Energy minimizing maps with free boundaries

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Background

Energy minimizing maps

Let $D \subset \mathbb{R}^n$, be a bounded domain, and consider the functional

$$E(\mathbf{u}) := \int_D \left(|\nabla \mathbf{u}|^2 + 2|\mathbf{u}|\right) dx,$$

gives rise to the singular Euler equations

$$\Delta \mathbf{u} = \frac{\mathbf{u}}{|\mathbf{u}|}\chi_{\{|\mathbf{u}|>0\}}, \qquad \mathbf{u} = (u_1, \cdots, u_m) \,.$$

This is standard and one can do it using first variation along with standard L^p-theory.

Background

Application

This is the equilibrium state of a cooperative reaction-diffusion system

$$u_t - \Delta u = -\frac{u}{\sqrt{u^2 + v^2}},$$
$$v_t - \Delta v = -\frac{v}{\sqrt{u^2 + v^2}}.$$

Considering the concentrations u and v of two species (reactants), each species (reactant) slows down the extinction (reaction) of the other species.

Background

Scalar case

The above system may also be seen as one of the simplest extensions of the classical *obstacle problem* to the vector-valued case:
Solutions of the classical obstacle problem are minimisers of the energy

$$\int_D \left(\frac{1}{2}|\nabla u|^2 + \max(u,0)\right) dx,$$

where $u : \mathbb{R}^n \supset D \to \mathbb{R}$.

Background

Two-phase scalar case

In the scalar case ($m = 1$), one recovers the two phase free boundary problem

$$\Delta u = \chi_{\{u>0\}} - \chi_{\{u<0\}},$$

which is a well-studied problem.

An example

Case $m = 2$

The real and imaginary parts of the function

$$S(z) = z^2 \log|z| \qquad (z = x + \mathbf{i}y)$$

satisfy the **unstable** equation (up to a multiplicative constant) and they have singularities at the origin:

$$\Delta u_i = \frac{-u_i}{|\mathbf{u}|}, \quad i = 1, 2, \ldots$$

Hence optimal $C^{1,1}$ regularity is lost for the unstable problem!

Examples

One dimensional examples are the following class of solutions

1. $u_i = \alpha_i P(x)$, with $P(x) \geq 0$, $\Delta P(x) = 1$, and $\sum_{i=1}^m \alpha_i^2 = 1$,
2. $u_i = \alpha_i (x_1^+)^2/2 + \beta_i (x_1^-)^2/2$, (two-phase) $\sum_{i=1}^m \alpha_i^2 = 1$, $\sum_{i=1}^m \beta_i^2 = 1$.
3. $u_i = \alpha_i (x_1^+)^2/2$, (one-phase case) $\sum_{i=1}^m \alpha_i^2 = 1$,

Examples

We shall be interested in the class of solutions **u** that asymptotically, near a free boundary point, behave like

$$\frac{\max(x \cdot \nu, 0)^2}{2}\mathbf{e}$$

where ν is a unit vector in $\mathbb{R}^n$ and **e** is a unit vector in $\mathbb{R}^m$.

Also denote by $\mathbb{H}$ the class of all these Half-space solutions.

Properties of the solution

One can work out many properties of the solution $\mathbf{u}$ to our problem:

- **Uniqueness:** use first variation by $\phi := \mathbf{u} - \mathbf{v}$, both having the same boundary data.
- **Bounds:**
  $\sup_{B_{3/4}} |\mathbf{u}| + \sup_{B_{3/4}} |\nabla \mathbf{u}| \le C_1(n,m) \left(\|\mathbf{u}\|_{L^1(B_1;\mathbb{R}^m)} + 1 \right).$
- **Stability:** $\mathbf{u}_k \to \mathbf{u}$ weakly in $W^{1,2}(D;\mathbb{R}^m)$ then Rellich's theorem together with the fact that $D^2\mathbf{u} = 0$ a.e. in $\{\mathbf{u} = 0\}$, implies that $\mathbf{u}$ is a solution, too.

Further properties

- **Non-Degeneracy:**

$$\sup_{B_r(x^0)} |\mathbf{u}| \geq \frac{1}{2n} r^2, \qquad \forall x^0 \in \overline{\{|\mathbf{u}| > 0\}}.$$

  Use the fact that $\Delta|\mathbf{u}| \geq 1$ in the set $\{|\mathbf{u}| > 0\}$.

- ***L^1*-closeness implies geometric closeness:** If $\|\mathbf{u} - \mathbf{h}\|_{L^1(B_1;\mathbb{R}^m)} \leq \epsilon < 1$, where $\mathbf{h} := \frac{\max(x_n,0)^2}{2}\mathbf{e}^1$. Then

$$B_{1/2}(0) \cap \textbf{supp}\, \mathbf{u} \subset \left\{x_n > -C\epsilon^{\frac{1}{2n+2}}\right\}$$

  with a constant $C = C(n, m)$.

A balanced energy functional

We define a new energy functional

$$\mathcal{M}(\mathbf{v}, x^0, r) := \frac{1}{r^{n+2}} \int_{B_r(x^0)} (|\nabla \mathbf{v}|^2 + 2|\mathbf{v}|) - \frac{2}{r^{n+3}} \int_{\partial B_r(x^0)} |\mathbf{v}|^2 \, d\mathcal{H}^{n-1},$$

which will be used to prove both the growth of **u** from the free boundary and also the behavior of the free boundary at good points.

For $\mathbf{v}_r(x) := r^{-2}\mathbf{v}(rx + x^0)$, we have

$$\mathcal{M}(\mathbf{v}, x^0, r) = \mathcal{M}(\mathbf{v}_r, 0, 1) =: \mathcal{M}(\mathbf{v}_r),$$

Important properties of $\mathcal{M}$

Monotonicity

The balanced energy functional $\mathcal{M}$ is monotone in r:

$$\frac{d\mathcal{M}(\mathbf{v}, x^0, r)}{dr} \geq 0.$$

We also define

$$\frac{\alpha_n}{2} := \mathcal{M}(\frac{\max(x \cdot \nu, 0)^2}{2}\mathbf{e}),$$

and one can show that $\alpha_n = 2\mathcal{M}(\mathbf{h})$ for all $\mathbf{h} \in \mathbb{H}$.

Important properties of $\mathcal{M}$

Homogeneous global solutions, and energy level

- $\mathcal{M}(\mathbf{u}) \geq \frac{\alpha_n}{2}$ for all 2-homogeneous global solutions, and with equality if and only if $\mathbf{u} \in \mathbb{H}$.
- In the $L^1(B_1(0); \mathbb{R}^m)$- topology, $\mathbb{H}$ is isolated within the class of homogeneous solutions of degree 2.

The free boundary

Regular points

We define

$$\Gamma(\mathbf{u}) := D \cap \partial\{x \in D : |\mathbf{u}(x)| > 0\},$$

$$x \in \Gamma_0(\mathbf{u}) := \Gamma(\mathbf{u}) \cap \{x : \nabla\mathbf{u}(x) = 0\}$$

A point x is a regular free boundary point for **u** if:

$$x \in \Gamma_0(\mathbf{u}) \qquad \text{and} \qquad \lim_{r\to 0} \mathcal{M}(\mathbf{u}, x, r) = \frac{\alpha_n}{2}.$$

We denote by $\mathcal{R}_u$ the set of all regular free boundary points of u in B_1.

The free boundary

The set of Regular points are open in Γ_0

From the upper semicontinuity of $\mathcal{M}(\mathbf{u}, x, r)$ in $\mathbf{u}$, and the isolated property of $\mathbb{H}$ we can conclude that the set of regular free boundary points $\mathcal{R}_u$ is open relative to $\Gamma_0(\mathbf{u})$.

Properties of $\mathbf{u}$

Quadratic Grwoth

Any solution $\mathbf{u}$ to our system in $B_1(0)$ satisfies

$$|\mathbf{u}(x)| \leq C\mathbf{dist}^2(x, \Gamma_0(u))$$

and

$$|\nabla \mathbf{u}(x)| \leq C\mathbf{dist}(x, \Gamma_0(\mathbf{u})) \text{ for every } x \in B_{1/2}(0),$$

where the constant C depends only on n and

$$E(\mathbf{u}, 0, 1) = \int_{B_1(0)} (|\nabla \mathbf{u}|^2 + 2|\mathbf{u}|).$$

Properties of $\mathbf{u}$

Quadratic Grwoth

Using the monotonicity formula one can show (elementary)

$$\frac{2}{r^{n+2}}\int_{B_r}|\mathbf{u}| \le E(\mathbf{u},0,1) + \frac{2}{r^{n+3}}\int_{\partial B_r}|\mathbf{u}-\mathbf{p}|^2\, d\mathcal{H}^{n-1}$$

$$\le C_0 + C_1(\mathbf{p})|D^2\mathbf{u}|_{BMO(B_{1/2})} \le C_2$$

for each $\mathbf{p} = (p_1,\dots,p_m)$ such that each component p_j is a homogeneous harmonic polynomial of second order.

Regularity of the free boundary

$\mathcal{R}_u$ is locally in D a $C^{1,\beta}$-surface.

One shows there exist $\beta' > 0$, $r_0 > 0$ and $C < \infty$:

$$\int_{\partial B_1(0)} \left| \frac{\mathbf{u}(x^0 + rx)}{r^2} - \frac{1}{2}\mathbf{e}(x^0) \max(x \cdot \nu(x^0), 0)^2 \right| d\mathcal{H}^{n-1} \leq C\, r^{\beta'}$$

for every $x^0 \in \mathcal{R}_u$ and every $r \leq r_0$.

Here $\nu(x^0)$ depends on the blow-up of $\mathbf{u}$ at x^0.

A main technical tool

Epiperimetric inequality

There exists $\kappa \in (0, 1)$ and $\delta > 0$ such that if $\mathbf{c}$ is a homogeneous function of degree 2 satisfying $\|\mathbf{c} - \mathbf{h}\|_{W^{1,2}(B_1;\mathbb{R}^m)} + \|\mathbf{c} - \mathbf{h}\|_{L^\infty(B_1;\mathbb{R}^m)} \leq \delta$ for some $\mathbf{h} \in \mathbb{H}$, then there is a $\mathbf{v} \in W^{1,2}(B_1;\mathbb{R}^m)$ such that $\mathbf{v} = \mathbf{c}$ on ∂B_1 and

$$\mathcal{M}(\mathbf{v}) \leq (1 - \kappa)\mathcal{M}(\mathbf{c}) + \kappa \frac{\alpha_n}{2}.$$