - \epsilon(-1)^*\omega_{-b}\Big|\begin{pmatrix} 1 & \frac{s\eta}{\ell^m} \\ 0 & 1 \end{pmatrix}$$

is a boundary symbol modulo $\mathcal{B}^e$. Especially when we choose $s = 0$, we obtain the theorem. $\square$

As a corollary, we are able to deduce a variation of Washington's theorem.

Corollary 5.2 ([19]). *For a periodic λ, $L(0, \lambda\chi) \equiv 0 \,(\bmod \mathcal{B}^e)$ for infinitely many $\chi \in \Xi_\kappa$ if and only if $\lambda(n) - \epsilon\lambda(-n) \equiv 0 \,(\bmod \mathcal{B}^e)$ for each n.*

Proof. Recall that $L(0, \lambda\chi) = \frac{1}{NG(\chi)} L(\omega(R_{\widehat{\lambda}}), \chi^{-1})$. Observe that

$$\omega(R_{\widehat{\lambda}})_b - \epsilon(-1)^* \omega(R_{\widehat{\lambda}})_{-b} = \omega(R'),$$

where

$$R'(q) = \sum_{t=0}^{\ell-1} \zeta_\ell^{bt} R_{\widehat{\lambda}}(\zeta_\ell^t q) - \epsilon \zeta_\ell^{-bt} R_{\widehat{\lambda}}(\zeta_\ell^t q^{-1}) = \sum_{r \equiv -b(\ell)} \frac{(\widehat{\lambda}(r) - \epsilon\widehat{\lambda}(-r)) q^r}{1 - q^{\ell N}}.$$

From this, we have

$$\omega(R')(c_r) = \ell^{-1}(\lambda(-\overline{\ell} r) - \epsilon\lambda(\overline{\ell} r)) \zeta_N^{r\overline{\ell}(-b)_\ell} \zeta_{\ell N}^{r(-b)_\ell}$$

for each $0 \le r < N\ell$. From Theorem 5.1, we conclude the corollary. $\square$

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