Gromov-Witten Invariants and Symplectic Vortices

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Abstract

In 1985, Gromov introduced the notion of J-holomorphic curves into symplectic geometry, for which they have proved to be a powerful tool. The Gromov-Witten invariants of a symplectic manifold (M, ω) are given by the number of J-holomorphic curves that represent a fixed homology class and at fixed or varying points pass through given submanifolds of M. Here J is an ω-compatible almost complex structure. The symplectic vortex invariants on the other hand are associated to a symplectic manifold (M, ω) with a Lie group G acting on M in a Hamiltonian way.

In this overview article I will describe both invariants and I will explain what the relation between the two is.

Contents

1 Overview

The main objects of study of symplectic geometry are symplectic manifolds. J-holomorphic curves and the *Gromov-Witten invariants (GW's)* are one particular tool to investigate (rather general) symplectic manifolds. Vortices on the other hand have first appeared in physics as minimizers of the *Ginzburg-Landau functional*, which is used as a model for superconductivity. However, they also turn up as one tries to understand the GW's of a special kind of a symplectic manifold, namely a symplectic quotient. This is a symplectic manifold $(\bar{M}, \bar{\omega})$ arising from some symplectic manifold (M, ω) and a Hamiltonian action of a Lie group G on M.

This paper is split into three parts: In section 2 symplectic manifolds will be introduced. Furthermore, the use of the theory of J-holomorphic curves and GW's is illustrated by a list of applications. These applications are intended to serve as a motivation. Since the statements of some of the theorems can already be understood at that point, two of them will be discussed in more detail. Good references for this part are [11, 13]. Section 3 is concerned with the definition of J-holomorphic curves and the GW's (see also [12, 13]). Finally in section 4 we will focus our attention on symplectic quotients. I will define *symplectic vortices* and the invariants associated to them. Finally, I will explain how these invariants are related to the Gromov-Witten invariants of the symplectic quotient. Interested readers may also have a look at the overview article [2] and at [13, 3, 5, 6, 16, 17]. A standard reference for symplectic geometry in general is [11]. Parts of this article are not rigorous (for example the definition of the GW's and the symplectic vortex invariants). However, if something is not rigorous this will be pointed out and references to articles will be given, in which the reader can find technically correct statements. The reason for giving nonrigorous definitions is that the technicalities necessary for strict definitions are rather involved and I did not want to get lost in them. My intention was in fact to explain the geometric ideas lying behind the GW's and the vortex invariants. Let me also remark that one of my aims was that this article should be as self-contained as possible.

The present paper arose from a talk given December 3, 2003 at the *Graduate Colloquium* at ETH Zürich. However, the presented material has grown considerably since then. I would like to thank Urs Frauenfelder very much for proof-reading and Dietmar Salamon for his feedback to this article. Any further suggestions or corrections are always welcome.

2 Introduction

2.1 Symplectic Manifolds

The notion of a symplectic manifold is built on the following.

Definition 2.1 *A* symplectic vector space *is a real vector space V with a bilinear form $\omega : V \times V \to \mathbb{R}$, which is skew symmetric, i.e. $\omega(v,w) = -\omega(w,v)$ for all $v, w \in V$, and nondegenerate, i.e. if $v \in V$ such that $\omega(v,w) = 0$ for every $w \in V$ then $v = 0$. Such an ω is called a* linear symplectic form *on V.*

Example 2.2 The *standard linear symplectic form ω_{st} on $V := \mathbb{R}^{2n}$* is given as follows. Let $\zeta, \widetilde{\zeta} \in \mathbb{R}^{2n}$. Writing $\zeta =: (\xi^1, \eta_1, \dots, \xi^n, \eta_n)$, $\widetilde{\zeta} =: (\widetilde{\xi}^1, \widetilde{\eta}_1, \dots, \widetilde{\xi}^n, \widetilde{\eta}_n)$ we define

$$\omega_{\mathrm{st}}(\zeta, \widetilde{\zeta}) := \sum_{i=1}^{n} \xi^i \widetilde{\eta}_i - \widetilde{\xi}^i \eta_i.$$

Definition 2.3 *Let M be a manifold and $\omega \in \Omega^2(M)$ be a 2-form. ω is called a* symplectic structure *iff it is closed, i.e. $d\omega = 0$, and nondegenerate, i.e. for every $p \in M$ the map*

$$T_pM \times T_pM \to \mathbb{R}, \qquad (v,w) \to \omega_p(v,w)$$

is a nondegenerate bilinear map.

The pair (M, ω) is called a symplectic manifold.

If (M, ω) is a symplectic manifold, then for each $p \in M$, the tangent space T_pM together with the restriction of ω to $T_pM \times T_pM$ is a symplectic vector space.

Example 2.4 Let $M := \mathbb{R}^{2n}$. We define the *standard symplectic form on* $\mathbb{R}^{2n}$ (as a manifold) by

$$\omega_0 := \sum_{i=1}^{n} dx^i \wedge dy_i.$$

Under the canonical identification $T_{(x,y)}\mathbb{R}^{2n} = \mathbb{R}^{2n}$ we have $(\omega_0)_{(x,y)} = \omega_{st}$. Now let $\mathbb{T}^{2n} := \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be the torus. The form ω_0 descends to the quotient $\mathbb{T}^{2n}$, i.e. there is a unique symplectic form ω on $\mathbb{T}^{2n}$ such that

$$\pi^*\omega = \omega_0,$$

where $\pi : \mathbb{R}^{2n} \to \mathbb{T}^{2n}$ is the canonical projection. (Recall that for $k \in \mathbb{N}_0$ the pullback of a k-form $\omega \in \Omega^k(N)$ by a map $f : M \to N$ is defined by $(f^*\omega)_p(v_1, \ldots, v_k) := \omega_{f(p)}(df(p)v_1, \ldots, df(p)v_k)$, for $v_i \in T_pM$, $i = 1, \ldots, k$, $p \in M$.)

Example 2.5 Let $M := S^2$. We define the *standard volume form* $\omega := \mathrm{dvol}_{S^2}$ *on* S^2 by $(\mathrm{dvol}_{S^2})_x(v, w) := \langle x, v \times w\rangle$, for any $x \in S^2 \subseteq \mathbb{R}^3$, $v, w \in T_xS^2$. It is a symplectic form.

Example 2.6 Let L be a manifold and $M := T^*L$ be the cotangent bundle. There is a unique 1-form λ_{can} on T^*L which satisfies

$$\alpha^*\lambda_{\mathrm{can}} = \alpha,$$

for every $\alpha \in \Omega^1(L)$. Here on the left hand side α is interpreted as a section of T^*L, i.e. as a map $\alpha : L \to T^*L$ and $\alpha^*\lambda_{\mathrm{can}} \in \Omega^1(L)$ is the pullback 1-form. The *canonical symplectic 2-form on* T^*L is defined by

$$\omega_{\mathrm{can}} := -d\lambda_{\mathrm{can}} \in \Omega^2(T^*L).$$

Example 2.7 Let (M_i, ω_i), $i = 1, 2$ be two symplectic manifolds. There is an induced symplectic structure ω on the product $M_1 \times M_2$. It is given by

$$\omega_{(p_1,p_2)}((v_1, v_2), (w_1, w_2)) := (\omega_1)_{p_1}(v_1, w_1) + (\omega_2)_{p_2}(v_2, w_2),$$

for $v_i \in T_{p_i}M_i$, $p_i \in M_i$, $i = 1, 2$. Denoting the projection to the i-th factor by $\mathrm{pr}_i : M_1 \times M_2 \to M_i$, we can write $\omega = \mathrm{pr}_1^*\,\omega_1 + \mathrm{pr}_2^*\,\omega_2$.

2.2 What are J-holomorphic curves and GW's good for?

This paragraph is intended to be a motivation. Readers not interested in this paragraph may directly move on to Part II, since nothing of the material presented here is used at a later stage.

There are the following basic questions about symplectic structures:

Question 1: Given a manifold M, is there a symplectic structure ω on M?

Question 2: Given two symplectic manifolds (M_1, ω_1), (M_2, ω_2), are they "symplectically equal"?

I am not going to talk about **Question 1** in this paper. However, **Question 2** will serve as a motivation.

Let us make sense of the words "symplectically equal":

Definition 2.8 *Let (M_1, ω_1), (M_2, ω_2) be symplectic manifolds. Let $\psi : M_1 \to M_2$ be a smooth map. ψ is called a* symplectic map *iff*

$$\psi^* \omega_2 = \omega_1.$$

ψ is called a symplectomorphism*, iff in addition it is a diffeomorphism. (M_1, ω_1) and (M_2, ω_2) are called symplectomorphic ("symplectically equal") iff there is a symplectomorphism $\psi : M_1 \to M_2$.*

A *symplectic invariant* is a "quantity" Q, defined for each symplectic manifold, that is "left unchanged" under symplectomorphisms. Here is one way to give a negative answer to Question 2: Given two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) suppose that we find a symplectic invariant that takes different values for (M_1, ω_1) and (M_2, ω_2). Then the two manifolds are necessarily not symplectomorphic.

Since the GW's are symplectic invariants, we have found one answer to the question asked in the title of this subsection: the GW's can be used to distinguish symplectic manifolds.

Yet, J-holomorphic curves and the GW's have many other applications in symplectic geometry. Let me make a list of some of them (see [13] for details):

- a nonsqueezing theorem for symplectic embeddings
- a nonexistence theorem for Lagrangian embeddings
- a result stating that under certain circumstances a closed Lagrangian submanifold cannot be removed from itself by a Hamiltonian isotopy
- an existence theorem for periodic orbits of Hamiltonian systems

- partial results in the classification of symplectic 4-manifolds
- results about the group of symplectomorphisms

I would like to briefly discuss the first two applications:

Recall that a smooth map $\psi : M_1 \to M_2$ is called an *embedding*, iff it is a diffeomorphism onto its image. Let $\overline{B}_r^m \subseteq \mathbb{R}^m$ denote the closed ball of radius $r > 0$ around 0. The *Nonsqueezing Theorem* says the following.

Theorem 2.9 (Gromov) *Let $r_1, r_2 > 0$ and $\psi : \overline{B}_{r_1}^{2n} \to \mathbb{R}^{2n}$ be a symplectic embedding, where $\mathbb{R}^{2n}$ is endowed with the standard structure ω_0. Suppose that*

$$\psi(\overline{B}_{r_1}^{2n}) \subseteq \overline{B}_{r_2}^{2} \times \mathbb{R}^{2n-2}.$$

Then $r_1 \leq r_2$.

The point of this theorem is the following. Given a symplectic manifold (M, ω) (in our case $(\mathbb{R}^{2n}, \omega_0)$), there is an induced volume form $\Omega := \frac{1}{n!}\omega^n \in \Omega^{2n}(M)$. *Liouville's theorem* states that if $\psi : M \to M$ is a symplectic map, then the volume form Ω is preserved by ψ, i.e. $\psi^*\Omega = \Omega$. (This theorem follows by induction from the definition of a symplectic map and the property $\psi^*(\omega_1 \wedge \omega_2) = \psi^*\omega_1 \wedge \psi^*\omega_2)$). Now if $n = 1$, then by definition, preserving the volume form $\Omega = \omega$ is equivalent to being symplectic. This is however not true for $n \geq 2$. So in this case we ask how big the difference for a map is between being volume preserving and being symplectic. What kind of things that can be done using volume preserving maps, can also be done by symplectic maps? One example is the following. Let $M := \mathbb{R}^{2n}$ and $r > 0$. Then there is a volume preserving map $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that the ball $\overline{B}_r^{2n}$ is mapped to an arbitrarily thin (but long) region in $\mathbb{R}^{2n}$. In particular, for all $r_1, r_2 > 0$ there is ψ such that $\psi^*\Omega = \Omega$ and

$$\psi(\overline{B}_{r_1}^{2n}) \subseteq \overline{B}_{r_2}^{2} \times \mathbb{R}^{2n-2}.$$

In sharp contrast to this, Theorem 2.9 says that there is no such ψ that in addition is symplectic, unless $r_1 \leq r_2$. (In the case $r_1 \leq r_2$ such a ψ exists, take for example $\psi := \mathrm{id}$.) So this theorem imposes a geometric restriction on symplectic maps.

As already indicated, one of the proofs of this theorem uses J-holomorphic curves. (see [13])

The second application I want to discuss involves the following definition.

Definition 2.10 *Let (M^{2n}, ω) be a symplectic manifold and $L \subseteq M$ be a submanifold of dimension n. L is called* Lagrangian *iff for every $p \in L$ and every $v, w \in T_pL$ we have*

$$\omega_p(v, w) = 0,$$

i.e. the two form ω vanishes on L.

Example 2.11 If $n = 1$ then every 1-dimensional submanifold $L \subseteq M^2$ is Lagrangian. This follows immediately from the fact that ω is skew symmetric.

Example 2.12 Let (M_i, ω_i), $i = 1, 2$ be two symplectic manifolds and consider the product structure $\omega := \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2$ on $M := M_1 \times M_2$ as in Example 2.7. Let $L_i \subseteq M_i$, $i = 1, 2$ be Lagrangian submanifolds. Then the product $L := L_1 \times L_2 \subseteq M$ is a Lagrangian submanifold w.r.t. ω.
By induction and Example 2.11, this implies that the torus $\mathbb{T}^n \cong (S^1)^n \subseteq \mathbb{R}^{2n}$ is a Lagrangian submanifold w.r.t. ω_0.

Consider again the standard symplectic structure ω_0 on $\mathbb{R}^{2n}$. The second application is the following.

Theorem 2.13 *Let L be a compact manifold without boundary of dimension n. Assume that the first de Rham cohomology group of L vanishes. Then there is no embedding $\psi : L \to \mathbb{R}^{2n}$ such that $\psi(L) \subseteq \mathbb{R}^{2n}$ is Lagrangian.*

In particular, this theorem says that there is no Lagrangian embedding of $L := S^n$ into $\mathbb{R}^{2n}$ in the case $n \geq 2$, since $H^1(S^n) = 0$ for $n \geq 2$. The proof of this theorem relies on the following result, which in turn is based on the theory of J-holomorphic curves (see [13]).

Theorem 2.14 (Gromov) *Let $L \subseteq \mathbb{R}^{2n} \cong \mathbb{C}^n$ be a compact Lagrangian submanifold. Then there is a nonconstant holomorphic map $u : \overline{B}_1(0) \subseteq \mathbb{C} \to \mathbb{C}^n$ such that $u(S^1) \subseteq L$.*

Remark 2.15 (Riemann Mapping Theorem) This theorem generalizes the Riemann Mapping Theorem (RMT) in the following sense. The RMT states that for every connected simply connected open subset $\Omega \subsetneq \mathbb{C}$ there is a biholomorphic map $u : B_1(0) \to \Omega$, where $B_1(0) \subseteq \mathbb{C}$ is the open unit disk. Let now $n = 1$ and let $L^1 \subseteq \mathbb{C}$ be any compact connected 1-dimensional submanifold. (Then L is diffeomorphic to S^1.) Recall from Example 2.11 that L automatically is Lagrangian. Let $\Omega \subseteq \mathbb{C}$ be the open region inside L. Then we can apply Theorem 2.14 to conclude that there is a nonconstant holomorphic map $u : \overline{B}_1(0) \subseteq \mathbb{C} \to \mathbb{C}$ such that $u(S^1) \subseteq L$. For any regular

$w \in \mathbb{C} \setminus L \subseteq \mathbb{C} \setminus u(S^1)$ let $\deg(u, w)$ denote the degree of u with respect to w, i.e. the number of preimages of w. $\deg(u, w)$ only depends on u and the connected component of $\mathbb{C} \setminus u(S^1)$ containing w. Let $w \in \Omega$ be a regular value and denote $d := \deg(u, w)$. If $d = 1$ then one can show that $u|B_1(0) \to \Omega$ is biholomorphic, so it satisfies the requirement of the Riemann Mapping Theorem. There is indeed a sharpening of Theorem 2.14 which says that there is a map u as in Theorem 2.14 with $d = 1$. So in the case where Ω is the region inside a compact connected 1-dimensional submanifold $L \subseteq \mathbb{C}$ the RMT follows from this sharpening of Theorem 2.14.

J-holomorphic curves and GW's are related e.g. to Floer homology, Hamiltonian dynamical systems and group theory. They play an important role in mirror symmetry. In this paper, I will however explain some relation with gauge theory, namely with symplectic vortices.

3 Gromov-Witten Invariants

In this section the Gromov-Witten Invariants with fixed Riemann surface $(\Sigma, j) := (S^2, J_0)$ and fixed marked points are introduced. As a warm-up to the general definition (Definition 3.26) we will define the GW's with no marked points at all. In order to define the invariants we need the following auxiliary object.

Definition 3.1 (almost complex structure) *Let M be a manifold and $J : TM \to TM$ an automorphism of the tangent bundle, i.e. for every $p \in M$ the restricted map $J|_{T_pM}$ is a vector space isomorphism from T_pM to T_pM and the coefficients of J in local coordinates depend smoothly on p. J is called an almost complex structure iff for each $p \in M$*

$$J^2 = -\mathrm{id} : T_pM \to T_pM.$$

Example 3.2 Consider $M := \mathbb{R}^{2n}$. Identifying $\mathbb{R}^{2n} \cong \mathbb{C}^n$ the imaginary unit $i \in \mathbb{C}$ gives rise to the *standard almost complex structure* J_{st} on $\mathbb{R}^{2n}$ by the formula

$$J_{\mathrm{st}}(\zeta_1, \ldots, \zeta_n) := (i\zeta_1, \ldots, i\zeta_n),$$

for $(\zeta_1, \ldots, \zeta_n) \in T_{(z_1,\ldots,z_n)}\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$, $(z_1, \ldots, z_n) \in \mathbb{R}^{2n}$.

Remark 3.3 Every almost complex structure j on a 2-manifold Σ is integrable, i.e. it arises from an atlas with holomorphic transition maps. The pair (Σ, j) is called a *Riemann surface*.

Example 3.4 $(\Sigma, j) := (\mathbb{R}^2, J_{\rm st}) = (\mathbb{C}, i)$ is an example of a Riemann surface.

Example 3.5 The *standard almost complex structure* J_0 *on* $\Sigma := S^2$ is given by $J_0 v := x \times v$, for $v \in T_x S^2$, $x \in S^2 \subseteq \mathbb{R}^3$. Let $N := (0,0,1) \in S^2$ denote the north pole. Then on $S^2 \setminus \{N\}$ J_0 is given by the pullback of the standard complex structure on $\mathbb{R}^2$ under stereographic projection from $\{N\}$. Under the identification $S^2 \cong \mathbb{C}\mathrm{P}^1$ J_0 corresponds to the standard complex structure on $\mathbb{C}\mathrm{P}^1$.

Definition 3.6 *Let* (M, ω) *be a symplectic manifold and* J *an almost complex structure on* M*.* J *is called* ω-compatible *iff the map*

$$TM \oplus TM \ni (x; v, w) \mapsto (g_{\omega,J})_x(v,w) := \omega_x(v, Jw)$$

is a Riemannian metric on M*. We denote by* $\mathcal{J}(M,\omega)$ *the set of all* ω*-compatible almost complex structures.*

Example 3.7 Let $(M, \omega) := (\mathbb{R}^{2n}, \omega_0)$ be as in Example 2.4 and let $J := J_{\rm st}$ be as in Example 3.2. Then $J_{\rm st}$ is compatible with ω_0. Furthermore, the standard complex structure J_0 on S^2 given by Example 3.5 is compatible with the symplectic structure dvol_{S^2} from Example 2.5.

Definition 3.8 *Let* (M_1, J_1) *and* (M_2, J_2) *be almost complex manifolds. A smooth map* $u : M_1 \to M_2$ *is called* (J_1, J_2)*-holomorphic iff it satisfies the* Cauchy-Riemann equations (CR)

$$(\bar\partial_{J_1,J_2}(u))_x(v) := \frac{1}{2}\left(du(x)v + J_2(u(x))du(x)J_1(x)v\right) = 0, \qquad \forall x \in M_1,\ v \in T_x M_1. \tag{1}$$

In the case $\dim_{\mathbb{R}} M_1 = 2$ *such a map* u *is called a* (J_1, J_2)*-holomorphic (or shortly* J_2*-holomorphic)* curve*.*

Example 3.9 Let $(M_1, J_1) := (M_2, J_2) := (\mathbb{R}^2, J_{\rm st})$. Then the CR equations (1) are the usual CR equations from complex analysis of one complex variable. So a map $u : \mathbb{C} = \mathbb{R}^2 \to \mathbb{C} = \mathbb{R}^2$ is $(J_{\rm st}, J_{\rm st})$-holomorphic iff it is holomorphic.

Let (M, ω) be a symplectic manifold, let $g \in \mathbb{N}_0$ be a natural number (including 0). Furthermore, let J be an ω-compatible almost complex structure on M and let (Σ, j) be a closed (compact without boundary) Riemann surface of genus g. The basic idea of the *Gromov-Witten invariants of genus* g is to count J-holomorphic curves $u : \Sigma \to M$ that represent a given second homology class $B \in H_2(M, \mathbb{Z})$. We will see below why we can not just count *all* J-holomorphic curves, but only those in a fixed class $B \in H_2(M, \mathbb{Z})$. There are different versions of GW's, in some of which the complex structure j on Σ is fixed, whereas in some other versions it is allowed to vary. Now in the case $g = 0$ there is up to isomorphy only one complex structure on $\Sigma = S^2$, so in this case there is no real difference between fixed and varying j. However for $g > 0$ this is not true. For simplicity, I will only consider the case $g = 0$. So let $\Sigma := S^2$ and let $j := J_0$ be as in example 3.5. In order to make the number of J-holomorphic curves that we count finite and therefore to get meaningful invariants, in general we have to impose additional conditions on the J-holomorphic curves $u : S^2 \to M$. For this we will have to fix finitely many distinct points z_i, $i = 1, \ldots, k$ and closed oriented submanifolds $N_i \subseteq M$ $(i = 1, \ldots, k)$. The additional conditions are that $u(z_i) \in N_i$ for $i = 1, \ldots, k$. The points z_i are called marked points. By counting all the J-holomorphic curves $u : S^2 \to M$ such that $[u] = B$ and $u(z_i) \in N_i$, $i = 1, \ldots, k$ we will get a map that assigns an integer to every collection $N_i \subseteq M$, $i = 1, \ldots, k$ of closed oriented submanifolds. Here $[u] \in H_2(M, \mathbb{Z})$ denotes the homology class induced by the map u. It will turn out that this number only depends on the homology classes $[N_i]$ represented by N_i $(i = 1, \ldots, k)$. Therefore this induces a map $\mathrm{GW}^{M,\omega}_{k,B} : (H_*(M, \mathbb{Z}))^k \to \mathbb{Z}$, which comprises the *genus 0 Gromov-Witten invariants of* (M, ω, B) *with k fixed marked points*. (See the Working definition 3.26.) There is a different version of the GW's in which the points z_i, $i = 1, \ldots, k$ are not fixed, but they are part of the data which is counted. This means that we count tuples $(u; z_1, \ldots, z_k)$ of J-holomorphic curves together with distinct points $z_i \in S^2$, $i = 1, \ldots, k$ rather than J-holomorphic curves alone. For simplicity, I will not discuss that version, but will constrain myself to the case of fixed marked points.

As a warm-up I will now first discuss the GW's with *no marked points* at all. So in this case we do not impose any condition corresponding to a submanifold $N \subseteq M$. Let me remark that in many cases these invariants are 0 by dimensional reasons and therefore uninteresting. However, they seem conceptually easier to me than those with marked points. We will see below what the words "dimensional reasons" mean.

3.1 GW's with no marked points (warm-up)

Let (M, ω) be a symplectic manifold, $J \in \mathcal{J}(M, \omega)$ an ω-compatible almost complex structure and $B \in H_2(M, \mathbb{Z})$ be a second homology class. As mentioned above, the Gromov-Witten invariant of (M, ω, B) with no marked points should basically be the number of J-holomorphic spheres $u : S^2 \to M$ such that $[u] = B$, where $[u] \in H_2(M, \mathbb{Z})$ denotes the homology class induced by u. Actually, we can not quite define the GW's like that. Namely, assume that $B \neq 0$ and that there is at least one J-holomorphic sphere representing B. Then as I will explain below, there is in fact a 6-dimensional family of J-holomorphic spheres in the same homology class B. So the number of such spheres is not finite, but in fact uncountable. The way out of this problem is to count *equivalence classes* of J-holomorphic spheres instead of individual J-holomorphic spheres. Let me now make all this precise.

Fact 3.10 Let (M_i, J_i) be an almost complex manifold, $i = 1, 2, 3$ and let $u_1 : M_1 \to M_2$ and $u_2 : M_2 \to M_3$ be holomorphic w.r.t. the corresponding almost complex structures. Then the composition $u_2 \circ u_1 : M_1 \to M_3$ is (J_1, J_3)-holomorphic.

Now let $u : S^2 \to M$ be J-holomorphic and let $\phi : S^2 \to S^2$ be holomorphic. Then by Fact 3.10 $u \circ \phi : S^2 \to M$ is also J-holomorphic. Assume moreover that ϕ is bijective, i.e. a Möbius transformation. Then $[u] = [u \circ \phi] \in H_2(M, \mathbb{Z})$.

Fact 3.11 The set $\mathrm{PSL}(2, \mathbb{C})$ of Möbius transformations is a Lie group of real dimension 6. If $u : S^2 \to M$ is a *nonconstant* J-holomorphic sphere then there is a neighbourhood U of $\mathrm{id} \in \mathrm{PSL}(2, \mathbb{C})$ such that for every $\mathrm{id} \neq \phi \in U$ we have $u \circ \phi \neq u$.

Now let $B \in H_2(M, \mathbb{Z})$ be a second homology class. We define

$$\widetilde{\mathcal{M}}_{J,B} := \{u \in C^\infty(S^2, M) | u \text{ is } J\text{-holomorphic, } [u] = B\}. \tag{2}$$

Assume that $B \neq 0$. This condition is equivalent to the condition that every $u \in \widetilde{\mathcal{M}}_{J,B}$ is nonconstant. So by Fact 3.11 if $\widetilde{\mathcal{M}}_{J,B} \neq \emptyset$ then for any $u \in \widetilde{\mathcal{M}}_{J,B}$ the whole 6-dimensional family $\{u \circ \phi\}_{\phi \in U}$ belongs to $\widetilde{\mathcal{M}}_{J,B}$, where U is as in Fact 3.11. So the number $\#\widetilde{\mathcal{M}}_{J,B}$ is not finite. Let us therefore consider the quotient

$$\mathcal{M}_{J,B} := \widetilde{\mathcal{M}}_{J,B} / \sim,$$

where the equivalence relation $\sim$ is defined by

$$u_1 \sim u_2 :\iff \exists \phi \in \mathrm{PSL}(2, \mathbb{C}) : \, u_2 = u_1 \circ \phi. \tag{3}$$

In favorable cases, the set $\widetilde{\mathcal{M}}_{J,B}$ carries a natural structure of a smooth manifold. Its dimension is given by the Riemann-Roch formula,

$$\dim(\widetilde{\mathcal{M}}_{J,B}) = \dim M + 2\langle c_1(TM, J), B\rangle. \tag{4}$$

Here $c_1(TM, J)$ is the first Chern class of the complex vector bundle (TM, J). It turns out that this class does not depend on the ω-compatible almost complex structure J, so we can define $c_1(TM, \omega) := c_1(TM, J)$, where $J \in \mathcal{J}(M, \omega)$ is arbitrary. We define

$$d(M, \omega, B) := \dim M + 2\langle c_1(TM, \omega), B\rangle. \tag{5}$$

Furthermore, $\mathcal{M}_{J,B}$ carries a natural smooth manifold structure of dimension

$$\dim(\mathcal{M}_{J,B}) = \dim(\widetilde{\mathcal{M}}_{J,B}) - 6 = d(M, \omega, B) - 6.$$

Remark 3.12 A short comment on the words "favorable cases": We have to impose a certain condition on the almost complex structure J, namely that it is *regular w.r.t.* B (see [13]). Then the subset $\widetilde{\mathcal{M}}^*_{J,B} \subseteq \widetilde{\mathcal{M}}_{J,B}$ of *simple* J-holomorphic curves in the class B carries a natural structure of a smooth manifold without boundary of dimension (4). (But the same is not true for the bigger set $\widetilde{\mathcal{M}}_{J,B}$.) There is always at least one regular J, so this imposes no restriction on the symplectic manifold. (In fact, there are always many regular J's.) In some very special cases every J-holomorphic curve is simple, in these cases what I said above about $\widetilde{\mathcal{M}}_{J,B}$ and $\mathcal{M}_{J,B}$ is literally true if J is regular. In general however, one would have to replace $\widetilde{\mathcal{M}}_{J,B}$ by the subset $\widetilde{\mathcal{M}}^*_{J,B}$ and $\mathcal{M}_{J,B}$ by $\mathcal{M}^*_{J,B} := \widetilde{\mathcal{M}}^*_{J,B}/\sim$ and should assume that J is regular. Since the purpose of this paper is to explain the basic ideas of the definition of the GW's and the vortex invariants and not to give rigorous definitions, from now on I will ignore these technicalities. They are all very well explained in [13]. I have called the definition below a "working definition" to account for the lack of rigour.

Let $0 \neq B \in H_2(M, \mathbb{Z})$ be such that $d(M, \omega, B) - 6 = 0$ and let $J \in \mathcal{J}(M, \omega)$. Let us pretend that the manifold $\mathcal{M}_{J,B}$ was compact. Then $\mathcal{M}_{J,B}$ is a finite set. As I will explain in Remark 3.14, each point in $\mathcal{M}_{J,B}$ carries a natural sign. In the following definition, $\#\mathcal{M}_{J,B}$ means the number of elements of $\mathcal{M}_{J,B}$ counted with signs. As we will see below, this number does not depend on $J \in \mathcal{J}(M, \omega)$.

Working definition 3.13 *Let (M, ω) be a closed (compact without boundary) symplectic manifold and let $0 \neq B \in H_2(M, \mathbb{Z})$. We define the* genus 0 Gromov-Witten invariant of (M, ω, B) with no marked points *as follows. If*

$$d(M, \omega, B) - 6 = \dim M + 2\langle c_1(TM, \omega), B\rangle - 6 = 0 \tag{6}$$

then

$$\mathrm{GW}(M,\omega,B) := \#\mathcal{M}_{J,B},$$

where $J \in \mathcal{J}(M,\omega)$ *is any* ω*-compatible almost complex structure. Otherwise* $\mathrm{GW}(M,\omega,B) := 0$.

We consider now the case in which $d(M,\omega,B) = 0$.

Remark 3.14 The almost complex structure J induces an orientation on $\widetilde{\mathcal{M}}_{J,B}$ and therefore on $\mathcal{M}_{J,B}$. An orientation on a 0-dimensional manifold is the same as a collection of signs ($+1$ or -1), one for each point in the manifold. In Definition 3.13 the number $\#\mathcal{M}_{J,B}$ is really to be understood as the *signed* number of elements of $\mathcal{M}_{J,B}$. If we just counted without signs then this number would depend on J and not only on M, ω, B.

Example 3.15 Let $(M,\omega) := (S^2, \mathrm{dvol}_{S^2})$, where dvol_{S^2} is the symplectic structure of Example 2.5. Let $[S^2] \in H_2(S^2,\mathbb{Z})$ denote the generator corresponding to the orientation induced by dvol_{S^2}. Then the genus 0 GW's are given by

$$\mathrm{GW}(S^2, \mathrm{dvol}_{S^2}, [S^2]) = 1 \tag{7}$$

and $\mathrm{GW}(S^2, \mathrm{dvol}_{S^2}, B) = 0$ for $B \neq [S^2]$. This is seen as follows. In order to have nonzero invariants, the second homology class $B \in H_2(S^2)$ has to satisfy the dimensional condition (6), i.e. $2\langle c_1(TS^2, \mathrm{dvol}_{S^2}), B\rangle = 6 - \dim S^2 = 4$. Since $\langle c_1(TS^2, \mathrm{dvol}_{S^2}), [S^2]\rangle = 2$ this is equivalent to $B = [S^2]$. So if $B \neq [S^2]$ then the invariant is 0. Now let $B := [S^2]$. We show the equality (7). Let $J := J_0$ be the standard almost complex structure, cf. Example 3.5. J_0 is compatible with dvol_{S^2}. So $\mathrm{GW}(S^2, \mathrm{dvol}_{S^2}, [S^2])$ equals the number of equivalence classes of holomorphic maps $u : S^2 \cong \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1 \cong S^2$ that represent the homology class $[\mathbb{C}\mathrm{P}^1]$. Recall that the equivalence relation is given by (3). Now every Möbius transformation u is a (J_0-)holomorphic map from $\mathbb{C}\mathrm{P}^1$ to $\mathbb{C}\mathrm{P}^1$ that satisfies $[u] = [\mathbb{C}\mathrm{P}^1]$. Conversely by Fact 3.16 below if $u : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ is holomorphic such that $[u] = [\mathbb{C}\mathrm{P}^1]$ then u is biholomorphic, i.e. a Möbius transformation. This implies that there is exactly one equivalence class of such maps u, namely the class containing the identity $\mathrm{id}_{\mathbb{C}\mathrm{P}^1}$. This yields (7).

Fact 3.16 Let (X, j_X) and (Y, j_Y) be connected closed Riemann surfaces and let $u : X \to Y$ be a holomorphic map. Denote by $[Y] \in H_2(Y,\mathbb{Z})$ the fundamental class corresponding to the orientation induced by j_Y. Assume that $[u] = [Y]$. Then u is biholomorphic.

Remark 3.17 (Noncompactness) In general the manifold $\mathcal{M}_{J,B}$ is not compact. To have any chance for it to be compact, we have to assume that M is compact or satisfies some *convexity at* ∞ *condition*. But even then $\mathcal{M}_{J,B}$ will in most cases not be compact. One reason for this is the phenomenon of *bubbling*. In the case in which $\mathcal{M}_{J,B}$ is not compact the theory of *pseudocycles* is used to define the invariant $\mathrm{GW}(M, \omega, B)$. (See [13].)

Let us now try to understand why $\mathrm{GW}(M, \omega, B)$ does indeed only depend on M, ω, B, but not on $J \in \mathcal{J}(M, \omega)$.

Reason for the independence of $\#\mathcal{M}_{J,B}$ on $J \in \mathcal{J}(M, \omega)$: Let $J_0, J_1 \in \mathcal{J}(M, \omega)$ be ω-compatible almost complex structures. We have to show that

$$\#\mathcal{M}_{J_0,B} = \#\mathcal{M}_{J_1,B}. \tag{8}$$

Fact 3.18 $\mathcal{J}(M, \omega)$ is contractible.

So let $[0, 1] \ni \lambda \mapsto J_\lambda \in \mathcal{J}(M, \omega)$ be a smooth homotopy. Let us look at the set

$$\mathcal{W}(J_\cdot, B) := \{(\lambda, [u]) | u \text{ is } J_\lambda\text{-holomorphic}\},$$

where $[u]$ denotes the equivalence class of u under the relation (3). In good cases this is a smooth compact oriented 1-dimensional manifold with boundary

$$\partial\mathcal{W}(J_\cdot, B) = -\mathcal{M}(J_0, B) \amalg \mathcal{M}(J_1, B).$$

The following fact implies (8), so the $\mathrm{GW}(M, \omega, B)$ is indeed independent of $J \in \mathcal{J}(M, \omega)$. □

Fact 3.19 The number of points of the boundary of any smooth compact oriented 1-dimensional manifold equals 0, *if counted with signs.*

Now let us consider the case in which the dimensional condition (6) is not satisfied. We have defined the GW's with no marked points to be 0 in this case. To get nonzero invariants, we now introduce marked points.

3.2 GW's with fixed marked points

Let (M, ω) be a closed (compact without boundary) symplectic manifold, let $B \in H_2(M, \mathbb{Z})$ be a second homology class and let $k \in \mathbb{N}_0$ be a natural number (including 0). Furthermore let $J \in \mathcal{J}(M, \omega)$ be a ω-compatible almost complex structure. The idea of the genus 0 GW's with fixed marked points of (M, ω, B) is to count J-holomorphic spheres representing B and satisfying the conditions $u(z_i) \in N_i$, $i = 1, \ldots, k$. Here $z_i \in S^2$, $i = 1, \ldots, k$ are fixed distinct points and $N_i \subseteq M$ is a fixed closed oriented submanifold of M ($i = 1, \ldots, k$). The point is to impose "as many conditions as are necessary to make the number of such spheres finite". In order to see what this means, we take on the following more sophisticated viewpoint. Recall from (2) that $\widetilde{\mathcal{M}}_{J,B}$ is the set of J-holomorphic spheres in the class $B \in H_2(M, \mathbb{Z})$. We abbreviate $\mathbf{z} := (z_1, \ldots, z_k) \in (S^2)^k$. Let $\mathrm{ev}_{J,B,\mathbf{z}} : \widetilde{\mathcal{M}}_{J,B} \to M^k$ be the *evaluation map* given by

$$\mathrm{ev}_{J,B,\mathbf{z}}(u) := (u(z_1), \ldots, u(z_k)). \tag{9}$$

Let $N_i \subseteq M$ be a submanifold, for $i = 1, \ldots, k$. We denote

$$\begin{aligned} \widetilde{\mathcal{M}}_{J,B,\mathbf{z},N_1\times\ldots\times N_k} &:= \{u \in \widetilde{\mathcal{M}}_{J,B} \,|\, u(z_i) \in N_i,\ i = 1, \ldots, k\} \\ &= \mathrm{ev}^{-1}_{J,B,\mathbf{z}}(N_1 \times \ldots \times N_k) \subseteq \widetilde{\mathcal{M}}_{J,B}. \end{aligned} \tag{10}$$

Assume now that $\mathrm{ev}_{J,B,\mathbf{z}}$ is a smooth map and that it is transverse to $N_1 \times \ldots \times N_k \subseteq M^k$. The latter condition means that

$$\mathrm{im}(d\,\mathrm{ev}_{J,B,\mathbf{z}}(u)) + T_{\mathrm{ev}_{J,B,\mathbf{z}}(u)}(N_1 \times \ldots \times N_k) = T_{\mathrm{ev}_{J,B,\mathbf{z}}(u)}M^k, \tag{11}$$

for every $u \in \widetilde{\mathcal{M}}_{J,B}$. Then the implicit function theorem says that $\widetilde{\mathcal{M}}_{J,B,\mathbf{z},N_1\times\ldots\times N_k} \subseteq \widetilde{\mathcal{M}}_{J,B}$ is a submanifold of dimension

$$\begin{aligned} &\dim(\widetilde{\mathcal{M}}_{J,B,\mathbf{z},N_1\times\ldots\times N_k}) \\ &\quad = \dim(\widetilde{\mathcal{M}}_{J,B}) - \mathrm{codim}_{M^k}(N_1 \times \ldots \times N_k) \\ &\quad = d(M, \omega, B) - k \dim M + \sum_{i=1}^{k} \dim N_i \end{aligned} \tag{12}$$

Let us again pretend that the manifold $\widetilde{\mathcal{M}}_{J,B,\mathbf{z},N_1\times\ldots\times N_k}$ was compact. Assume that it is nonempty. Then it consists of finitely many elements if and only if it is 0-dimensional. So in order for the invariants to be well defined, we have to impose the condition (12)=0. Now as in the case of no marked points, by a bordism argument, the *signed* number of points $\#\widetilde{\mathcal{M}}_{J,B,\mathbf{z},N_1\times\ldots\times N_k}$ (see Remark 3.21) does not depend on the ω-compatible almost complex structure J. Furthermore, it does not depend on the k-tuple of distinct points $(z_1, \ldots, z_k) \in (S^2)^k$ either. Finally, there is always a $J \in \mathcal{J}(M, \omega)$ such that (11) is satisfied. Therefore we can define the following.

Working definition 3.20 (first version of $\mathrm{GW}^{M,\omega}_{k,B}$) *Let (M,ω) be a closed symplectic manifold, $B \in H_2(M,\mathbb{Z})$ and $k \in \mathbb{N}_0$ be a natural number. We define the map*

$$\widetilde{\mathrm{GW}}^{M,\omega}_{k,B} : \{\textit{closed oriented submanifolds of } M\}^k \to \mathbb{Z}$$

as follows. If

$$\sum_{i=1}^k \dim N_i = k \dim M - d(M,\omega,B) \tag{13}$$

then

$$\widetilde{\mathrm{GW}}^{M,\omega}_{k,B}(N_1,\ldots,N_k) := \#\widetilde{\mathcal{M}}_{J,B,\mathbf{z},N_1\times\ldots\times N_k}.$$

Otherwise we define $\widetilde{\mathrm{GW}}^{M,\omega}_{k,B}(N_1,\ldots,N_k) := 0$. Here $\mathbf{z} := (z_1,\ldots,z_k) \in (S^2)^k$ is any k-tuple of distinct points, $J \in \mathcal{J}(M,\omega)$ is any ω-compatible almost complex structure such that $\mathrm{ev}_{J,B,\mathbf{z}}$ is transverse to $N_1\times\ldots\times N_k \subseteq M^k$ and $\widetilde{\mathcal{M}}_{J,B,\mathbf{z},N_1\times\ldots\times N_k}$ is given by (10).

Remark 3.21 Assume that (13) holds. Then as in the case of no marked points, # really means the *signed* number of points of a set. For each point in $\widetilde{\mathcal{M}}_{J,B,\mathbf{z},N_1\times\ldots\times N_k}$ we get a sign as follows. Recall from Remark (3.14) that there is a canonical orientation on $\widetilde{\mathcal{M}}_{J,B}$ coming from J. Furthermore, ω induces a canonical orientation on M. Together with the given orientations on N_i, $i=1,\ldots,k$ this induces an orientation on $\mathcal{M}_{J,B,\mathbf{z},N_1\times\ldots\times N_k}$. This is the same as a sign for every point in this set. We will call a set endowed with a sign for each of its points a *signed set*.

Remark 3.22 Note that in contrast to the case of no marked points, $\widetilde{\mathrm{GW}}^{M,\omega}_{k,B}(N_1,\ldots,N_k)$ does not count *equivalence classes*, but individual J-holomorphic spheres. In particular, for $k=0$ $\widetilde{\mathrm{GW}}^{M,\omega}_{0,B}$ does not coincide with $\mathrm{GW}(M,\omega,B)$. The point of counting individual spheres is that in the case $B \neq 0$ the conditions $u(z_i) \in N_i$, $i=1,\ldots,k$ are not invariant under reparametrization, so we cannot define an evaluation map on the quotient $\mathcal{M}_{J,B} = \widetilde{\mathcal{M}}_{J,B}/\sim$ but only on $\widetilde{\mathcal{M}}_{J,B}$. Assume for example that $k=3$, $N_i = \{x_i\}$ for some $x_i \in M$, for $i=1,2,3$, and that $B \neq 0$. Suppose that there is a J-holomorphic sphere u such that $[u]=B$ and $u(z_i)=x_i$, $i=1,2,3$. Then there is a Möbius transformation $\phi \in \mathrm{PSL}(2,\mathbb{C})$ such that $u\circ\phi(z_i) \neq x_i$ for all i. So $u\circ\phi$ does not satisfy the conditions corresponding to N_i for any $i=1,2,3$.

Note also that in the case $k<3$ $\widetilde{\mathrm{GW}}^{M,\omega}_{k,B} = 0$, unless $B=0$. (For the case $B=0$ see Example 3.23.) The reason is the following. Assume $k<3$, $B\neq 0$

and that $J \in \mathcal{J}(M,\omega)$ is such that $\mathrm{ev}_{J,B,\mathbf{z}}$ is transverse to $N_1 \times \ldots \times N_k \subseteq M^k$. Recall that this implies that $\widetilde{\mathcal{M}}_{J,B,\mathbf{z},N_1\times\ldots\times N_k}$ is a 0-dimensional manifold. We claim that there is no J-holomorphic sphere $u : S^2 \to M$ such that $u(z_i) \in N_i$, for $i \leq k$. Assume by contradiction that there was such a u. Then there is a $(6-2k)$-dimensional family of Möbius transformations $U \subseteq \mathrm{PSL}(2,\mathbb{C})$ such that for every $\phi \in U$ we have $\phi(z_i) = z_i$ for $i \leq k$ and $u \circ \phi \neq u$. In particular, $\widetilde{\mathcal{M}}_{J,B,\mathbf{z},N_1\times\ldots\times N_k}$ is not a 0-dimensional manifold. This is a contradiction. (See also Fact 3.11.)

Example 3.23 Let (M,ω) be a closed symplectic manifold, $k \in \mathbb{N}_0$, $B := 0$ and let $N_i \subseteq M$, $i = 1,\ldots,k$ be closed oriented submanifolds. Assume that

$$\sum_{i=1}^k \operatorname{codim} N_i := \sum_{i=1}^k (\dim M - \dim N_i) = \dim M,$$

so the dimensional condition (13) holds. Furthermore, assume that for each $i = 1,\ldots,k$ and $x \in N_1 \cap \ldots \cap N_i$ we have $(T_xN_1 \cap \ldots \cap T_xN_{i-1}) + T_xN_i = T_xM$. Then $N_1 \cap \ldots \cap N_k \subseteq M$ is an oriented compact 0-dimensional submanifold, i.e. a finite signed set. We have

$$\widetilde{\mathrm{GW}}^{M,\omega}_{k,0}(N_1,\ldots,N_k) = \#(N_1 \cap \ldots \cap N_k), \tag{14}$$

counting with signs. To see this let $J \in \mathcal{J}(M,\omega)$ and let $z_1,\ldots,z_k \in S^2$ be pairwise distinct. For any $x \in N_1 \cap \ldots \cap N_k$ the constant map $u_x :\equiv x : S^2 \to M$ is J-holomorphic and satisfies $[u_x] = B = 0$ and $u(z_i) \in N_1 \cap \ldots \cap N_k$, $i = 1,\ldots,k$. The sign of the point $u \in \widetilde{\mathcal{M}}_{J,0,\mathbf{z},N_1\times\ldots\times N_k}$ is the same as the sign of the point $x_0 \in N_1 \cap \ldots \cap N_k$. Conversely, if $u : S^2 \to M$ is J-holomorphic and $[u] = B = 0$ then u is constant. So $\widetilde{\mathcal{M}}_{J,0,\mathbf{z},N_1\times\ldots\times N_k} = \{u_x \,|\, x \in N_1 \cap \ldots \cap N_k\}$ and this implies (14).

Example 3.24 Let $k := 3$, $(M,\omega) := (S^2, \mathrm{dvol}_{S^2})$ and let x_1, x_2, x_3 be distinct points. Then

$$\widetilde{\mathrm{GW}}^{S^2,\mathrm{dvol}_{S^2}}_{3,[S^2]}(\{x_1\},\{x_2\},\{x_3\}) = 1, \tag{15}$$

where $[S^2] \in H_2(S^2,\mathbb{Z})$ is the generator corresponding to the orientation of S^2 induced by dvol_{S^2}. To see this we first check the dimensional condition (13) with $N_i := \{x_i\}$, $i = 1,\ldots,k$. This condition holds since $\langle c_1(TS^2, \mathrm{dvol}_{S^2}), [S^2]\rangle = 2$. Let now $J := J_0$ be the standard structure on S^2, cf. Example 3.5. The J_0-holomorphic maps $u : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ such that $[u] = [\mathbb{C}\mathrm{P}^1]$ are exactly the Möbius transformation (see also Example 3.15). Let $z_1 := 0$, $z_2 := 1$, $z_3 := \infty$. There is a unique Möbius transformation $\phi \in \mathrm{PSL}(2,\mathbb{C})$ such that $\phi(z_i) = x_i$. This implies (15).

As already mentioned, $\widetilde{\mathrm{GW}}^{M,\omega}_{k,B}(N_1,\dots,N_k)$ only depends on the classes $[N_i] \in H_{\dim N_i}(M,\mathbb{Z})$. Now let $a_i \in H_*(M,\mathbb{Z})$ be a homology class, for $i = 1,\dots,k$. If there are closed oriented submanifolds $N_i \subseteq M$ such that $[N_i] = a_i$ then we define $\mathrm{GW}^{M,\omega}_{k,B}(a_1,\dots,a_k) := \widetilde{\mathrm{GW}}^{M,\omega}_{k,B}(N_1,\dots,N_k)$. In general, there are no such submanifolds. However, we have the following fact.

Fact 3.25 Let M be a manifold and $a \in H_*(M,\mathbb{Z})$ be a class of degree $0 \leq \deg a \leq \dim M - 1$. Then by a theorem of Thom [18] there is an integer $0 \neq m \in \mathbb{Z}$ and a closed oriented submanifold $N \subseteq M$ (possibly nonconnected) such that $ma = [N]$.

The case $\deg a = \dim M$ is left out in the above fact because of the following. Assume that M is a closed, oriented and connected manifold. Denote by $[M] \in H_{\dim M}(M,\mathbb{Z})$ the fundamental cycle. Let $d \in \mathbb{Z}$ be an integer such that $d \neq 0, 1, -1$ and define $a := d[M]$. Then a can not be represented by closed oriented submanifold of M. The reason is that any such submanifold equals M (possibly with reversed orientation). But a/d can be represented by M. Since $[M]$ is a generator of $H_{\dim M}(M,\mathbb{Z}) \cong \mathbb{Z}$ any $a \in H_{\dim M}(M,\mathbb{Z})$ is an integer multiple of the fundamental class $[M]$. Summarizing we see therefore that given a class $a \in H_*(M,\mathbb{Z})$ of any degree there is always an oriented closed submanifold $N \subseteq M$ and a rational number $r \in \mathbb{Q}$ such that $a = r[N]$. We are now ready for the following.

Working definition 3.26 *Let (M,ω) be a connected closed symplectic manifold, $B \in H_2(M,\mathbb{Z})$ and $k \in \mathbb{N}_0$ be a natural number. We define a k-linear map $\mathrm{GW}^{M,\omega}_{k,B} : (H_*(M,\mathbb{Z}))^k \to \mathbb{Q}$ as follows. If $(a_1,\dots,a_k) \in (H_*(M,\mathbb{Z}))^k$ satisfies the dimensional condition*

$$\sum_{i=1}^{k} \deg a_i = k \dim M - d(M,\omega,B), \tag{16}$$

where $d(M,\omega,B)$ is given by (5) then we define

$$\mathrm{GW}^{M,\omega}_{k,B}(a_1,\dots,a_k) := r_1 \cdots r_k \cdot \#\widetilde{\mathcal{M}}_{J,B,\mathbf{z},N_1\times\ldots\times N_k}.$$

Otherwise $\mathrm{GW}^{M,\omega}_{k,B}(a_1,\dots,a_k) := 0$. Here for $i = 1,\dots,k$ $N_i \subseteq M$ is a closed oriented submanifold, $r_i \in \mathbb{Q}$ is a rational number such that $r_i[N_i] = a_i$ and $\widetilde{\mathcal{M}}_{J,B,\mathbf{z},N_1\times\ldots\times N_k}$ is defined as in (10). We also define a map $\mathrm{GW}^{M,\omega}_{k,B} : (H^*(M,\mathbb{Z}))^k \to \mathbb{Q}$ (same symbol, different domain) by the formula

$$\mathrm{GW}^{M,\omega}_{k,B}(\alpha_1,\dots,\alpha_k) := \mathrm{GW}^{M,\omega}_{k,B}(\mathrm{PD}(\alpha_1),\dots,\mathrm{PD}(\alpha_k)).$$

Here for $0 \le m \le 2n$ $\mathrm{PD} : H^m(M,\mathbb{Z}) \to H_{2n-m}(M,\mathbb{Z})$ *denotes Poincaré duality. Both maps, the one on homology and the one on cohomology, are referred to as the* genus 0 Gromov-Witten invariants of (M, ω, B) with k fixed marked points.

Since Poincaré duality is an isomorphism, both maps, the one on homology and the one on cohomology, contain the same information. It will be clear from the context which one of both is meant.

Remark 3.27 A rigorous definition of the GW's uses *pseudocycles*. These are a generalization of smooth cycles in singular homology. It follows then immediately that the invariants in fact take values in the integers, and not just in $\mathbb{Q}$. Fact (3.25) is not used in that definition. The rigorous definition involves the same technical issues as in the case of no marked points. Moreover, one cannot really work with *one* fixed $J \in \mathcal{J}(M,\omega)$, but has to take a whole family of almost complex structures $\{J_z\}_{z\in S^2} \subseteq \mathcal{J}(M,\omega)$. The Cauchy-Riemann equations (1) get replaced by

$$du(z) + J_z(u(z))\, du(z)\, j_z = 0, \qquad \forall z \in S^2.$$

Example 3.28 Let (M, ω) be a connected closed symplectic manifold, $k \in \mathbb{N}_0$, $B := 0$ and let $a_i \in H_*(M,\mathbb{Z})$, $i = 1, \ldots, k$. Assume that the dimensional condition (16) holds. Then

$$\mathrm{GW}^{M,\omega}_{k,0}(a_1, \ldots, a_k) = a_1 \cdot \ldots \cdot a_k. \tag{17}$$

Here $\cdot : H_\ell(M,\mathbb{Z}) \times H_m(M,\mathbb{Z}) \to H_{\ell+m-2n}(M,\mathbb{Z})$ denotes the intersection product, for $\ell, m = 0, \ldots, 2n$. Furthermore, we identify any element $a \in H_0(M,\mathbb{Z})$ with the integer $N \in \mathbb{Z}$ defined by $a = N[\mathrm{pt}]$, where $\mathrm{pt} \in M$ is any point. Formula (17) follows from Example 3.23.

Example 3.29 Let $(M, \omega) := (S^2, \mathrm{dvol}_{S^2})$, $k := 3$, $B := [S^2]$ and let $\mathrm{pt} \in S^2$ be any point. Then

$$\mathrm{GW}^{S^2,\mathrm{dvol}_{S^2}}_{3,[S^2]}([\mathrm{pt}], [\mathrm{pt}], [\mathrm{pt}]) = 1. \tag{18}$$

This follows from Example 3.24.

Example 3.30 Let $M := \mathbb{C}\mathrm{P}^n$ and let $\omega := \omega_{FS}$ be the Fubini Studi form on $\mathbb{C}\mathrm{P}^n$. It is defined to be the canonical quotient form coming from the standard form ω_0 on $\mathbb{R}^{2n+2}$ and the diagonal Hamiltonian group action of $S^1 \subseteq \mathbb{C}$ on $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. (See Example 4.9. There are other ways to define ω_{FS}, see e.g. [9].) Note that in the case $n = 1$ $\omega_{FS} = 1/4\,\mathrm{dvol}_{S^2}$, where dvol_{S^2} is the standard volume form on S^2, cf. Example 2.5. Let $k := 3$. Let $a_1 := a_2 := [\mathrm{pt}] \in H_0(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$ be the homology class of any point $\mathrm{pt} \in \mathbb{C}\mathrm{P}^n$. For $i = 1, \dots, n$ we identify $\mathbb{C}\mathrm{P}^i$ with the submanifold $\mathbb{C}\mathrm{P}^i \cong \{[z_0, \dots, z_i, 0, \dots, 0] \in \mathbb{C}\mathrm{P}^n \,|\, [z_0, \dots, z_i] \in \mathbb{C}\mathrm{P}^i\} \subseteq \mathbb{C}\mathrm{P}^n$. Let $B := [\mathbb{C}\mathrm{P}^1] \in H_2(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$ and let $a_3 := [\mathbb{C}\mathrm{P}^{n-1}] \in H_{2n-2}(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$. Then

$$\mathrm{GW}^{\mathbb{C}\mathrm{P}^n, \omega_{FS}}_{3,[\mathbb{C}\mathrm{P}^1]}([\mathrm{pt}], [\mathrm{pt}], [\mathbb{C}\mathrm{P}^{n-1}]) = 1. \tag{19}$$

To see this let J_0 be the standard almost complex structure on $\mathbb{C}\mathrm{P}^n$. It is uniquely determined by the condition $\phi_i^* J_0 = J_{\mathrm{st}}$. Here $\phi_i : \mathbb{C}^n \to \mathbb{C}\mathrm{P}^n$ denotes the i-th standard coordinate chart defined by $\phi_i(z_1, \dots, z_n) := [z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n]$. Moreover, J_{st} is the standard almost complex structure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as in Example 3.2. Since the structure J_{st} on $\mathbb{R}^{2n+2}$ is compatible with ω_0, J_0 is also compatible with ω_{FS}. Let now $z_1 := [1, 1]$, $z_2 := [0, 1]$, $z_3 := [1, 0] \in \mathbb{C}\mathrm{P}^1$ and let $\mathrm{pt}_1 := [1, 0, \dots, 0, 1]$ and $\mathrm{pt}_2 := [0, \dots, 0, 1]$. Let $u : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ be defined by $u([z^0, z^1]) := [z^0, 0, \dots, 0, z^1]$. Then u is holomorphic and represents the class $B = [\mathbb{C}\mathrm{P}^1]$. Furthermore, $u(z_1) = \mathrm{pt}_1$, $u(z_2) = \mathrm{pt}_2$ and $u(z_3) \in \mathbb{C}\mathrm{P}^{n-1} \subseteq \mathbb{C}\mathrm{P}^n$. So $u \in \widetilde{\mathcal{M}}_{J_0,[\mathbb{C}\mathrm{P}^1],\mathbf{z},\mathrm{pt}_1 \times \mathrm{pt}_2 \times \mathbb{C}\mathrm{P}^{n-1}}$. One can show that u is the only element of $\widetilde{\mathcal{M}}_{J_0,[\mathbb{C}\mathrm{P}^1],\mathbf{z},\mathrm{pt}_1 \times \mathrm{pt}_2 \times \mathbb{C}\mathrm{P}^{n-1}}$. This implies (19).

There is the following geometric interpretation of (19). Namely, since the points pt_1, $\mathrm{pt}_2 \in \mathbb{C}\mathrm{P}^n$ are distinct, there is exactly one complex projective line $L \subseteq \mathbb{C}\mathrm{P}^n$ through pt_1 and pt_2. Furthermore, since pt_1 and pt_2 are not both contained in the hyperplane $\mathbb{C}\mathrm{P}^{n-1} \subseteq \mathbb{C}\mathrm{P}^n$ L intersects $\mathbb{C}\mathrm{P}^{n-1}$ in exactly one point. The map $u \in \widetilde{\mathcal{M}}_{J_0,[\mathbb{C}\mathrm{P}^{n-1}]}$ parametrizes L. Since none of the points pt_1 and pt_2 lie in $\mathbb{C}\mathrm{P}^{n-1}$, this parametrization is uniquely determined by the condition that u passes through pt_i at the point z_i, $i = 1, 2$ and it passes through the hyperplane $\mathbb{C}\mathrm{P}^{n-1} \subseteq \mathbb{C}\mathrm{P}^n$ at the point z_3.

4 Symplectic Reduction and the Symplectic Vortex Equations

In this part of the paper I will describe a way how to construct a symplectic manifold $(\bar{M}, \bar{\omega})$ out of an other symplectic manifold (M, ω), on which a Lie group acts in a Hamiltonian way. This procedure is called *symplectic reduction* and the resulting manifold is called the *symplectic quotient*. This will be introduced in subsection 4.1. As an example, every symplectic manifold is the quotient of itself under the action of the trivial Lie group $\{e\}$. In contrast to this not so interesting example, there are examples of complicated symplectic manifolds that arise as symplectic quotients of simpler symplectic manifolds. (As an example, $\mathbb{C}\mathrm{P}^n$ with the Fubini-Studi form ω_{FS} is obtained via symplectic reduction from $\mathbb{C}^{n+1}$ with the standard form ω_0. In this case the Lie group is $S^1 \subseteq \mathbb{C}$, acting on $\mathbb{C}^{n+1}$ by multiplication on each component. See Example 4.9.) In these cases, one can hope to achieve the following:

Motivational Goal: Given a symplectic manifold $(\bar{M}, \bar{\omega})$ that is the symplectic quotient of a symplectic manifold (M, ω) under the action of a Lie group G, compute its genus 0 Gromov-Witten invariants with fixed marked points by analyzing the action of G on M.

A first step toward achieving this motivational goal will be described in subsection 4.2. It is to lift the Cauchy-Riemann equations for a map $f : S^2 \to \bar{M}$ to an equation for a map u that takes values in some submanifold of M. In subsection 4.3, I will discuss the *symplectic vortex equations*. These equations have been studied by physicists since the 1950's. Their solutions are minimizers of the Yang-Mills-Higgs functional. I will describe the symplectic vortex invariants, which are built on these equations. Finally, in subsection 4.4 we will see how the Gromov-Witten invariants of $(\bar{M}, \bar{\omega})$ are related to the symplectic vortex invariants. This relation is the content of Theorem 4.33. I will indicate in a very rough way the general idea of proof of this Theorem.

4.1 Symplectic Reduction

Let G be a Lie group with neutral element $e \in G$ and Lie algebra $\mathfrak{g} := T_eG$. We denote by $\mathrm{ad} : G \times \mathfrak{g} \to \mathfrak{g}$ the adjoint action of G on $\mathfrak{g}$ given by

$$\mathrm{ad}(g, \eta) := \mathrm{ad}_g \eta := g\eta g^{-1}.$$

Let $\langle ., . \rangle_{\mathfrak{g}}$ be an inner product on $\mathfrak{g}$ that is invariant under this action. This means that

$$\langle \mathrm{ad}_g \xi, \mathrm{ad}_g \eta \rangle = \langle \xi, \eta \rangle,$$

for every $\xi, \eta \in \mathfrak{g}$, $g \in G$. Let M be a manifold and $\psi : G \times M \to M$ be a smooth left action of G on M. We use the notation $gx := \psi_g(x) := \psi(g, x) \in M$ for $g \in G$, $x \in M$. Furthermore for $\eta \in \mathfrak{g}$ the *infinitesimal action of* η on M is the vector field $X_\eta \in \mathrm{Vect}(M)$ given by

$$X_\eta(x) := \left.\frac{d}{dt}\right|_{t=0} \exp(t\eta)x \in T_xM.$$

Let now (M, ω) be a symplectic manifold and G a Lie group that acts smoothly on M.

Definition 4.1 *The action of G on M is called* Hamiltonian *iff there is a smooth map $\mu : M \to \mathfrak{g}$ such that the following conditions hold.*

$$\langle d\mu(x)v, \eta\rangle = \omega(X_\eta(x), v), \qquad \forall v \in T_xM,\, x \in M \tag{20}$$

$$\langle \mu(x), [\xi, \eta]\rangle = \omega(X_\xi(x), X_\eta(x)), \qquad \forall x \in M,\, \xi, \eta \in \mathfrak{g}.$$

The map $\mu : M \to \mathfrak{g}$ is called a moment map *for the action.*

Example 4.2 Assume that M is closed and let $H : M \to \mathbb{R}$ be a smooth function. This gives rise to a Hamiltonian action of $G := \mathbb{R}$ on M as follows. Since ω is nondegenerate, there is a unique vector field $X_H \in \mathrm{Vect}(M)$ satisfying

$$dH = \omega(X_H, .).$$

H is called a *Hamiltonian* and X_H is called the *vector field generated by* H. Since M is closed, the integral curves of X_H exists for all times. This means that for every $x_0 \in M$ there is a solution $x : \mathbb{R} \to M$ of the equations $X_H(x(t)) = \dot{x}(t)$, $x(0) = x_0$. Let $\{\phi^t\}_{t\in\mathbb{R}}$ be the flow of X_H. It is uniquely determined by the conditions

$$\begin{aligned} \phi^0 &= \mathrm{id}, \\ \frac{d}{dt}\phi^t(x) &= X_H(\phi^t(x)), \qquad \forall x \in M. \end{aligned}$$

The action of $G := \mathbb{R}$ on M is given by

$$\mathbb{R} \times M \to M, \qquad (t, x) \mapsto \phi^t(x).$$

This is a Hamiltonian action with moment map $\mu := H : M \to \mathbb{R}$. Here the invariant inner product on the Lie algebra $\mathbb{R} = \mathrm{Lie}(\mathbb{R})$ is given by ordinary multiplication.

In the following let G be a Lie group and $\psi : G \times M \to M$ be a Hamiltonian action with moment map $\mu : M \to \mathfrak{g}$.

Fact 4.3 For any $g \in G$ the diffeomorphism ψ_g of M is a symplectomorphism, i.e. $\psi_g^*\omega = \omega$.

Remark 4.4 We define the *Poisson bracket* $\{.,.\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ by

$$\{F, G\} := \omega(X_F, X_G),$$

where X_F and X_G are as in Example 4.2. The pair $(C^\infty(M), \{.,.\})$ is a Lie algebra. For $\eta \in \mathfrak{g}$ we define the function $H_\eta : M \to \mathbb{R}$ by

$$H_\eta(x) := \langle \mu(x), \eta \rangle.$$

Let $[.,.]$ be the usual Lie bracket on $\mathfrak{g}$. Then the map

$$\mathfrak{g} \ni \eta \mapsto \langle \mu, \eta \rangle \in C^\infty(M, \mathbb{R})$$

is a Lie algebra homomorphism w.r.t. $[.,.]$ and $\{.,.\}$.

Remark 4.5 If $\tau \in \mathfrak{g}$ is in the center under the adjoint action, then $\mu' := \mu + \tau$ is a moment map for the same action. Thus moment maps are in general not unique.

Fact 4.6 If G is connected then μ is equivariant w.r.t. the adjoint action of G on $\mathfrak{g}$, i.e.

$$\mu(gx) = \mathrm{ad}_g \mu(x) := g\mu(x)g^{-1}, \qquad \forall x \in M,\ g \in G.$$

Let G be a group that acts from the left on a set X. Then the action is called *free* iff the equality $gx = x$ implies $g = e$, for every $x \in X$ and $g \in G$. Assume now that G is a topological group (for example a Lie group) and X is a topological space and that G acts continuously on X. Then the action is called *proper* iff the map

$$G \times X \to X \times X, \qquad (g, x) \mapsto (x, gx)$$

is proper, i.e. the preimage of every compact set $K \subseteq X \times X$ is compact. Let (M, ω) be a symplectic manifold and G be a Lie group that acts in a Hamiltonian way on M, with equivariant moment map $\mu : M \to \mathfrak{g}$. (By Fact 4.6 μ is equivariant if G is connected.) Let us look at $\mu^{-1}(0) \subseteq M$. Since μ is equivariant, for every $x \in \mu^{-1}(0)$ and $g \in G$ the point gx lies again in $\mu^{-1}(0)$, i.e. the action of G on M restricts to an action on $\mu^{-1}(0)$. If 0 is a regular value of μ then $\mu^{-1}(0) \subseteq M$ is a submanifold. Assume that G acts freely on $\mu^{-1}(0)$. Then 0 is a regular value of μ. Suppose in addition that the action of G on $\mu^{-1}(0)$ is proper. (This is satisfied if for example G is compact.) Under these assumptions the quotient $\bar{M} := \mu^{-1}(0)/G$ carries a canonical manifold structure (see Example 4.16 below).

Definition 4.7 *The canonical symplectic form* $\bar{\omega} \in \Omega^2(\bar{M})$ *on the quotient* $\bar{M}$ *is given as follows. Denote by* $\pi_0 : \mu^{-1}(0) \to \bar{M}$ *the canonical projection. Let* $\bar{x} := Gx \in \bar{M}$ *and* $\bar{v}_i \in T_{\bar{x}}\bar{M}$, $i = 1, 2$. *Let* $v_i \in T_x(\mu^{-1}(0)) \cong \ker d\mu(x)$ *such that* $d\pi_0(x)v_i = \bar{v}_i$. *Then*

$$\bar{\omega}(\bar{v}_1, \bar{v}_2) := \omega(v_1, v_2),$$

The pair $(\bar{M}, \bar{\omega})$ *is called the* symplectic quotient *of* (M, ω) *under the action of* G.

Remark 4.8 $\dim \mu^{-1}(0) = \dim M - \dim G$, $\dim \bar{M} = \dim M - 2\dim G$.

Example 4.9 Let $M := \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$, $\omega := \omega_0 := \sum_{i=0}^n dx^i \wedge dy_i$ the standard symplectic structure on $\mathbb{C}^{n+1}$ and let $G := S^1 \subseteq \mathbb{C}$ act on $\mathbb{C}^{n+1}$ by multiplication on each component, i.e.

$$g(z_0, \ldots, z_n) := (gz_0, \ldots, gz_n),$$

for $g \in S^1$ and $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$. This action is Hamiltonian. Denoting an element of $\mathbb{C}^{n+1}$ by $\mathbf{z} := (z_0, \ldots, z_n)$ a moment map of this action is given by $\mu_0 : \mathbb{C}^{n+1} \to i\mathbb{R} = \mathrm{Lie}(S^1)$,

$$\mu_0(\mathbf{z}) := -\frac{i}{2}|\mathbf{z}|^2 := -\frac{i}{2}\sum_{j=0}^{n} |z_j|^2.$$

For $R > 0$ we define $\mu_R := \mu_0 + \frac{iR^2}{2}$. Then

$$\mu_R^{-1}(0) = S_R^{2n+1} := \{\mathbf{z} \in \mathbb{C}^{n+1} \,|\, |\mathbf{z}| = R\}.$$

For $R := 1$ the symplectic quotient $\bar{M}$ is $\mathbb{C}\mathrm{P}^n$ with the Fubini-Studi form $\bar{\omega} = \omega_{FS}$. In the case $n = 0$ the quotient $\bar{M}$ is a single point, in the case $n = 1$ $\bar{M} = \mathbb{C}\mathrm{P}^1 \cong S^2$ and $\bar{\omega} = \omega_{FS} = 1/4\,\mathrm{dvol}_{S^2}$, where dvol_{S^2} is as in Example 2.5.

Remark 4.10 One can generalize the above action to actions of the k-dimensional torus $\mathbb{T}^k = (S^1)^k$ on $\mathbb{C}^n$. There are many ways in which $\mathbb{T}^k$ can act on $\mathbb{C}^n$. So there are a lot of interesting examples of symplectic manifolds that are symplectic quotients of the "simple" symplectic manifold $(\mathbb{C}^n, \omega_0)$.

4.2 Lifting the $\bar{\partial}_{\bar{J}}$-equation

Let (M,ω) be a symplectic manifold and G be a compact connected Lie group that acts on M in a Hamiltonian way with moment map μ. Assume that $\mu : M \to \mathfrak{g}$ is proper. (This means that the preimage of any compact set is compact. In particular, $\mu^{-1}(0)$ is compact). Assume furthermore that the restricted action of G on $\mu^{-1}(0)$ is free. Under these conditions the symplectic quotient $(\bar{M},\bar{\omega})$ is well defined and $\bar{M}$ is compact. In this subsection we will do a first step toward achieving our motivational goal, which is to compute the genus 0 Gromov-Witten invariants with fixed marked points of $(\bar{M},\bar{\omega})$. For simplicity we only consider GW's with Riemann surface equal to (S^2, J_0) and with fixed marked points.
Let $\bar{B} \in H_2(\bar{M},\mathbb{Z})$ be a second homology class, $k \in \mathbb{N}_0$ be a natural number, $z_i \in S^2$, $i = 1,\dots,k$ be distinct points and let $\hat{J} \in \mathcal{J}(\bar{M},\bar{\omega})$ be an $\bar{\omega}$-compatible almost complex structure. For $i = 1,\dots,k$ let $\bar{\alpha}_i \in H^*(\bar{M})$ be a cohomology class whose Poincaré dual $PD^{\bar{M}}(\bar{\alpha}_i)$ is represented by a closed oriented submanifold $\bar{N}_i \subseteq \bar{M}$. The genus 0 Gromov-Witten invariants of $(\bar{M},\bar{\omega},\bar{B})$ with k fixed marked points count $(J_0,\hat{J})$-holomorphic curves $v : S^2 \to \bar{M}$ such that $[v] = \bar{B}$ and $v(z_i) \in \bar{N}_i$ for $i = 1,\dots,k$. We now make a special choice for $\hat{J}$. For $x \in M$, $v \in T_xM$ and $g \in G$ we denote $gv := d\psi_g(x)v \in T_{\psi_g(x)}M$. An almost complex structure J on M is said to be G-invariant iff $Jgv = gJv$, for every $g \in G$, $v \in T_xM$, $x \in M$. For $x \in M$ we define the linear map $L_x : \mathfrak{g} \to T_xM$ by

$$L_x\eta := X_\eta(x).$$

Recall from Definition 3.6 that the map $TM \oplus TM \ni (x; v, w) \mapsto (g_{\omega,J})_x(v,w) := \omega_x(v, Jw)$ is a Riemannian metric on M. Let $\perp$ denote the orthogonal complement w.r.t. this metric. Let $x \in M$. We define

$$H_x := \ker d\mu(x) \cap \mathrm{im} L_x^\perp.$$

Let $v \in H_x$. Then by assertion (ii) of Lemma 4.27 below $Jv \in H_x$. This enables us to make the following definition.

Definition 4.11 *Let J be a G-invariant almost complex structure on M. Let $\bar{x} \in \bar{M}$ and $\bar{v} \in T_{\bar{x}}\bar{M}$. Let $x \in \pi_0^{-1}(\bar{x}) \subseteq \mu^{-1}(0)$. Let $v \in H_x$ be the unique vector such that $d\pi_0(x)v = \bar{v}$. We define $\bar{J}\bar{v} := d\pi_0(x)Jv$.*

Since J is G-invariant, this definition does not depend on the choice of x. Assume now that J is G-invariant and ω-compatible. (Such a J exists.) Then $\bar{J}$ is $\bar{\omega}$-compatible. Since the GW's do not depend on $\hat{J} \in \mathcal{J}(\bar{M},\bar{\omega})$, we can and will from now on assume that $\hat{J} = \bar{J}$ for some G-invariant and

ω-compatible J on M.

Let (Σ, j) be a Riemann surface and $f : \Sigma \to \bar{M}$ be a smooth map. Recall that f is called $\bar{J}$-holomorphic iff it solves the Cauchy-Riemann equations (1). These equations now read

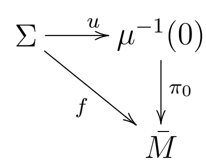
$$(\bar{\partial}_{\bar{J}}(f))_z\zeta := \frac{1}{2}\big(df(z)\zeta + \bar{J}(u(z))\, df(z)\, j(z)\zeta\big) = 0, \qquad \forall z \in \Sigma,\, \zeta \in T_z\Sigma. \tag{21}$$

The task at hand is now the following.

Task Find a manifold P and a map u with domain P and range $\mu^{-1}(0)$ that lifts f in a suitable sense. Furthermore, transform the equation for f, (21) into an equation for the map u.

As we will see below, $f : \Sigma \to \bar{M}$ can be lifted to a map $u_f : P := P_f \to \mu^{-1}(0)$, where P_f is a principal G-bundle over Σ constructed from f. A principal G-bundle over a manifold X consists of X, another manifold P, a smooth surjective map $\pi : P \to X$ and a smooth action of a Lie group G on P satisfying a certain condition (see Definition 4.13). A map $u : P \to \mu^{-1}(0)$ is called a lift of $f : X \to \bar{M}$ iff $\pi_0 \circ u = f \circ \pi$. Our task will be accomplished by Proposition 4.22. There we will see that equation (21) corresponds to an equation for a map u from a principal fixed G-bundle P_0 over Σ to $\mu^{-1}(0)$.

Let us return to the problem of lifting the map $f : \Sigma \to \bar{M}$. The naive approach is to try to find a map $u : \Sigma \to \mu^{-1}(0)$ such that the following diagram commutes



Unfortunately, such a map need not exist for topological reasons. As an example let $M := \mathbb{C}^2$, $G := S^1$ acting on $\mathbb{C}^2$ as in example 4.9, let $\Sigma := S^2$ and $f := \mathrm{id}$. Recall that $\mu^{-1}(0) = S^3$ (with $R := 1$). The orbits S^1x, for $x \in \mu^{-1}(0)$ are the fibres of the Hopf fibration $S^1 \hookrightarrow S^3 \twoheadrightarrow \mathbb{C}\mathrm{P}^1 \cong S^2$. If there was a lift u as above then the Hopf fibration would be trivial, so the naive approach doesn't work.

Remark 4.12 (An easier example) The following example seems to me easier to understand than the previous one. However, we consider here a more general setup. Let G be a Lie group acting properly and freely on a manifold X. Then X/G carries a canonical manifold structure. Denote the canonical projection by $\pi_0 : X \to X/G$. Furthermore, let Y be another manifold and $f : Y \to X/G$ be a smooth map. The following shows that there need not be a smooth map $u : Y \to X$ such that $f = \pi_0 \circ u$. Namely, we consider the usual action of $G := \mathbb{Z}$ on $M := \mathbb{R}$ with quotient $\mathbb{R}/\mathbb{Z} \cong S^1$. Now let $Y := S^1$ and $f := \mathrm{id} : S^1 \to S^1$. Then there is no smooth $u : S^1 \to \mathbb{R}$ such that $f = \pi_0 \circ u$.

Since the naive approach does not work, let us try to lift f as follows:

$$\begin{array}{ccc} P & \xrightarrow{u} & \mu^{-1}(0) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_0} \\ \Sigma & \xrightarrow[f]{} & \bar{M} \end{array} \tag{22}$$

What should P be?

Since the action of G on $\mu^{-1}(0)$ is proper and free, the triple $(\mu^{-1}(0), \pi_0, \bar{M})$ together with the action of G on $\mu^{-1}(0)$ is a *principal G-bundle*. This means the following.

Definition 4.13 *Let G be a Lie group, P and X be manifolds and $\pi : P \to X$ be a smooth surjective map. Furthermore let $\cdot : P \times G \to P$*

$$(p, g) \mapsto p \cdot g =: pg,$$

be a smooth right action. The quadruple $(P, \pi, X, \cdot)$ is called a (right) principal G-bundle *iff it is locally equivariantly trivial. This means that for every x in X there is a neighborhood $U \subseteq X$ of x and a smooth map $g : \pi^{-1}(U) \to G$ such that the following is satisfied. g is equivariant, i.e.*

$$g(ph) = g(p)h, \qquad \forall p \in \pi^{-1}(U),\ h \in G.$$

Furthermore the map $\Phi : \pi^{-1}(U) \to U \times G$ defined by $\Phi(p) = (\pi(p), g(p))$, is a diffeomorphism.
Let P, P' be right principal G-bundles and let $u : P \to P'$ be a map. u is called G-equivariant *iff for every $p \in P$, $g \in G$*

$$u(pg) = u(p)g.$$

If $(P, \pi, X, \cdot)$ is a principal G-bundle, then P is called the *total space*, X is called the *base* and for $x \in X$ $\pi^{-1}(x) \subseteq P$ is called the *fibre over x*. We say that P is a principal G-bundle *over X*.

Remark 4.14 Let $(P, \pi, X, \cdot)$ be a principal G-bundle. Then for any $p \in P$ and $g \in G$ we have $\pi(pg) = \pi(p)$. Let $x \in X$ be a point in the base. Then the action of G on P restricts to an action on the fibre $\pi^{-1}(x) \subseteq P$. The restricted action is free and for any $p \in \pi^{-1}(x)$ the orbit pG equals $\pi^{-1}(x)$.

Example 4.15 Let X be a manifold and G a Lie group. Let $P := X \times G$, $\pi : X \times G \to X$ be the projection to the first factor and let $\cdot : P \times G \to P$ be given by $(x, h) \cdot g := (x, hg)$. This is a principal G-bundle over X called the *trivial bundle.*

Example 4.16 Let P be a manifold of dimension n, G be a k-dimensional Lie group, $\cdot : P \times G \to P$ be a free and proper right action and let $\pi : P \to X := P/G$ be the projection to the quotient. There is an induced structure of a smooth $(n-k)$-manifold on X and $(P, \pi, X, \cdot)$ is a principal G-bundle. The manifold structure on X is given as follows. Let $x_0 \in X$. Let $p_0 \in P$ such that $\pi(p_0) = x_0$. The local slice theorem (see e.g. [7], appendix B) says that there is a G-invariant open neighbourhood $p_0 \in V \subseteq P$ and an equivariant diffeomorphism $\phi : V \to \mathbb{R}^{n-k} \times G$. That V is G-invariant means that for every $p \in V$, $g \in G$ we have $pg \in V$. Denote by $\mathrm{pr}_1, \mathrm{pr}_2$ the projections to the first and second factor in the product $\mathbb{R}^{n-k} \times G$. Let $U := \pi(V) \subseteq X$ and define $\bar{\phi} : U \to \mathbb{R}^{n-k}$ as follows. Let $x \in U$ and let $p \in \pi^{-1}(x)$. We define $\bar{\phi}(x) := \mathrm{pr}_1 \circ \phi(p)$. $\bar{\phi}(x)$ does not depend on the point $p \in \pi^{-1}(x)$, so $\bar{\phi}$ is well defined. It is a chart for X around the point $x_0 \in X$. Each two such charts $\bar{\phi}_1, \bar{\phi}_2$ around points $x_1, x_2 \in X$ are smoothly compatible, so the system of all such charts defines a manifold structure of dimension $n-k$ on X. Moreover, let $x \in X$ and let $p \in \pi^{-1}(x)$. Let $L_p : \mathfrak{g} \to T_pP$ be the linear map $L_p(\xi) := X_\xi(p)$, where $X_\xi \in \mathrm{Vect}(P)$ is the infinitesimal action corresponding to $\xi \in \mathfrak{g}$. Then the tangent space T_xX can canonically be identified with $T_pP/\mathrm{im}L_p$. Namely, consider the map $d\pi(p) : T_pP \to T_xX$. It is surjective with kernel $\ker d\pi(p) = \mathrm{im}L_p$ and therefore induces an isomorphism $T_pP/\mathrm{im}L_p \cong T_xX$. Let us now check that $(P, \pi, X, \cdot)$ is locally equivariantly trivial. For this let $x_0 \in X$, let $p_0 \in \pi^{-1}(x_0)$ and let $\phi : V \subseteq P \to \mathbb{R}^{n-k} \times G$ be as above. We define $g := \mathrm{pr}_2 \circ \phi : V = \pi^{-1}(U) \to G$. This satisfies the requirements of definition 4.13.

Let now $\cdot : \mu^{-1}(0) \times G \to \mu^{-1}(0)$ be defined by $x \cdot g := g^{-1}x$. This is a right action on $\mu^{-1}(0)$. By Example 4.16 $(\mu^{-1}(0), \pi_0, \bar{M}, \cdot)$ is a principal G-bundle. Let again $f : \Sigma \to \bar{M}$ be a smooth map. We define the pullback bundle $(P_f, \pi, \Sigma, \cdot)$ by

$$P_f := f^*\mu^{-1}(0) := \{(z, x) \in \Sigma \times \mu^{-1}(0) \,|\, x \in \pi_0^{-1}(f(z))\}, \tag{23}$$

$$\pi(z,x) := z,$$

$$(z,x)\cdot g := (z, x\cdot g) = (z, g^{-1}x).$$

This is a principal G-bundle. Furthermore we define a map $u_f : P \to \mu^{-1}(0)$ by

$$u_f(z,x) := x.$$

Then u_f makes the diagram (22) commute, so u_f is a lift of the map $f : \Sigma \to \bar{M}$. Note that u_f is *equivariant*, i.e. $u_f(pg) = g^{-1}u_f(p)$ for every $p \in P$, $g \in G$. This follows immediately from its definition. So we have accomplished part of our task. There is however a problem. Namely, we would like to have *one fixed* domain P for the map $u : P \to \mu^{-1}(0)$. But P_f depends on the map f. We will solve this problem by fixing a homology class $\bar{B}_0 \in H_2(\bar{M},\mathbb{Z})$ and a map $f_0 : \Sigma \to \bar{M}$ such that $[f_0] = \bar{B}_0$ and defining $P_0 := f_0^*\mu^{-1}(0)$.

After these motivational considerations let me now introduce the setup to formulate the precise correspondence of $\bar{J}$-holomorphic curves and maps from a principal G-bundle to $\mu^{-1}(0)$ solving a certain equation. This correspondence will be the content of Proposition 4.22.

Let (P,π,X) be a principal G-bundle. For $x \in M$ and a vector subspace $V \leq T_xM$ let $\mathrm{Pr}^V : T_xM \to V$ denote the orthogonal projection w.r.t. $(g_{\omega,J})_x$ onto V. Recall that for $x \in M$ $L_x : \mathfrak{g} \to T_xM$ is defined by $L_x\eta := X_\eta(x)$. Since G acts freely on $\mu^{-1}(0)$ the map L_x is injective for any $x \in \mu^{-1}(0)$. We can therefore make the following definition.

Definition 4.17 *Let $u : P \to \mu^{-1}(0)$ be any smooth equivariant map. We define a smooth map $A^u : TP \to \mathfrak{g}$ as follows. For $p \in P$ and $v \in T_pP$ let $A^u_p(v) := A^u(p,v) \in \mathfrak{g}$ be the unique element such that $L_u(p)A^u_p(v) = -\mathrm{Pr}^{\mathrm{im}L_{u(p)}}(du(p)v)$.*

Let u be as in the definition. Then for each $p \in P$ the map $A^u_p : T_pP \to \mathfrak{g}$ is linear. This means that A^u is a 1-form on P with values in $\mathfrak{g}$. Moreover, A^u meets the requirements of the following definition.

Definition 4.18 *A 1-form A on P with values in $\mathfrak{g}$ is called a* connection 1-form *iff the following conditions hold. A is equivariant w.r.t. the adjoint action on $\mathfrak{g}$, i.e.*

$$A_{pg}(vg) = g^{-1}A_p(v)g,$$

for every $p \in P$, $v \in T_pP$, $g \in G$. Furthermore, A is canonical along the fibres, *i.e.*

$$A_p(p\eta) = \eta,$$

for every $p \in P$, $\eta \in \mathfrak{g}$. Here $p\eta := X^P_\eta(p)$ denotes the infinitesimal action of η on P at the point $p \in P$.
We denote by $\mathcal{A}(P)$ the set of connection 1-forms on P.

Remark 4.19 *$\mathcal{A}(P)$ is an affine space.*

Fact 4.20 For every equivariant map $u : P \to \mu^{-1}(0)$ the 1-form A^u is a connection 1-form.

Let $u : P \to \mu^{-1}(0)$ be a smooth equivariant map and $A \in \mathcal{A}(P)$ be a connection 1-form. Let u^*TM denote the pullback bundle. It is a vector bundle of rank $2n = \dim M$ over P. We define d_Au to be the 1-form on P with values in u^*TM given by

$$(d_Au)_p(v) := du(p)v + X_{A_p(v)}(u(p)),$$

for $v \in T_pP$ and $p \in P$.

Fact 4.21 d_Au is equivariant, i.e. $(d_Au)_{pg}(vg) = g^{-1}(d_Au)_p(v)$, for $v \in T_pP$ and $p \in P$. Furthermore, it is horizontal, i.e. $(d_Au)_p(p\eta) = 0$.

Let (Σ, j) be a Riemann surface and P a principal G-bundle over Σ. Let α be a 1-form on P with values in u^*TM that is equivariant and horizontal. We define the *complex antilinear part of* α to be the following 1-form $\alpha^{(0,1)}$ on P with values in u^*TM. Let $p \in P$ and $v \in T_pP$. Let $w \in T_pP$ be any vector such that $d\pi(p)w = j(\pi(p))d\pi(p)v$. We define

$$(\alpha^{(0,1)})_p(v) := \frac{1}{2}(\alpha_p(v) + J(u(p))\alpha_p(w)).$$

Since α is horizontal, this does not depend on the lift w of $j(\pi(p))d\pi(p)v$. Moreover, $\alpha^{(0,1)}$ is again equivariant and horizontal. Let now $\alpha := d_Au$. We denote

$$\bar{\partial}_{J,A}(u) := (d_Au)^{(0,1)}.$$

A smooth map $g : P \to G$ is called a *gauge transformation* iff it is equivariant w.r.t. the adjoint action of G on G, i.e. iff

$$g(ph) = h^{-1}g(p)h, \qquad \forall p \in P,\ h \in G.$$

Let $g : P \to G$ be a gauge transformation and let $u : P \to \mu^{-1}(0)$ be smooth and equivariant. We use the notation $g^{-1}u$ to denote the map $g^{-1}u : P \to G$ given by $(g^{-1}u)(p) := g(p)^{-1}u(p)$. Now let $A \in \mathcal{A}(P)$ be a connection 1-form. We define

$$(g^*A)_p(v) := g(p)^{-1}dg(p)v + g(p)^{-1}A_p(v)g(p),$$

for $p \in P$, $v \in T_pP$, and

$$g^*(u, A) := (g^{-1}u, g^*A). \tag{24}$$

The set of all gauge transformations forms a group called the *gauge group* and denoted by $\mathcal{G}(P)$. The neutral element is given by the constant map $e : P \to G$. We denote the set of all smooth G-equivariant maps from P to $\mu^{-1}(0)$ by $C^\infty_G(P, \mu^{-1}(0))$. Formula (24) defines an action of $\mathcal{G}(P)$ on $C^\infty_G(P, \mu^{-1}(0)) \times \mathcal{A}(P)$. Let now $u : P \to \mu^{-1}(0)$ be an equivariant map. Then u induces the map

$$\bar{u} : \Sigma \to \bar{M}, \qquad \bar{u}(z) := \pi_0 \circ u(p), \tag{25}$$

where $p \in \pi^{-1}(z)$ is arbitrary. Let $\bar{B} \in H_2(\bar{M}, \mathbb{Z})$. We define

$$\widehat{\mathcal{M}}^0_{J,\bar{B},P} := \{(u, A) \in C^\infty_G(P, \mu^{-1}(0)) \times \mathcal{A}(P) \,|\, \bar{\partial}_{J,A}(u) = 0,\ [\bar{u}] = \bar{B}\}. \tag{26}$$

Let $(u, A) \in C^\infty_G(P, \mu^{-1}(0)) \times \mathcal{A}(P)$ and $g \in \mathcal{G}(P)$. Then

$$(\bar{\partial}_{J,g^*A}(g^{-1}u))_p(v) = g(p)^{-1}(\bar{\partial}_{J,A}(u))_p(v),$$

for any $p \in P$, $v \in T_pP$. Therefore, if $\bar{\partial}_{J,A}(u) = 0$ then also $\bar{\partial}_{J,g^*A}(g^{-1}u) = 0$. Moreover, $\overline{g^{-1}u} = \bar{u}$. So the action of $\mathcal{G}(P)$ on $C^\infty_G(P, \mu^{-1}(0)) \times \mathcal{A}(P)$ restricts to an action on $\widehat{\mathcal{M}}^0_{J,\bar{B},P}$. Let

$$\mathcal{M}^0_{J,\bar{B},P} := \widehat{\mathcal{M}}^0_{J,\bar{B},P}/\mathcal{G}(P) \tag{27}$$

denote the quotient.

After all this preparation, we are now able to formulate the following proposition. Note that the assumptions of this proposition imply that the symplectic quotient $(\bar{M}, \bar{\omega})$ is well defined.

Proposition 4.22 *Let (M, ω) be a symplectic manifold and G be a compact connected Lie group that acts on M in a Hamiltonian way with moment map μ. Assume that the restriction of the action of G to $\mu^{-1}(0)$ is free. Let (Σ, j) be a Riemann surface. Then the following statements hold.*

(A) *Let P be a principal G-bundle over Σ, $u : P \to \mu^{-1}(0)$ be a smooth equivariant map and let $\bar{u} : \Sigma \to \bar{M}$ be the induced map. Then the equation*

$$\bar{\partial}_{\bar{J}}(\bar{u}) = 0$$

holds if and only if

$$\bar{\partial}_{J,A^u}(u) = 0. \tag{28}$$

(B) *Let P and u be as in* **(A)** *and let $A \in \mathcal{A}(P)$. If $\bar{\partial}_{J,A}(u) = 0$ then $A = A^u$.*

(C) *Assume that Σ is closed and connected. Let $\bar{B}_0 \in H_2(\bar{M}, \mathbb{Z})$ be a second homology class such that there is a smooth map $f_0 : \Sigma \to \bar{M}$ with $[f_0] = \bar{B}_0$. We abbreviate $P_0 := f_0^* \mu^{-1}(0)$. Then the map*

$$F : \mathcal{M}^0_{J,\bar{B}_0,P_0} \longrightarrow \widetilde{\mathcal{M}}_{J,\bar{B}_0} = \{f \in C^\infty(\Sigma, \bar{M}) \,|\, \bar{\partial}_{\bar{J}}(f) = 0, [f] = \bar{B}_0\}$$

given by

$$F([u, A]) := \bar{u}$$

is a bijection.

Remark 4.23 A consequence of part (B) and (C) of this proposition is that the map

$$\{u \in C^\infty_G(P, \mu^{-1}(0)) \,|\, \bar{\partial}_{J,A^u}(u) = 0,\ [\bar{u}] = \bar{B}_0\}/\mathcal{G}(P_0) \ni [u] \longmapsto \bar{u} \in \widetilde{\mathcal{M}}_{J,\bar{B}_0}$$

is a bijection.

Remark 4.24 Note that the preceding remark does not state a bijection between the set of $\bar{J}$-holomorphic curves f such that $[f] = \bar{B}_0$ and the set of all smooth maps $u : P_0 \to \mu^{-1}(0)$ such that $\bar{\partial}_{J,A^u}(u) = 0$ and $\pi_0 \circ u = f \circ \pi$. It only says that after restricting to *equivariant* maps and identifying gauge equivalent maps we get a bijection. In fact, assume that $\dim G > 0$. Then the preceding remark implies that given a $\bar{J}$-holomorphic map $f : \Sigma \to \bar{M}$ such that $[f] = \bar{B}_0$ there is a huge amount of smooth (equivariant) maps $u : P \to \mu^{-1}(0)$ such that $\pi_0 \circ u = f \circ \pi$ and $\bar{\partial}_{J,A^u}(u) = 0$. All these maps differ by the action of a gauge transformation. To understand the meaning of equivariance and gauge equivalence better let us consider the case $P = \Sigma \times G$. Then specifying an equivariant map $u : \Sigma \times G \to \mu^{-1}(0)$ is equivalent to specifying a map $u' : \Sigma \to \mu^{-1}(0)$. Namely, for each equivariant $u : \Sigma \times G \to \mu^{-1}(0)$ let $u' : \Sigma \to \mu^{-1}(0)$ be defined by $u'(z) := u(z, e)$. Conversely, given $u' : \Sigma \to \mu^{-1}(0)$ let $u : \Sigma \times G \to \mu^{-1}(0)$ be defined by $u(z, g) := g^{-1}u'(z)$. The two processes are inverses of each other. Similarly, the set of (equivariant) gauge transformations $g : \Sigma \times G \to G$ corresponds bijectively to the set of maps $g' : \Sigma \to G$. (See also the next remark.) Now let $f : \Sigma \to \bar{M}$ be a smooth map and let $u, \tilde{u} : \Sigma \times G \to \mu^{-1}(0)$ be two equivariant lifts of f. Then the corresponding maps $u', \tilde{u}' : \Sigma \to \mu^{-1}(0)$ satisfy $\pi_0 \circ u' = f \circ \pi = \pi_0 \circ \tilde{u}'$. Therefore, u' differs from $\tilde{u}'$ by the action of some "gauge transformation" $g' : \Sigma \to G$, meaning that $\tilde{u}' = g'^{-1}u'$. This is a direct proof of injectivity of the map of Remark 4.23 in the case $P_0 = \Sigma \times G$, assuming that this map is well defined (which follows from Proposition 4.22(A)). It explains the importance of imposing equivariance for the map $u : P_0 \to \mu^{-1}(0)$ and of dividing by the gauge group.

Remark 4.25 Assume that $\dim G > 0$. Then the gauge group $\mathcal{G}(P_0) = \{g \in C^\infty(P_0, G) | g(ph) = h^{-1}g(p)h\}$ is infinite dimensional. (This statement can be made precise by exhibiting a canonical structure of a smooth infinite dimensional Fréchet manifold on $\mathcal{G}(P_0)$.) This is easiest to see if $P = \Sigma \times G$. Then specifying a gauge transformation on P is the same as giving a map $g' : \Sigma \to G$. Namely, for any such map g' we can define $g \in \mathcal{G}(P_0)$ by $g(z, h) := h^{-1}g'(z)h$. On the other hand, given $g \in \mathcal{G}(P_0)$ we define $g' : \Sigma \to G$ by $g'(z) := g(z, e)$. The two processes are inverses of each other. Now since $C^\infty(\Sigma, G)$ is infinite dimensional the same holds for $\mathcal{G}(P_0)$. Assume that $\widetilde{\mathcal{M}}_{J,\bar{B}_0}$ is nonempty. There is a canonical structure of a smooth finite dimensional manifold on $\mathcal{M}^0_{J,\bar{B}_0,P_0}$. It is induced from the manifold structure of $\widetilde{\mathcal{M}}_{J,\bar{B}_0}$ by the bijection of Proposition 4.22. The space $\widehat{\mathcal{M}}^0_{J,\bar{B}_0,P_0}$ of solutions (u, A) of $\bar{\partial}_{J,A}(u) = 0$ with $[\bar{u}] = \bar{B}_0$ is infinite dimensional. The reason for this is the following. As we will see below, $\mathcal{G}(P_0)$ acts freely on $\widehat{\mathcal{M}}^0_{J,\bar{B}_0,P_0}$. Therefore heuristically, we can view $\widehat{\mathcal{M}}^0_{J,\bar{B}_0,P_0}$ as a principal $\mathcal{G}(P_0)$-bundle over $\mathcal{M}^0_{J,\bar{B}_0,P_0} = \widehat{\mathcal{M}}^0_{J,\bar{B}_0,P_0}/\mathcal{G}(P_0)$. So the dimension of $\widehat{\mathcal{M}}^0_{J,\bar{B}_0,P_0}$ is the sum of the dimensions of $\mathcal{M}^0_{J,\bar{B}_0,P_0} \cong \widetilde{\mathcal{M}}_{J,\bar{B}_0}$ and $\mathcal{G}(P_0)$ and therefore infinite. We conclude that the finite dimensional space $\mathcal{M}^0_{J,\bar{B}_0,P_0}$ is the quotient of the two infinite dimensional spaces $\widehat{\mathcal{M}}^0_{J,\bar{B}_0,P_0}$ and $\mathcal{G}(P_0)$. To see that $\mathcal{G}(P_0)$ acts freely on $\widehat{\mathcal{M}}^0_{J,\bar{B}_0,P_0}$ let $(u, A) \in \widehat{\mathcal{M}}^0_{J,\bar{B}_0,P_0}$ and let $g \in \mathcal{G}(P_0)$. Assume that $g^*(u, A) = (g^{-1}u, g^*A) = (u, A)$. Then for every point $p \in P$ $g(p)^{-1}u(p) = u(p)$. Since $u(p) \in \mu^{-1}(0)$ and G acts freely on $\mu^{-1}(0)$ we conclude that $g(p) = e$ and therefore $g \equiv e$.

The proof of Proposition 4.22 uses the following.

Definition 4.26 *Let P_0, P_1 be principal G-bundles over the same base manifold X and let $\Phi : P_0 \to P_1$ be a diffeomorphism. Φ is called an* isomorphism of principal G-bundles *iff it is equivariant, i.e.* $\Phi(pg) = \Phi(p)g$, *and* $\pi_1 \circ \Phi = \pi_0$.

Recall that the Riemannian metric $g_{\omega,J}$ on M is defined by $(g_{\omega,J})_x(v,w) := \omega_x(v, Jw)$, for $x \in M$, $v, w \in T_xM$. Let $x \in M$. For any linear map $L : \mathfrak{g} \to T_xM$ we denote by $L^* : T_xM \to \mathfrak{g}$ the adjoint w.r.t. the inner product $(g_{\omega,J})_x$ on T_xM and the inner product $\langle .,. \rangle$ on $\mathfrak{g}$. Recall that for $x \in M$ $L_x : \mathfrak{g} \to T_xM$ is defined by $L_x\eta := X_\eta(x)$.

Lemma 4.27 *Let (M, ω) be a symplectic manifold and G be a connected Lie group that acts in a Hamiltonian way on M with moment map μ. Then the following holds.*

(i) For any $x \in M$

$$d\mu(x)J = L_x^*.$$

(ii) For any $x \in M$ and any $v \in H_x := \ker d\mu(x) \cap (\mathrm{im}L_x)^\perp$ we have $Jv \in H_x$.

(iii) For any $x \in \mu^{-1}(0)$ and $\eta \in \mathfrak{g}$ we have

$$X_\eta(x) \in \ker d\mu(x).$$

(iv) Let (P, π, X) be a principal G-bundle, $u : P \to \mu^{-1}(0)$ be an equivariant map and let A^u be as in Definition 4.17. Then for any $p \in P$ and $v \in T_pP$

$$d_{A^u}u(p)v \in H_{u(p)}$$

and

$$\bar{\partial}_{J,A^u}(u)(p)v \in H_{u(p)}.$$

Proof: To prove **(i)** let $v \in T_xM$ and $\xi \in \mathfrak{g}$. Then

$$\langle d\mu(x)Jv, \xi \rangle = \omega(X_\xi(x), Jv) = g_{\omega,J}(X_\xi(x), v) = g_{\omega,J}(L_x\xi, v) = \langle \xi, L_x^*v \rangle.$$

Since this holds for any v and ξ statement **(i)** follows.

Proof of (ii): Let $x \in M$ and $v \in H_x$. Since $v \in (\mathrm{im}L_x)^\perp = \ker L_x^*$ we have $L_x^*v = 0$ and therefore by (i) $\ker d\mu(x)Jv = L_x^*v = 0$. Furthermore, since $v \in \ker d\mu(x)$ we have $L_x^*Jv = d\mu(x)J^2v = -d\mu(x)v = 0$ and therefore $Jv \in (\mathrm{im}L_x)^\perp$. This implies **(ii)**.
Proof of (iii): Let $x \in \mu^{-1}(0)$ and $\xi, \eta \in \mathfrak{g}$. Then

$$\langle d\mu(x)X_\eta(x), \xi \rangle = \omega(X_\xi(x), X_\eta(x)) = \langle \mu(x), [\xi, \eta] \rangle = 0.$$

The first and the second equality follow from the definition of μ. This implies **(iii)**.

Proof of (iv): Let $p \in P$ and $v \in T_pP$. Then by definition

$$\begin{aligned} d_{A^u}u(p)v &= du(p)v + X_{A^u_p(v)}(u(p)) \\ &= du(p)v - \mathrm{Pr}^{\mathrm{im}L_{(u(p))}}\,du(p)v \in (\mathrm{im}L_{u(p)})^\perp. \end{aligned} \tag{29}$$

Since $u : P \to \mu^{-1}(0)$ we have $du(p)v \in T_{u(p)}\mu^{-1}(0) = \ker d\mu(u(p))$. Furthermore, by (iii) also $X_{A^u_p(v)}(u(p))) \in \ker d\mu(u(p))$. This implies $d_{A^u}u(p)v \in \ker d\mu(u(p))$. Together with (29) this implies $d_{A^u}u(p)v \in H_{u(p)}$. Now let $w \in T_pP$ be such that $d\pi(p)w = j\,d\pi(p)v$. Then $d_{A^u}u(p)w \in H_{u(p)}$ by what we have just proved and therefore by (ii) $J\,d_{A^u}u(p)w \in H_{u(p)}$. This implies

$$\bar\partial_{J,A^u}(u)(p)v = \frac{1}{2}(d_{A^u}u(p)v + J\,d_{A^u}u(p)w) \in H_{u(p)}.$$

This concludes the proof of **(iv)** and of the lemma. □

Fact 4.28 Let G be a compact and connected Lie group and (Σ, j) be a closed connected Riemann surface. Let P_0, P_1 be principal G-bundles over Σ and let $u_i : P_i \to \mu^{-1}(0)$, $i = 0, 1$ be smooth equivariant maps such that $[\bar u_0] = [\bar u_1] \in H_2(\bar M, \mathbb{Z})$. Then P_0 is isomorphic to P_1. (See [2], Proposition 2.1.)

Proof of Proposition 4.22, see also [2, 6]: We prove **(A)**. Let $u : P \to \mu^{-1}(0)$ be a smooth equivariant map. Recall that $\pi_0 : \mu^{-1}(0) \to \bar M$ denotes the canonical projection.

Claim For every $p \in P$ and $v \in T_pP$

$$d\pi_0\bar\partial_{J,A^u}(u)(p)v = \bar\partial_{\bar J}(\bar u)d\pi(p)v. \tag{30}$$

Assume the claim to be true. We will deduce statement (A) from the claim. Suppose that $\bar\partial_{J,A^u}(u) = 0$. Since for each $p \in P$ $d\pi(p) : T_pP \to T_{\pi(p)}\Sigma$ is surjective, (30) implies that $\bar\partial_{\bar J}(\bar u) = 0$. Now suppose conversely that $\bar\partial_{\bar J}(\bar u) = 0$. Let $p \in P$ and $v \in T_pP$. Then by (30) $d\pi_0\bar\partial_{J,A^u}(u)(p)v = 0$, i.e. $\bar\partial_{J,A^u}(u)(p)v \in \mathrm{im}L_{u(p)}$. By assertion (iv) of Lemma 4.27 we also have $\bar\partial_{J,A^u}(u)(p)v \in \mathrm{im}L^\perp_{u(p)}$ and therefore $\bar\partial_{J,A^u}(u)(p)v = 0$. So we have deduced statement (A) from the claim.

Proof of the Claim: Note that $\pi_0 \circ u = \bar{u} \circ \pi$ and therefore $d\pi_0 du = d\bar{u}\, d\pi$. Let $p \in P$ and $v \in T_pP$. Then

$$d\pi_0\, d_{A^u} u(p)v = d\pi_0(du(p)v + X_{A^u_p(v)}(u(p))) = d\bar{u}\, d\pi(p)v. \tag{31}$$

The last equality uses the fact that $X_{A^u_p(v)}(u(p)) \in \mathrm{im}L_{u(p)} = \ker d\pi_0(u(p))$. Let now $p \in P$ and $v \in T_pP$ and let $w \in T_pP$ be such that

$$d\pi(p)w = j d\pi(p)v. \tag{32}$$

Then by assertion (iv) of Lemma 4.27 $d_{A^u}u(p)w \in H_{u(p)}$ and therefore by definition of $\bar{J}$ (cf. Definition 4.11)

$$d\pi_0 J\, d_{A^u} u\, w = \bar{J}\, d\pi_0 d_{A^u} u\, w = \bar{J} d\bar{u}\, d\pi w = \bar{J} d\bar{u}\, j\, d\pi v.$$

The second equality follows from (31) and the last one from (32). This implies

$$\begin{aligned} d\pi_0 \bar{\partial}_{J,A^u}(u)v &= \frac{1}{2}(d\pi_0 d_{A^u} u\, v + d\pi_0 J d_{A^u} u\, w) \\ &= \frac{1}{2}(d\bar{u}\, d\pi v + \bar{J}\, d\bar{u}\, j\, d\pi v) \\ &= \bar{\partial}_{\bar{J}}(\bar{u}) d\pi\, v. \end{aligned}$$

This proves the claim and completes the proof of **(A)**.

Proof of (B): Let $u : P \to \mu^{-1}(0)$ be equivariant such that $\bar{\partial}_{J,A}(u) = 0$. Let $p \in P$, $v \in T_pP$ and let $w \in T_pP$ be such that $d\pi(p)w = j\, d\pi(p)v$. Then by assertion (iv) of Lemma 4.27 $d_{A^u}u(p)w \in H_{u(p)}$ and by (ii) $J\, d_{A^u}u(p)w \in H_{u(p)}$. Therefore $\mathrm{Pr}^{\mathrm{im}L_{u(p)}}(J\, d_{A^u}u(p)w) = 0$. From the assumption $\bar{\partial}_{J,A}(u) = 0$ it follows that (dropping the p)

$$\begin{aligned} 0 &= 2\,\mathrm{Pr}^{\mathrm{im}L_{u(p)}}(\bar{\partial}_{J,A}(u)v) \\ &= \mathrm{Pr}^{\mathrm{im}L_u}(d_A u\, v + J d_A u\, w) \\ &= \mathrm{Pr}^{\mathrm{im}L_u}\, d_A u\, v \\ &= \mathrm{Pr}^{\mathrm{im}L_u}(du\, v + X_{Av}(u)) \\ &= \mathrm{Pr}^{\mathrm{im}L_u}(du\, v) + X_{Av}(u). \end{aligned}$$

This implies $X_{A_p(v)}(u(p)) = -\,\mathrm{Pr}^{\mathrm{im}L_{u(p)}}\, du\, v$ and therefore $A_p(v) = A^u_p(v)$. This proves **(B)**.

Proof of (C): Let $\bar{B}_0$, f_0 and P_0 be as in the hypothesis. We show that F is injective. Assume that $[u, A]$, $[u', A'] \in \mathcal{M}^0_{J,\bar{B}_0,P_0}$ are equivalence classes of solutions of (28) such that $\bar{u} = \overline{u'}$. Then for every $p \in P$ $\pi_0(u(p)) = \pi_0(u'(p))$, i.e. there is a unique $g_p \in G$ such that $u(p) = g_p u'(p)$. The map $g : P \to G$, $p \mapsto g_p$ is smooth and G-equivariant, i.e. a gauge transformation, and $u' = g^{-1}u$. By (B) we have $A = A^u$ and $A' = A^{u'}$. A calculation shows that $A^{u'} = A^{g^{-1}u} = g^* A^u$. Putting things together we have

$$(u', A') = (g^{-1}u, A^{u'}) = (g^{-1}u, g^* A^u) = g^*(u, A),$$

and therefore $[u, A] = [u', A']$. This proves injectivity of F.
To prove surjectivity let $f \in \widetilde{\mathcal{M}}_{\bar{J},\bar{B}_0}$ be a $\bar{J}$-holomorphic curve. We have to find a pair $(u, A) \in \widehat{\mathcal{M}}^0_{J,\bar{B}_0,P_0}$ such that $F([u, A]) = \bar{u} = f$. Consider the pullback bundle $P_f := f^*\mu^{-1}(0)$ over Σ as in (23) and let $u_f : P_f \to \mu^{-1}(0)$ again be defined by $u_f(z, x) := x$. Then $\overline{u_f} = f$ and therefore $[\overline{u_f}] = [f] = \bar{B}_0 = [f_0] = [\overline{u_{f0}}]$. By Fact 4.28 there is an isomorphism $\Phi : P_0 \to P_f$. We define $u := u_f \circ \Phi : P_0 \to \mu^{-1}(0)$ and $A := A^u$. Then $\bar{u} = f$ and therefore by part (A) $\bar{\partial}_{J,A}(u) = 0$, so $[u, A] \in \mathcal{M}^0_{J,\bar{B}_0,P_0}$. This proves that the map F is surjective and therefore concludes the prove of **(C)** and of the proposition. □

4.3 The Symplectic Vortex Equations and Vortex Invariants

In this subsection the Yang-Mills-Higgs functional is introduced. Solutions of the symplectic vortex equations are minimizers of this functional. These solutions are used to define the symplectic vortex invariants.

Let (M, ω) be a symplectic manifold, let G be a compact connected Lie group that acts on M in a Hamiltonian way and let J be a G-invariant ω-compatible almost complex structure. Let (Σ, j) be a closed connected Riemann surface, let dvol_Σ be a volume form on Σ that is compatible with j and let P be a principal G-bundle over Σ. Note that from now on, the map u will no longer take values in the submanifold $\mu^{-1}(0) \subseteq M$, but in the whole manifold M. Let $C^\infty_G(P, M)$ denote the set of smooth equivariant maps from P to M. As we will see below, the symplectic vortex equations are the equations for a minimizer of the following functional. The *Yang-Mills-Higgs-functional* $\mathcal{YMH} : C^\infty_G(P, M) \times \mathcal{A}(P) \to [0, \infty)$ is defined by

$$\mathcal{YMH}(u, A) := \frac{1}{2} \int_\Sigma \left(|d_A u|^2 + |F_A|^2 + |\mu \circ u|^2 \right) \mathrm{dvol}_\Sigma .$$

Here F_A is the curvature of the connection 1-form $A \in \mathcal{A}(P)$, i.e. the 2-form on P with values in $\mathfrak{g}$ given by

$$F_A := dA + \frac{1}{2}[A \wedge A],$$

Here for any $p \in P$, $v, w \in T_pP$

$$[A \wedge A]_p(v,w) := [A_p(v), A_p(w)] - [A_p(w), A_p(v)] = 2[A_p(v), A_p(w)].$$

F_A is equivariant w.r.t. the adjoint action on $\mathfrak{g}$, i.e. for any $p \in P$, $v_1, v_2 \in T_pP$ and $g \in G$ $(F_A)_{pg}(v_1 g, v_2 g) = g^{-1}((F_A)_p(v_1, v_2))g$. Furthermore, F_A is horizontal in the sense $(F_A)_p(v, p\xi) = 0$ for any $p \in P$ and $v \in T_pP$, $\xi \in \mathfrak{g}$. The functions $|d_Au|^2$, $|F_A|^2$ and $|\mu|^2 : \Sigma \to [0, \infty)$ are to be understood as follows. Let $z \in \Sigma$. Let $p \in \pi^{-1}(z)$. Then $|\mu \circ u|(z)$ is given by

$$|\mu \circ u|(z) := |\mu \circ u(p)|_{\mathfrak{g}} := \sqrt{\langle \mu \circ u(p), \mu \circ u(p) \rangle}.$$

Since $\mu \circ u$ is equivariant and $\langle ., . \rangle$ is an invariant inner product on $\mathfrak{g}$ this does not depend on the choice of $p \in \pi^{-1}(z)$. Now let ζ_1, ζ_2 be an orthonormal basis of $T_z\Sigma$ w.r.t. the metric $\mathrm{dvol}_\Sigma(., j.)$ and let $v_i \in T_pP$ be such that $d\pi(p)v_i = \zeta_i$. Then $|d_Au|(z)$ is given by

$$|d_Au|(z) := \sqrt{|d_Au\, v_1|^2_{\omega,J} + |d_Au\, v_2|^2_{\omega,J}}.$$

Here for any $x \in M$ and $w \in T_xM$, $|w|_{\omega,J} := \sqrt{g_{\omega,J}(w,w)} = \sqrt{\omega(w, Jw)}$. Since d_Au is equivariant and horizontal, this definition does not depend on the point $p \in \pi(z)$ nor on the lifts v_1, v_2 of ζ_1, ζ_2. It does not depend on the choice of the orthonormal basis ζ_1, ζ_2 either. Similarly,

$$|F_A|(z) := |(F_A)_p(v_1, v_2)|_{\mathfrak{g}} := \sqrt{\langle (F_A)_p(v_1, v_2), (F_A)_p(v_1, v_2) \rangle},$$

where p and v_i, $i = 1, 2$ are as above.

In the case $M := \mathbb{C}$, $\omega = \omega_0 = dx \wedge dy$, $G := S^1$ acting on $\mathbb{C}$ in the standard way as in example 4.9, $\Sigma = \mathbb{C}$, $J := j := J_0$, $P := \mathbb{C} \times S^1$ and $\mathrm{dvol}_{\mathbb{C}} := dx \wedge dy$, the functional $\mathcal{YMH}$ appeared in physics in a paper by Ginzburg and Landau [8]. It is used as a model for superconductivity. The generalization to $M := \mathbb{C}^n$ and G a Lie subgroup of $U(n)$ is known in the physics literature (starting with [19]) as *gauged sigma models.*

Let me now introduce the setup for the definition of the symplectic vortex invariants.

For $k \in \mathbb{N}_0$ let $\Omega^k_{\rm ad}(P, \mathfrak{g})$ denote the vector space of equivariant and horizontal k-forms on P with values in $\mathfrak{g}$. Note that $\Omega^0_{\rm ad}(P, \mathfrak{g}) = C^\infty_G(P, \mathfrak{g})$. We define the *Hodge-*-operator* $* : \Omega^2_{\rm ad}(P, \mathfrak{g}) \to \Omega^0(P, \mathfrak{g})$ as follows. Let $\Omega \in \Omega^2_{\rm ad}(P, \mathfrak{g})$ and $p \in P$. Let $v_1, v_2 \in T_pP$ be such that $d\pi(p)v_1, d\pi(p)v_2$ is a positively oriented orthonormal basis of $T_z\Sigma$ w.r.t. $\mathrm{dvol}_\Sigma(., j.)$. We define $(*\Omega)(p) := \Omega_p(v_1, v_2) \in \mathfrak{g}$. Let EG be a contractible topological space on which G acts continuously and freely from the left. (Such a space exists for any topological group and therefore for any Lie group. For the case that G is a Lie subgroup of $U(k)$ for some $k \in \mathbb{N}$ see e.g. [10].) Let Y be a topological space on which G acts continuously. Then the action of G on Y and EG defines a "simultaneous" action on $Y \times \mathrm{EG}$. We define $Y \times_G \mathrm{EG} := (Y \times \mathrm{EG})/G$ and denote the equivalence class of an element $(x, e) \in Y \times \mathrm{EG}$ by $[x, e] \in Y \times_G \mathrm{EG}$. The equivariant homology and cohomology of M are defined by

$$H^G_*(M, \mathbb{Z}) := H_*(M \times_G \mathrm{EG}, \mathbb{Z}), \qquad H^*_G(M, \mathbb{Z}) := H^*(M \times_G \mathrm{EG}, \mathbb{Z}).$$

This definition does not depend on EG in the sense that given two contractible spaces EG_1, EG_2 on which G acts freely then $H_*(M \times_G \mathrm{EG}_1, \mathbb{Z}) \cong H_*(M \times_G \mathrm{EG}_2, \mathbb{Z})$ and similarly for H^*. Let $\theta^P : P \to \mathrm{EG}$ be a continuous equivariant map. (Such a map exists, it is a lift of a classifying map of P.) Let $u : P \to M$ be an equivariant map. We define $u_G : \Sigma \to M \times_G \mathrm{EG}$ by $u_G(z) := [u(p), \theta^P(p)]$, for $z \in \Sigma$, where $p \in \pi^{-1}(z)$ is arbitrary. The *equivariant homology class of* u is defined by

$$[u]_G := [u_G] \in H^G_2(M, \mathbb{Z}).$$

The functional $\mathcal{YMH}$ can be rewritten as

$$\mathcal{YMH}(u, A) = \int_\Sigma \left(|\bar\partial_{J,A}(u)|^2 + \frac{1}{2} | * F_A + \mu \circ u|^2 \right) \mathrm{dvol}_\Sigma + \text{topological term}. \tag{33}$$

Here $*F_A \in \Omega^0_{\rm ad}(P, \mathfrak{g}) = C^\infty_G(P, \mathfrak{g})$ is defined as above. The "topological term" is a term depending only on ω, μ and $[u]_G$. (It is the contraction of the *equivariant cohomology class* $[\omega - \mu]$ and $[u]_G$, cf. [2].) Consider the *symplectic vortex equations*

$$\bar\partial_{J,A}(u) = 0, \tag{34}$$

$$*F_A + \mu \circ u = 0. \tag{35}$$

Since the integrand in (33) is always nonnegative, any solution of (34,35) is a minimizer of $\mathcal{YMH}$ in its equivariant homology class. A solution (u, A) of these equations is called a *symplectic vortex*.

Remark 4.29 Recall statement (A) of Proposition 4.22 which says that the Cauchy-Riemann equations for a map $f : \Sigma \to \bar{M}$ are equivalent to equation (28) for a lift $u : P \to \mu^{-1}(0)$ of f. Consider now u as a map from P to M and interpret equation (28) as an equation for a pair $(u, A) \in C^\infty_G(P, M) \times \mathcal{A}(P)$. Then (28) corresponds to the two equations

$$\begin{aligned} \bar{\partial}_{J,A}(u) &= 0, && (36) \\ \mu(u) &= 0 && (37) \end{aligned}$$

Note that equations (34,35) are almost the same as equations (36,37). The only difference is the curvature term $*F_A$ in the second equation.

Let $B \in H_2^G(M, \mathbb{Z})$ be an equivariant homology class and let $b \in H_2(BG)$ be the pushforward of B under the map $M \times_G \mathrm{EG} \to \mathrm{BG}$, $[x, e] \mapsto [e]$. There is a principal G-bundle P over Σ and a continuous equivariant map $u : P \to \mathrm{EG}$, such that for the induced map $\bar{u} : \Sigma \to \mathrm{BG}$ we have $[\bar{u}] = b$. Any two bundles P for which such a map u exists, are isomorphic. (See [2], proof of Proposition 2.1.) Let P be such a bundle. We define

$$\widehat{\mathcal{M}}^1_{J,B,P} := \{(u, A) \in C^\infty_G(P, M) \times \mathcal{A}(P) \,|\, (34), (35), [u]_G = B\}.$$

The gauge group $\mathcal{G}(P)$ acts on $\widehat{\mathcal{M}}^1_{J,B,P}$ as in (24). We denote the quotient by

$$\mathcal{M}^1_{J,B,P} := \widehat{\mathcal{M}}^1_{J,B,P}/\mathcal{G}(P).$$

In favorable cases, $\mathcal{M}^1_{J,B,P}$ carries a natural structure of a smooth closed oriented manifold. (In general, one has to modify the definition of $\widehat{\mathcal{M}}^1_{J,B,P}$ to make this true. Namely, we have to replace J by a family $\{J_z\}_{z \in \Sigma}$ or alternatively introduce a Hamiltonian perturbation. Furthermore we have to restrict to the subset of *irreducible* vortices (u, A). (See [3].)) The dimension of $\mathcal{M}^1_{J,B,P}$ depends only on $M, \omega, \mu, \Sigma, B$, but not on J. We denote this dimension by $d(M, \omega, \mu, \Sigma, B)$. (For a formula for $d(M, \omega, \mu, \Sigma, B)$ see [3, 5].) Let $p_0 \in P$ and let $\mathcal{G}_{p_0}(P)$ be the set of all gauge transformations $g : P \to G$ such that $g(p_0) = e$. For $(u, A) \in \widehat{\mathcal{M}}_{J,B,P}$ we denote the orbit of (u, A) under the action of $\mathcal{G}_{p_0}(P)$ by $[u, A]_{p_0} := \mathcal{G}_{p_0}(P)^*(u, A)$. Heuristically, the quotient $\widehat{\mathcal{M}}^1_{J,B,P}/\mathcal{G}_{p_0}(P)$ is a principal G-bundle over $\mathcal{M}^1_{J,B,P}$. Let $\Theta : \widehat{\mathcal{M}}^1_{J,B,P}/\mathcal{G}_{p_0}(P) \to \mathrm{EG}$ be a continuous equivariant map. (Such a map exists, since it is a lift of a classifying map of the bundle $\widehat{\mathcal{M}}^1_{J,B,P}/\mathcal{G}_{p_0}(P)$.) We define the evaluation map $\mathrm{ev}_{J,B,P,\Theta} : \mathcal{M}^1_{J,B,P} \to M \times_G \mathrm{EG}$ by

$$\mathrm{ev}_{J,B,P,\Theta}([u, A]) := [u(p_0), \Theta([u, A]_{p_0})].$$

We denote by $[\mathcal{M}^1_{J,B,P}] \in H_{d(M,\omega,\mu,B)}(\mathcal{M}^1_{J,B,P}, \mathbb{Z})$ the fundamental class and by $\smile: H^\ell(\mathcal{M}^1_{J,B,P}) \times H^m(\mathcal{M}^1_{J,B,P}) \to H^{\ell+m}(\mathcal{M}^1_{J,B,P})$ the cup product. Note that for any $\alpha \in H^*_G(M,\mathbb{Z})$ the pullback cohomology class $(\mathrm{ev}_{J,B,P,\Theta})^*\alpha \in H^*(\mathcal{M}^1_{J,B,P}, \mathbb{Z})$ is independent of Θ and of $p_0 \in P$.

Working definition 4.30 *Let (M,ω) be a symplectic manifold, let G be a compact connected Lie group which acts on M in a Hamiltonian way with moment map μ, let (Σ, j) be a closed connected Riemann surface and let $k \in N_0$. Furthermore, let $B \in H^G_2(M,\mathbb{Z})$ and let P be a principal G-bundle over Σ whose isomorphism class is determined by B as above. The* symplectic vortex invariants of (M,ω,μ,B) with k fixed marked points *are the map $\Phi^{M,\omega,\mu}_{k,B,\Sigma} : (H^*_G(M,\mathbb{Z}))^k \to \mathbb{Z}$ defined as follows. For $i = 1,\dots,k$ let $\alpha_i \in H^*_G(M,\mathbb{Z})$ be an equivariant cohomology class. If $\sum_{i=1}^k \deg \alpha_i = d(M,\omega,\mu,\Sigma,B)$ then we define*

$$\Phi^{M,\omega,\mu}_{k,B,\Sigma}(\alpha_1,\dots,\alpha_k) := \langle (\mathrm{ev}_{J,B,P,\Theta})^*\alpha_1 \smile \dots \smile (\mathrm{ev}_{J,B,P,\Theta})^*\alpha_k, [\mathcal{M}^1_{J,B,P}]\rangle. \tag{38}$$

Otherwise $\Phi^{M,\omega,\mu}_{k,B,\Sigma}(\alpha_1,\dots,\alpha_k) := 0$. Here J is any G-invariant ω-compatible almost complex structure on M, $\Theta : \widehat{\mathcal{M}}^1_{J,B,P}/\mathcal{G}_{p_0}(P) \to \mathrm{EG}$ is an equivariant map for some $p_0 \in P$ and $\mathrm{ev}_{J,B,P,\Theta} : \mathcal{M}^1_{J,B,P} \to M \times_G \mathrm{EG}$ is defined as above.

Remark 4.31 As for the Gromov-Witten invariants, some assumptions have to be made to make this definition rigorous. Additionally, one has to replace J by a *regular family* $\{J_z\}_{z\in\Sigma}$ or introduce a regular Hamiltonian perturbation (see [3]). With these modifications the invariants is well defined, since by a bordism argument as for the Gromov-Witten invariants the right hand side of (38) does not depend on J.

Remark 4.32 The symplectic vortex invariants can be interpreted as follows. For $c > 0$ let M^c be the manifold with boundary

$$M^c := \{x \in M \mid |\mu(x)| \le c\}.$$

Under some "convexity at infinity" assumption there is a constant $c_0 > 0$ such that for any solution (u, A) of (34,35) the image $u(P)$ is contained in M^{c_0}. We assume that $c > c_0$. There is a closed finite dimensional manifold $\widetilde{\mathrm{EG}}$ such that the following holds. There exists an injective continuous map $\iota : \widetilde{\mathrm{EG}} \to \mathrm{EG}$, G acts smoothly and freely on $\widetilde{\mathrm{EG}}$ and for any point $p_0 \in P$ there is a smooth equivariant map $\widetilde{\Theta}_{p_0} : \mathcal{M}^1_{J,B,P}/\mathcal{G}_{p_0}(P) \to \widetilde{\mathrm{EG}}$. ($\widetilde{\mathrm{EG}}$ can be thought of as a finite dimensional approximation of EG. Note that unlike EG, $\widetilde{\mathrm{EG}}$ is not contractible.) Let $z_i \in \Sigma$, $i = 1,\dots,k$ be distinct points. Let

$i = 1, \ldots, k$. Let $p_i \in \pi^{-1}(z_i)$ and let $\widetilde{\Theta}_i$ be as above with $p_0 = p_i$. We define the evaluation map $\widetilde{\mathrm{ev}}_i : \mathcal{M}^1_{J,B,P} \to M^c \times_G \widetilde{\mathrm{EG}}$ by

$$\widetilde{\mathrm{ev}}_i([u,A]) := [u(p_i), \widetilde{\Theta}_i([u,A]_{p_i})].$$

Let now $\alpha_i \in H^*_G(M, \mathbb{Z})$ be such that $\sum_{i=1}^k \deg \alpha_i = d(M, \omega, \mu, \Sigma, B)$. We denote by $\iota^c_G : M^c \times_G \widetilde{\mathrm{EG}} \to M \times_G \mathrm{EG}$ the map induced by the inclusions $M^c \subseteq M$ and $\iota : \widetilde{\mathrm{EG}} \to \mathrm{EG}$. For $i = 1, \ldots, k$ let $N_i \subseteq M^c \times \widetilde{\mathrm{EG}}$ be a compact oriented submanifold (with boundary) that represents the Poincaré dual of $(\iota^c_G)^* \alpha_i$. (By replacing α_i by a rational multiple, analogously to fact 3.25, we can assume that there is such an N_i.) Assume that $\widetilde{\mathrm{ev}}_i$ is transverse to N_i, for $i = 1, \ldots, k$. Then

$$\Phi^{M,\omega,\mu}_{k,B,\Sigma}(\alpha_1, \ldots, \alpha_k) = \#\{(u,A) \in \mathcal{M}^1_{J,B,P} \,|\, \widetilde{\mathrm{ev}}_i([u,A]) \in N_i,\ i = 1, \ldots, k\}. \tag{39}$$

Here # means the number of elements with an appropriate sign. This is seen as follows. Let $i = 1, \ldots, k$. We define

$$\mathrm{ev}_i := \iota^c_G \circ \widetilde{\mathrm{ev}}_i : \mathcal{M}^1_{J,B,P} \to M \times_G \mathrm{EG}.$$

Consider the closed oriented submanifold $\widetilde{\mathrm{ev}}_i^{-1}(N_i) \subseteq \mathcal{M}^1_{J,B,P}$. Since $[N_i] = \mathrm{PD}((\iota^c_G)^* \alpha_i)$ the Poincaré dual of $[\widetilde{\mathrm{ev}}_i^{-1}(N_i)] \in H_*(\mathcal{M}^1_{J,B,P}, \mathbb{Z})$ agrees with $\widetilde{\mathrm{ev}}_i^*(\iota^c_G)^* \alpha_i \in H^*(\mathcal{M}^1_{J,B,P}, \mathbb{Z})$ modulo torsion (see for example [1]). Therefore

$$\begin{aligned}
\Phi^{M,\omega,\mu}_{k,B,\Sigma}(\alpha_1, \ldots, \alpha_k) &= \langle \mathrm{ev}_1^* \alpha_1 \smile \ldots \smile \mathrm{ev}_k^* \alpha_k, [\mathcal{M}^1_{J,B,P}] \rangle \\
&= \langle \widetilde{\mathrm{ev}}_1^* (\iota^c_G)^* \alpha_1 \smile \ldots \smile \widetilde{\mathrm{ev}}_k^* (\iota^c_G)^* \alpha_k, [\mathcal{M}^1_{J,B,P}] \rangle \\
&= \langle \mathrm{PD}(\widetilde{\mathrm{ev}}_1^{-1}(N_1)) \smile \ldots \smile \mathrm{PD}(\widetilde{\mathrm{ev}}_k^{-1}(N_k)), [\mathcal{M}^1_{J,B,P}] \rangle \\
&= \# \,\mathrm{PD}(\mathrm{PD}(\widetilde{\mathrm{ev}}_1^{-1}(N_1)) \smile \ldots \smile \mathrm{PD}(\widetilde{\mathrm{ev}}_k^{-1}(N_k))) \\
&= \#([\widetilde{\mathrm{ev}}_1^{-1}(N_1)] \cdot \ldots \cdot [\widetilde{\mathrm{ev}}_k^{-1}(N_k)]) \\
&= \#(\widetilde{\mathrm{ev}}_1^{-1}(N_1) \cap \ldots \cap \widetilde{\mathrm{ev}}_k^{-1}(N_k)).
\end{aligned}$$

In the forth and the fifth line $\# : H_0(\mathcal{M}^1_{J,B,P}, \mathbb{Z}) \to \mathbb{Z}$ is defined by $\#a := \#S$, where $S \subseteq \mathcal{M}^1_{J,B,P}$ is any finite signed set representing a. This proves (39) under the assumption that $\mathcal{M}^1_{J,B,P}$ is a smooth closed oriented manifold.

4.4 Relation between the Vortex Invariants and the Gromov-Witten Invariants

Let (M, ω) be a symplectic manifold, let G be a compact connected Lie group that acts on M in a Hamiltonian way, let J a G-invariant ω-compatible almost complex structure and let $\Sigma = S^2$. Assume that $\mu : M \to \mathfrak{g}$ is proper, so $\mu^{-1}(0) \subseteq M$ is a compact submanifold. Furthermore suppose that G acts freely on $\mu^{-1}(0)$, so the symplectic quotient $(\bar{M}, \bar{\omega})$ is well defined and $\bar{M}$ is compact. Let $k \in \mathbb{N}_0$. Recall that our motivational goal is to compute the Gromov-Witten invariants with k marked points of the symplectic quotient $(\bar{M}, \bar{\omega})$. These invariants count $\bar{J}$-holomorphic curves in $\bar{M}$ that satisfy certain conditions. In subsection 4.3 on the other hand I have introduced invariants of the Hamiltonian group action that count equivalence classes of symplectic vortices satisfying some conditions (see Remark 4.32). In this subsection we will see how the two invariants are related to each other and how the symplectic vortex invariants of (M, ω, μ) can be used to compute the GW's of $(\bar{M}, \bar{\omega})$. Finally, I will shortly explain the geometric idea of the proof of this relation. It uses the correspondence established in subsection 4.2. Its "hard" ingredient is an adiabatic limit argument.

Let $\iota : \mu^{-1}(0) \to M$ be the inclusion map and let $\theta^0 : \mu^{-1}(0) \to \mathrm{EG}$ be a continuous equivariant map. We define

$$\iota_G : \bar{M} \to M \times_G \mathrm{EG} = (M \times \mathrm{EG})/G, \qquad \iota_G(\bar{x}) := [\iota(x), \theta^0(x)],$$

where $x \in \pi_0^{-1}(\bar{x})$. The induced maps $\kappa_{.} := (\iota_G)_* : H_*(\bar{M}, \mathbb{Z}) \to H_*^G(M, \mathbb{Z})$ and $\kappa^{.} := \iota_G^* : H_G^*(M, \mathbb{Z}) \to H^*(\bar{M}, \mathbb{Z})$ are called the *Kirwan homomorphisms*. The following theorem establishes the correspondence between the Gromov-Witten invariants of $\bar{M}$ and the symplectic vortex invariants.

Theorem 4.33 *Let $\bar{B} \in H_2(\bar{M}, \mathbb{Z})$ and $\alpha_1, \ldots, \alpha_k \in H_G^*(M; \mathbb{Z})$. We define $B := \kappa_{.}\bar{B}$ and $\bar{\alpha}_i := \kappa^{.}\alpha_i$, $i = 1, \ldots, k$. Then under some assumptions on (M, ω, μ) and on the degrees of α_i, $i = 1, \ldots, k$ (cf. [5]) we have*

$$\Phi^{M,\omega,\mu}_{k,B,S^2}(\alpha_1, \ldots, \alpha_k) = \mathrm{GW}^{\bar{M},\bar{\omega}}_{k,\bar{B}}(\bar{\alpha}_1, \ldots, \bar{\alpha}_k). \tag{40}$$

Remark 4.34 Recall that $d(M, \omega, \mu, S^2, B)$ equals the dimension of the manifold $\mathcal{M}^1_{J,B,P}$ (pretending that it is a manifold), where J is any G-invariant ω-compatible almost complex structure. In order for the left hand side of (40) not to vanish, the cohomology classes α_i have therefore to satisfy the dimensional condition $\sum_{i=1}^k \deg \alpha_i = d(M, \omega, \mu, S^2, B)$. Similarly, the right hand side of (40) can only be nonzero if $\sum_{i=1}^k \deg \bar{\alpha}_i = d(\bar{M}, \bar{\omega}, \bar{B})$, where $d(\bar{M}, \bar{\omega}, \bar{B}) = \dim \bar{M} + 2\langle c_1(T\bar{M}, \bar{\omega}), \bar{B}\rangle$, cf. formula (5). It turns out that if $\kappa_{.}\bar{B} = B$ then

$$d(M, \omega, \mu, S^2, B) = d(\bar{M}, \bar{\omega}, \bar{B}).$$

Therefore, the dimensional condition for the left hand side of (40) is satisfied if and only if the dimensional condition for the right hand side is satisfied. So we can already see that the theorem holds in the case that the dimensional condition for one of the two sides of (40) is not satisfied, since then both sides are automatically 0.

Kirwan proved that $\kappa_{\cdot} : H_*(\bar{M},\mathbb{Z}) \to H^G_*(M,\mathbb{Z})$ is injective and $\kappa^{\cdot} : H^*_G(M,\mathbb{Z}) \to H^*(\bar{M},\mathbb{Z})$ is surjective (see [14]). Therefore, if the assumptions of Theorem 4.33 are satisfied and if we are able to compute the vortex invariants of (M,ω,μ,S^2) then we can use this theorem to obtain the Gromov Witten invariants of $(\bar{M},\bar{\omega})$. In the case of certain torus actions on $\mathbb{C}^n$, this program has been carried out in [4].

The idea of proof of Theorem 4.33 is the following. Let $\bar{B} \in H_2(\bar{M},\mathbb{Z})$, $B := \kappa_{\cdot}\bar{B}$ and let P_0 be a principal G-bundle whose isomorphism class is determined by B (see the proof of Proposition 2.1. in [2]). For $\epsilon \geq 0$ consider the equations

$$\bar{\partial}_{J,A}(u) = 0 \tag{41}$$

$$\epsilon * F_A + \mu \circ u = 0. \tag{42}$$

For $\epsilon = 0$ these are the equations (36,37), whereas for $\epsilon = 1$ we get the symplectic vortex equations (34,35). Let $0 \leq \epsilon \leq 1$. We define $\widehat{\mathcal{M}}^\epsilon_{J,B,P_0}$ to be the set of solutions of (41,42) such that $[u]_G = B$ and denote $\mathcal{M}^\epsilon_{J,B,P_0} := \widehat{\mathcal{M}}^\epsilon_{J,B,P_0}/\mathcal{G}(P_0)$. Note that for $\epsilon = 0$ it follows from the fact that $\kappa_{\cdot} : H_2(\bar{M},\mathbb{Z}) \to H^G_2(M,\mathbb{Z})$ is injective that

$$\widehat{\mathcal{M}}^0_{J,B,P_0} = \widehat{\mathcal{M}}^0_{J,\bar{B},P_0},$$

where $\widehat{\mathcal{M}}^0_{J,\bar{B},P_0}$ is defined as in (26). Therefore

$$\mathcal{M}^0_{J,B,P_0} = \widehat{\mathcal{M}}^0_{J,B,P_0}/\mathcal{G}(P_0) = \widehat{\mathcal{M}}^0_{J,\bar{B},P_0}/\mathcal{G}(P_0) = \mathcal{M}^0_{J,\bar{B},P_0}.$$

There are invariants $\Phi^\epsilon_{k,B} : (H^*_G(M,\mathbb{Z}))^k \to \mathbb{Z}$ associated to $\widehat{\mathcal{M}}^\epsilon_{J,B,P_0}$. They are defined analogously to $\Phi^{M,\omega,\mu}_{k,B,S^2}$ (Working definition 4.30), but with $\mathcal{M}^1_{J,B,P_0}$ replaced by $\mathcal{M}^\epsilon_{J,B,P_0}$. Let $\alpha_i \in H^*_G(M,\mathbb{Z})$, $i = 1,\ldots,k$ be such that $\sum_{i=1}^k \deg\alpha_i = d(M,\omega,\mu,S^2,B)$. For $\epsilon,\epsilon' > 0$ a bordism argument shows that

$$\Phi^\epsilon_{k,B}(\alpha_1,\ldots,\alpha_k) = \Phi^{\epsilon'}_{k,B}(\alpha_1,\ldots,\alpha_k). \tag{43}$$

Now let $\bar{\alpha}_i := \kappa^{\cdot}\alpha_i$, $i = 1,\ldots,k$. Then

$$\mathrm{GW}^{\bar{M}}_{k,\bar{B}}(\bar{\alpha}_1,\ldots,\bar{\alpha}_k) = \Phi^0_{k,B}(\alpha_1,\ldots,\alpha_k). \tag{44}$$

This is seen as follows. Let $\bar{\mathrm{ev}}_i : \mathcal{M}_{\bar{J},\bar{B}} \to \bar{M}$ be defined by $\bar{\mathrm{ev}}_i(f) := f(z_i)$. Assume w.l.o.g. that the Poincaré dual of $\bar{\alpha}_i$ can be represented by a closed oriented submanifold $\bar{N}_i \subseteq \bar{M}$. Consider the closed oriented submanifold $\bar{\mathrm{ev}}_i^{-1}(\bar{N}_i) \subseteq \widetilde{\mathcal{M}}_{\bar{J},\bar{B}}$. Modulo torsion, its homology class is Poincaré dual to $\bar{\mathrm{ev}}_i^*\bar{\alpha}_i \in H^*(\widetilde{\mathcal{M}}_{\bar{J},\bar{B}}, \mathbb{Z})$. Therefore

$$
\begin{aligned}
\mathrm{GW}^{\bar{M}}_{\bar{J},\bar{B}}(\bar{\alpha}_1,\ldots,\bar{\alpha}_k) &= \#(\bar{\mathrm{ev}}_1^{-1}(\bar{N}_1) \cap \ldots \cap \bar{\mathrm{ev}}_k^{-1}(\bar{N}_k)) \\
&= \#([\bar{\mathrm{ev}}_1^{-1}(N_1)] \cdot \ldots \cdot [\bar{\mathrm{ev}}_k^{-1}(N_k)]) \\
&= \#(\mathrm{PD}(\bar{\mathrm{ev}}_1^*\bar{\alpha}_1) \cdot \ldots \cdot \mathrm{PD}(\bar{\mathrm{ev}}_k^*\bar{\alpha}_k)) \\
&= \#\,\mathrm{PD}(\bar{\mathrm{ev}}_1^*\bar{\alpha}_1 \smile \ldots \smile \bar{\mathrm{ev}}_k^*\bar{\alpha}_k) \\
&= \langle \bar{\mathrm{ev}}_1^*\bar{\alpha}_1 \smile \ldots \smile \bar{\mathrm{ev}}_k^*\bar{\alpha}_k, [\widetilde{\mathcal{M}}_{\bar{J},\bar{B}}]\rangle. \qquad (45)
\end{aligned}
$$

Here $\# : H_0(\widetilde{\mathcal{M}}_{\bar{J},\bar{B}}, \mathbb{Z}) \to \mathbb{Z}$ is defined by $\#a := \#S$, where S is any finite set representing a. Let $\theta^0 : \mu^{-1}(0) \to \mathrm{EG}$ be a continuous G-equivariant map. Denote by $\iota : \mu^{-1}(0) \to M$ the inclusion and by $\iota_G : \bar{M} \to M \times_G \mathrm{EG}$ the map induced by ι and θ^0. Let $z_i \in \Sigma$, $i = 1,\ldots,k$ be distinct points and choose $p_i \in \pi^{-1}(z_i)$, $i = 1,\ldots,k$. Let $i = 1,\ldots,k$. We define $\Theta_i : \widehat{\mathcal{M}}^0_{J,B,P_0}/\mathcal{G}_{p_i}(P_0) \to \mathrm{EG}$ by $\Theta_i([u,A]_{p_i}) := \theta^0(u(p_i))$. Furthermore, we define $\mathrm{ev}_i^0 : \mathcal{M}^0_{J,B,P} \to M \times_G \mathrm{EG}$ by $\mathrm{ev}_i^0([u,A]) := [u(p_i), \Theta_i([u,A]_{p_i})]$. By Proposition 4.22 the map $F : \mathcal{M}^0_{J,\bar{B},P_0} = \mathcal{M}^0_{J,B,P_0} \to \widetilde{\mathcal{M}}_{\bar{B},P_0}$ given by $[u,A] \mapsto \bar{u}$ is a bijection. Assume that it is in fact a diffeomorphism. Then PD $F^* = F^*$ PD. Furthermore,

$$\mathrm{ev}_i^0 = \iota_G \circ \bar{\mathrm{ev}}_i \circ F \qquad (46)$$

and therefore $F^*\bar{\mathrm{ev}}_i^*\iota_G^* = (\iota_G \circ \bar{\mathrm{ev}}_i \circ F)^* = (\mathrm{ev}_i^0)^* : H^*_G(M,\mathbb{Z}) \to H^*(\mathcal{M}^0_{J,B,P})$. Recall also that $\bar{\alpha}_i = \kappa^{\cdot}\alpha_i = \iota_G^*\alpha_i$. By (45) we have

$$
\begin{aligned}
\mathrm{GW}^{\bar{M}}_{\bar{J},\bar{B}}(\bar{\alpha}_1,\ldots,\bar{\alpha}_k) &= \langle \bar{\mathrm{ev}}_1^*\bar{\alpha}_1 \smile \ldots \smile \bar{\mathrm{ev}}_k^*\bar{\alpha}_k, [\widetilde{\mathcal{M}}_{\bar{J},\bar{B}}]\rangle \\
&= \langle F^*(\bar{\mathrm{ev}}_1^*\bar{\alpha}_1 \smile \ldots \smile \bar{\mathrm{ev}}_k^*\bar{\alpha}_k), [\mathcal{M}^0_{J,B,P_0}]\rangle \\
&= \langle F^*\bar{\mathrm{ev}}_1^*\bar{\alpha}_1 \smile \ldots \smile F^*\bar{\mathrm{ev}}_k^*\bar{\alpha}_k, [\mathcal{M}^0_{J,B,P_0}]\rangle \\
&= \langle F^*\bar{\mathrm{ev}}_1^*\iota_G^*\alpha_1 \smile \ldots \smile F^*\bar{\mathrm{ev}}_k^*\iota_G^*\alpha_k, [\mathcal{M}^0_{J,B,P_0}]\rangle \\
&= \langle (\mathrm{ev}_1^0)^*\alpha_1 \smile \ldots \smile (\mathrm{ev}_k^0)^*\alpha_k, [\mathcal{M}^0_{J,B,P_0}]\rangle \\
&= \Phi^0_{k,B}(\alpha_1,\ldots,\alpha_k). \qquad (47)
\end{aligned}
$$

This implies (44).

The main step in the proof of Theorem 4.33 is now to prove that for some $\epsilon > 0$

$$\Phi^0_{k,B}(\alpha_1,\ldots,\alpha_k) = \Phi^\epsilon_{k,B}(\alpha_1,\ldots,\alpha_k). \qquad (48)$$

If this is true then (44) and (43) imply

$$\begin{aligned}\mathrm{GW}^{\bar{M}}_{k,\bar{B}}(\bar{\alpha}_1,\ldots,\bar{\alpha}_k) &= \Phi^0_{k,B}(\alpha_1,\ldots,\alpha_k)\\ &= \Phi^\epsilon_{k,B}(\alpha_1,\ldots,\alpha_k)\\ &= \Phi^1_{k,B}(\alpha_1,\ldots,\alpha_k)\end{aligned}$$

and this will imply the theorem. The very general idea of proof of (48) is the following. Let $0 \le \epsilon \le 1$. Then as in Remark 4.32 there are equivariant maps $\widetilde{\Theta}^\epsilon_i : \widehat{\mathcal{M}}^\epsilon_{J,B,P_0}/\mathcal{G}_{p_i}(P_0) \to \widetilde{\mathrm{EG}}$, $i = 1,\ldots,k$ and there is an evaluation map $\widetilde{\mathrm{ev}}^\epsilon_{J,B,P_0,\{\Theta^\epsilon_i\}} : \mathcal{M}^\epsilon_{J,B,P_0} \to M^c \times_G \widetilde{\mathrm{EG}}$. Here for $c > 0$ $M^c := \{x \in M \,|\, |\mu(x)| \le c\}$. The constant $c > 0$ and the manifold $\widetilde{\mathrm{EG}}$ can be chosen independently of $0 \le \epsilon \le 1$. Let $\iota^c_G : M^c \times_G \widetilde{\mathrm{EG}} \to M \times_G \mathrm{EG}$ be the map induced by the two inclusions. Assume that for $i = 1,\ldots,k$ there is an oriented submanifold (with boundary) $N_i \subseteq M^c \times_G \widetilde{\mathrm{EG}}$ that represents the Poincaré dual of $\iota^*\alpha_i$ and is transverse to $\widetilde{\mathrm{ev}}^\epsilon_i$. Let

$$\mathcal{M}^\epsilon_{N_1\times\ldots\times N_k} := (\widetilde{\mathrm{ev}}^\epsilon_1)^{-1}(N_1) \cap \ldots \cap (\widetilde{\mathrm{ev}}^\epsilon_k)^{-1}(N_k) \subseteq \mathcal{M}^\epsilon_{J,B,P_0}.$$

Then as in Remark 4.32

$$\Phi^\epsilon_{k,B}(\alpha_1,\ldots,\alpha_k) = \#\mathcal{M}^\epsilon_{N_1\times\ldots\times N_k}.$$

So it suffices to show that there is a (sign preserving) bijection $\mathcal{M}^0_{N_1\times\ldots\times N_k} \to \mathcal{M}^\epsilon_{N_1\times\ldots\times N_k}$. This bijection is established by proving that for $\epsilon > 0$ small enough for every $[u, A] \in \mathcal{M}^0_{N_1\times\ldots\times N_k}$ there is a unique "near" class $[u, A] \in \mathcal{M}^\epsilon_{N_1\times\ldots\times N_k}$ and vice versa. (One can choose the maps $\widetilde{\Theta}^\epsilon_i : \widehat{\mathcal{M}}^\epsilon_{J,B,P_0}/\mathcal{G}_{p_i}(P_0) \to \widetilde{\mathrm{EG}}$ for $\epsilon \ge 0$ in a "compatible" way so that $\widetilde{\Theta}^\epsilon_i$ is "near" to $\widetilde{\Theta}^0_i$ for small $\epsilon > 0$. Note that the domains of $\widetilde{\Theta}^0_i$ and $\widetilde{\Theta}^\epsilon_i$ are different.) The argument to establish the above bijection is called an *adiabatic limit argument*. It has been carried out by R. Gaio in her PhD thesis [6].

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