On the uniqueness of quadrature surfaces

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Introduction

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- This is a special case of the mean value formula

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applied to $h(y) = \frac{1}{|x-y|^{N-2}}$.

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- Actually, the problem is UNIQUENESS!

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Remark The uniqueness of a quadrature surface $\partial\Omega$ holds in the case where $\mu = \omega_N \delta_0$ by the method of moving planes.

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$\mathbf{\Omega_1}$: sub & $\mathbf{\Omega_2}$: super & $\mathbf{\Omega_1} \subset \mathbf{\Omega_2} \Rightarrow \mathbf{\Omega_1} \subset \exists \mathbf{\Omega} \subset \mathbf{\Omega_2}$.

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Variational method

$\partial\Omega$ is characterized as the set $\partial\{u > 0\}$, where u is a critical point of

$$J(u) := \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla u|^2 - \mu u + \frac{1}{2}\chi_{\{u>0\}} \right) dx.$$

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QD is unique for $\mu(t)$, but QS is not, in general.
We will show that $\partial\Omega(t)$, **continuously deformable from $\partial\Omega(0)$, is unique!**

Main results

Geometric flow

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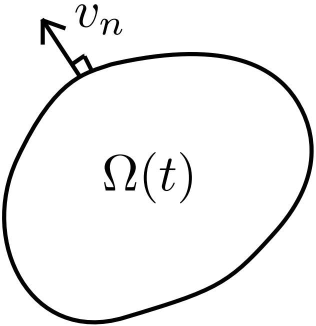
$$v_n = p \quad \text{on } \partial\Omega(t)$$

$$\text{where} \begin{cases} -\Delta p = \delta_0 & \text{in } \Omega(t) \\ \partial_n p + Hp = 0 & \text{on } \partial\Omega(t) \end{cases}$$

H: mean curvature of $\partial\Omega(t)$



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Comparison with Hele-Shaw flow

$$v_n = -\partial_n p \quad \text{on } \partial\Omega(t)$$

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$\because$ If $p(x_0) = \min_{\partial\Omega(t)} p \leq 0$ for $x_0 \in \partial\Omega(t)$,
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- $\{\partial\Omega(t)\}_{0 \leq t < T}$ is called *a $C^{3+\alpha}$ family of surfaces* if $\partial\Omega(t)$ is locally represented as the graph of a $C^{3+\alpha}$ function and its time derivative is of $C^{2+\alpha}$.

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- $\{\partial\Omega(t)\}_{0\leq t<T}$ is called *a* $C^{3+\alpha}$ *family of surfaces* if $\partial\Omega(t)$ is locally represented as the graph of a $C^{3+\alpha}$ function and its time derivative is of $C^{2+\alpha}$.

  Why $C^{3+\alpha}$? Because $H \in C^{1+\alpha}$ is required to solve the elliptic equation, and hence $\partial\Omega(t) \in C^{3+\alpha}$. Then, $v_n = p \in C^{2+\alpha}$.

Theorems

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$$\int_{\partial\Omega(t)} h\,d\mathcal{H}^{N-1} = \int_{\partial\Omega(0)} h\,d\mathcal{H}^{N-1} + t h(0) \quad \left(\forall h \in H(\overline{\Omega(t)})\right).$$

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Theorem (Unique solvability of the geometric flow)

Let $\partial\Omega(0)$ be an $h^{3+\alpha}$ closed surface with positive mean curvature. Then, there exists a unique $h^{3+\alpha}$ solution $\{\partial\Omega(t)\}_{0\le t<T}$ to the geometric flow.

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- Why $h^{3+\alpha}$? Since the equation turns out to be fully-nonlinear, we need to use the maximal regularity of Da Prato and Grisvard, in which the phase space must be a continuous interpolation space.

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Corollary (Uniqueness of a family of QS)

Let $\{\Gamma(t)\}_{0\le t<T}$ be an $h^{3+\alpha}$ family of quadrature surfaces of $\{\mu(t)\}$. If each $\Gamma(t)$ has positive mean curvature, then $\Gamma(t) = \partial\Omega(t)$.

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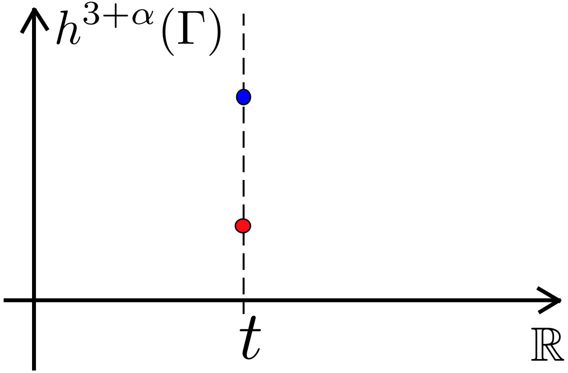

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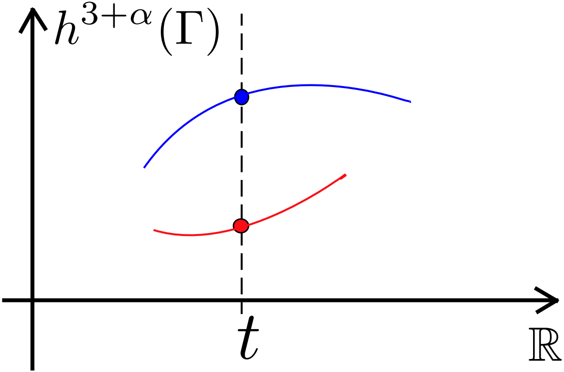

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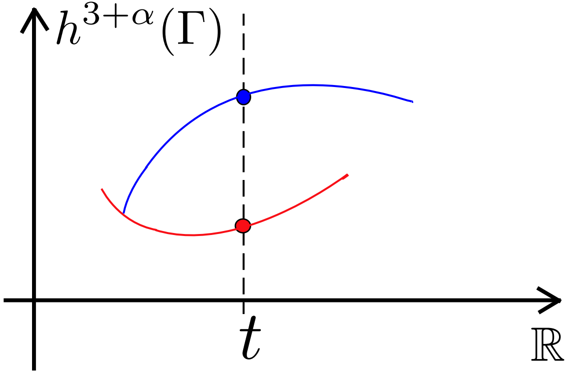

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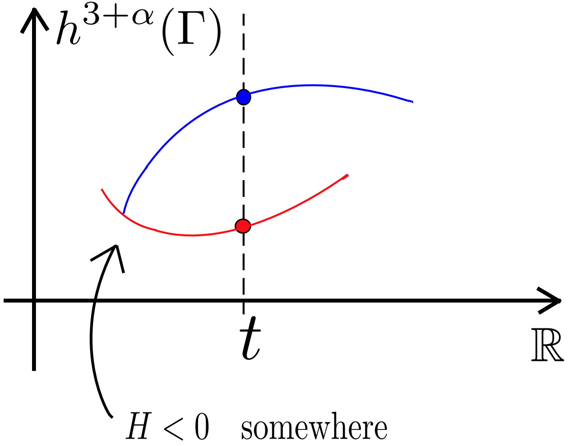

Outline of proof

Characterization of a family of QS

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(Proof of (GF) $\Rightarrow$ (QS))

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$$\therefore \quad \int_{\partial\Omega(t)} h\, d\sigma = \int_{\partial\Omega(0)} h\, d\sigma + th(0).$$

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By combining the above equalities,

$$\int_{\partial\Omega(t)} (hH + \partial_n h)(v_n - p) \, d\sigma = 0 \quad (\forall h\text{: harmonic}).$$

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Hence, $v_n = p$ follows from the solvability of the equation

$$\begin{cases} -\Delta h = 0 & \text{in a nb'd of } \overline{\Omega(t)}, \\ hH + \partial_n h = v_n - p & \text{on } \partial\Omega(t). \end{cases}$$

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1. Reformulation of problem into ODE in a fixed Banach space:

$$t \mapsto \rho(t) \in h^{3+\alpha}(\Gamma) \text{ s.t. } \begin{cases} \partial_t \rho = F(\rho) = L\rho + G(\rho), \\ \rho(0) = \rho_0. \end{cases}$$

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$$L: \text{ sectorial } \Rightarrow \exists \{e^{tL}\}_{t \geq 0}: \text{ analytic semigroup}$$

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3. Theory of maximal regularity deduces the existence:

$\exists$ a fixed point $\rho(\cdot) \in C([0,T); h^{3+\alpha}(\Gamma))$ of

$$\Phi(\rho) := e^{tL}\rho_0 + \int_0^t e^{(t-s)L} G(\rho(s))\, ds.$$

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$L \sim -M_1 T_\Gamma M_2(-\Delta_\Gamma) \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{2+\alpha}(\Gamma))$.
Here, T_Γ is the Robin-to-Dirichlet operator defined by $T_\Gamma \varphi = v$, where

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To prove that L is sectorial,

Step 1. Localize L near each point on Γ.

Step 2. $L \sim$ a constant coefficient operator $\mathcal{L}_0$ on $\mathbb{R}^{N-1}$.

Step 3. Show that $\mathcal{L}_0$ is a pseudo-differential operator of first order. In fact, the symbol is of monomial type of degree one, which implies $\mathcal{L}_0$ is sectorial.

Summary

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A. TRUE under the geometric condition $H > 0$

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Thank you for your kind attention!