Isospectral surfaces of genus two and three

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Abstract

We give examples of isospectral non-isometric surfaces of genus two and three with variable curvature, as well as hyperbolic orbifolds of genus two. We apply the first result to construct isospectral potentials on a Riemann surface of genus two.

1. *Introduction*

Much work has been done on the spectral determination of geometry since Kac [**15**] popularised the topic in the sixties. For a recent survey of this area, see Gordon [**13**]. However, certain details remain open: for the spectrum of the Laplacian acting on smooth functions, the problem that concerns us here is the construction of pairs of isospectral non-isometric surfaces of low genus. By a surface, we understand a compact two-dimensional Riemannian manifold without boundary, reserving the term Riemann surface for one with constant curvature -1 and referring to its metric as hyperbolic. The first examples of such pairs of Riemann surfaces, due to Vignéras [**19**], had rather high genera. Later, in [**17**], Sunada developed a technique for producing isospectral manifolds using certain group theoretic properties that, when applied to surfaces, gave examples of genus $g = 8n+17$ for $n = 0, 1, 2, \ldots$. Buser [**9**] analysed and developed Sunada's method to obtain examples of genus 5 and genera ≥ 7. Brooks and Tse [**6**] added two of the missing genera, 4 and 6, of Riemann surfaces and also constructed an example of such a pair of genus 3 with non-constant curvature.

In this paper we obtain three results for genus two: isospectral non-isometric surfaces; isospectral potentials on a Riemann surface and isospectral but non-isometric Riemannian orbifolds. We also obtain isospectral non-isometric surfaces of genus three, distinct from those of [**6**]. In all cases there are infinite families of examples. This, of course, still leaves open the question of whether there are isospectral non-isometric Riemann surfaces of genus two or three.

The paper is arranged as follows. In Section 2, we recall the necessary definitions for, and give a statement of, Sunada's theorem, for proofs and further details of which we refer to Buser [**9**]. In Section 3, we show how to construct surfaces to which Sunada's theorem will apply and, in Section 4, we describe the groups that give rise to our specific examples of isospectral non-isometric surfaces. In Section 5, we give isometric hyperbolic metrics on the genus two surfaces from Section 4 and, hence, obtain isospectral potentials by applying Sunada's theorem to Schrödinger operators. Finally, in Section 6, we describe isospectral non-isometric orbisurfaces of genus two that have the hyperbolic metric except at their singularities. Throughout the paper manifolds and, in particular, surfaces will be compact and without boundary.

2. *Sunada's theorem*

In order to state Sunada's theorem, we require the following group theoretic concepts.

DEFINITION 2·1. *Let T be a finite group, and let U_1 and U_2 be subgroups of T. Then*

(*i*) *U_1 and U_2 are almost conjugate or Gassmann equivalent if for all $g \in T$,*

$$\big|[g] \cap U_1\big| = \big|[g] \cap U_2\big|,$$

where $[g]$ denotes the conjugacy class of g in T, and $|X|$ the number of elements of the set X;

(*ii*) *(T, U_1, U_2) is called a Sunada triple if, in addition to (i), U_1 and U_2 are non-conjugate in T. We refer to the index of the subgroups U_i in T as the index of the Sunada triple.*

The following version of Sunada's theorem may be found in [**9**].

THEOREM 2·2 (Sunada). *If T is a finite group of isometries of a compact Riemannian manifold M with Gassmann equivalent subgroups U_1 and U_2 that act freely on M, then the quotient manifolds $M_i = U_i \backslash M$, when given the quotient metrics, are isospectral for the Laplacian acting on smooth functions.*

We require that U_1 and U_2 not be conjugate in T since, if they were, then M_1 and M_2 would be isometric. Unfortunately non-conjugacy is not a sufficient condition for non-isometry. Indeed our examples illustrate that insufficiency: we shall see that, if the covered manifolds are given metrics of constant curvature -1, making them Riemann surfaces, then they are isometric; a fact of which we make use in Section 5. However Sunada also showed that it is possible to ensure that M_1 and M_2 are not isometric, while still remaining isospectral, by giving them the metrics induced from a 'bumpy metric' on $M_0 = T \backslash M$, provided T itself acts freely on M so that M_0 is indeed a differential manifold. A 'bumpy metric' is one such that no two points have isometric neighbourhoods, and Sunada showed that, given any metric, there exists a bumpy metric which is arbitrarily close in the C^∞-topology to the given one. In our case T, necessarily for our purpose, will not act freely on the surfaces we consider. However all its isotropy groups will be (finite) cyclic so that M_0 will be a topological surface which may be imbued with a differential structure and a bumpy metric. The argument that non-isometry of the M_i then follows from the non-conjugacy of U_1 and U_2 may be applied to the complement of the images in the M_i of the (isolated) points in M with non-trivial isotropy groups.

3. *Isospectral surfaces*

In order to obtain the surfaces to which we shall apply Sunada's theorem using a given Sunada triple (T, U_1, U_2), we first construct a surface M on which T acts as a group of diffeomorphisms: when we choose a metric on M, these diffeomorphisms will be isometries. The non-trivial isotropy groups of this action will all be point stabilisers, which we can describe explicitly in order to ensure that U_1 and U_2 act freely.

3·1. *Construction of the surfaces*

The following well-known construction works for any number of generators but, since both our groups in Section 4 are two-generator groups, we describe only that case. The adaption to the three generator group that we use in section 6 is straightforward.

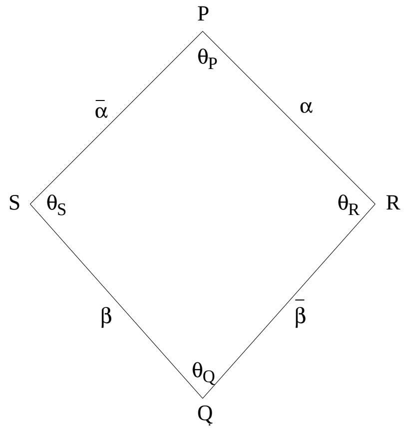


Fig. 1. The quadrangular domain $\mathcal{D}$

Given generators a, b of T, we choose a planar quadrangular domain $\mathcal{D}$, illustrated in Figure 1, with its boundary edges being given the induced orientation from that of the plane and α and $\overline{\alpha}$ having the same length, as do β and $\overline{\beta}$. We then take the disjoint union of copies $\mathcal{D}_g$ of $\mathcal{D}$, one for each element g of T, and for each $g \in T$ we identify the copy, α_g, of the edge α on $\mathcal{D}_g$, by an orientation reversing isometry, with the copy, $\overline{\alpha}_{ga}$, of the edge $\overline{\alpha}$ on $\mathcal{D}_{ga}$ and similarly identify the edge β_g with the edge $\overline{\beta}_{gb}$.

The left action of $h \in T$ on the resulting surface M is defined by identifying each copy, $\mathcal{D}_g$, of $\mathcal{D}$ in M with the copy $\mathcal{D}_{hg}$ in the natural way. This respects the identifications we have made and so is well-defined on M. Moreover T acts on M with only the vertices as potential fixed points.

In forming M, the vertex P_g in $\mathcal{D}_g$ is identified with $|a|-1$ other copies P_{ga}, P_{ga^2}, ... to form a vertex $\widetilde{P}_g$ in M, where $|a|$ denotes the order of a. Then $a^g := gag^{-1} \in T$ maps P_{ga^r} to $P_{ga^{r+1}}$ and so fixes $\widetilde{P}_g$ which, since $\widetilde{P}_{ga} = \widetilde{P}_g$, etc., is also fixed by each $(a^r)^g$. However no other elements of T can fix it.

Similarly, other copies P_h of P, where $h \neq ga^r$, are identified in cycles of length $|a|$ to form other vertices $\widetilde{P}_h$ in M with stabilisers generated by a^h. Analogously the copies Q_g of the vertex Q are identified in cycles of length $|b|$ to give vertices in M with stabilisers generated by conjugates of b. However, the vertex R_g of $\mathcal{D}_g$ is identified with S_{ga} in $\mathcal{D}_{ga}$ and then with R_{gab} in $\mathcal{D}_{gab}$ etc., so that it takes $|ab|$ copies of each of R and S to form a vertex in M with stabiliser generated by a conjugate of ab. The free left action of U_i on M identifies the copies of $\mathcal{D}$ forming M in sets of size $|U_i|$, one for each cost of U_i in T. Since, as noted above, this left action respects the identifications made in forming M, this has the effect that M_i may be constructed from $|T|/|U_i|$ copies of $\mathcal{D}$, labelled by the cosets $U_i g$ with corresponding edges in the different copies identified, as before, according to the right action of the chosen generators on these cosets: the edge α in the copy of $\mathcal{D}$ labelled by $U_i g$ is identified, by an orientation reversing isometry, with the copy of $\bar{\alpha}$ in the copy of $\mathcal{D}$ labelled by $U_i ga$, etc. See [**6**], [**8**] or [**9**] for full details.

Note that, if we wish M to remain smooth at the vertices when it inherits a Riemannian metric from the various copies of $\mathcal{D}$, the angle θ_P at P must be $2\pi/|a|$, θ_Q must be $2\pi/|b|$ and $\theta_R + \theta_S$ must be $2\pi/|ab|$. Such a quadrangle may not be realisable in the Euclidean plane, but those that we require will always be realisable in the hyperbolic plane and, for any metric of constant curvature, the symmetry will imply that $\theta_R = \theta_S$.

Since the vertices in copies of $\mathcal{D}$ are identified in sets of size $|a|$, $|b|$ and $|ab|$, respectively, and the quotient has $|T|$ faces and $2|T|$ edges, we see that

$$\chi(M) = |T| \left(\frac{1}{|a|} + \frac{1}{|b|} + \frac{1}{|ab|} - 1 \right).$$

Then, provided the Gassman subgroups U_i of the Sunada triple (T, U_1, U_2) avoid all the stabilisers in T described above, the U_i will act freely on M and so both quotients M_i will have Euler characteristic

$$\chi(M_i) = k \left(\frac{1}{|a|} + \frac{1}{|b|} + \frac{1}{|ab|} - 1 \right),$$

where k is the common index of the subgroups U_i in T.

4. *Examples of isospectral non-isometric surfaces*

4·1. *Isospectral surfaces of genus two*

In [**5**], Bosma and de Smit proved that, up to isomorphism, there are exactly 19 Sunada triples of index at most 15, and the group T that we consider here is the one they denote by $G_{2,2,3} \cap A_{12}$. Their notation reflects the fact that $G_{2,2,3}$ is one of the 3-step abelian groups $G_{p,m,k}$ constructed in [**11**].

In [**16**], the properties required of T and its subgroups were deduced, using the computer algebra package MAGMA, from two particular generators, while in [**12**] an account is given closely following the original description of T in [**11**]. Here instead we shall give an explicit construction of T directly as a subgroup of the alternating group on twelve symbols. This will have the advantage of making it possible for the reader to make an elementary check of our claims.

We start with the dihedral group D of order eight, realised as the subgroup of the permutation group on the set $\{0, 1, 2, 3\}$ generated by $\phi = (0\ 1\ 2\ 3)$ and $\rho = (1\ 3)$. We also name the elements $\sigma = \phi\rho = (0\ 1)(2\ 3)$, $\kappa = \phi^2 = (0\ 2)(1\ 3)$ and the identity ι, noting that κ is central.

We then consider, in the group of permutations on the twelve symbols $0, 1, \cdots, 11$, the subgroup L comprising those permutations that preserve the congruence classes modulo 3 and, on each class, act in the obvious way as a member of D; for example, κ acts on the class of symbols congruent to 1 modulo 3 as $(1\ 7)(4\ 10)$. We denote by $(\theta_0, \theta_1, \theta_2)$ the element of L that acts as $\theta_i \in D$ on the class of symbols congruent to i modulo 3.

Then the elements $a = (\kappa, \iota, \iota)$, $b = (\iota, \kappa, \iota)$ and $c = (\iota, \iota, \kappa)$ generate a subgroup K of L of exponent 2 and order 8 that, together with its cosets determined by $d = (\rho, \phi, \sigma)$, $e = (\sigma, \rho, \phi)$ and $f = (\phi, \sigma, \rho)$, forms a group H of order 32: it suffices to observe that, since κ is central in D, each of a, b and c commutes with d, e and f; that $d^2 = b$, $e^2 = c$, $f^2 = a$ and that, since $\sigma\phi\rho = \iota$ and $\rho\phi\sigma = \kappa$, def is the identity and $fed = abc$, which element we shall denote by z. Then the finite set $H = K \cup Kd \cup Ke \cup Kf$ can be seen to be closed under composition and so be a subgroup of L.

We now introduce the element $s = (0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)(9\ 10\ 11)$ that cycles the three conjugacy classes modulo 3. Since $d^s := sds^{-1} = e$, $e^s = f$ and $a = f^2$ etc., it follows that H is a subgroup of the group $T = \langle d, s\rangle$ generated by d and s. Indeed, since $H^s = H$, H is normal in T and so T is the union $H \cup Hs \cup Hs^2$ and has order 96. Note that, although K is central in H, only the element z, and of course the identity, remains central in T.

In order to identify suitable Gassmann equivalent subgroups of T, we need to know certain of its conjugacy classes. First note that H, being normal in T, must be a union of conjugacy classes. The elements of order 2 are all in K and form the classes $\{a, b, c\}$, $\{bc, ca, ab\}$ and $\{z\}$. The remaining elements of H are all of order 4 and, from the relations stated above, we find $d^e = f^{-1}abce^{-1} = zd$ and so $(cd)^e = cd^e = z(cd)$. These identities, together with their cyclically permuted versions and their inverses, give us the following four conjugacy classes in T of elements of order 4:

$$\begin{gathered} d, e, f, zd, ze, zf; \\ bd, ce, af, zbd, zce, zaf; \\ cd, ae, bf, zcd, zae, zbf; \\ ad, be, cf, zad, zbe, zcf. \end{gathered}$$

From these results it is clear that the groups $U_1 = \langle c, d\rangle$ and $U_2 = \langle b, e\rangle$ are Gassmann equivalent subgroups of T of order 8, each the direct product of cyclic groups of order 2 and order 4. They are not conjugate since both c and d, and hence all elements of U_1, fix 0 and 6, whereas b and e have no common fixed point.

To produce surfaces of genus 2, and so Euler characteristic -2, from our chosen quadrangular domain $\mathcal{D}$ and this Sunada triple of index 12, we choose generators of order 3 with a product of order 6 since $12(1/3 + 1/3 + 1/6 - 1) = -2$. In fact, $T = \langle s, ds\rangle$ and $(ds)^3 = dsds^2s^2ds = def = id$ and $(sds)^3 = sds^2ds^2ds = edf = z$ of order 2. In the resulting surfaces, the vertex stabilisers under the action of T are generated by conjugates of non-trivial powers of s, ds and sds. All these are of orders 3 or 6 and so not in U_1 or U_2, except possibly for $(sds)^3 = z$. However, this last is central and so not conjugate to any element of U_1 or U_2. Hence the Gassmann equivalent subgroups U_i act freely on M as required for Sunada's theorem, and so the resulting surfaces M_i will be isospectral. As explained in Section 2, when given the metrics induced from a bumpy metric on the surface that they both cover, they will not be isometric.

Since Sunada's result is that bumpy metrics are dense with respect to the C^∞-topology on the space of all metrics, we easily obtain results such as the following.

Theorem 4·1. *There is an infinite sequence of pairs of Riemannian metrics* $(g_{1,k}, g_{2,k})$, $k = 1, 2, \cdots$, *on the topological surface* N *of genus two, such that no two distinct surfaces* $(N, g_{i,k})$ *and* $(N, g_{j,l})$ *are isometric and no two are isospectral except that, for each* k, $(N, g_{1,k})$ *and* $(N, g_{2,k})$ *are isospectral. Each sequence* $g_{i,k}$, $k = 1, 2, \cdots$, *converges with respect to the* C^∞*-topology to the hyperbolic metric* g.

Proof. The surface N is that denoted M_1 or M_2 above and the hyperbolic metric g on N arises from that on $\mathcal{D}$ or, except at its singular vertices, its quotient M_0. Since N has genus two, with the hyperbolic metric it has area 4π. Then, with the conformal metric $(1+1/k)g$, it would have area $4(1+1/k)\pi$. So we choose the metrics $g_{1,k}$ and $g_{2,k}$

to be those induced from a bumpy metric on M_0 sufficiently close to $(1+1/k)g$ for the sequence $A_k = area(N, g_{1,k}) = area(N, g_{2,k})$ to be strictly decreasing as $k \to \infty$. Then, by the construction, $(N, g_{1,k})$ and $(N, g_{2,k})$ are, for each k, isospectral but not isometric. However, since the area is a spectral invariant, no other pair can be isospectral. □

Remark. As we shall see in Section 5, the surfaces constructed in this subsection are isometric whenever they are endowed with the hyperbolic metric. While this thwarts our aim for non-isometric Riemann surfaces, it will allow us to construct pairs of distinct isospectral potentials on the resulting unique Riemann surface.

4·2. *Isospectral surfaces of genus three*

Isospectral non-isometric surfaces of genus three have been constructed by Brooks and Tse [**6**] using the group $SL(3,\mathbb{Z}/2\mathbb{Z})$. Here we present further examples using the group $T := GL(2,\mathbb{Z}/4\mathbb{Z})$, which has order 96. Define subgroups

$$U_1 := \{x \in T \,|\, xe_1 = e_1\},$$
$$U_2 := \{x^\tau \,|\, x \in U_1\},$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and write x^τ for the transposed matrix of x. Then, to fix e_1, the first column of each matrix in U_1 must be e_1, so that, writing the elements of $\mathbb{Z}/4\mathbb{Z}$ as $0, \pm 1, 2$,

$$U_1 = \left\{ x = \begin{pmatrix} 1 & n \\ 0 & m \end{pmatrix} \middle| m = \pm 1, n \in \mathbb{Z}/4\mathbb{Z} \right\} = \langle u, w\rangle \qquad \text{and} \qquad U_2 = \langle u^\tau, w\rangle$$

where

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since u and u^τ are of order 4 while, writing u^w for the conjugate wuw^{-1} of u, $u^w = u^{-1}$ and $(u^\tau)^w = (u^\tau)^{-1}$, it follows that both U_1 and U_2 are dihedral groups of order 8 and so index 12 in T.

We shall see that U_1 and U_2 are Gassmann equivalent but non-conjugate and obtain generators a, b of T, both of order 4 and with product of order 6. Then, using the quadrangular domain $\mathcal{D}$ again, we shall obtain isospectral surfaces that, granted they are non-singular, have Euler characteristic $12(1/4+1/4+1/6-1) = -4$ and so are of genus 3 as required. Let

$$a := \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad c := ab = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix},$$

$$s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad v := \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $u^s = u^\tau$, $(wu)^v = (wu)^\tau$, $(wu^2)^c = (wu^2)^\tau$ and $(wu^3)^v = (wu^3)^\tau$ show directly that U_1 and U_2 are Gassmann equivalent. In order to see that they are not conjugate, one may observe that the elements of U_1 have a common eigenvector corresponding to the eigenvalue 1, whereas members of U_2 have no common eigenvector. Non-conjugacy may also be seen from the representation, which will be seen to be faithful, of T as permutations of the right cosets of U_2 that we shall use to show that the actions of U_1 and U_2 on the surfaces that we shall construct are free.

Note that b^2 is the central element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and c has order 6. Then, writing $-x$ for b^2x, the elements $\pm c^k$, $k = 0, 1, ..., 5$, form a group of order 12 isomorphic with $C_6 \times C_2$. None of these elements, other than the identity, lie in U_1 or U_2, so we may use them as coset representatives.

Denoting the coset U_2c^k by k and $-U_2c^k$ by $\overline{k}$, the permutation representation ρ_x of $x \in T$ on these cosets is defined by $U_2c^kx = U_2c^{\rho_x(k)}$. Note that each element of U_2c^k has its first row equal to that of c^k. So U_2c^ka is determined by the right action of a on the first row of c^k. The first rows of c^k, $k = 0, ..., 5$, are

$$(1\ 0),\ (1\ 1),\ (0\ -1),\ (1\ 2),\ (-1\ 1),\ (2\ 1),$$

respectively. Hence the permutation representations of a and b on the cosets of U_2 are, respectively

$$\rho_a = (0\ 3\ \overline{0}\ \overline{3})(1\ \overline{5}\ \overline{4}\ \overline{2})(\overline{1}\ 5\ 4\ 2)$$
$$\rho_b = (0\ \overline{4}\ \overline{0}\ 4)(1\ \overline{3}\ \overline{1}\ 3)(2\ 5\ \overline{2}\ \overline{5}),$$

confirming that that of $c = ab$ is

$$\rho_c = \rho_b \circ \rho_a = (0\ 1\ 2\ 3\ 4\ 5)(\overline{0}\ \overline{1}\ \overline{2}\ \overline{3}\ \overline{4}\ \overline{5}).$$

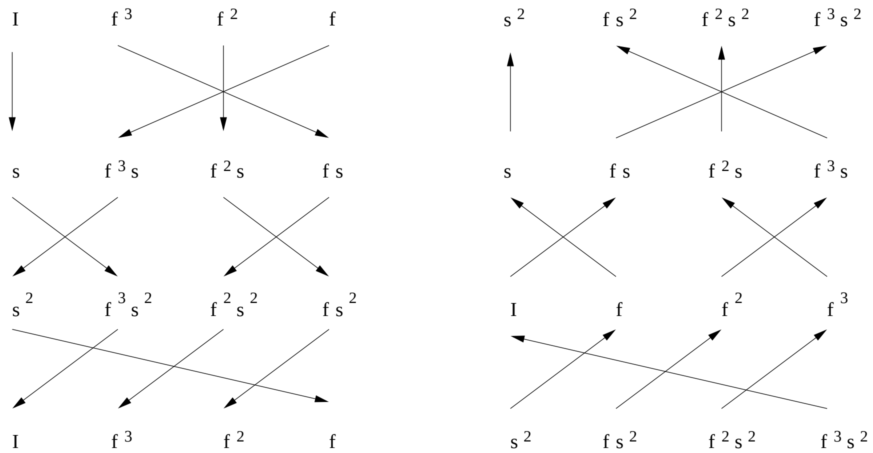
By the definition of the right action on the cosets, it is immediate that every element of U_2 must fix the identity coset, 0, and hence also $\overline{0}$. Then every element of U_1, since it is conjugate to some element of U_2, must also have fixed points. Recall that, using the quadrangular domain $\mathcal{D}$ of Figure 1 and the generators a, b of T to construct a surface M on which T acts, the only possible non-trivial stabilisers of points of M will be conjugates of non-trivial powers of a, b or c. However, by the above, none of these have fixed points so cannot lie in U_1 or U_2. Thus the subgroups U_1 and U_2 will act freely on M and Sunada's theorem will ensure that the resulting quotient surfaces M_i are isospectral. Once again, since U_1 and U_2 are not conjugate, when given the metrics induced from bumpy metrics on the surface M_0 that they both cover, the surfaces M_i will not be isometric.

The resulting metrics on M_i will not quite be bumpy since all the $|T|/|U_i|$ points of a fibre over a non-singular point in M_0 will have isometric neighbourhoods. In our case above these are sets of 12 points. In the Brooks-Tse example there are 7 points in each set with isometric neighbourhoods, so their surfaces cannot be isometric with ours.

Once again we have sequences of pairs of Riemannian metrics $(g_{1,k}, g_{2,k})$ on the topological surface of genus three with properties analogous to those of the metrics in Theorem 4·1.

5. *Isospectral potentials on surfaces of genus two*

We would have liked our surfaces in the previous section to be Riemann surfaces. Indeed we can make them so: however, then they are isometric. For the genus two case, we take our fundamental quadrangle to be hyperbolic with $\theta_P = \theta_Q = 2\pi/3$ and $\theta_R + \theta_S = \pi/3$. Such a quadrangle is easily shown to exist with $\theta_R = \theta_S = \pi/6$. To check whether the resulting Riemann surfaces are isometric we look at the Cayley-Schreier diagrams for the action of our generators s and ds on the cosets of U_1 and U_2. We can take coset representatives for both subgroups to be the twelve distinct elements f^rs^t. The



(a) The action on cosets of U_1 (b) The action on cosets of U_2

Fig. 2. The action of ds on cosets of U_1 and U_2

calculations are then straightforward: using the relations in Section 4.1, extract and replace d, respectively e, from the subgroups as appropriate. For example, $sd = es$, $de = f^{-1}$, $fe = zef$ and $z = abc$ give

$$(U_1s)ds = U_1es^2 = U_1des^2 = U_1f^3s^2;$$
$$\text{and } (U_2fs)ds = U_2fes^2 = U_2zefs^2 = U_2abcfs^2 = U_2f^3s^2,$$

which are as long as any calculation required in either case. We obtain the diagrams (a) and (b) in Figure 2 where the arrows indicate the action of ds on the various cosets: the action of s, which is not shown, cycles each column vertically down for the cosets of U_1 and vertically up for the cosets of U_2. The top row is repeated at the bottom in each case for clarity.

Note that there is a "colour preserving", orientation reversing isomorphism between the two Cayley-Schreier diagrams, that is, a bijection between the cosets of U_1 and those of U_2 under which the actions of s and ds on the former correspond to those of s^{-1} and $(ds)^{-1}$ on the latter. Since there is also an, orientation reversing, isometry of our fundamental domain interchanging the edges α and β with $\overline{\alpha}$ and $\overline{\beta}$ respectively, it follows that, if M_1 and M_2 are given the hyperbolic metrics inherited from $\mathcal{D}$, then there is an orientation reserving isometry between them.

Two smooth functions ϕ_1 and ϕ_2 on a smooth manifold M are said to be isospectral potentials if the spectrum of $\Delta + \phi_1$ is identical to that of $\Delta + \phi_2$. Guillemin and Kazdan [**14**] showed that for compact, negatively curved 2-manifolds with simple length spectrum, the spectrum of $\Delta + \phi$ uniquely determines the potential ϕ and later Croke and Sharafutdinov [**10**] showed that this is also true for such manifolds of any dimension. But it was pointed out by Brooks [**4**] that for a metric of constant negative curvature, the length spectrum is not necessarily simple and so the spectral rigidity of the potential does not necessarily hold for Riemann surfaces. Indeed, Brooks constructed isospectral potentials on a surface of genus three and we do the same for our surface of genus two.

Let, then, M_1 and M_2 be our Riemann surfaces of genus two and $\psi : M_1 \longrightarrow M_2$ be the orientation reversing isometry between them. If $M_0 = T\backslash M$ and $\pi_1 : M_1 \longrightarrow M_0$, $\pi_2 : M_2 \longrightarrow M_0$ are the natural projections, let ϕ be a function on M_0 such that for

$\phi_1 = \phi \circ \pi_1$ and $\phi_2 = \phi \circ \pi_2$, we have $\phi_1 \neq \phi_2 \circ \psi$. Clearly, uncountably many such functions exist. As proved by Brooks [**3**], Sunada's result may be extended to show that $\Delta + \phi_1$ on M_1 and $\Delta + \phi_2$ on M_2 are isospectral; thus ϕ_1 and $\phi_2 \circ \psi$ are distinct isospectral potentials on M_1:

THEOREM 5·1. *There is a Riemann surface of genus two carrying uncountably many pairs of distinct isospectral potentials.*

6. *Isospectral orbifolds of genus two*

If we relax our requirement for isospectral non-isometric Riemann surfaces to their being Riemannian orbifolds, then we can achieve examples of genus two. See [**18**] for relevant details of orbifolds and, for the extension of Sunada's result to this context see, for example, [**1**] and [**2**]. The Sunada triple below was first used for isospectral constructions by Sunada in [**17**] and next by Buser in [**7**].

We take the group T of order 32 which is the semi-direct product of the additive group of the ring $\mathbb{Z}/8\mathbb{Z}$ with its multiplicative group of units. Writing the elements of T as (m, n) where $m \in \{1, 3, 5, 7\}$ is a unit and $n \in \{0, 1, 2, 3, 4, 5, 6, 7\}$ is an arbitrary element in $\mathbb{Z}/8\mathbb{Z}$, we note that T has a convenient representation as the group of 2×2 upper triangular matrices of the form $\begin{pmatrix} 1 & n \\ 0 & m \end{pmatrix}$. From the general conjugacy relation

$$(m, n)^{(p,q)} = (m, p(n + q(m - 1))),$$

it may be checked that

$$U_1 = \{(1, 0), (3, 0), (5, 0), (7, 0)\}$$
$$\text{and } U_2 = \{(1, 0), (3, 4), (5, 4), (7, 0)\}$$

are Gassmann equivalent non-conjugate subgroups of T so that (T, U_1, U_2) is a Sunada triple of index 8: $(3, 0)^{(p,q)} = (3, 4)$ if and only if q is 2 or 6 whereas $(5, 0)^{(p,q)} = (5, 4)$ if and only if q is odd.

The product formula $(m, n)(p, q) = (mp, np + q)$ shows that $a = (3, 2)$, $b = (7, 1)$ and $c = (7, 2)$ are each of order 2 with product $abc = (3, 3)$ of order 4. Since $d := bc = (1, 1)$ generates the normal subgroup isomorphic with $\mathbb{Z}/8\mathbb{Z}$ and $(1, 1)^b = (1, 7) = (1, 1)^{-1}$, the group $\langle b, c\rangle$ is dihedral of order 16 and, as it does not contain $(3, 2)$, $\langle a, b, c\rangle$ is the entire group T. Thus we have the required data for the construction of isospectral surfaces except that, if we impose a hyperbolic metric, we shall have singular vertices.

The appropriate domain is a hexagon with successive edges $\alpha, \overline{\alpha}, \beta, \overline{\beta}, \gamma, \overline{\gamma}$ where, since these edges will correspond to generators of order 2, the vertex angles between $\alpha, \overline{\alpha}$, and $\beta, \overline{\beta}$, and $\gamma, \overline{\gamma}$ must be π. Thus in effect we have an arbitrary hyperbolic triangle as in Figure 3 with additional vertices P, Q, R at the centre of each edge that will, potentially, give rise to singular vertices. In order to exclude further singular vertices, since, analogously to the case for the quadrangular domain, the remaining isotropy groups are generated by powers of conjugates of abc, which has order 4, we require the vertex angles of the hyperbolic triangle to sum to $\pi/2$. Note however that, since a hyperbolic triangle is determined by its angles, this gives us a continuous two-parameter family of examples.

Since no power of d, other than $(1, 0)$, lies in U_1 or U_2, we may take these powers as coset representatives. Then, for each generator x, writing $U_i d^k x = U_i d^{x_i(k)}$, we note that

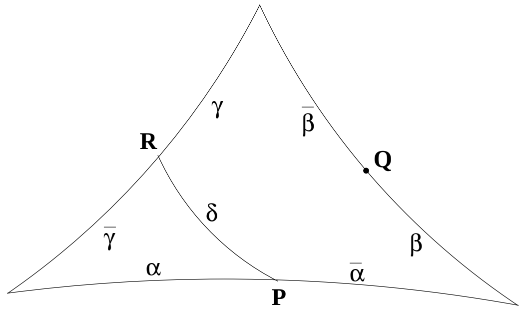
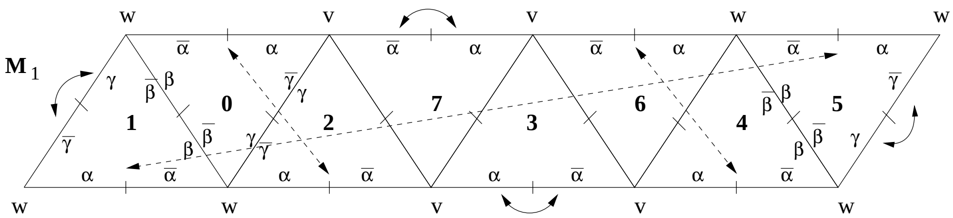


Fig. 3. The fundamental hyperbolic domain $\mathcal{D}$

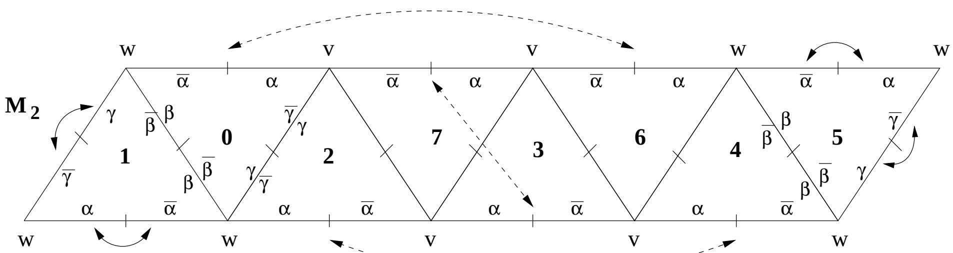
for $x = (m, n)$, if (m, p_i) is the unique element of U_i with m as its first coordinate, then $p_i \in \{0, 4\}$ and $x_i(k) = km + n - p_i$. Then the permutations x_i are:

$$\begin{aligned} a_1 &= (0\ 2)(1\ 5)(4\ 6)(3)(7) \\ a_2 &= (0\ 6)(1)(2\ 4)(3\ 7)(5) \\ b_1 = b_2 &= (0\ 1)(2\ 7)(3\ 6)(4\ 5) \\ c_1 = c_2 &= (0\ 2)(1)(3\ 7)(4\ 6)(5). \end{aligned}$$

Identifying the eight copies $\mathcal{D}_k$ of our fundamental domain accordingly we obtain the



(a) The orbisurface arising from U_1



(b) The orbisurface arising from U_2

Fig. 4. Isospectral non-isometric orbisurfaces of genus 2

orbifolds illustrated in Figure 4, where the hyperbolic domain $\mathcal{D}_k$ is mis-represented by a flat equilateral triangle and labelled simply k, with certain of the identifications of the edges having already been carried out. The remaining identifications are of two types: the dotted arrows indicate two pairs of edges from two different domains that should be identified; whereas each of the solid arrows indicates two edges from a single domain that should be identified, giving rise to a singularity of order two. Each of M_1 and M_2 has four such singularities, but the pair of singularities on the boundary of (the quotient of) $\mathcal{D}_1$ in M_2, as well as those on the boundary of $\mathcal{D}_5$, are closer together than any pair in M_1. If δ is the geodesic in $\mathcal{D}$ joining P to R, as illustrated in Figure 3, the distance between the singularities formed in M_2 from the identifications on the boundary of $\mathcal{D}_1$ will be $l(\delta)$, the length of δ. Similarly the pair formed by identifications on the boundary of $\mathcal{D}_5$ will also be $l(\delta)$ apart.

To see that the singularities in M_1 are further apart than this, note that the identifications lead to just two quotient vertices, other than the singular ones, in each of M_1 and M_2 and these are indicated by v and w in the diagrams. In M_2 all four singularities are adjacent to w, while in M_1 two are adjacent to v and two to w. In the latter surface, the two singularities p_1 and p_2 adjacent to v are joined by geodesics of length $2l(\delta)$ and $2l(\alpha)$. The former is the obvious one internal to the diagram. For the latter we note that, when the copy of $\overline{\alpha}$ in $\mathcal{D}_6$ is identified with the copy of α in $\mathcal{D}_4$, since the sum of the interior angles of $\mathcal{D}$ is $\pi/2$ and one of each occurs both in passing from α in $\mathcal{D}_7$ to $\overline{\alpha}$ in $\mathcal{D}_6$ and from α in $\mathcal{D}_4$ to $\overline{\alpha}$ in $\mathcal{D}_3$, α in $\mathcal{D}_7$ followed by $\overline{\alpha}$ in $\mathcal{D}_3$ form an unbroken geodesic from p_1 to p_2 of the stated length.

The singularities adjacent to w similarly have geodesics of length $2l(\delta)$ and $2l(\gamma)$ between them. It is clear from the diagram that there is no shorter geodesic joining the singularities adjacent to the same vertex and that those joining members of different pairs are longer still, as v and w are $2\min\{l(\alpha), l(\gamma)\}$ apart. Thus M_1 and M_2 cannot be isometric, certainly provided $l(\delta) < 2\min\{l(\alpha), l(\gamma)\}$.

Since M_i, which are topological surfaces of genus one, as can be seen from Figure 4, have four singularities of order two, the orbisurface Euler characteristic of each is

$$\chi(M_i) = 0 - \sum_{i=1}^{4}(1 - \frac{1}{2}) = -2$$

so that they are, as claimed, orbisurfaces of genus two. Note also that, since the area of the fundamental domain is $\pi - \pi/2$, the area of each orbifold is 4π, in accord with the Gauss-Bonnet theorem for orbifolds ([**18**]).

THEOREM 6·1. *There exists a two real parameter family of pairs of isospectral non-isometric hyperbolic orbisurfaces of genus two which are topological tori, each with four cone points of order* 2.

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