When are potentials optimally regular?

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Background

Potentials

Let U^D denote the potential of a (bounded) domain D in $\mathbb{R}^n$

$$U^D(x) = c_n \int_D |x - y|^{2-n} dy \qquad (n \geq 3)$$

and for $n = 2$ we have the logarithmic potential. Here c_n is a normalization factor. In general we shall consider smooth densities f and the weighted potential

$$U^{D,f}(x) = c_n \int_D \frac{f(y)dy}{|x - y|^{n-2}} dy$$

but for simplicity of notation we shall always write U for the potentials with density $f\chi_D$.

Background

Regularity of potentials

It is well known that a potential U satisfies (in the distributional sense)

$$\Delta U = f\chi_D.$$

In particular U has bounded Laplacian, and one obtains from well-known classical Schauder theory that $U \in C^{1,\alpha}$ for any $\alpha < 1$.
Note that U is not C^2 across the boundary of D.

Background

Regularity of potentials

Since we are interested in local regularity properties of U, we shall always consider U around a given boundary point $x^0 \in \partial D$, say in $B_r(x^0)$.

For simplicity we assume $x^0 = 0$, and $r = 1$. We thus henceforth will denote by U a solution to

$$\Delta U = f\chi_D \qquad \text{in } B_1(0),$$

regardless of how it behaves outside B_1.

Regularity of potentials

Motivations

In elliptic (parabolic) pde one is in general interested in optimal regularity of solutions to a given equation.

What is optimal regularity in general?

Regularity of potentials

Motivations

In free boundaries, and the theory of quadrature domains scaling is an effective tool. The scaling

$$U_r(x) = \frac{U(rx) - U(0) - rx \cdot \nabla U(0)}{r^2}$$

has "invariant" Laplacian

$$\Delta U_r(x) = (\Delta U)(rx) = f(rx)\chi_{D_r}$$

where D_r is a scaled version of D. This is part of the technique for local analysis of free boundaries.

Regularity of potentials

Motivations

In particular one needs U_r to be bounded, that is

$$\sup_{B_1} |U_r(x)| \leq C$$

which amounts to

$$\sup_{B_r} |U(x) - U(0) - x \cdot \nabla U(0)| \leq Cr^2.$$

In other words one would need $C^{1,1}$-regularity at the origin.

Regularity of potentials

Examples

Suppose ∂D is reasonably smooth to allow one-sided ($C^{1,1}$) solution to Cauchy problem

$$\Delta u = f \quad \text{in } D \cap B_1 \qquad u = \nabla u = 0 \quad \text{on } \partial D \cap B_1.$$

Now the function $w = U - u$ is h harmonic in B_1 and hence C^2. Therefore $U = w + u$ is $C^{1,1}$.
A similar situation applies if ∂D allows a solution to Cauchy problem from the exterior of D.

Regularity of potentials

Examples

Next we consider the case of D^c being very *thin* at the origin such that

$$|D^2 U^{D^c}(0)| \approx \int_{D^c} \frac{dx}{|x|^n} \leq C < \infty.$$

In particular U^{D^c} will be $C^{1,1}$ at the origin.

Now $w = U^D + U^{D^c}$ satisfy $\Delta w = f$ in B_1 and hence it is C^2. Therefore $U^D = w - U^{D^c}$ is $C^{1,1}$ at the origin.

We thus raise the question of finding reasonable conditions that make the potential U^D to be $C^{1,1}$ at the origin.

Regularity of potentials

General cases

Suppose

$$U(x) = h(x) \qquad \text{in } D^c \cap B_1$$

for some harmonic function h in B_1.

This would then imply that $u = U - h$ satisfies a one-sided Cauchy problem in D.

In case D^c is thick at the origin (a capacity density is enough) then one can deduce that u, and hence U is $C^{1,1}$ in $B_{1/2}$.

Other variants of assumptions exists, e.g. if $u \geq 0$ in B_1 then optimal smoothness follows easily.

Regularity of potentials

General cases

Other examples are when

$$\nabla U = \nabla h \qquad \text{in } D^c \cap B_1,$$

(h harmonic in B_1) or even when $|D^2 U|$ is bounded in D^c.

Probably the ultimate case would be when we have $U^D(x)$ is $C^{1,1}$ at the origin, from one side (say from D^c), and we ask whether the same holds from the other side.

The answer to the first questions are given, but not to the last one.

Regularity of potentials

Results so far

Case $\nabla U = \nabla h$ in D^c, with constant density for U, was settled by Caffarelli-Karp-Sh 2000. Use of monotonicity formula and blow-up techniques!

For Lipschitz densities, it was settled by Sh- 2003, use of monotonicity formulas.

For the more general case $D^2 U = 0$ in D^c and with Dini density, it was settled by Andersson-Lindgren-Sh CPAM-2013. Use of Harmonic analysis technique.

Regularity of potentials

Fully nonlinear cases

The nonlinear theory, i.e. replacing the Laplacian with $F(D^2u)$ has also been done recently (Figalli-Sh.) as well as parabolic versions of these.

The p-Laplacian operator, Monge-Ampere, and many other operators, are yet not treated. One may also look at potentials of fractional order.

Other variants that one may consider

$$L(u_p) = |u_p|^p$$

and what happens when $p \to 0$.

Regularity of potentials

Heuristics behind Harm-Anal technique

Let us assume D^2U is uniformly bounded outside D. Since U is $W^{2,p}(B_1)$ $(p > n)$ then it is C^2 almost everywhere in B_1.

We want to show $D^2U(x)$ is universally bounded in $B_{1/2}$.
Fix any point $z \in B_{1/2}$ and set

$$\bar{U} := U(x) - U(z) - (x - z) \cdot \nabla U(z).$$

$$S_r = \sup_{B_r(z)} |\bar{U}(x)|/r^2.$$

Need to show that for a universal C_0

$$S_r \leq C_0 \qquad \forall r < 1/2.$$

Regularity of potentials

Heuristics behind Harm-Anal technique

Define

$$\lambda_r = |B_1(0) \setminus D_r|,$$

where $D_r := \frac{1}{r}(D - z)$.

Proposition: (J. Andersson) There is a universal constant M such that for any $0 < r < 1$ either of the following hold

- $S_r \leq M$, this is what we want
- $\lambda_r \leq \frac{1}{2}\lambda_{2r}$, this says the complement is thin.

Regularity of potentials

Heuristics behind Harm-Anal technique

The first observation is that

$$\lambda_r \leq \frac{1}{2}\lambda_{2r} \qquad \Longrightarrow \qquad \lambda_r \lessapprox r,$$

i.e.

$$|D^c \cap B_r| \leq Cr^{n+1}.$$

We thus arrive at

$$D^2 U^{D^c}(z) \approx \int_{D^c \cap B_1(z)} \frac{dy}{|y-z|^n} \leq C.$$

Let us see how we use this estimate.

Regularity of potentials

Heuristics behind Harm-Anal technique

Set $r_k = 2^{-k}$, and consider now two cases:

- $\liminf_k S_{r_k} \leq 3M$,
- $\liminf_k S_{r_k} > 3M$.

In the first case we obtain

$$|D^2 U(z)| = |D^2 \overline{U}(0)| \leq \liminf_{k\to\infty} \sup_{B_{2^{-k}}(0)} \frac{2|\overline{U}|}{2^{-2k}} \leq 2(C_1 + 3M).$$

(Recall $\bar{U} := U(x) - U(z) - (x - z) \cdot \nabla U(z)$.)

Regularity of potentials

Heuristics behind Harm-Anal technique

In the second case, there exists k_0 such that:

$$S_{r_{k_0}} \leq 3M, \quad \text{and} \quad S_{r_k} > 3M, \qquad \forall k \geq k_0.$$

(That there is a k_0, depends on $S_1 \leq 3M$.)

In particular by Andersson's proposition one has

$$\lambda_r \leq Cr \qquad \forall r \leq 2^{-k_0}.$$

Now set $U_{r_{k_0}}(x) := 2^{-2k_0}\bar{U}(2^{-k_0}x + z)$. Then

$$\left|\bar{U}_{r_{k_0}}\right|(x) \leq 3M \qquad \text{in } B_1(0),$$

Regularity of potentials

Heuristics behind Harm-Anal technique

We may write

$$\bar{U}_{r_{k_0}}(x) = w(x) - \bar{U}^{D^c_{r_{k_0}}}(x),$$

where now $\Delta w = f(2^{-k_0}x + z)$, $|w| \le C$ in B_1, and f is Dini.

In particular $|D^2 w(0)| \le C$. From here we arrive at

$$D^2 U(z) = D^2 \bar{U}_{r_{k_0}}(0) = D^2 w(0) + D^2 \bar{U}^{D^c_{r_{k_0}}}(0).$$

It remains to prove bound for the last term.

Heuristics behind Harm-Anal technique

The last term can be rewritten in terms of an integral, and we have

$$|D^2 \bar{U}^{D^c_{r_{k_0}}}(0)| \le \int_{D^c_{r_{k_0}} \cap B_1} \frac{dx}{|x|^n} \le C,$$

where we used

$$\lambda_r \le Cr \qquad \forall r \le 2^{-k_0}.$$

This gives the result.

Regularity of potentials

John Andersson's Dichotomy: Main idea

Proposition: There is a universal constant M such that either of the following hold

- $S_r \leq M$
- $\lambda_r \leq \frac{1}{2}\lambda_{2r}$.

Regularity of potentials

John Andersson's Proposition: Main idea

Set $\overline{U}_r(x) = \overline{U}(rx)/r^2$, and let v_r be such that

$$\Delta v_r = -f(rx)\chi_{B_1 \setminus D_r}, \qquad v_r = 0 \text{ on } \partial B_1.$$

Then $\overline{U}_r = w_r + v_r$, with $\Delta w_r = f(rx)$ and w_r has the information of supnorm of $\overline{U}_r$ on ∂B_1. Also

$$\int_{B_{1/2}} |D^2 v_r|^2 \leq C|D_r^c \cap B_1| = C\lambda_r.$$

John Andersson's Proposition: Main idea

For clarity we assume $D^2\overline{U} = 0$ in D^c. Next

$$0 = \int_{D_r^c \cap B_{1/2}} |D^2 \overline{U}_r|^2 = \int_{D_r^c \cap B_{1/2}} |D^2 w_r + v_r|^2.$$

In particular (by triangle ineq. and previous slide)

$$\int_{D_r^c \cap B_{1/2}} |D^2 w_r|^2 \leq \int_{D_r^c \cap B_{1/2}} |D^2 v_r|^2 \leq C\lambda_r.$$

Regularity of potentials

John Andersson's Proposition: Main idea

Let now $\tilde{w}_r = w_r / S_r$, then (for $S_r \geq M$ large) we have

$$\int_{D_r^c \cap B_{1/2}} |D^2 \tilde{w}_r|^2 \leq \frac{C}{S_r^2} \lambda_r \leq \frac{C}{M^2} \lambda_r,$$

with $\tilde{w}_r$ solving $\Delta \tilde{w}_r = f(rx)/S_r$ and $\sup_{B_1} |\tilde{w}_r| = 1$.

Now we need

$$C_1 \lambda_{r/2} \leq \int_{D_r^c \cap B_{1/2}} |D^2 \tilde{w}_r|^2 \leq \frac{C}{M^2} \lambda_r,$$

which would give $2\lambda_{r/2} \leq \lambda_r$, if M is large enough.

Regularity of potentials

John Andersson's Proposition: Main idea

This is another tricky part!

We need a kind of non-degeneracy for $|D^2\tilde{w}_r|^2$ on the set $D_r^c \cap B_{1/2}$. Here is how we do it.

For the first inequality above we may now split $\tilde{w}_r$ into two parts:

$$\tilde{w}_r = h_r + g_r$$

where h_r is homogeneous harmonic polynomial of degree two and g_r satisfies

$$\Delta g_r = f(rx)/S_r \text{ and } \quad g_r = 0 \text{ on } \partial B_1.$$

Regularity of potentials

John Andersson's Proposition: Main idea

In this way we get rid of g_r as it becomes uniformly C^2, since f is Dini (say).
For h_r we have $D^2 h_r$ is a constant matrix, so we obtain the volume

$$c_0 \lambda_{r/2} = \int_{D_r^c \cap B_{1/2}} |D^2 h_r|^2$$

Regularity of potentials

John Andersson's Proposition: Main idea

All to all we have

$$c_0\lambda_{r/2} - c_1\lambda_{r/2}/M = \int_{D_r^c\cap B_{1/2}} |D^2 h_r|^2 - \int_{D_r^c\cap B_{1/2}} |D^2 g_r|^2$$

$$\leq \int_{D_r^c\cap B_{1/2}} |D^2 \tilde{w}_r|^2 \leq \frac{C}{M^2}\lambda_r$$

For M large enough we have

$$2\lambda_{r/2} \leq \lambda_r.$$