

Relative Hofer Geometry and the Asymptotic Hofer-Lipschitz Constant

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Abstract

Let (M, ω) be a symplectic manifold and $U \subseteq M$ an open subset. I study the natural inclusion of the group of Hamiltonian diffeomorphisms of U into the group of Hamiltonian diffeomorphisms of M . The main result is an upper bound for this map in terms of the Hofer norms for U and M . Applications are upper bounds on the relative Hofer diameter of U and the asymptotic Hofer-Lipschitz constant, which are often sharp up to constant factors. As another consequence, the relative Hofer diameter of certain symplectic submanifolds vanishes.

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1 Results

1.1 Relative Hofer geometry of an open subset

Let (M, ω) be a symplectic manifold. (For simplicity all manifolds in this paper are assumed to have empty boundary.) The main result of this article is concerned with the following question. We denote by $\text{Ham}_c(M) := \text{Ham}_c(M, \omega)$ the group of Hamiltonian diffeomorphisms of M generated by a compactly supported time-dependent function, and by $\|\cdot\|_c^M := \|\cdot\|_c^{M, \omega}$ the Hofer norm on $\text{Ham}_c(M)$ (see Section 1.4). Let $U \subseteq M$ be an open subset. Consider the natural inclusion

$$\text{Ham}_c(U) \ni \varphi \mapsto \tilde{\varphi} \in \text{Ham}_c(M), \quad \tilde{\varphi}(x) := \begin{cases} \varphi(x), & \text{if } x \in U, \\ x, & \text{otherwise,} \end{cases} \quad (1)$$

Question *How much does this map fail to be an isometry with respect to the Hofer norms for U and M ?*

To give some answer to this question, for $a > 0$, we denote by $B^2(a), \overline{B}^2(a) \subseteq \mathbb{R}^2$ the open and closed balls of radius $\sqrt{a/\pi}$, around 0. We denote by ω_0 the standard symplectic form on \mathbb{R}^2 . The main result of this article is the following.

Theorem 1 (Relative Hofer geometry of an open subset) *For every $\varphi \in \text{Ham}_c(U)$ we have*

$$\|\tilde{\varphi}\|_c^M \leq \inf \left(8a + \frac{2\|\varphi\|_c^U}{N} \right), \quad (2)$$

where $a \in (0, \infty)$ and $N \in \mathbb{N}$ run over all numbers for which there exists a symplectic manifold (M', ω') and a symplectic embedding

$$\psi : B^2(3Na) \times M' \rightarrow M$$

(with respect to $\omega_0 \oplus \omega'$ and ω), satisfying

$$U \subseteq \psi(B^2(a) \times M').$$

This result gives an answer to the above question, which is often in two ways asymptotically sharp up to constant factors, as we will see below (Proposition 3 and Corollary 6). Its proof is based on a method by J.-C. Sikorav (see the remark on page 6).

Theorem 1 has the following direct application. We define the *extension relative Hofer diameter of U in M* to be

$$\text{Diam}_c(U, M) := \text{Diam}_c(U, M, \omega) := \sup \{ \|\tilde{\varphi}\|_c^M \mid \varphi \in \text{Ham}_c(U) \}. \quad (3)$$

Corollary 2 *Assume that there exists a symplectic manifold (M', ω') and a number $a > 0$, such that*

$$(M, U, \omega) = (\mathbb{R}^2 \times M', B^2(a) \times M', \omega_0 \oplus \omega').$$

Then we have

$$\text{Diam}_c(U, M) \leq 8a.$$

This corollary is closely related to a result by J.-C. Sikorav (see page 10 below) for the case $M = \mathbb{R}^{2n}$. The next result gives sufficient conditions under which this estimate is sharp up to a factor of 8. We call a symplectic manifold (M, ω) (*symplectically*) *aspherical* iff

$$\int_{S^2} u^* \omega = 0, \quad \forall u \in C^\infty(S^2, M). \quad (4)$$

For a definition of geometric boundedness see Section 1.4.

Proposition 3 *Assume that there exist (M', ω') and a as in Corollary 2. Suppose also that (M', ω') is aspherical and geometrically bounded, and there exists a closed symplectic manifold (X, σ) and an integer $n \geq 0$, such that*

$$2n + 2 + \dim X = \dim M', \quad B^2(2a) \times (B^2(a))^n \times X \subseteq M'.$$

Then we have

$$\text{Diam}_c(U, M) \geq a. \quad (5)$$

The proof of this result is based on a leafwise fixed point theorem for coisotropic submanifolds proved by the author in [Zi].

Asymptotic Hofer-Lipschitz constant

Another immediate consequence of Theorem 1 is the following. We define the *asymptotic Hofer-Lipschitz constant* of (M, U, ω) to be

$$\text{Lip}^\infty(M, U) := \text{Lip}^\infty(M, U, \omega) := \quad (6)$$

$$\lim_{C \rightarrow \infty} \sup \left\{ \frac{\|\tilde{\varphi}\|_c^M}{\|\varphi\|_c^U} \mid \varphi \in \text{Ham}_c(U) : \|\varphi\|_c^U > C \right\}.$$

(Here our convention is that $\sup \emptyset := 0$.) This number can be understood as the asymptotic (for large distances) Lipschitz constant of the inclusion (1), with respect to the Hofer distances for U and M . It is the simplest interesting quantity comparing the two Hofer geometries, if M is closed. (See the remark on page 6.)

Corollary 4 *Assume that there exists $a > 0$, $N \in \mathbb{N} \cup \{\infty\}$, and a symplectic manifold (M', ω') , such that, defining $c := 3Na$, we have*

$$M = B^2(c) \times M', \quad \omega = \omega_0 \oplus \omega', \quad U = B^2(a) \times M'. \quad (7)$$

(Here for $c = \infty$ we define $B^2(\infty) := \mathbb{R}^2$.) Then we have

$$\text{Lip}^\infty(M, U) \leq \frac{2}{N} = \frac{6a}{c}. \quad (8)$$

In particular, we have $\text{Lip}^\infty(M, U) = 0$, if $N = \infty$. Note that the obvious extension of the estimate (8) to a general triple (M, ω, U) is wrong, hence the hypothesis that M, ω and U are products, cannot be dropped. (See the discussion on page 7 below.)

The next result provides a sufficient criterion under which the estimate (8) is sharp up to a factor of 6. We call a symplectic manifold (M, ω) *strongly (symplectically) aspherical* iff it is (symplectically) aspherical, and the contraction of the first Chern class of (M, ω) with every element of $\pi_2(M)$ vanishes. We denote $2n := \dim M$.

Theorem 5 *Let (M, ω) be a strongly aspherical closed symplectic manifold, and $U \subseteq M$ an open subset that is displaceable in a Hamiltonian way. Then we have*

$$\text{Lip}^\infty(M, U) \geq \frac{\int_U \omega^n}{\int_M \omega^n}. \quad (9)$$

The proof of this result is based on the argument of the proof of Theorem 1.1. in the paper [Os] by Y. Ostrover. Its key ingredient is a result by M. Schwarz about action selectors. Theorem 5 has the following consequence.

Corollary 6 *Assume that there exist numbers $a > 0$ and $c \geq 2a$, and a closed and strongly aspherical symplectic manifold (M', ω') , such that (7) holds. Then we have*

$$\text{Lip}^\infty(M, U) \geq \frac{a}{c}. \quad (10)$$

It follows that under the hypotheses of this corollary, the inequality (8) is sharp up to a factor of 6.

1.2 Relative Hofer geometry of a closed subset

Let (M, ω) be a symplectic manifold and $X \subseteq M$ a closed subset. Then X carries natural absolute and relative Hofer geometries. As an application of

Theorem 1, the corresponding relative Hofer diameter of X vanishes, if X is a symplectic submanifold of positive codimension, which arises as a product.

To explain this, we define the set of “*compactly supported*” Hamiltonian diffeomorphisms of X , $\text{Ham}_c(X, \omega)$ as follows. Let $V : [0, 1] \times M \rightarrow TM$ be a smooth compactly supported time-dependent vector field on M . For every $t \in [0, 1]$ we denote by φ_V^t the time- t -flow of V . We say that V is X -compatible iff $\varphi_V^t(X) = X$, for every $t \in [0, 1]$. For every function $H \in C^\infty([0, 1] \times M, \mathbb{R})$ we denote by X_H its time-dependent Hamiltonian vector field, and we abbreviate $\varphi_H^t := \varphi_{X_H}^t$. We define

$$\mathcal{H}_c(M, \omega, X) := \{H \in C_c^\infty([0, 1] \times M, \mathbb{R}) \mid X_H \text{ is } X\text{-compatible}\},$$

$$\text{Ham}_c(X) := \text{Ham}_c(X, M, \omega) := \{\varphi_H^1|_X \mid H \in \mathcal{H}_c(M, \omega, X)\}.$$

It follows from an argument as in the proof of [SZ2, Proposition 1] that $\text{Ham}_c(X)$ is a subgroup of the group of homeomorphisms of X , and that $\text{Ham}_c(X) = \text{Ham}_c(X, \omega|_X)$ if X is a symplectic submanifold of M . (Here the right hand side denotes the usual group of Hamiltonian diffeomorphisms of X .) Hence $\text{Ham}_c(X)$ is a natural generalization of $\text{Ham}_c(M)$.

We define the (“*compactly supported*”) *Hofer semi-norm on $\text{Ham}_c(X)$ relative to M* to be the map

$$\begin{aligned} \|\cdot\|_{X,c}^M : \text{Ham}_c(X) &\rightarrow [0, \infty), \\ \|\varphi\|_{X,c}^M &:= \inf \{\|\psi\|_c^M \mid \psi \in \text{Ham}_c(M) : \psi|_X = \varphi\}. \end{aligned} \tag{11}$$

This map measures how short a Hamiltonian path on X can be made in M . It is an invariant semi-norm (in the sense explained on page 9 below). (This follows from an argument as in the proof of [SZ2, Proposition 4].) In the case $X = M$ it equals the (absolute) Hofer norm $\|\cdot\|_c^M$. It gives rise to the “*compactly supported*” *Hofer diameter of X relative to M* , defined as

$$\text{diam}_c(X, M) := \text{diam}_c(X, M, \omega) := \sup \{\|\varphi\|_{X,c}^M \mid \varphi \in \text{Ham}_c(X)\}. \tag{12}$$

In [SZ2, Theorem 6] we gave examples in which this diameter is positive and finite. As a consequence of Theorem 1, the relative Hofer diameter has the surprising property that it vanishes for a large class of symplectic submanifolds:

Corollary 7 (Relative Hofer geometry of a symplectic submanifold)

Let (M, ω) and (M', ω') be connected symplectic manifolds and $X' \subseteq M'$ a finite subset. Assume that M' has positive dimension. Then we have

$$\text{diam}_c(M \times X', M \times M') = 0. \tag{13}$$

This result puts the “restriction relative Hofer diameter” diam_c into sharp contrast with the “extension relative Hofer diameter” Diam_c (defined in (3)). Namely, assume that (M, ω) is closed and strongly aspherical, and let $U \subseteq M$ be a non-empty open subset. Then it follows from Theorem 5 that

$$\text{Diam}_c(U, M) = \infty.$$

1.3 Remarks

On the proof of Theorem 1

As mentioned above, the proof of Theorem 1 is an adaption of the proof of a result by J.-C. Sikorav. The idea is to write $\tilde{\varphi}$ as a composition of two maps, each of which is the composition of flows $\varphi_{H_1}^1, \dots, \varphi_{H_N}^1$. The functions H_i are chosen to have small Hofer norm and support in $[0, 1] \times X_i$, where X_1, \dots, X_N are disjoint subsets of M . Such flows satisfy the key inequality

$$\|\varphi_{H_1}^1 \circ \dots \circ \varphi_{H_N}^1\|_c^M \leq c \max_i \|H_i\|, \quad (14)$$

where $c = 2$ in general, and $c = 1$, if all H_i 's are non-negative (or non-positive). (See Proposition 8 below). Inequality (2) is a consequence of this estimate. A crucial point in the proof of (14) is to suitably reparametrize the functions H_i in time.

In order to chop $\tilde{\varphi}$ into pieces, we choose a certain collection of subsets $U_i \subseteq B^2(3Na)$, $i = 1, \dots, 2N$ containing the disjoint union of $\overline{B}^2(a)$ and a subset X_i that is symplectomorphic to $\overline{B}^2(a)$. (See Lemma 18 below.) The functions H_i are later chosen to have support in open subsets of M constructed from the U_i 's. The collection (U_i) is chosen in a careful way, so that the supports of the H_i 's can be made disjoint.

That the functions H_i as above can be chosen with small Hofer norms relies on the fact that given $b > a$, $\overline{B}^2(a)$ can be moved to X_i inside U_i with Hofer energy at most b . (See Proposition 11 below.) This in turn is based on the fact that $\overline{B}^2(a)$ and X_i can be made nice by some area preserving map. (See Proposition 12 below.) The proof of this uses a relative version of Moser's theorem involving a hypersurface.

On Hofer-Lipschitz constants

Let (M, ω) be a symplectic manifold and $U \subseteq M$ an open subset. Instead of $\text{Lip}^\infty(M, U)$ (as defined in (6)), consider the *Hofer-Lipschitz constant* of (M, U, ω) , which we define as

$$\text{Lip}(M, U) := \text{Lip}(M, U, \omega) := \sup \left\{ \frac{\|\tilde{\varphi}\|_c^M}{\|\varphi\|_c^U} \mid \text{id} \neq \varphi \in \text{Ham}_c(U) \right\}. \quad (15)$$

This can be viewed as the Lipschitz constant of the natural inclusion (1). Note that every $\text{id} \neq \varphi \in \text{Ham}_c(U)$ has positive Hofer norm on U , by a result by D. McDuff and F. Lalonde [LM, Theorem 1.1]. Hence this definition makes sense. However, if M is closed and $U \neq \emptyset$ then

$$\text{Lip}(M, U) = 1, \quad (16)$$

hence this number is uninteresting. To see that (16) holds, note that without loss of generality, we may assume that M is connected. By definition, we have $\text{Lip}(M, U) \leq 1$. Furthermore¹, let $H \in C_c^\infty(U, \mathbb{R})$ be a non-constant function. We define $\tilde{H} : M \rightarrow \mathbb{R}$ by $\tilde{H}(x) := H(x)$, if $x \in U$, and $\tilde{H}(x) := 0$, otherwise. It follows from Theorem 1.6(i) in the article [McD2] by D. McDuff that there exists $t_0 > 0$ such that

$$\|\varphi_{\tilde{H}}^{t_0}\|_c^M = t_0 \|H\| \geq \|\varphi_H^{t_0}\|_c^U.$$

It follows that $\text{Lip}(M, U) \geq 1$, and therefore, equality (16) holds.

On Corollary 4 and Theorem 5

Let (M, ω) be a symplectic manifold of finite volume. We denote $2n := \dim M$. In view of the estimate (8), it is natural to ask the following question.

Question *Does there exist a constant $C > 0$ such that for every open subset $U \subseteq M$, we have*

$$\text{Lip}^\infty(M, U) \leq C \frac{\int_U \omega^n}{\int_M \omega^n}? \quad (17)$$

The answer is “no” in the following two examples, which are due to L. Polterovich.

Example A *Let M be a (real) closed connected surface of positive genus, ω an area form on M , and $U \subseteq M$ an open neighborhood of some non-contractible embedded circle in M .*

Example B *Let $n \in \mathbb{N}$, (M, ω) be the complex projective space \mathbb{CP}^n together with the Fubini-Study form, and $U \subseteq M$ an open neighborhood of the real projective space \mathbb{RP}^n (embedded in \mathbb{CP}^n in the standard way).*

We will show below that in both examples, we have

$$\text{Lip}^\infty(M, U) = 1. \quad (18)$$

Since we may choose U to have arbitrary small volume in these examples, it follows that the bound (17) does not hold.

¹I was made aware of the following argument by F. Schlenk.

Note that the set U in these examples is non-displaceable, since the same holds for the circle and $\mathbb{R}P^n$, respectively. Hence the statement of Theorem 5 continues to hold for some non-aspherical symplectic manifolds and some small non-displaceable subsets U . The examples also show that the inequality (9) cannot be sharp for *general* open subsets U (given that it is true for such sets).

Let now (M, ω) and U be as in either example above. In order to prove equality (18), it suffices to prove the inequality

$$\text{Lip}^\infty(M, U) \geq 1. \quad (19)$$

We denote by L the non-contractible circle in the surface M , or $L := \mathbb{R}P^n \subseteq \mathbb{C}P^n$, respectively. Let $C \in (0, \infty)$. We choose a function $H \in C_c^\infty(U, \mathbb{R})$ such that

$$\int_U H \omega^n = 0, \quad -1 \leq H \leq C, \quad H = C \text{ on } L.$$

It follows that

$$\|\varphi_H^1\|_c^U \leq \|H\| \leq C + 1. \quad (20)$$

We claim that

$$\|\widetilde{\varphi_H^1}\|_c^M \geq C, \quad (21)$$

unless (M, ω) and U are as in Example B and $n = 1$, in which case we have

$$\|\widetilde{\varphi_H^1}\|_c^M \geq C - \pi. \quad (22)$$

Since $C > 0$ is arbitrary, the inequality (19) will be a consequence of (20) and this claim.

Consider the case of Example A. Then L is stably non-displaceable and $\text{Ham}_c(M, \omega)$ is simply connected. (For the latter see for example [Po, Section 7.2].) Hence inequality (21) follows from [Po, Theorem 7.4.A], using the facts $\int_U H \omega^n = 0$ and $H = C$ on L .

Consider the case of Example B with $n = 1$. Then again L is stably non-displaceable. (This follows e.g. from [EP3, Theorems 1.8 and 1.4].) Furthermore, the fundamental group of $\text{Ham}(\mathbb{C}P^1)$ is isomorphic to \mathbb{Z}_2 . Hence inequality (22) follows from Theorem 7.4.A (using the facts $\int_U H \omega^n = 0$ and $H = C$ on L), the corollary on p. 66, and Definition 7.3.A in the book [Po].

Consider now the case of Example B, with $n \geq 2$. To see that (21) holds, we denote by

$$\mu : \text{Ham}_c(\mathbb{C}P^n) \rightarrow \mathbb{R}$$

the Floer homological Calabi quasi-morphism of $\mathbb{C}P^n$ associated with the fundamental class $[\mathbb{C}P^n]$. (See [EP1, Sections 3.4 and 4.3].) It satisfies the

bound

$$|\mu(\varphi)| \leq \|\varphi\|_c^M \int_{\mathbb{CP}^n} \omega^n, \quad \forall \varphi \in \text{Ham}_c(\mathbb{CP}^n). \quad (23)$$

(See [EP1, Corollary 3.6].) We define

$$\zeta : C^\infty(\mathbb{CP}^n, \mathbb{R}) \rightarrow \mathbb{R}, \quad \zeta(F) := \frac{\int_{\mathbb{CP}^n} F \omega^n - \mu(\varphi_F^1)}{\int_{\mathbb{CP}^n} \omega^n}. \quad (24)$$

This is a symplectic quasi-state. (See [EP2], definition (4) and the discussion afterwards.)

\mathbb{RP}^n is a closed monotone Lagrangian submanifold of \mathbb{CP}^n . Furthermore, since $n \geq 2$, it satisfies the Albers condition (see [EP3, Section 1.2.2]). (This follows from an argument involving the second Stiefel-Whitney class of the tautological bundle of \mathbb{CP}^n .) Therefore, by [EP3, Theorem 1.17] \mathbb{RP}^n is a ζ -heavy subset. (See [EP3, Definition 1.3, p. 779].)

We define $\tilde{H} : \mathbb{CP}^n \rightarrow \mathbb{R}$ by $\tilde{H}(x) := H(x)$, if $x \in U$, and $\tilde{H}(x) := 0$, otherwise. Since \mathbb{RP}^n is ζ -heavy, $\tilde{H} = C$ on \mathbb{RP}^n , and ζ is homogeneous, it follows that $\zeta(\tilde{H}) \geq C$. Combining this with the equality $\int_U H \omega^n = 0$ and the definition (24) of ζ , we obtain

$$C \int_{\mathbb{CP}^n} \omega^n \leq -\mu(\varphi_{\tilde{H}}^1).$$

Combining this with the bound (23), inequality (21) follows.

This completes the proof of inequality (19) and hence equality (18) in all cases.

On the relative Hofer diameters

The diameter of a pseudo-distance function d on a set X is by definition the number

$$\text{diam}(d) := \sup \{d(x, y) \mid x, y \in X\}.$$

Let (M, ω) be a symplectic manifold, $X \subseteq M$ a closed subset, and $U \subseteq M$ and open subset. We can view $\text{diam}_c(X, M)$ and $\text{Diam}_c(U, M)$ (defined in (12) and (3) as such diameters, as follows. Let G be a group. By a *semi-norm* on G we mean a map $\|\cdot\| : G \rightarrow [0, \infty]$ such that

$$\begin{aligned} \|\mathbf{1}\| &= 0, \\ \|g^{-1}\| &= \|g\|, \\ \|gh\| &\leq \|g\| + \|h\|, \end{aligned}$$

for every $g, h \in G$. We call the last of these conditions the *triangle inequality*. We call $\|\cdot\|$ a *norm* iff also

$$\|g\| = 0 \implies g = \mathbf{1}.$$

We call $\|\cdot\|$ *invariant* iff

$$\|ghg^{-1}\| = \|h\|, \quad \forall g, h \in G.$$

Every semi-norm $\|\cdot\|$ on G gives rise to a pseudo-distance function $d(\|\cdot\|)$ on G via

$$d(\|\cdot\|)(g, h) := \|g^{-1}h\|.$$

The diameter of $d(\|\cdot\|)$ is given by

$$\text{diam}(d(\|\cdot\|)) = \sup_{g \in G} \|g\|.$$

We can now interpret the “restriction relative Hofer diameter” as

$$\text{diam}_c(X, M) = \text{diam}(d(\|\cdot\|_{X,c}^M)),$$

where $\|\cdot\|_{X,c}^M$ is defined as in (11). Consider now the canonical extension homomorphism $E : \text{Ham}_c(U) \rightarrow \text{Ham}_c(M)$ given by (1). The map $\|\cdot\|_c^M \circ E : \text{Ham}_c(U) \rightarrow [0, \infty)$ is a semi-norm, and we have

$$\text{Diam}_c(U, M) = \text{diam}(d(\|\cdot\|_c^M \circ E)).$$

The “restriction” and “extension” relative Hofer diameters are related as follows. Let $U \subseteq M$ be an open subset and $X \subseteq U$ a compact subset. Then we have

$$\text{Diam}_c(U, M) \geq \text{diam}_c(X, M).$$

(This follows from an argument as in the proof of [SZ2, Proposition 8].)

Open question *Does there exist a symplectic manifold (M, ω) such that for every non-empty open subset $U \subseteq M$ we have $\text{Diam}_c(U, M) = \infty$, but there exists a closed subset $X \subseteq M$ such that $0 < \text{diam}_c(X, M) < \infty$?*

Related work

About Corollary 2: This result is closely related to a result by J.-C. Sikorav, which states that for every open subset $U \subseteq \mathbb{R}^{2n}$ and every function $H \in C^\infty([0, 1] \times \mathbb{R}^{2n}, \mathbb{R})$ with compact support contained in $[0, 1] \times U$, we have

$$\|\varphi_H^1\|_c^{\mathbb{R}^{2n}} \leq 16e_p(U, \mathbb{R}^{2n}). \quad (25)$$

Here $e_p(U, \mathbb{R}^{2n})$ denotes the proper displacement energy of U . (See [Si] or Theorem 10, Section 5.6 in the book [HZ].) If

$$U = B^2(a) \times \mathbb{R}^{2n-2}$$

for some $a > 0$, then

$$e_p(U, \mathbb{R}^{2n}) = a.$$

Hence in this case Corollary 2 implies inequality (25), improving the constant 16 by a factor of two. This factor is saved in the proof of Theorem 1 by using a version of the Key Inequality involving only positive (negative) functions (Proposition 8 below).

About Corollary 7: In [SZ2, Theorem 6(i)] we proved that

$$\text{diam}_c(S^{2n-1}, \mathbb{R}^{2n}) \geq \frac{\pi}{2}.$$

We also showed [SZ2, Theorem 6(ii)] that for every $n \geq 2$ and $d \geq n$ there exists a compact subset $X \subseteq \overline{B}^{2n}$, of Hausdorff dimension at most $d+1$, such that

$$\text{diam}_c(X, \mathbb{R}^{2n}) \geq \frac{\pi}{k(n, d)},$$

where $k(n, d) \in \{1, \dots, 2n - d\}$ is a combinatorial expression in n and d .

Note also that the *absolute* Hofer diameter $\text{diam}_c(M) = \text{Diam}_c(M, M)$ has been calculated for many symplectic manifolds. In all known examples it is infinite. For a recent overview and references, see the article by D. McDuff [McD3].

1.4 Organization, background, notation

Organization of the article

In Section 2 the key ingredients of the proof of Theorem 1 are stated and proved: Proposition 8 (Disjoint supports and small Hofer norm), Proposition 11 (Moving disks with small Hofer energy) and Lemma 18 (Nice subsets of the disk). Based on these results, Theorem 1 is proved.

In Section 3 a result by M. Schwarz is reformulated (Theorem 21), and based on this result, Theorem 5 and Corollary 6 are proved. Section 4 contains the proofs of Proposition 3 and Corollary 7.

The appendix contains the proofs of two auxiliary results (Lemmas 9 and 13) that are used in the proofs of Propositions 8 and 11.

Background

Let (M, ω) be a symplectic manifold. We denote by $C_c^\infty([0, 1] \times M, \mathbb{R})$ the space of all smooth functions on $[0, 1] \times M$ with compact support. The set of “*compactly supported*” *Hamiltonian diffeomorphisms* is by definition given by

$$\text{Ham}_c(M) := \text{Ham}_c(M, \omega) := \{\varphi_H^1 \mid H \in C_c^\infty([0, 1] \times M, \mathbb{R})\}.$$

This is a subgroup of the group of diffeomorphisms of M . It carries the following natural norm. We define

$$\begin{aligned} \|\cdot\| &:= \|\cdot\|_M : C_c^\infty([0, 1] \times M, \mathbb{R}) \rightarrow [0, \infty), \\ \|H\| &:= \int_0^1 (\max_M H^t - \min_M H^t) dt. \end{aligned}$$

The *Hofer norm* is defined to be the map

$$\begin{aligned} \|\cdot\|_c^M &:= \|\cdot\|_c^M : \text{Ham}_c(M) \rightarrow [0, \infty), \\ \|\varphi\|_c^M &:= \inf \{ \|H\| \mid H \in C_c^\infty([0, 1] \times M, \mathbb{R}) : \varphi_H^1 = \varphi \}. \end{aligned}$$

We call (M, ω) (*geometrically*) *bounded* iff there exist an almost complex structure J on M and a complete Riemannian metric g such that the following conditions hold:

- The sectional curvature of g is bounded and $\inf_{x \in M} \iota_x^g > 0$, where ι_x^g denotes the injectivity radius of g at the point $x \in M$.
- There exists a constant $C \in (0, \infty)$ such that

$$|\omega(v, w)| \leq C|v||w|, \quad \omega(v, Jv) \geq C^{-1}|v|^2,$$

for all $v, w \in T_x M$ and $x \in M$. Here $|v| := \sqrt{g(v, v)}$.

Notation

To simplify notation, in the rest of this article we will drop the subscript c (for “compact support”), and write

$$\text{Ham}(M) := \text{Ham}_c(M), \quad \|\cdot\|^M := \|\cdot\|_c^M.$$

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2 Proof of Theorem 1

2.1 Disjoint supports and small Hofer norms

A key ingredient for the proof of Theorem 1 is the following. Let (M, ω) be a symplectic manifold, $\varphi \in \text{Ham}(M) = \text{Ham}_c(M)$, and $X \subseteq M$ a subset. We define

$$c_{\pm} := 1, \quad c_0 := 2,$$

and for $\nu \in \{+, -, 0\}$, we define

$$\|\varphi\|^{X, \nu} := \|\varphi\|^{M, X, \nu} := \tag{26}$$

$$c_{\nu} \inf \{ \|H\| \mid H \in C^{\infty}([0, 1] \times M, \mathbb{R}) : \varphi_H^1 = \varphi, \text{supp } H \subseteq [0, 1] \times X, \nu H \geq 0 \}.$$

(Here our convention is that $\inf \emptyset = \infty$.)

Proposition 8 (Disjoint supports and small Hofer norm) *Let $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in \text{Ham}(M)$, $X_1, \dots, X_N \subseteq M$ be (pairwise) disjoint subsets, and $\nu \in \{+, -, 0\}$. Then we have*

$$\|\varphi_1 \circ \dots \circ \varphi_N\|^M \leq \max_{i=1, \dots, N} \|\varphi_i\|^{X_i, \nu}. \tag{27}$$

(Recall that we simplified notation by writing $\|\cdot\|^M := \|\cdot\|_c^M$.) The proof of this result is an adaption of the proof of [HZ, Lemma 9 of Chapter 5.6, p. 176]: Consider the case $\nu = 0$. For $i = 1, \dots, N$, fix a time-dependent function H_i that generates φ_i , has support in X_i at each time and Hofer norm close to the Hofer norm of φ_i on X_i . The rough idea is to consider the sum

$$H := \sum_i H_i.$$

This generates the composition $\varphi_1 \circ \dots \circ \varphi_N$, since the sets X_1, \dots, X_N are disjoint. However, its norm need not be close the right hand side of (27). In

order to achieve this condition, we need to reparametrize the function H_i in time in a suitable way. This is possible by the following lemma.

Lemma 9 *Let $N \in \mathbb{N}$, $f_1, \dots, f_N : [0, 1] \rightarrow [0, \infty)$ be measurable functions, and $\varepsilon > 0$. Then there exist orientation preserving diffeomorphisms $\varphi_1, \dots, \varphi_N$ of $[0, 1]$, such that*

$$\int_0^1 \max_i (f_i \circ \varphi_i(\tilde{t})) \dot{\varphi}_i(\tilde{t}) d\tilde{t} \leq \max_i \int_0^1 f_i(t) dt + \varepsilon. \quad (28)$$

The proof of this lemma is given in the appendix on page 39.

Proof of Proposition 8: Without loss of generality we may assume that $N \geq 2$, $X_i \neq \emptyset$, for every i , and the right hand side of (27) is finite. Let $\varepsilon > 0$. Then for every $i = 1, \dots, N$ there exists a function $H_i \in C_c^\infty([0, 1] \times M, \mathbb{R})$ such that

$$\varphi_{H_i}^1 = \varphi_i, \text{ supp } H_i \subseteq [0, 1] \times X_i, \nu H_i \geq 0, \|H_i\| < \|\varphi_i\|^{X_i, \nu} + \varepsilon. \quad (29)$$

For $i = 1, \dots, N$ we define

$$f_i : [0, 1] \rightarrow \mathbb{R}, \quad f_i(t) := \max_M H_i^t - \min_M H_i^t. \quad (30)$$

Since H_i has compact support, the function f_i is continuous, and hence measurable. Hence we may apply Lemma 9, obtaining diffeomorphisms φ_i of $[0, 1]$ as in the conclusion of that lemma. Let $i = 1, \dots, N$. We define

$$\begin{aligned} \tilde{H}_i : [0, 1] \times M &\rightarrow \mathbb{R}, \quad \tilde{H}_i^{\tilde{t}}(x) := \tilde{H}_i(\tilde{t}, x) := \dot{\varphi}_i(\tilde{t}) H_i^{\varphi_i(\tilde{t})}(x), \\ \tilde{H} : [0, 1] \times M &\rightarrow \mathbb{R}, \quad \tilde{H}^{\tilde{t}} := \sum_i \tilde{H}_i^{\tilde{t}}. \end{aligned} \quad (31)$$

Let $\tilde{t} \in [0, 1]$. Let $j = 1, \dots, N$. Since $\text{supp } \tilde{H}_j \subseteq [0, 1] \times X_j$, we have $d\tilde{H}_j^{\tilde{t}} = 0$ on $M \setminus X_j$. Since $X_j \cap X_i = \emptyset$ if $j \neq i$, it follows that

$$d\tilde{H}^{\tilde{t}} = d\tilde{H}_i^{\tilde{t}} \text{ on } X_i, \quad d\tilde{H}^{\tilde{t}} = 0 \text{ on } M \setminus \bigcup_i X_i.$$

It follows that

$$\varphi_{\tilde{H}}^1 = \varphi_{\tilde{H}_i}^1 \text{ on } X_i, \forall i,$$

and therefore, using $\varphi_{\tilde{H}_i}^1 = \varphi_i$,

$$\varphi_{\tilde{H}}^1 = \varphi_1 \circ \dots \circ \varphi_N. \quad (32)$$

Recall that $c_\pm = 1$ and $c_0 = 2$. Let $\tilde{t} \in [0, 1]$. Since $\tilde{H}_i^{\tilde{t}} = 0$ on $M \setminus X_i$, and the sets X_i are disjoint, we have

$$\max_M \tilde{H}^{\tilde{t}} - \min_M \tilde{H}^{\tilde{t}} \leq c_\nu \max_i \left(\max_M \tilde{H}_i^{\tilde{t}} - \min_M \tilde{H}_i^{\tilde{t}} \right). \quad (33)$$

Furthermore, using (30,31), we have

$$\max_M \tilde{H}_i^t - \min_M \tilde{H}_i^t = (f_i \circ \varphi_i)(\tilde{t}) \dot{\varphi}_i(\tilde{t}), \quad \forall i.$$

Combining this and (33) with (28,29), we obtain

$$\begin{aligned} \|\tilde{H}\| &\leq c_\nu \left(\max_i \int_0^1 f_i(t) dt + \varepsilon \right) \\ &= c_\nu \left(\max_i \|H_i\| + \varepsilon \right) \\ &< c_\nu \left(\max_i \|\varphi_i\|^{X_i, \nu} + 2\varepsilon \right). \end{aligned}$$

Using (32) and the fact that $\varepsilon > 0$ is arbitrary, inequality (27) follows. This proves Proposition 8. \square

Remark 10 *In the above proof, in the case $\nu = 0$, we really need the factor $c_0 = 2$ in inequality (33). As an example, consider $N = 2$, assume that $\tilde{H}_1 \geq 0$ and $\tilde{H}_1 \not\equiv 0$, and that $\tilde{H}_2 = -\tilde{H}_1$. Then without the factor $c_0 = 2$, inequality (33) is wrong.*

2.2 Moving disks with small Hofer energy

The proof of Theorem 1 relies on the fact that balls inside two dimensional symplectic manifolds can easily be moved around. This is the content of the following result.

Proposition 11 (Moving disks with small Hofer energy) *Let (M, ω) be connected two-dimensional symplectic manifold, $c \in \mathbb{R}$, X_0, X_1 disjoint images of embeddings $\overline{B}^2 \rightarrow M$, and $\nu = \pm$. Assume that*

$$\int_{X_0} \omega = \int_{X_1} \omega < c.$$

Then there exists a function $H \in C_c^\infty(M, \mathbb{R})$ such that

$$\begin{aligned} \varphi_H^1(X_0) &= X_1, \\ \nu H &\geq 0, \\ \|H\| &< c. \end{aligned}$$

The proof of this proposition will be given on page 18. It is based on the following flexibility result for balls in manifolds with volume forms. Let M

be an oriented manifold (without boundary). By a *volume form* on M we mean a top degree form Ω that induces the orientation of M . (Such a form does not vanish anywhere.) By the *support* of a map $\varphi : M \rightarrow M$ we mean the set

$$\text{supp } \varphi := \overline{\{x \in M \mid \varphi(x) \neq x\}}.$$

For $r > 0$ we denote by $B_r^n, \overline{B}_r^n \subseteq \mathbb{R}^n$ the open and closed ball of radius r , around 0, and we abbreviate $B^n := B_1^n, \overline{B}^n := \overline{B}_1^n$.

Proposition 12 *Assume that M is connected and of dimension at least 2. Let Ω be a volume form on M , $k \in \mathbb{N}$, and for $i = 0, 1$ let $X_i^1, \dots, X_i^k \subseteq M$ be a collection of disjoint images of smooth embeddings of the closed ball \overline{B}^n into M . If*

$$\int_{X_0^j} \Omega = \int_{X_1^j} \Omega, \quad \forall j \in \{1, \dots, k\}, \quad (34)$$

then there exists a diffeomorphism $\psi : M \rightarrow M$ with compact support, such that

$$\psi^* \Omega = \Omega, \quad \psi(X_0^j) = X_1^j, \quad \forall j \in \{1, \dots, k\}. \quad (35)$$

The proof of this result was outlined to me (for $k = 1$) by Professor Tatsuhiko Yagasaki. It is given on page 18. We need the following.

Lemma 13 *Let M be a connected manifold (without boundary) of dimension at least 2, $k \in \mathbb{N}$, and for $i = 0, 1$ and $j = 1, \dots, k$, let X_i^j be the image of a smooth embedding of \overline{B}^n into M . Then there exists a diffeomorphism $\psi : M \rightarrow M$ with compact support, such that*

$$\psi(X_0^j) = X_1^j, \quad \forall j = 1, \dots, k.$$

The proof of this lemma is given in the appendix, on page 41. The main ingredient of the proof of Proposition 12 is the following result, which is a relative version of Moser's Theorem involving a hypersurface. Here by the support of a differential form ω on a manifold M we mean the set

$$\overline{\{x \in M \mid \omega_x \neq 0\}}.$$

Proposition 14 *Let M be an oriented manifold, Ω_0 and Ω_1 volume forms on M , and $N \subseteq M$ a closed subset that is an oriented (real) codimension one submanifold. Assume that the support of $\Omega_1 - \Omega_0$ is compact and for every connected component M' of $M \setminus N$, we have*

$$\int_{M'} (\Omega_1 - \Omega_0) = 0. \quad (36)$$

Then there exists a diffeomorphism $\psi : M \rightarrow M$ with compact support, such that

$$\begin{aligned}\psi^*\Omega_1 &= \Omega_0, \\ \psi(M') &= M', \quad \forall \text{ connected component } M' \subseteq M \setminus N.\end{aligned}$$

Lemma 15 *The statement of Proposition 14 holds if $N = \emptyset$.*

Proof of Lemma 15: This follows from the argument by J. Moser [Mo] used to prove the statement for a closed manifold M . \square

For the proof of Proposition 14 in the case $N \neq \emptyset$ need the following.

Lemma 16 *Let M be an oriented manifold, and Ω_0 and Ω_1 volume forms on $\mathbb{R} \times M$, such that $\Omega_1 - \Omega_0$ has compact support. Then there exists a diffeomorphism ψ on $\mathbb{R} \times M$ and a neighborhood $U \subseteq \mathbb{R} \times M$ of $\{0\} \times M$, such that ψ has compact support, and*

$$\psi^*\Omega_1 = \Omega_0 \text{ on } U, \quad \psi = \text{id} \text{ on } \{0\} \times M.$$

Proof of Lemma 16: This follows from the proof of [McD1, Lemma A2]. \square

Proof of Proposition 14: Without loss of generality, we may assume that M is connected. We choose a tubular neighborhood $U_0 \subseteq M$ of N . (By definition this is the image of some embedding $\varphi : \mathbb{R} \times N \rightarrow M$, satisfying $\varphi(0, x) = x$, for every $x \in N$.) It follows from Lemma 16 that there exists a diffeomorphism $\psi_0 : U_0 \rightarrow U_0$ with compact support, and a neighborhood U of N such that

$$\psi_0^*\Omega_1 = \Omega_0 \text{ on } U, \quad \psi_0 = \text{id} \text{ on } N. \quad (37)$$

We extend ψ_0 to M by defining $\psi_0(x) := x$, for $x \in M \setminus U_0$. Let M' be a connected component of $M \setminus N$. An elementary argument shows that $\psi_0(M') = M'$. Since the supports of $\Omega_1 - \Omega_0$ and ψ_0 are compact, the same holds for the support of $\psi_0^*\Omega_1 - \Omega_0$. Using (36), it follows that

$$\int_{M'} (\psi_0^*\Omega_1 - \Omega_0) = \int_{M'} (\Omega_1 - \Omega_0) = 0.$$

Furthermore, the first equality in (37) implies that $\text{supp}(\psi_0^*\Omega_1 - \Omega_0) \subseteq M \setminus N$. Hence we may apply Lemma 15 with M, Ω_1 replaced by $M \setminus N, \psi_0^*\Omega_1$, to conclude that there exists a diffeomorphism ψ_1 of $M \setminus N$ with compact support, such that

$$\psi_1^*\psi_0^*\Omega_1 = \Omega_0.$$

We extend ψ_1 to M by defining $\psi_1(x) := x$ for $x \in N$. The map $\psi := \psi_0 \circ \psi_1$ has the desired properties. This completes the proof of Proposition 14. \square

Proof of Proposition 12: By Lemma 13 there exists a diffeomorphism ψ_0 of M , with compact support, such that $\psi_0(X_0^j) = X_1^j$, for every $j = 1, \dots, k$. Since $n \geq 2$, the connected components of $M \setminus N$ are

$$\text{Int } X_0^1, \dots, \text{Int } X_0^k, \quad M \setminus \bigcup_j X_0^j,$$

where $\text{Int } X$ denotes the interior of a subset $X \subseteq M$. Therefore, the hypothesis (34) implies that the condition (36) is satisfied with $\Omega_0 := \Omega$, $\Omega_1 := \psi_0^* \Omega$, $N := \bigcup_j \partial X_0^j$. Hence applying Proposition 14, there exists a diffeomorphism (denoted by ψ_1) as in the statement of that proposition. The map $\psi := \psi_0 \circ \psi_1$ has compact support and satisfies conditions (35). This proves Proposition 12. \square

For the proof of Proposition 11 we need the following.

Proposition 17 *Let $n \in \mathbb{N}$, and Ω_0, Ω_1 be volume forms on \mathbb{R}^n (equipped with the standard orientation), such that*

$$\int_{\mathbb{R}^n} \Omega_0 = \int_{\mathbb{R}^n} \Omega_1.$$

Then there exists a diffeomorphism ψ of \mathbb{R}^n such that $\psi^ \Omega_1 = \Omega_0$.*

Proof of Proposition 17: This is a special case of a theorem by R. Greene and K. Shiohama ([GS, Theorem 1]). \square

We are now ready for the proof of Proposition 11.

Proof of Proposition 11: Consider the case $\nu = +$. (The case $\nu = -$ is treated analogously.) Since M is connected, there exists an open subset $U \subseteq M$ that is diffeomorphic to \mathbb{R}^2 , contains $X_0 \cup X_1$, and satisfies

$$a := \frac{1}{2} \int_U \omega \leq c.$$

Without loss of generality, we may assume that

$$M = U = (-1, 1) \times (0, a).$$

Furthermore, by Proposition 17, we may assume without loss of generality that ω equals the standard structure ω_0 . We choose a subset X that is the image of an embedding $\overline{B}^2 \rightarrow (-1, 0) \times (0, a)$ and has area $\int_{X_0} \omega_0$. By Proposition 12, we may assume without loss of generality that

$$X_0 = X, \quad X_1 = X + (1, 0).$$

We choose a function $\rho \in C_c^\infty(M, [0, 1])$ such that

$$\rho(x) = 1, \quad \forall x \in \bigcup_{t \in [0, 1]} (X_0 + (t, 0)).$$

We define

$$H : M \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}, \quad H(q, p) := \rho(q, p)p.$$

The map $\psi := \varphi_H^1$ has the required properties. This proves Proposition 11. \square

2.3 Finding nice subsets of the disk

The last ingredient of the proof of Theorem 1 is the following result, which roughly states that there exists a nice collection of open subsets $U_i \subseteq \mathbb{R}^2$ containing $\overline{B}^2(a)$. The factors in the decomposition of $\tilde{\varphi}$ (where φ is the given Hamiltonian diffeomorphism) will have support in certain sets constructed from the U_i 's.

Lemma 18 (Nice subsets of the disk) *Let M be a manifold diffeomorphic to \mathbb{R}^2 , $a > 0$, $N \in \mathbb{N}$, and ω a symplectic form on M . Assume that*

$$\int_M \omega > 3Na.$$

Then there exist collections $(U_i)_{i=1, \dots, 2N}$, $(X_i)_{i=0, \dots, 2N}$, and $(\chi_j)_{j=1, \dots, N}$, where the U_i 's are open and connected subsets of M , the X_i 's are images of embeddings $\overline{B}^2 \rightarrow M$, and the χ_j 's are symplectomorphisms of M , such that for every $i, i' \in \{1, \dots, 2N\}$ and $j, j' \in \{1, \dots, N\}$, we have

$$X_0 \cup X_i \subseteq U_i, \tag{38}$$

$$\int_{X_i} \omega = a, \tag{39}$$

$$X_0 \cap X_i = \emptyset, \tag{40}$$

$$\chi_j|_{X_i} = \text{id}, \tag{41}$$

$$\chi_j(U_i) \cap \chi_{j'}(U_{i'}) = \emptyset \text{ if } i > i', j > j'. \tag{42}$$

In the proof of Theorem 1 we will choose collections $(U_i), (X_i), (\chi_j)$ as in this lemma, and apply Proposition 11 with M, X_1 replaced by U_i, X_i . The condition (42) will be used to make the supports of certain functions disjoint. Furthermore, condition (41) will ensure that composing these functions with $\chi_j \times \text{id}_{M'}$ does not change the Hamiltonian diffeomorphisms they generate.

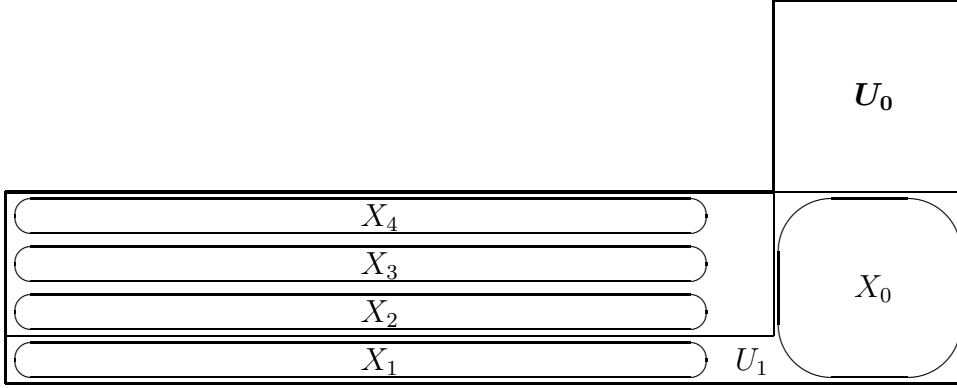


Figure 1: Some of the sets used in the proof, with $N = 2$. The set U_0 contains the other depicted sets and is bounded by the thick line. The set U_1 is the region bounded by thin and thick lines and containing X_0 and X_1 .

Proof of Lemma 18: We choose numbers

$$c \in (3Na, \int_M \omega), \quad (43)$$

$$b \in (a, \frac{c}{N} - 2a). \quad (44)$$

(Since $c > 3Na$, there is such a b .) We denote

$$d := c - Nb. \quad (45)$$

It follows that the set

$$U_0 := ((0, b) \times (0, N)) \cup ((-d, 0] \times (0, 1))$$

(see Figure 1) has standard area equal to c . Hence using (43), there exists an open subset $U \subseteq \mathbb{R}^2$ that is diffeomorphic to \mathbb{R}^2 , has standard area equal to $\int_M \omega$, and contains $\overline{U_0}$. We may assume without loss of generality that

$$M = U.$$

We denote by ω_0 the standard symplectic form on \mathbb{R}^2 . Using Proposition 17, we may assume without loss of generality that $\omega = \omega_0$. Since $b > a$, there exists a subset

$$X_0 \subseteq (0, b) \times (0, 1)$$

with ω_0 -area equal to a , which is the image of an embedding of \overline{B}^2 . Let $i = 1, \dots, N$. We define

$$U_i := (0, b) \times (0, 1) \cup \left((-d, 0] \times \left(\frac{i-1}{2N}, \frac{i}{2N} \right) \right).$$

It follows from (44,45) that $d/2N > a$. Hence we may choose a subset

$$X_i \subseteq (-d, 0) \times \left(\frac{i-1}{2N}, \frac{i}{2N}\right) \quad (46)$$

that is the image of an embedding of \overline{B}^2 and has (standard) area equal to a . We choose $\varepsilon > 0$ so small that

$$[-\varepsilon, 0] \times [0, N] \subseteq M \setminus \bigcup_{i=1}^{2N} X_i. \quad (47)$$

We also choose a function $f \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$f(x) = \frac{\varepsilon}{2}, \quad \forall x \in (-\infty, -\varepsilon], \quad (48)$$

$$f(x) = -x, \quad \forall x \in [0, \infty), \quad (49)$$

$$-1 \leq f'(x) \leq 0, \quad \forall x \in [-\varepsilon, 0]. \quad (50)$$

Furthermore, we define

$$K := ([-d, 0] \times [0, 1]) \cup ([-\varepsilon, b] \times [0, N]),$$

and we choose a function $\rho \in C_c^\infty(M, \mathbb{R})$ such

$$\rho = 1 \quad \text{on } K. \quad (51)$$

We define

$$H : M \rightarrow \mathbb{R}, \quad H(q, p) := \rho(q, p)f(q), \quad (52)$$

$$\chi_j := \varphi_{(j-1)H}^1, \quad \forall j = 1, \dots, N. \quad (53)$$

Claim 1 *The sets X_i, U_i and the maps χ_j satisfy the requirements of Lemma 18.*

Proof of Claim 1: The sets U_i are clearly open and connected, and they satisfy (38). Conditions (39,40) are satisfied by construction. Condition (41) follows from (46,47,48,51,52,53).

We show that condition (42) holds. (See Figure 2.) Let $j \in \{1, \dots, N\}$. It follows from (49,50,51,52) that

$$X_H(q, p) = (0, -f'(q)) \in \mathbb{R} \times [0, 1], \quad \forall (q, p) \in [-\varepsilon, b] \times [0, N].$$

Therefore, using (53), we have

$$\chi_j(q, p) = (q, p - (j-1)f'(q)), \quad \forall (q, p) \in [-\varepsilon, b] \times [0, 1]. \quad (54)$$

On the other hand, it follows from (48) that $\chi_j = \text{id}$ on $[-d, -\varepsilon] \times [0, 1]$. Combining this with (54) and using (49,50), condition (42) follows. Hence the sets X_i, U_i and the maps χ_j have the required properties. This proves Claim 1, and completes the proof of Lemma 18. \square

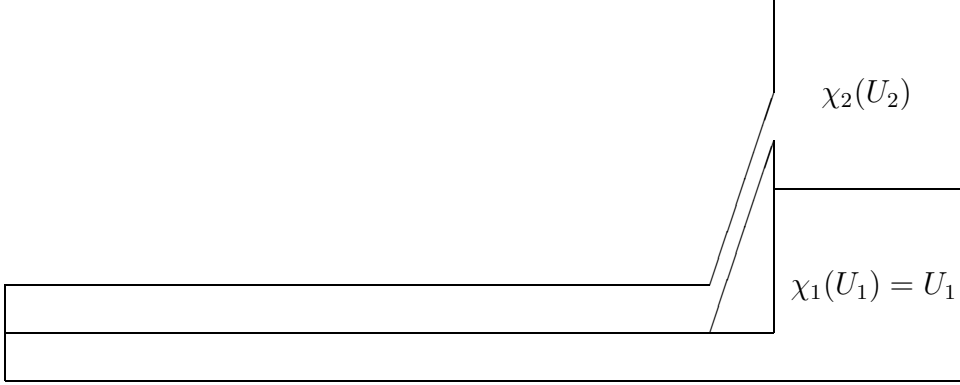


Figure 2: The sets $\chi_1(U_1) = U_1$ and $\chi_2(U_2)$.

2.4 Proof of Theorem 1 (Relative Hofer geometry of an open subset)

Using the results of the previous sections, we are now able to prove the main result of this article. We will use the following.

Remark 19 *Let (M, ω) be a symplectic manifold, $X \subseteq M$ a subset, $\varphi, \varphi' \in \text{Ham}(M)$, $\psi : M \rightarrow M$ a symplectomorphism, and $\nu \in \{+, -, 0\}$. Recall the definition (26). We have*

$$\begin{aligned} \|\varphi^{-1}\|^{X, -\nu} &= \|\varphi\|^{X, \nu} = \|\psi \circ \varphi \circ \psi^{-1}\|^{\psi(X), \nu}, \\ \|\varphi \circ \varphi'\|^{X, \nu} &\leq \|\varphi\|^{X, \nu} + \|\varphi'\|^{X, \nu} \end{aligned}$$

(Here our convention is $-+ = -$, $-- = +$, $-0 = 0$.) These assertions follow from elementary arguments, using [HZ, Chapter 5, Proposition 1].

For simplicity, in the proof of Theorem 1, for the composition of two maps φ and ψ we write

$$\varphi\psi := \varphi \circ \psi.$$

Proof of Theorem 1: It suffice to prove the following inequality. Let $H \in C^\infty([0, 1] \times M, \mathbb{R})$ be a function with compact support contained in $[0, 1] \times U$. Assume that $a \in (0, \infty)$, $N \in \mathbb{N}$, (M', ω') is a symplectic manifold, and $\psi : B^2(3Na) \times M' \rightarrow M$ is a symplectic embedding, satisfying

$$U \subseteq \psi(B^2(a) \times M').$$

Then we have

$$\|\varphi_H^1\|^M \leq 8a + \frac{2}{N}\|H\|. \quad (55)$$

To prove this inequality, note that without loss of generality, we may assume that

$$M = B^2(3Na) \times M', \quad \omega = \omega_0 \oplus \omega', \quad \psi = \text{id}.$$

Since the set

$$\text{Supp } H := \bigcup_{t \in [0,1]} \text{supp } H^t$$

is compact and contained in $B^2(a) \times M'$, there exists $a' < a$ such that

$$\text{Supp } H \subseteq B^2(a') \times M'. \quad (56)$$

We choose numbers $0 < t_1 < \dots < t_{2N-1} < 1$ such that, defining $t_0 := 0$, $t_{2N} := 1$, we have

$$\int_{t_{i-1}}^{t_i} \left(\max_M H^t - \min_M H^t \right) dt = \frac{1}{2N} \|H\|, \quad (57)$$

for every $i \in \{1, \dots, 2N\}$. We define

$$\varphi_i := \varphi_H^{t_i}, \quad \forall i \in \{0, \dots, 2N\}. \quad (58)$$

Applying Lemma 18 with M, ω, a replaced by $B^2(3Na), \omega_0, a'$, there exist collections $(U_i)_{i=1, \dots, 2N}$, $(X_i)_{i=0, \dots, 2N}$, and $(\chi_j)_{j=1, \dots, N}$, where $U_i \subseteq B^2(3Na)$ is an open and connected subset, X_i is the image of some embedding $\overline{B}^2 \rightarrow B^2(3Na)$, and χ_j is a symplectomorphism of $B^2(3Na)$ (with respect to ω_0), such that

$$\int_{X_i} \omega_0 = a', \quad \forall i \in \{0, \dots, 2N\}, \quad (59)$$

and the conditions (38,40,41,42) are satisfied. Using Proposition 12, we may assume without loss of generality that

$$X_0 = \overline{B}^2(a').$$

Let $i = 1, \dots, 2N$. We define

$$\nu_i := \begin{cases} +, & \text{if } i \text{ is even,} \\ -, & \text{if } i \text{ is odd.} \end{cases} \quad (60)$$

Using (38,40,59), the hypotheses of Proposition 11 are satisfied, with $c := a$, and M, ω, X_1 replaced by U_i, ω_0, X_i and $\nu := \nu_i$. Applying this Proposition, there exists a map $\psi_i \in \text{Ham}(M)$ such that

$$\psi_i(\overline{B}^2(a')) = X_i, \quad (61)$$

$$\|\psi_i\|^{U_i, \nu_i} < a, \quad (62)$$

where $\|\psi_i\|^{U_i, \nu_i}$ is defined as in (26). (Note that the condition (62) implies that ψ_i is generated by some function with support in $[0, 1] \times U_i$.) We define

$$\psi_0 := \text{id}.$$

Let $i = 0, \dots, 2N$. We define

$$\tilde{U}_i := U_i \times M'.$$

The conditions (38,56) imply that $\text{Supp } H \subseteq \tilde{U}_i$. Hence it follows from (62) and a cutoff argument that there exists a map $\tilde{\psi}_i \in \text{Ham}(M)$ such that

$$\tilde{\psi}_i = \psi_i \times \text{id}_{M'} \text{ on } \text{Supp } H, \quad (63)$$

$$\|\tilde{\psi}_i\|^{\tilde{U}_i, \nu_i} < a. \quad (64)$$

For every pair of maps $\varphi, \psi : M \rightarrow M$ we denote

$$\text{ad}_\varphi \psi := \varphi \psi \varphi^{-1}.$$

We define

$$\varphi'_i := \begin{cases} \text{ad}_{\tilde{\psi}_i} \varphi_i, & \text{if } i \text{ is even,} \\ \text{ad}_{\tilde{\psi}_i} \varphi_i^{-1}, & \text{if } i \text{ is odd.} \end{cases} \quad (65)$$

Using (58), we have

$$\varphi_H^1 = \text{ad}_{\tilde{\psi}_{2N}^{-1}} \varphi'_{2N}. \quad (66)$$

By the triangle inequality, we have

$$\|\varphi'_{2N}\|^M \leq \|\varphi'_{2N} \cdots \varphi'_1\|^M + \|(\varphi'_{2N-1} \cdots \varphi'_1)^{-1}\|^M. \quad (67)$$

Claim 1 *For every $m \in \{1, \dots, 2N\}$, we have*

$$\|\varphi'_m \cdots \varphi'_1\|^M \leq 4a + \frac{1}{N} \|H\|. \quad (68)$$

Proof of Claim 1: For a number $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the biggest integer $\leq x$, and we define

$$\ell := \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Let $j = 0, \dots, \ell$. We define

$$\begin{aligned} U_0 &:= \emptyset, \\ V_j &:= U_{m-2j} \cup U_{m-2j-1}, \\ \widetilde{V}_j &:= V_j \times M', \\ \widetilde{W}_j &:= \chi_{\ell-j+1}(V_j) \times M', \\ \widetilde{\chi}_j &:= \chi_{\ell-j+1} \times \text{id}_{M'}, \end{aligned}$$

$$\Psi_j^1 := \text{ad}_{\widetilde{\chi}_j} \widetilde{\psi}_{m-2j}, \quad (69)$$

$$\Psi_j^2 := \text{ad}_{\widetilde{\chi}_j \varphi_{m-2j}^{\nu_m}} (\widetilde{\psi}_{m-2j}^{-1} \widetilde{\psi}_{m-2j-1}), \quad (70)$$

$$\Psi_j^3 := \text{ad}_{\widetilde{\chi}_j} (\varphi_{m-2j}^{\nu_m} \varphi_{m-2j-1}^{-\nu_m}), \quad (71)$$

$$\Psi_j^4 := \text{ad}_{\widetilde{\chi}_j} \widetilde{\psi}_{m-2j-1}^{-1}. \quad (72)$$

Claim 2 *We have*

$$\varphi'_m \cdots \varphi'_1 = \Psi_0^1 \cdots \Psi_\ell^1 \cdots \Psi_0^4 \cdots \Psi_\ell^4. \quad (73)$$

For the proof of this claim, we need the following.

Claim 3 *We have*

$$\text{supp } \Psi_j^k \subseteq \widetilde{W}_j, \quad \forall k \in \{1, \dots, 4\}, j \in \{1, \dots, \ell\}. \quad (74)$$

Proof of Claim 3: Let $i = 0, \dots, 2N$ and recall the notation $\widetilde{U}_i = U_i \times M'$. Inequality (64) implies that there exists a function with support in $[0, 1] \times \widetilde{U}_i$, which generates $\widetilde{\psi}_i$. Hence we have

$$\text{supp } \widetilde{\psi}_i \subseteq \widetilde{U}_i. \quad (75)$$

Combining this with the fact $\text{supp}(\text{ad}_\psi \varphi) = \psi(\text{supp } \varphi)$, it follows that (74) holds for $k = 1$ and 4 . To see that it holds for $k = 2, 3$, note that condition (56) and definition (58) imply that

$$\text{supp } \varphi_i \subseteq B^2(a') \times M'. \quad (76)$$

Since $X_0 = \overline{B}^2(a')$, (38) implies that $\overline{B}^2(a') \times M' \subseteq \widetilde{U}_i$. Hence by (76), we have

$$\text{supp } \varphi_i \subseteq \widetilde{U}_i. \quad (77)$$

Using (75), it follows that for $j = 1, \dots, \ell$, we have

$$\text{supp}(\text{ad}_{\varphi_{m-2j}^{\nu_m}}(\tilde{\psi}_{m-2j}^{-1}\tilde{\psi}_{m-2j-1})) \subseteq \tilde{V}_j.$$

Combining this with the fact $\text{supp}(\text{ad}_\psi\varphi) = \psi(\text{supp}\varphi)$, it follows that (74) holds for $k = 2$. Finally, (77) implies (74) for $k = 3$. This proves Claim 3. \square

Proof of Claim 2: Let $j \in \{1, \dots, \ell\}$, and note that

$$\Psi_j^1 \cdots \Psi_j^4 = \text{ad}_{\tilde{\chi}_j}(\varphi'_{m-2j}\varphi'_{m-2j-1}). \quad (78)$$

Furthermore, using (65,76,63,61) and the fact $\text{supp}(\text{ad}_\psi\varphi) = \psi(\text{supp}\varphi)$, it follows that

$$\text{supp}\varphi'_i = \tilde{\psi}_i(\text{supp}\varphi_i) \subseteq \tilde{X}_i := X_i \times M', \quad \forall i \in \{1, \dots, 2N\}. \quad (79)$$

It follows from (41) that $\tilde{\chi}_j = \text{id}$ on \tilde{X}_i for every $i = 1, \dots, 2N$. Combining this with (79), it follows that

$$\text{ad}_{\tilde{\chi}_j}(\varphi'_{m-2j}\varphi'_{m-2j-1}) = \varphi'_{m-2j}\varphi'_{m-2j-1}. \quad (80)$$

(Here for odd m and $j = \ell$ we use that $\varphi'_0 = \text{id}$.) It follows from (42) that the sets \tilde{W}_j , $j = 1, \dots, \ell$, are disjoint. Therefore, using (74), it follows that the maps Ψ_j^k and $\Psi_{j'}^{k'}$ commute, if $j, j' \in \{1, \dots, \ell\}$ are such that $j \neq j'$, and $k, k' \in \{1, \dots, 4\}$. Combining this with (78,80), equality (73) follows. This proves Claim 2. \square

It follows from Claim 2 that

$$\|\varphi'_m \cdots \varphi'_1\|^M \leq \sum_{k=1}^4 \|\Psi_0^k \cdots \Psi_\ell^k\|^M. \quad (81)$$

Recall the definition (26). We define

$$\begin{aligned} \mu_1^0 &:= +, & \mu_2^0 &:= -, & \mu_3^0 &:= 0, & \mu_4^0 &:= +, \\ \mu_k &:= \begin{cases} +\mu_k^0, & \text{if } m \text{ is even,} \\ -\mu_k^0, & \text{if } m \text{ is odd,} \end{cases} & \forall k \in \{1, 2, 3, 4\}. \end{aligned}$$

(Here our convention is $++ = -- = +$, $+- = -+ = -$ and $\pm 0 = 0$.) Let $k \in \{1, 2, 3, 4\}$. Since the sets \tilde{W}_j , $j = 0, \dots, \ell$, are disjoint, we may apply Proposition 8, to conclude that

$$\|\Psi_0^k \cdots \Psi_\ell^k\|^M \leq \max_{j=0, \dots, \ell} \|\Psi_j^k\|^{\tilde{W}_j, \mu_k}. \quad (82)$$

Consider now the **case** $k \neq 2$ and let $j \in \{0, \dots, \ell\}$. Since $\widetilde{W}_j = \widetilde{\chi}_j(\widetilde{V}_j)$, by Remark 19, we have

$$\|\Psi_j^k\|_{\widetilde{W}_j, \mu_k} = \|\text{ad}_{\widetilde{\chi}_j^{-1}} \Psi_j^k\|_{\widetilde{V}_j, \mu_k}.$$

Using (69,71,72) and the facts $\widetilde{U}_{m-2j}, \widetilde{U}_{m-2j-1}, \overline{B}^2(a') \times M' \subseteq \widetilde{V}_j$, it follows that

$$\|\Psi_j^1\|_{\widetilde{W}_j, \mu_1} \leq \|\widetilde{\psi}_{m-2j}\|_{\widetilde{U}_{m-2j}, \mu_1}, \quad (83)$$

$$\|\Psi_j^3\|_{\widetilde{W}_j, 0} \leq \|\varphi_{m-2j} \varphi_{m-2j-1}^{-1}\|_{\overline{B}^2(a') \times M', 0}, \quad (84)$$

$$\|\Psi_j^4\|_{\widetilde{W}_j, \mu_4} \leq \|\widetilde{\psi}_{m-2j-1}^{-1}\|_{\widetilde{U}_{m-2j-1}, \mu_4}. \quad (85)$$

Note that $\mu_1 = \nu_{m-2j}$. Combining this with (64,83), it follows that

$$\|\Psi_j^1\|_{\widetilde{W}_j, \mu_1} < a. \quad (86)$$

Furthermore, we have $\mu_4 = -\nu_{m-2j-1}$. Hence using Remark 19, (64,85) imply that

$$\|\Psi_j^4\|_{\widetilde{W}_j, \mu_4} < a. \quad (87)$$

Moreover, by (58) the map $\varphi_{m-2j} \varphi_{m-2j-1}^{-1}$ is generated in a Hamiltonian way by family of functions H^t , with $t \in [t_{m-2j-1}, t_{m-2j}]$. Since $\text{Supp } H \subseteq \overline{B}^2(a') \times M'$, using (84,57), it follows that

$$\|\Psi_j^3\|_{\widetilde{W}_j, 0} \leq 2 \int_{t_{m-2j-1}}^{t_{m-2j}} (\max_M H^t - \min_M H^t) dt = \frac{1}{N} \|H\|. \quad (88)$$

(Recall here that in the definition of the left hand side there is a factor $c_0 = 2$.) Consider now the **case** $k = 2$. Since $\text{supp } \varphi_{m-2j} \subseteq \overline{B}^2(a') \times M' \subseteq \widetilde{V}_j$, we have $\widetilde{\chi}_j \varphi_{m-2j}(\widetilde{V}_j) = \widetilde{W}_j$. Therefore, by Remark 19, we have

$$\|\Psi_j^2\|_{\widetilde{W}_j, \mu_2} = \|\text{ad}_{(\widetilde{\chi}_j \varphi_{m-2j})^{-1}} \Psi_j^2\|_{\widetilde{V}_j, \mu_2}. \quad (89)$$

Using (70), Remark 19, and (64), we have

$$\|\text{ad}_{(\widetilde{\chi}_j \varphi_{m-2j})^{-1}} \Psi_j^2\|_{\widetilde{V}_j, \mu_2} \leq \|\widetilde{\psi}_{m-2j}\|_{\widetilde{U}_{m-2j}, \mu_1} + \|\widetilde{\psi}_{m-2j-1}\|_{\widetilde{U}_{m-2j-1}, \mu_2} < 2a.$$

Combining this with (89), it follows that $\|\Psi_j^2\|_{\widetilde{W}_j, \mu_2} < 2a$. Combining this with (81,82,86,87,88), inequality (68) follows. This proves Claim 1. \square

Combining (66,67) and Claim 1, the claimed inequality (55) follows. This completes the proof of Theorem 1. \square

3 Proofs of Theorem 5 and Corollary 6

In this section we prove Theorem 5 and Corollary 6, adapting the proof of [Os, Theorem 1.1], which is based on a result by M. Schwarz.

Let (M, ω) be an aspherical symplectic manifold (i.e., (4) holds) and $H \in C_c^\infty([0, 1] \times M, \mathbb{R})$. We define the *action spectrum* Σ_H as follows. We denote by $\mathbb{D} \subseteq \mathbb{C}$ the closed unit disk, and define the set of *contractible H -periodic points* to be

$$\mathcal{P}^\circ(H) := \{x_0 \in M \mid \exists u \in C^\infty(\mathbb{D}, M) : \varphi_H^t(x_0) = u(e^{2\pi it}), \forall t \in [0, 1]\}.$$

We define the *H -twisted symplectic action* of $x_0 \in \mathcal{P}^\circ(H)$ to be

$$\mathcal{A}_H(x_0) := - \int_{\mathbb{D}} u^* \omega - \int_0^1 H(t, \varphi_H^t(x_0)) dt,$$

where $u \in C^\infty(\mathbb{D}, M)$ is any map satisfying $\varphi_H^t(x_0) = u(e^{2\pi it})$, for every $t \in [0, 1]$. It follows from asphericity of (M, ω) that this number does not depend on the choice of u and hence is well-defined. We define

$$\Sigma_H := \mathcal{A}_H(\mathcal{P}^\circ(H)) \subseteq \mathbb{R}.$$

Proposition 20 *Assume that M is closed. Then the spectrum Σ_H is compact.*

Proof of Proposition 20: This is part of the statement of [Sch, Proposition 3.7]. \square

The proof of Theorem 5 is based on the following result, which is a consequence of an argument by M. Schwarz.

Theorem 21 *Assume that (M, ω) is closed, connected, and strongly aspherical. Then we have*

$$\|\varphi_H^1\|^M \geq \min \Sigma_H + \frac{\int_0^1 (\int_M H^t \omega^n) dt}{\int_M \omega^n}. \quad (90)$$

Proof of Theorem 21: Assume first that H is *normalized*, i.e.,

$$\int_M H^t \omega^n = 0, \quad \forall t \in [0, 1]. \quad (91)$$

We denote by

$$C_0^\infty([0, 1] \times M, \mathbb{R})$$

the set of all normalized functions. We call a function $F \in C^\infty([0, 1] \times M, \mathbb{R})$ *regular* iff every fixed point x_0 of φ_F^1 is non-degenerate, i.e., 1 is not an eigenvalue of $d\varphi_F^1(x_0)$. It follows from the proof of Theorem 1.2 in the paper [Sch] by M. Schwarz, that there exists a map $c : C_0^\infty([0, 1] \times M, \mathbb{R}) \rightarrow \mathbb{R}$ that is continuous with respect to the Hofer norm, such that

$$c(F) \in \Sigma_F, \text{ and if } F \text{ regular} \implies c(F) \leq - \int_0^1 \min_M F^t dt, \quad (92)$$

for every $F \in C_0^\infty([0, 1] \times M, \mathbb{R})$. (For the estimate see the argument on p. 429 in [Sch]. Here we use the hypothesis that M is closed and ω and $c_1(M, \omega)$ vanish on $\pi_2(M)$.) The set of regular normalized functions is dense in $C_0^\infty([0, 1] \times M, \mathbb{R})$ with respect to the Hofer norm, see [Sa]. Using the second condition in (92) and continuity of c , it follows that

$$c(H) \leq - \int_0^1 \min_M H^t dt. \quad (93)$$

Since H is normalized, we have

$$\max_M H^t \geq 0, \quad \forall t \in [0, 1].$$

Combining this with (93), it follows that

$$c(H) \leq \|H\|. \quad (94)$$

By the first condition in (92) we have $c(H) \in \Sigma_H$. Hence inequality (90) follows from (91,94) and the fact $\|\varphi_H^1\|^M \geq \|H\|$.

In the general situation, we define

$$f : [0, 1] \rightarrow \mathbb{R}, \quad f(t) := \frac{\int_M H^t \omega^n}{\int_M \omega^n},$$

and $F : [0, 1] \times M \rightarrow \mathbb{R}$ by $F^t(x) := F(t, x) := H^t(x) - f(t)$. By straightforward arguments this function is normalized, generates φ_H^1 , and satisfies

$$\Sigma_F = \Sigma_H + \int_0^1 f(t) dt.$$

Therefore, inequality (90) follows from what we already proved. This proves Theorem 21. \square

In the proof of Theorem 5 we will also use the following. Let (M, ω) be a symplectic manifold and $H, H' \in C_c^\infty([0, 1] \times M, \mathbb{R})$. We denote by $H \# H' : [0, 1] \times M \rightarrow \mathbb{R}$ the time-concatenation of H and H' , given by

$$(H \# H')^t := \begin{cases} 2H^{2t}, & \text{if } t \in [0, \frac{1}{2}], \\ 2H'^{2t-1}, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Proposition 22 Assume $H^t, H'^t = 0$ for t in some neighborhood of $\{0, 1\}$, and defining $X := \bigcup_{t \in [0, 1]} \text{supp } H^t$, we have

$$\varphi_{H'}^1(X) \cap X = \emptyset. \quad (95)$$

Then

$$\mathcal{P}^\circ(H \# H') = \mathcal{P}^\circ(H'). \quad (96)$$

Furthermore, if (M, ω) is aspherical then we have

$$\mathcal{A}_{H \# H'}(x_0) = \mathcal{A}_{H'}(x_0), \quad \forall x_0 \in \mathcal{P}^\circ(H'). \quad (97)$$

For the proof of this result, we need the following. For a function $H \in C_c^\infty([0, 1] \times M, \mathbb{R})$ and $x_0 \in M$ we denote

$$\mathbb{D}_{x_0}^H := \{u \in C^\infty(\mathbb{D}, M) \mid u(e^{2\pi i t}) = \varphi_H^t(x_0), \forall t \in [0, 1]\}. \quad (98)$$

Lemma 23 Assume $H^t, H'^t = 0$ for t in some neighborhood of $\{0, 1\}$, and defining $X := \bigcup_{t \in [0, 1]} \text{supp } H^t$, condition (95) is satisfied. Let $x_0 \in \text{Fix}(\varphi_{H'}^1)$. Then there exists a bijection

$$\Phi : \mathbb{D}_{x_0}^{H'} \rightarrow \mathbb{D}_{x_0}^{H \# H'}, \quad (99)$$

such that

$$\int_{\mathbb{D}} \Phi(u')^* \omega = \int_{\mathbb{D}} u'^* \omega, \quad \forall u' \in \mathbb{D}_{x_0}^{H'}. \quad (100)$$

Proof of Lemma 23: We define the map Φ as follows. By hypothesis, there exists $\varepsilon > 0$ such that $H^t, H'^t = 0$ for $t \in [0, 2\varepsilon] \cup [1 - 2\varepsilon, 1]$. We choose a diffeomorphism $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ such that

$$\varphi(e^{2\pi i t}) = e^{2\pi i(2t-1)}, \quad \forall t \in [\frac{1}{2} + \varepsilon, 1 - \varepsilon]. \quad (101)$$

Let $u' \in \mathbb{D}_{x_0}^{H'}$. We define

$$v := \Phi(u') := u' \circ \varphi : \mathbb{D} \rightarrow M. \quad (102)$$

Claim 1 We have $v \in \mathbb{D}_{x_0}^{H \# H'}$.

Proof of Claim 1: Since $H'^t = 0$ for $t \in [0, 2\varepsilon] \cup [1 - 2\varepsilon, 1]$, we have

$$u'(e^{2\pi it}) = \varphi_{H'}^t(x_0) = x_0, \quad \forall t \in [-2\varepsilon, 2\varepsilon]. \quad (103)$$

It follows from (101) that

$$\varphi(e^{2\pi it}) \in \{e^{2\pi it'} \mid t' \in [-2\varepsilon, 2\varepsilon]\}, \quad \forall t \in [0, \frac{1}{2} + \varepsilon] \cup [1 - \varepsilon, 1].$$

Combining this with (102,103), it follows that

$$v(e^{2\pi it}) = x_0, \quad \forall t \in [0, \frac{1}{2} + \varepsilon] \cup [1 - \varepsilon, 1]. \quad (104)$$

Therefore, using again (103), we have

$$v(e^{2\pi it}) = \varphi_{H'}^{2t-1}(x_0) = \varphi_{H\#H'}^t(x_0), \quad \forall t \in [\frac{1}{2}, \frac{1}{2} + \varepsilon] \cup [1 - \varepsilon, 1]. \quad (105)$$

Furthermore, it follows from (95) that

$$x_0 \notin X = \bigcup_{t \in [0,1]} \text{supp } H^t. \quad (106)$$

This implies that

$$\varphi_H^t(x_0) = x_0, \quad \forall t \in [0, 1]. \quad (107)$$

Combining this with (104), it follows that

$$v(e^{2\pi it}) = \varphi_H^{2t}(x_0) = \varphi_{H\#H'}^t(x_0), \quad \forall t \in [0, \frac{1}{2}]. \quad (108)$$

Finally, it follows from (101,102) and the fact $u' \in \mathbb{D}_{x_0}^{H'}$, that

$$v(e^{2\pi it}) = \varphi_{H'}^{2t-1}(x_0) = \varphi_{H\#H'}^t(x_0), \quad \forall t \in [\frac{1}{2} + \varepsilon, 1 - \varepsilon].$$

Combining this with (105,108), it follows that $v \in \mathbb{D}_{x_0}^{H\#H'}$. This proves Claim 1. \square

A similar argument shows that the map

$$v \circ \varphi^{-1} \in \mathbb{D}_{x_0}^{H'}, \quad \forall v \in \mathbb{D}_{x_0}^{H\#H'}.$$

It follows that the map Φ is a bijection.

Equality (100) follows from (102), using that φ is orientation preserving. This proves Lemma 23. \square

Proof of Proposition 22: The equality (96) follows from Lemma 23.

We prove the **second statement**. Assume that the hypotheses of this part of the proposition are satisfied, and that $x_0 \in \mathcal{P}^\circ(H')$. It follows from the definition of $H \# H'$ that

$$\int_0^1 (H \# H')^t \circ \varphi_{H \# H'}^t(x_0) dt = \int_0^1 H^t \circ \varphi_H^t(x_0) dt + \int_0^1 H'^t \circ \varphi_{H'}^t(x_0) dt. \quad (109)$$

Furthermore, it follows from (95) that

$$x_0 \notin X = \bigcup_{t \in [0,1]} \text{supp } H^t. \quad (110)$$

This implies that $\varphi_H^t(x_0) = x_0$, for every $t \in [0, 1]$. Hence, using (110) again, it follows that

$$H^t \circ \varphi_H^t(x_0) = 0, \quad \forall t \in [0, 1]. \quad (111)$$

We choose a map Φ as in Lemma 23 and a map $u' \in \mathbb{D}_{x_0}^{H'}$. (defined as in (98)). The claimed equality (97) is a consequence of (109,111,100). This completes the proof of Proposition 22. \square

Proof of Theorem 5: Without loss of generality, we may assume that M is connected and $U \neq \emptyset$. For a measurable subset $X \subseteq M$ we write

$$|X| := \int_X \omega^n.$$

Let $C > 0$ and

$$c < c_0 := \frac{|U|}{|M|} \quad (112)$$

be a positive constant.

Claim 1 *There exists $\varphi \in \text{Ham}(U)$ such that*

$$\|\tilde{\varphi}\|^M \geq \max\{C, c\|\varphi\|^U\}. \quad (113)$$

Proof of Claim 1: By hypothesis there exists a function $F \in C^\infty([0, 1] \times M, \mathbb{R})$ such that

$$\varphi_F^1(U) \cap U = \emptyset. \quad (114)$$

Reparametrizing F , we may assume that $F^t = 0$ for t in some neighborhood of $\{0, 1\}$. Furthermore, replacing F^t by $F^t - \int_M F^t \omega^n / |M|$, we may assume that F is normalized, i.e.,

$$\int_M F^t \omega^n = 0, \quad \forall t \in [0, 1]. \quad (115)$$

We choose a compact subset $K \subseteq U$ such that

$$\frac{|K|}{|M|} > c. \quad (116)$$

Furthermore, we choose a function $H_0 \in C^\infty(U, [0, 1])$ such that

$$H_0|_K = 1. \quad (117)$$

It follows from Proposition 20 that the minimum $\min \Sigma_F$ exists. We define

$$t_0 := \max \left\{ \frac{\|F\| - \min \Sigma_F}{\frac{|K|}{|M|} - c}, \frac{C}{c} \right\}. \quad (118)$$

It follows from (116) that $t_0 < \infty$. We define

$$\varphi := \varphi_{t_0 H_0}^1.$$

Claim 2 *This map satisfies inequality (113).*

Proof of Claim 2: We choose a function $f \in C^\infty([0, 1], [0, 1])$ such that $f = i$ in a neighborhood of i , for $i = 0, 1$. We define

$$H : [0, 1] \times M \rightarrow \mathbb{R}, \quad H^t(x) := \begin{cases} f'(t)t_0 H_0(x), & \text{if } x \in U, \\ 0, & \text{otherwise.} \end{cases} \quad (119)$$

We have

$$\varphi_H^1 = \tilde{\varphi}.$$

Using (114), the second condition in (117) and asphericity of (M, ω) , the hypotheses of Proposition 22 with $H' := F$ are satisfied. Hence applying this proposition, it follows that

$$\Sigma_{H \# F} = \Sigma_F. \quad (120)$$

Applying Theorem 21, we have

$$\|\varphi_{H \# F}^1\|^M \geq \min \Sigma_{H \# F} + \frac{\int_0^1 \left(\int_M (H \# F)^t \omega^n \right) dt}{|M|}. \quad (121)$$

Using the triangle inequality and the fact $\|\varphi_F^1\|^M \leq \|F\|$, we have

$$\|\varphi_H^1\|^M \geq \|\varphi_{H \# F}^1\|^M - \|F\|. \quad (122)$$

Furthermore, using (119,115,117), we have

$$\int_0^1 \left(\int_M (H \# F)^t \omega^n \right) dt \geq t_0 |K|.$$

Combining this with (121,122,120) and (118), we obtain

$$\|\tilde{\varphi}\|^M = \|\varphi_H^1\|^M \geq \min \Sigma_F - \|F\| + t_0 \frac{|K|}{|M|} \geq ct_0. \quad (123)$$

Using again (118), it follows that

$$\|\tilde{\varphi}\|^M \geq C. \quad (124)$$

Furthermore, using (119) and the fact $0 \leq H_0 \leq 1$, we have

$$\|\varphi\|^U \leq \|H\| = t_0.$$

Combining this with (123) and (124), inequality (113) follows. This proves Claim 2 and hence Claim 1. \square

We choose a map φ as in Claim 1. Inequality (113) and the fact $\|\varphi\|^U \geq \|\tilde{\varphi}\|^M$ imply that $\|\varphi\|^U \geq C$. Inequality (113) also implies that $\|\tilde{\varphi}\|^M / \|\varphi\|^U \geq c$. It follows that

$$\text{Lip}^\infty(M, U) \geq c.$$

Since $c < c_0$ (as defined in (112)) is arbitrary, the estimate (9) follows. This completes the proof of Theorem 5. \square

Proof of Corollary 6: Let $\varepsilon > 0$. We choose an area form σ on the two-torus \mathbb{T}^2 such that $\int_{\mathbb{T}^2} \sigma = c + \varepsilon$, and a symplectic embedding $\psi : B^2(c) \rightarrow \mathbb{T}^2$. We define

$$(M, U, \omega) := (\mathbb{T}^2 \times M', \psi(B^2(a - \varepsilon)) \times M', \sigma \oplus \omega').$$

Then the hypotheses of Theorem 5 are satisfied. (That the subset $U \subseteq M$ is displaceable in a Hamiltonian way, follows for example from Proposition 11.) Therefore, applying this theorem, it follows that

$$\text{Lip}^\infty(M, U) \geq \frac{\int_U \omega^n}{\int_M \omega^n} = \frac{(a - \varepsilon) \int_{M'} \omega'^{n'}}{(c + \varepsilon) \int_{M'} \omega'^{n'}},$$

where $2n' := \dim M'$. Since $\varepsilon > 0$ is arbitrary, the claimed inequality (10) follows. This proves Corollary 6. \square

4 Proofs of Proposition 3 and Corollary 7

In the proof of Proposition 3 we will use the following definition. Let (M, ω) be a symplectic manifold and $N \subseteq M$ a coisotropic submanifold. We define the *action spectrum* and the *minimal area* of (M, ω, N) as

$$\begin{aligned} S(M, \omega, N) &:= \\ \{ \int_{\mathbb{D}} u^* \omega \mid u \in C^\infty(\mathbb{D}, M) : \exists \text{ isotropic leaf } F \subseteq N : u(S^1) \subseteq F \}, \\ A(M, \omega, N) &:= \inf (S(M, \omega, N) \cap (0, \infty)) \in [0, \infty]. \end{aligned}$$

Furthermore, for $n \in \mathbb{N}$ and $a > 0$ we denote by $S^{2n-1}(a) \subseteq \mathbb{R}^{2n}$ the sphere of radius $\sqrt{a/\pi}$, around 0.

Proof of Proposition 3: Let $\varepsilon > 0$. We define

$$N := S^1(a - \varepsilon) \times S^1(a - \varepsilon) \times S^{2n-1}(a - \varepsilon) \times X.$$

This is a closed and regular coisotropic submanifold of U . We choose a map $\varphi_0 \in \text{Ham}(B^2(2a))$ such that

$$\varphi_0(S^1(a - \varepsilon)) \cap S^1(a - \varepsilon) = \emptyset. \quad (125)$$

(The existence of such a map follows for example from Proposition 11.) Since N is compact, by a cutoff argument there exists a map $\varphi \in \text{Ham}(U)$ such that $\varphi = (\text{id}_{\mathbb{R}^2} \times \varphi_0 \times \text{id}_{(B^2(a))^n \times X})$ on N . (See for example [SZ1, Lemma 35].) It follows from (125) that

$$\varphi(N) \cap N = \emptyset. \quad (126)$$

We define $V := \mathbb{R}^2 \times B^2(2a) \times (B^2(a))^n \times X$. Since by hypothesis, (M', ω') is aspherical, the same holds for (X, σ) . Hence it follows from [SZ1, Remark 31, Lemma 30, and Proposition 34] that

$$A(V, \omega|_V, N) = a - \varepsilon. \quad (127)$$

Using again that (M', ω') is aspherical and [SZ1, Lemma 33], we have

$$A(V, \omega|_V, N) = A(M, \omega, N).$$

Combining this with (127) and using (126), it follows from [Zi, Theorem 1] that $\|\varphi\|^M \geq a - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that

$$\|\varphi\|^M \geq a.$$

The inequality (5) is a consequence of this. This proves Proposition 3. \square
For the proof of Corollary 7 we need the following.

Remark 24 Let (M, ω) be a symplectic manifold, $X \subseteq M$ a closed subset, and $U \subseteq M$ an open subset containing X . Then

$$\text{Ham}(X, M, \omega) = \text{Ham}(X, U, \omega|_U), \quad \|\cdot\|_X^M \leq \|\cdot\|_X^U.$$

These statements follow from elementary arguments.

Proof of Corollary 7: We denote

$$\widetilde{M} := M \times M', \quad \widetilde{X} := M \times X', \quad \widetilde{\omega} := \omega \oplus \omega', \quad 2n' := \dim M'.$$

We choose $c > 0$ and symplectic embeddings $\psi'_{x'} : B^2(c) \times B^{2n'-2}(c) \rightarrow M'$ (for $x' \in X'$), such that

$$\psi'_{x'}(0, 0) = x', \quad \text{and the images } \text{im}(\psi'_{x'}) \text{ are disjoint.}$$

We define $U' := \bigcup \text{im}(\psi'_{x'})$. By Remark 24, we may assume without loss of generality that $\widetilde{M} = M \times U'$. Furthermore, we may assume without loss of generality that X' consists of one point x'_0 , and

$$U' = B^2(c) \times B^{2n'-2}(c), \quad \psi_{x'_0} = \text{id}.$$

It follows that $x'_0 = 0$ and $\widetilde{X} = M \times \{0\}$. Let $\varphi \in \text{Ham}(\widetilde{X}, \widetilde{\omega})$. Since \widetilde{X} is a closed subset and a symplectic submanifold of \widetilde{M} , it follows from the proof of [SZ2, Proposition 1(ii)] that

$$\varphi \in \text{Ham}(\widetilde{X}, \widetilde{\omega}|_{\widetilde{X}}).$$

We choose a function $H \in C_c^\infty([0, 1] \times M, \mathbb{R})$ such that, identifying $\widetilde{X} = M$, we have

$$\varphi_H^1 = \varphi. \tag{128}$$

We fix $N \in \mathbb{N}$ and define $a := c/3N$. We claim that

$$\|\varphi\|_{\widetilde{X}}^{\widetilde{M}} \leq 8a + \frac{2}{N}\|H\|. \tag{129}$$

To see this, we choose a function $\rho \in C^\infty(B^2(c) \times B^{2n'-2}(c), [0, 1])$ with support in $B^2(a) \times B^{2n'-2}(c)$, such that ρ equals 1 in a neighborhood of $(0, 0)$. We define the function $\widetilde{H} : [0, 1] \times \widetilde{M} \rightarrow \mathbb{R}$ by

$$\widetilde{H}^t(x, x') := H^t(x) \cdot \rho(x').$$

Applying Theorem 1, it follows that

$$\|\varphi_H^1\|_{\widetilde{H}}^{\widetilde{M}} \leq 8a + \frac{2}{N}\|\widetilde{H}\|. \tag{130}$$

Furthermore, condition (128) implies that $\varphi_{\tilde{H}}^1|_{\tilde{X}} = \varphi$. Hence by definition, we have

$$\|\varphi\|_{\tilde{X}}^{\tilde{M}} \leq \|\varphi_{\tilde{H}}^1\|_{\tilde{M}}^{\tilde{M}}.$$

Combining this with (130) and the fact $\|\tilde{H}\| = \|H\|$, inequality (129) follows.

Since N is arbitrary, inequality (129) implies that $\|\varphi\|_{\tilde{X}}^{\tilde{M}} = 0$. The equality (13) follows. This proves Corollary 7. \square

A Proofs of Lemmas 9 and 13

In this section we prove Lemmas 9 and 13. Note that if $f_1, \dots, f_N : [0, 1] \rightarrow \mathbb{R}$ are smooth positive functions, then we may define

$$\psi_i : [0, 1] \rightarrow [0, 1], \quad \psi_i(t) := \frac{\int_0^t f_i(s) ds}{\int_0^1 f_i(s) ds}, \quad \varphi_i := \psi_i^{-1}.$$

The maps φ_i satisfy (28), making it an equality with $\varepsilon = 0$. In the general situation, we need the following result.

Lemma 25 *For every $f \in L^1([0, 1], [0, \infty))$ and $\varepsilon > 0$ there exists an orientation preserving diffeomorphism φ of $[0, 1]$, such that, defining*

$$\tilde{f} := (f \circ \varphi)\dot{\varphi}, \tag{131}$$

$$\tilde{X} := \{\tilde{t} \in [0, 1] \mid \tilde{f}(\tilde{t}) > \int_0^1 f(t) dt + 2\varepsilon\}, \tag{132}$$

we have

$$\int_{\tilde{X}} \tilde{f} \leq \varepsilon. \tag{133}$$

The proof of this lemma is based on the argument explained above and smooth approximation. We need the following.

Lemma 26 *Let $f \in L^1([0, 1], [0, \infty))$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that for every measurable subset $X \subseteq [0, 1]$ satisfying $|X| \leq \delta$, we have*

$$\int_X f < \varepsilon, \tag{134}$$

Proof of Lemma 26: Since f is integrable, there exists $C > 0$ such that

$$\int_{f^{-1}((C, \infty))} f < \frac{\varepsilon}{2}.$$

The number $\delta := \varepsilon/(2C)$ has the required property. This proves Lemma 26.

□

Proof of Lemma 25: We choose $C > 0$ so big that

$$\int_{f^{-1}([C, \infty))} f < \varepsilon.$$

Furthermore, we choose $\delta > 0$ such that

$$\delta < \min \{1, \varepsilon\},$$

and the condition of Lemma 26 is satisfied. We choose $g \in C^\infty([0, 1], [0, \infty))$ such that

$$\|f - g\|_1 < \delta\varepsilon. \quad (135)$$

We define

$$\psi(t) := \frac{\int_0^t (g(s) + \varepsilon) ds}{\int_0^1 g(s) ds + \varepsilon}. \quad (136)$$

This is an orientation preserving diffeomorphism of $[0, 1]$. We define

$$\varphi := \psi^{-1} : [0, 1] \rightarrow [0, 1].$$

Claim 1 *Inequality (133) holds.*

Proof of Claim 1: For $a \in \mathbb{R}$ we define

$$X_a := \{t \in [0, 1] \mid f(t) - g(t) > a\}.$$

We claim that

$$\varphi(\tilde{X}) \subseteq X_\varepsilon. \quad (137)$$

To see this, assume that $\tilde{t} \in \tilde{X}$. We define $t := \varphi(\tilde{t})$. Using (136), we have

$$\frac{1}{\dot{\varphi}(\tilde{t})} = \dot{\psi}(t) = \frac{g(t) + \varepsilon}{\int_0^1 g(s) ds + \varepsilon}. \quad (138)$$

By (135) and the fact $\delta < 1$, we have $\int_0^1 g(s) ds + \varepsilon < \int_0^1 f(s) ds + 2\varepsilon$. Combining this with (138, 131, 132), it follows that $f(t) > g(t) + \varepsilon$, and therefore, $t \in X_\varepsilon$. This proves (137).

By (135), we have

$$\varepsilon|X_\varepsilon| \leq \int_{X_\varepsilon} (f - g) < \delta\varepsilon,$$

and therefore, $|X_\varepsilon| < \delta$. Hence by the statement of Lemma 26, we have $\int_{X_\varepsilon} f < \varepsilon$. Furthermore, the substitution rule and (137) imply that

$$\int_{\tilde{X}} \tilde{f}(\tilde{t}) d\tilde{t} \leq \int_{X_\varepsilon} f(t) dt.$$

Combining this with the inequality $\int_{X_\varepsilon} f < \varepsilon$, inequality (133) follows. This proves Claim 1 and completes the proof of Lemma 25. \square

We are now ready for the proof of Lemma 9.

Proof of Lemma 9: We may assume without loss of generality that

$$0 < c_i := \int_0^1 f_i < \infty, \quad \forall i.$$

Let $\varepsilon > 0$, and fix $i = 1, \dots, N$. By Lemma 25 there exists an orientation preserving diffeomorphism φ_i of $[0, 1]$, such that, defining

$$\begin{aligned} \tilde{f}_i &:= (f_i \circ \varphi_i) \dot{\varphi}_i, \\ \tilde{X}_i &:= \{\tilde{t} \in [0, 1] \mid \tilde{f}_i(\tilde{t}) > c_i + 2\varepsilon\}, \end{aligned} \tag{139}$$

we have

$$\int_{\tilde{X}_i} \tilde{f}_i \leq \varepsilon. \tag{140}$$

We define

$$\tilde{X} := \bigcup_i \tilde{X}_i, \quad \tilde{f} : [0, 1] \rightarrow [0, \infty), \quad \tilde{f}(\tilde{t}) := \max_i \tilde{f}_i(\tilde{t}).$$

Using (139), we have

$$\int_{[0,1] \setminus \tilde{X}} \tilde{f} \leq \max_i c_i + 2\varepsilon. \tag{141}$$

We define

$$C_\varepsilon := \varepsilon \left(N + (\max_i c_i + 2\varepsilon) \sum_i \frac{1}{c_i} \right).$$

Claim 1 *We have*

$$\int_{\tilde{X}} \tilde{f} \leq C_\varepsilon. \tag{142}$$

Proof of Claim 1: We define

$$\begin{aligned} \tilde{Y} &:= \{\tilde{t} \in [0, 1] \mid \tilde{f}(\tilde{t}) > \max_i c_i + 2\varepsilon\}, \\ \tilde{Y}_i &:= \{\tilde{t} \in \tilde{Y} \mid \tilde{f}_i(\tilde{t}) = \tilde{f}(\tilde{t})\}, \quad i = 1, \dots, N. \end{aligned}$$

Note that $\tilde{Y} = \bigcup_i \tilde{Y}_i$, and therefore

$$\int_{\tilde{Y}} \tilde{f} \leq \sum_i \int_{\tilde{Y}_i} \tilde{f}_i. \quad (143)$$

Let $i = 1, \dots, N$. By (139) we have $\tilde{Y}_i \subseteq \tilde{X}_i$, and hence

$$\int_{\tilde{Y}_i} \tilde{f}_i \leq \int_{\tilde{X}_i} \tilde{f}_i.$$

Combining this with (143,140), we obtain

$$\int_{\tilde{Y}} \tilde{f} \leq N\varepsilon. \quad (144)$$

Furthermore, we have

$$\int_{\tilde{X} \setminus \tilde{Y}} \tilde{f} \leq (\max_i c_i + 2\varepsilon) |\tilde{X}|. \quad (145)$$

Let $i = 1, \dots, N$. Then we have $|\tilde{X}_i| c_i \leq \int_{\tilde{X}_i} \tilde{f}_i$. Combining this with (140), it follows that $|\tilde{X}_i| \leq \varepsilon/c_i$, and therefore

$$|\tilde{X}| \leq \varepsilon \sum_i \frac{1}{c_i}.$$

Combining this with (145,144), inequality (142) follows. This proves Claim 1. \square

It follows from (141) and Claim 1 that

$$\int_0^1 \max_i \tilde{f}_i \leq \max_i c_i + 2\varepsilon + C_\varepsilon.$$

Since $C_\varepsilon \rightarrow 0$, as $\varepsilon \rightarrow 0$, the statement of Lemma 9 follows. \square

Next we prove Lemma 13. We will use the following four results.

Lemma 27 *Let M be a connected manifold (without boundary) and $x_0, x_1 \in M$. Then there exists a diffeomorphism $\psi : M \rightarrow M$ with compact support, such that $\psi(x_0) = x_1$.*

Proof of Lemma 27: We choose a path $x \in C^\infty([0, 1], M)$ such that $x(i) = x_i$, for $i = 0, 1$, and a smooth time-dependent vector field X on M with compact support, such that $X^t \circ x(t) = \dot{x}(t)$, for every $t \in [0, 1]$. The map $\psi := \varphi_X^1$ has the required properties. This proves Lemma 27. \square

Lemma 28 *Let M be a manifold (without boundary), $X \subseteq M$ the image of an embedding $\overline{B}^n \rightarrow M$, and $U \subseteq X$ a non-empty open subset. Then there exists a diffeomorphism $\psi : M \rightarrow M$ with compact support, such that $\psi(X) \subseteq U$.*

Proof of Lemma 28: There exists an embedding $\mathbb{R}^n \rightarrow M$ that extends the given embedding $\overline{B}^n \rightarrow M$. Hence we may assume without loss of generality that $M = \mathbb{R}^n$ and $X = \overline{B}^n$. The existence of the claimed diffeomorphism of \mathbb{R}^n follows now from an elementary argument. This proves Lemma 28. \square
For $n \in \mathbb{N}$ and $r > 0$ we denote by $\overline{B}_r^n \subseteq \mathbb{R}^n$ the closed ball of radius r , around 0.

Lemma 29 *Let $n \in \mathbb{N}$ and $\varphi : B^n \rightarrow \mathbb{R}^n$ an embedding satisfying $\varphi(0) = 0$. Then there exists $r_0 \in (0, 1)$ and a diffeomorphism $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with compact support, such that*

$$\varphi(\overline{B}_{r_0}^n) = \psi(\overline{B}^n).$$

Proof of Lemma 29: It follows from an elementary argument that there exists an $r_0 \in (0, 1)$ and a function $f \in C^\infty(S^{n-1}, (0, \infty))$ such that

$$\varphi(\overline{B}_{r_0}^n) = \{rx \mid r \in [0, f(x)], x \in S^{n-1}\}.$$

We choose a smooth function $\tilde{f} : [0, \infty) \times S^{n-1} \rightarrow [0, \infty)$ with the following properties: $\tilde{f}(s, x) = s$ for (s, x) in some neighborhood of $(0, 0)$ and outside some compact set. Furthermore, $\tilde{f}(1, x) = f(x)$, for every $x \in S^{n-1}$, and $\partial_s \tilde{f} > 0$. The map

$$\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \psi(x) := \begin{cases} \tilde{f}(|x|, \frac{x}{|x|}) \frac{x}{|x|}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

has the required properties. \square

Lemma 30 *Let $K \subseteq \mathbb{R}^n$ be a compact subset and $v \in \mathbb{R}^n$. Then there exists a diffeomorphism φ of \mathbb{R}^n , with compact support, such that*

$$\varphi(x) = x + v, \quad \forall x \in K.$$

Proof of Lemma 30: This follows from an elementary argument. \square

We are now ready for the proof of Lemma 13.

Proof of Lemma 13: Consider the **case** $k = 1$ **and** $M = \mathbb{R}^n$. We may assume without loss of generality that $X_0^1 = \overline{B}^n$. Using Lemma 30, we may assume without loss of generality that there exists an embedding $\psi_1 :$

$\overline{B}^n \rightarrow \mathbb{R}^n$ with image X_1^1 , such that $\psi_1(0) = 0$. By Lemma 29 there exists a diffeomorphism $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with compact support, such that

$$\psi_1(\overline{B}_{r_0}^n) = \varphi(\overline{B}^n). \quad (146)$$

We choose a vector field V on \mathbb{R}^n with compact support, such that

$$\psi_1^* V(x) = \frac{(1 - r_0)x}{|x|}, \quad \forall x \in \overline{B}^n \setminus B_{r_0}^n. \quad (147)$$

We denote by φ_V^1 the time-one flow of V , and define

$$\psi := \varphi_V^1 \circ \varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

This is a diffeomorphism of \mathbb{R}^n with compact support.

Claim 1 *This map satisfies $\psi(\overline{B}^n) = X_1^1$.*

Proof of Claim 1: It follows from (146,147) that $\psi(\overline{B}^n) \subseteq \psi_1(\overline{B}^n) = X_1^1$. To see the opposite inclusion, observe that (146,147) imply that the restricted map

$$S^{n-1} \ni x \mapsto \psi_1^{-1} \circ \psi(x) \in S^{n-1}$$

is well-defined and an orientation preserving diffeomorphism, hence its degree equals one. It follows that

$$\overline{B}^n \subseteq \psi_1^{-1} \circ \psi(\overline{B}^n),$$

i.e., $X_1^1 = \psi_1(\overline{B}^n) \subseteq \psi(\overline{B}^n)$. This proves Claim 1 and therefore the statement of the lemma in case $k = 1$ and $M = \mathbb{R}^n$. \square

Consider now the **case $k = 1$ and M general**. For $i = 0, 1$ we choose a smooth embedding $\psi_i : \overline{B}^n \rightarrow M$ such that $\psi_i(\overline{B}^n) = X_i^1$. Since M is connected, by Lemma 27 there exists a diffeomorphism $\varphi : M \rightarrow M$ with compact support, such that $\varphi \circ \psi_0(0) = \psi_1(0)$.

We choose a neighborhood $U_0 \subseteq M$ of $\psi_0(0)$ that is diffeomorphic to \mathbb{R}^n , such that $\varphi(U_0) \subseteq \psi_1(\overline{B}^n)$. By Lemma 28 there exists a diffeomorphism $\varphi_0 : M \rightarrow M$ with compact support, such that $\varphi_0(X_0) \subseteq U_0$. We choose a neighborhood $U_1 \subseteq M$ of X_1 that is diffeomorphic to \mathbb{R}^n . (We may define U_1 to be the image of an embedding $\mathbb{R}^n \rightarrow M$ that extends ψ_1 .)

It follows from the statement of the lemma in the case $M = \mathbb{R}^n$ that there exists a diffeomorphism φ_1 of U_1 with compact support such that

$$\varphi_1 \circ \varphi \circ \varphi_0(X_0) = X_1.$$

We define $\tilde{\varphi}_1 : M \rightarrow M$ by $\tilde{\varphi}_1(x) := \varphi_1(x)$, if $x \in U_1$, and $\varphi_1(x) := x$, otherwise. The map

$$\psi := \tilde{\varphi}_1 \circ \varphi \circ \varphi_0$$

has the required properties.

For **general k and M** the claimed statement follows from what we just proved and a straight-forward induction argument. (We use the fact that the complement of finitely many disjoint balls in M is connected. This is ensured by our hypothesis $\dim M \geq 2$.) This proves Lemma 13. \square

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