Floer-Gromov-Compactness and Stable Connecting Orbits

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The goal of this diploma thesis is to prove Floer-Gromov-compactness for a sequence of Floer connecting cylinders in compact symplectic manifolds. Since we only assume an energy bound on such a sequence, bubbling can occur. This leads to the definition of a stable Floer connecting orbit, which is based on the definition of a stable map for pseudoholomorphic spheres given by H. Hofer and D. Salamon. We will also define the notion of Floer-Gromov-convergence to a stable Floer connecting orbit.

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1 Preliminaries

1.1 Notation

For a real number $\epsilon > 0$ and a subset $A \subseteq \mathbb{R}^2$ we define $U_\epsilon(A) := \bigcup_{(s,t)\in A} B_\epsilon(s,t)$ and for a (Lebesque-)measurable $A \subseteq \mathbb{R}^2$ we denote by $|A|$ its Lebesque measure.

If convenient, we will think of a point $(s,t) \in \mathbb{R}^2$ as a complex number and write $(s,t) = s + it \in \mathbb{C}$. Furthermore we denote by

$$S := [0,\infty) + i[0,1] = \{s + it | 0 \leq s < \infty,\ \ 0 \leq t \leq 1\}$$

the half strip and

$$\partial_0 S := [0,\infty), \qquad \partial_1 S := [0,\infty) + i.$$

Throughout my diploma thesis subsequences of sequences are indexed by the same index.

Let Σ, M be manifolds, $U \subseteq \Sigma$ an open subset and $u^\nu, u : \Sigma \to M$ be smooth functions. We say that the sequence $(u^\nu)_{\nu\in\mathbb{N}}$ converges to u, u.c.s. on U, if it converges to u in the C^∞-topology, i.e. uniformly with all derivatives, on compact subsets $Q \subseteq U$.

Let (M,ω) be a symplectic manifold and $H \in C^\infty(M)$ a smooth function. We denote by $X_H \in Ham(M,\omega)$ the Hamiltonian vector field generated by H, i.e. the vector field defined by

$$dH = i_{X_H}\omega$$

Furthermore for a smooth time dependent family of Hamiltonians $\{H_t\}_{t\in\mathbb{R}} \subseteq C^\infty(M)$ we define the flow $\{\psi_t^H\}_{t\in\mathbb{R}} \subseteq Diff(M,\omega)$ by

$$\psi_0^H := id, \tag{1}$$

$$\frac{d}{dt}\psi_t^H = X_t^H \circ \psi_t^H \tag{2}$$

An ω-compatible almost complex structure $J \in \mathcal{J}(M,\omega)$ induces a metric g_J on M defined by $g_J(\xi,\eta) := \omega(\xi, J\eta)$, $\xi,\eta \in T_pM$, $p \in M$. For a smooth family $\{J_t\}_{t\in\mathbb{R}} \subseteq \mathcal{J}(M,\omega)$ we abbreviate $g_t := g_{J_t}$, $|\xi|_t := \sqrt{g_t(\xi,\xi)}$ and denote by $d^t : M \times M \to [0,\infty]$ the distance function induced by g_t. Furthermore for a point $x_0 \in M$ we denote by $exp^t_{x_0}$ the exponential map. If it is clear from the context, what is meant by d, g, exp_{x_0}, we will drop the superscript t.

1.2 Introduction

This introduction follows closely the one in [13].

The Arnold conjecture

Let (M,ω) be a compact symplectic manifold and $H_t = H_{t+1} : M \to \mathbb{R}$ be a smooth time dependent 1-periodic family of Hamiltonian functions. Consider Hamilton's differential equation

$$\dot{x}(t) = X_{H_t}(x(t)), \quad \forall t \in \mathbb{R}. \tag{3}$$

The fixed points of the time-1-map ψ_1^H are in one-to-one correspondence with the 1-periodic solutions of (3) and we denote the set of such solutions by

$$\mathcal{P}(H) := \{x \in C^\infty(M) | x(t+1) = x(t), \ (3)\}$$

A periodic solution x is called **nondegenerate** if

$$\det(id_{x(0)} - d\psi_1^H(x(0))) \neq 0. \tag{4}$$

The Arnold conjecture asserts that in the nondegenerate case the number of 1-periodic solutions should be bounded below by the sum of the Betti numbers of M.

Conjecture 1.1 (Arnold) *Let (M,ω) be a compact symplectic manifold and $H_t = H_{t+1} : M \to \mathbb{R}$ be a smooth time dependent 1-periodic Hamiltonian. Suppose that the 1-periodic solutions of (3) are all nondegenerate. Then*

$$\#\mathcal{P}(H) \geq \sum_{i=0}^{2n} \dim H_i(M,\mathbb{Q}),$$

where $H_i(M,\mathbb{Q})$ denotes the singular homology of M with rational coefficients.

The Arnold conjecture (in the above form) has now been proved in full generality. In [1] Floer proved it for monotone symplectic manifolds. His proof was extended by Hofer-Salamon [3] and Ono [11] to the weakly monotone case, and by Fukaya-Ono [2], Liu-Tian [8] and Hofer-Salamon [4, 5, 6, 7] to the general case.
Floer defined some homology, which afterwards was named after him, to prove the Arnold conjecture in the monotone case. I will give the idea of this homology. Let us first define when (M, ω) is monotone:

Since the space $\mathcal{J}(M, \omega)$ of ω-compatible almost complex structures is nonempty and contractible, the first Chern class $c_1 := c_1(TM, \omega) := c_1(TM, J) \in H^2(M, \mathbb{Z})$ is independent of the choice of $J \in \mathcal{J}(M, \omega)$. We can therefore define:

Definition 1.2 *A compact symplectic manifold* (M, ω) *is called* ***monotone****, if there is a real number* $\tau > 0$ *such that*

$$\int_{S^2} v^* c_1 = \tau \int_{S^2} v^* \omega, \tag{5}$$

for every smooth map $v : S^2 \to M$.

Given a monotone symplectic manifold we can always attain $\tau = 1$ by multiplying ω by τ. Therefore we assume henceforth that for any monotone symplectic manifold the above constant τ equals 1.

Symplectic action

Let (M, ω) be a compact symplectic manifold that is monotone (with $\tau = 1$). By a loop in M we mean a smooth 1-periodic $x : \mathbb{R} \to M$. We denote by

$$\mathcal{L}_0 M := \{x \in C^\infty(\mathbb{R}, M) | x(t+1) = x(t), \ \ x \text{ contractible}\}$$

the space of contractible loops in M. A tangent vector ξ to $\mathcal{L}_0 M$ at such a loop x is a vector field along x, i.e. ξ is a smooth map $\xi : \mathbb{R} \to TM$ which satisfies $\xi(t) \in T_{x(t)} M$ and $\xi(t+1) = \xi(t)$ for $t \in \mathbb{R}$. We denote the tangent space of $\mathcal{L}_0 M$ at x by $T_x \mathcal{L}_0 M$. The contractible 1-periodic solutions of (3) can be interpreted as the critical points of the (circle valued) symplectic action functional on $\mathcal{L}_0 M$, which we will now define:
Let $H_t = H_{t+1} : M \to \mathbb{R}$ be a 1-periodic Hamiltonian. Then we define the 1-form $\Psi_H : T\mathcal{L}_0 M \to \mathbb{R}$ by

$$\Psi_H(x; \xi) := \int_0^1 \omega(\dot{x}(t) - X_{H_t}(x(t)), \xi(t)) dt$$

for $\xi \in T_x \mathcal{L}_0 M$. The zeros of this 1-form are precisely the 1-periodic solutions of (3). Ψ_H is closed but in general not exact. However, it is the differential of a circle valued function $a_H : \mathcal{L}_0 M \to \mathbb{R}/\mathbb{Z}$. Let $x \in \mathcal{L}_0 M$. Since x is contractible there is a smooth $u : B := \{z \in \mathbb{C} : |z| \leq 1\} \to M$ such that $u(e^{2\pi i t}) = x(t)$ for $t \in \mathbb{R}$. The symplectic action of x is now defined by

$$a_H(x) := -\int_B u^*\omega - \int_0^1 H_t(x(t))dt + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$$

Since we assume that (M, ω) is monotone with constant $\tau = 1$ by (5) we have $\int_{S^2} v^*\omega \in \mathbb{Z}$ for every smooth map $v : S^2 \to M$ and therefore $a_H(x)$ is welldefined. The following lemma is proved in appendix A.1.

Lemma 1.3 *Assume that (M, ω) is monotone. Then the 1-form $\Psi_H \in \Omega^1(\mathcal{L}_0 M)$ is the differential of a_H,*

$$\Psi_H = da_H.$$

Negative gradient flow lines and Floer connecting cylinders

Floer's idea is to carry out Morse theory for the symplectic action functional in analogy to the Morse-Smale-Witten complex in finite dimensional Morse theory. Therefore we want to study the gradient flow lines of the action functional $a_H : \mathcal{L}_0 M \to \mathbb{R}/\mathbb{Z}$. For this we must choose a metric on the loop space $\mathcal{L}_0 M$. Let $\{J_t\}_{t\in\mathbb{R}} \subseteq \mathcal{J}(M, \omega)$ be a smooth family of ω-compatible almost complex structures with $J_{t+1} = J_t$. Then for $x \in \mathcal{L}_0 M$ this induces a metric on the tangent space $T_x \mathcal{L}_0 M = C^\infty(\mathbb{R}/\mathbb{Z}, x^*TM)$ given by

$$\langle \xi, \eta \rangle := \int_0^1 g_t(\xi(t), \eta(t))dt, \quad \xi, \eta \in T_x \mathcal{L}_0 M. \tag{6}$$

The gradient of a_H with respect to this metric is given by

$$(\text{grad} a_H(x))(t) = J_t(x(t))\big(\dot{x}(t) - X_{H_t}(x(t))\big) \tag{7}$$

A negative gradient flow line of a_H is a smooth 1-parameter family of loops $\mathbb{R} \to \mathcal{L}_0 M : s \mapsto u_s$ that satisfies $\partial_s u_s + \text{grad} a_H(u_s) = 0$. We denote by

$$\mathcal{P}_0(H) := \{x \in C^\infty(\mathbb{R}, M) | x(t+1) = x(t), (3)\}$$

the set of critical points of a_H. The idea of Floer Homology is to define a chain complex

$$CF(H) := \oplus_{x \in \mathcal{P}_0(H)} \mathbb{Z}_2 x$$

and a boundary operator $\partial : CF(H) \to CF(H)$ by counting (modulo 2) gradient flow lines connecting two critical points x^- and x^+.
In view of the formula (7) for $\text{grad} a_H$ we can interpret gradient flow lines as solutions of the partial differential equation

$$\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0, \tag{8}$$

$$u(s, t+1) = u(s,t) \tag{9}$$

for a smooth map $u : \mathbb{R}^2 \to M$. In the case where $u(s,t) = x(t)$ is independent of s, this reduces to Hamilton's equations (3). In the case where $H_t \equiv 0$ and $J_t \equiv J$ this is the equation for a J-holomorphic curve.

In order to show that the boundary operator ∂ is welldefined, one needs the following compactness result: Given a sequence of solutions $u^\nu, \nu \in \mathbb{N}$ of (8), (9) with

$$\sup_{\nu \in \mathbb{N}} E(u^\nu) < \infty,$$

there is a subsequence that converges to some, till now undefined, object. The task of my diploma thesis is to define this object, which I will call a stable Floer connecting orbit, to define the notion of convergence to a stable Floer connecting orbit and to prove compactness in the above sense. This will be done in sections 2, 3 and 4.

Let us return to equations (8), (9). We can always get rid of the term $X_{H_t}(u)$ in (8) by the following trick: Let $K_t = K_{t+1} : M \to \mathbb{R}$ be a smooth family of functions, denote its flow by $\psi_t := \psi_t^K$ and define

$$u'(s,t) := \psi_t^{-1}(u(s,t)), \tag{10}$$

$$J_t' := \psi_t^* J_t, \tag{11}$$

$$H_t' := \psi_t^*(H_t - K_t). \tag{12}$$

Then u solves (15) iff u' solves

$$\partial_s u' + J_t'(\partial_t u' - X_{H_t'}(u')) = 0. \tag{13}$$

If we set $K_t := H_t$ then H_t' and therefore the term $X_{H_t'}(u')$ vanish. On the other hand the disadvantage of this trick is that u' is no longer periodic, but satisfies the twist condition

$$\psi_1^H(u'(s,t+1)) = u'(s,t), \tag{14}$$

Furthermore for a general K_t with $K_{t+1} = K_t$ the families $J_t', t \in \mathbb{R}$ and $H_t', t \in \mathbb{R}$ are not periodic, but the they satisfy the conditions $J_{t+1}' = \psi_1^{K*} J_t'$ and $H_{t+1}' = \psi_1^{K*} H_t'$.

Remark 1.4 Since $\psi_{t*}^K g_{J'_t} = g_t$ the symplectomorphism ψ_t^K pushes forward geodesics with respect to $g_{J'_t}$ to geodesics with respect to g_t and therefore the distance $d^{J'_t}(p', q')$, equals $d^t(\psi_t^K(p'), \psi_t^K(q'))$ for $p', q' \in M$.

In the rest of this diploma thesis we consider more generally Floer's equation with the twist condition given by any symplectomorphism. So let $\phi \in Diff(M, \omega)$ be a symplectomorphism and $J_t \in \mathcal{J}(M, \omega)$ and $H_t \in C^\infty(M, \mathbb{R})$ satisfy $J_{t+1} = \phi^* J_t$ and $H_{t+1} = \phi^* H_t$. We consider Floer's equation

$$\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0, \tag{15}$$

with the twist condition

$$\phi(u(s, t+1)) = u(s, t). \tag{16}$$

Furthermore we define

$$\mathcal{P}(H, \phi) := \{x \in C^\infty(\mathbb{R}, M) | \dot{x}(t) = X_{H_t}(x(t)),\ \ \phi(x(t+1)) = x(t)\}$$

For a solution x of Hamilton's equation (3) the following are equivalent:

(i) $\phi(x(t+1)) = x(t),\ \ t \in \mathbb{R}$.
(ii) $x(0) \in \mathrm{Fix}(\phi \circ \psi_1^H)$.

The nondegeneracy condition (4) is therefore replaced by the condition

$$\det(id_p - d_p(\phi \circ \psi_1^H)) \neq 0 \tag{17}$$

Throughout this Diploma thesis I will assume the
standing hypothesis (H1): The fixed points of $\phi \circ \psi_1^H$ are nondegenerate, i.e. if $p \in M$ with $\phi \circ \psi_1^H(p) = p$ then (17) holds.

Remark 1.5 Let (M, g) be a compact manifold with a Riemannian metric and $\phi \in \mathrm{Diff}(M)$ be a diffeomorphism. Denote by d_g the distance function with respect to g. Assume that the fixed points of ϕ are all nondegenerate. Then there is a constant $C := C(M, \phi) > 0$ such that for every $p \in M$ the following holds.

$$\begin{aligned} &\text{If } \min\{d_g(p, x) | x \in \mathrm{Fix}(\phi)\} < \infty, \text{ then} \\ &\min\{d_g(p, x) | x \in \mathrm{Fix}(\phi)\} \leq C d_g(p, \phi(p)). \end{aligned} \tag{18}$$

Let now x^- and $x^+ \in \mathcal{P}(H, \phi)$. A smooth function $u : \mathbb{R}^2 \to M$ is called a **Floer cylinder connecting x^- with x^+**, if it solves Floer's equation (15), (16) and in addition

$$\left.\begin{array}{l} u(s,t) \to x^{\pm}(t), \\ \partial_s u(s,t) \to 0, \end{array}\right\} s \to \pm\infty, \text{ uniformly in } t \in [0,1]. \tag{19}$$

We denote by $\mathcal{M}(x^-, x^+, J., H., \phi)$ the set of all Floer cylinders connecting x^- with x^+ and abbreviate $\mathcal{M}(x^-, x^+, J., \phi) := \mathcal{M}(x^-, x^+, J, 0, \phi)$.
On $\mathbb{R}^2$ we define the equivalence relation $(s,t) \sim (s',t') :\iff s = s',\ t - t' \in \mathbb{Z},\ (s,t),(s',t') \in \mathbb{R}^2$. We denote the equivalence class of (s,t) by $[(s,t)] = [s+it] = [s,t]$. There are natural diffeomorphisms

$$\begin{array}{rcl} \mathbb{R}^2/\sim \to & \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to & \mathbb{R} \times S^1 \\ [s,t] \mapsto & (s, t+\mathbb{Z}) \mapsto & (s, e^{2\pi i t}) \end{array}$$

Remark 1.6 If ϕ is the identity on M, then given a function $u : \mathbb{R}^2 \to M$ satisfying (16) the function $\tilde{u} : \mathbb{R} \times S^1 \cong \mathbb{R}^2/\sim,\ [s,t] \mapsto u(s,t)$ is welldefined. In the general case where ϕ may be different from the identity we modify this definition in the following way:

We define an equivalence relation on $M \times \mathbb{R}$ by $(\phi(p), t) \sim (p, t+1),\ p \in M,\ t \in \mathbb{R}$ and denote the quotient space by $\mathcal{M} := M \times \mathbb{R}/\sim$ and the equivalence class of (p,t) by $[p,t]$. Consider the action

$$(M \times \mathbb{R}) \times \mathbb{Z} \to M \times \mathbb{R},\ ((p,t),n) \mapsto (\phi^n(p), t-n)$$

then $\mathcal{M}_\phi$ is the orbit space $M \times \mathbb{R}/\mathbb{Z}$. For a function $x : \mathbb{R} \to M$ that solves the twist condition $\phi(x(t+1)) = x(t)$ the **induced function** $\tilde{x} : S^1 \cong \mathbb{R}/\mathbb{Z} \to \mathcal{M}_\phi$ with $\tilde{x}(t+\mathbb{Z}) := [x(t), t],\ t \in \mathbb{R}$ is welldefined. For $u : \mathbb{R}^2 \to M$ solving the twist condition (16) the **induced function** $\tilde{u} : \mathbb{R} \times S^1 \cong \mathbb{R}^2/\sim \to \mathcal{M}_\phi,\ \tilde{u}([s,t]) := [u(s,t), t]$ is welldefined.

1.3 J-holomorphic curves

Let Σ be a two-dimensional manifold, j a complex structure on Σ, M a manifold of even dimension, J an almost complex structure on M and $u : \Sigma \to M$ be a smooth function. Then u is called **J-holomorphic**, if

$$J(u(z)) \circ du(z) = du(z) \circ j(z), \quad z \in \Sigma. \tag{20}$$

1.4 Energy

Definition 1.7 (Energy of "cylinders") *Let $J_t \in \mathcal{J}(M,\omega),\ t \in \mathbb{R}$ be a smooth family of ω-compatible almost complex structures and $H_t \in C^\infty(M),\ t \in \mathbb{R}$ be a smooth family of Hamiltonians Let $u : \mathbb{R}^2 \to M$ be a smooth map and $U \subseteq \mathbb{R}^2$ be a measurable subset. The Energy of u on U is defined by*

$$E(u;U) := E_{J.H.}(u;U) := \frac{1}{2}\int_U \left(|\partial_s u|_t^2 + |\partial_t u - X_{H_t}(u)|_t^2\right) dtds \in [0,\infty] \tag{21}$$

Furthermore we define $E(u;I) := E(u;I\times(0,1))$ *for any measurable* $I\subseteq\mathbb{R}$ *and write* $E(u) := E(u;\mathbb{R}\times(0,1))$.

Remark 1.8 If $u:\mathbb{R}^2\to M$ solves Floer's equation (15), then its energy is

$$E(u;U) = \int_U |\partial_s u|_t^2 ds dt$$

Remark 1.9 Since $g_{t+1} = \phi^* g_t$ and $|\partial_s u(s,t)|_t = |\partial_s u(s,t+1)|_{t+1}$ for $u:\mathbb{R}^2\to M$ which satisfies the twist condition (16), the energy is invariant under translation in t-direction by an integer, i.e. if $U' := U + (0,n) := \{(s,t+n)\in\mathbb{R}^2 | (s,t)\in U\}$ for some $n\in\mathbb{N}$ then $E(u;U') = E(u;U)$.

Remark 1.10 Let $u:\mathbb{R}^2\to M$ be a smooth function. Then the energy of u' with respect to J', given by (10), (11) is equal to the energy of u with respect to J_t. For the case $K_t = H_t$ this is seen as follows:

$$\begin{aligned} E_J(u;U) &= \textstyle\int_U \omega(\partial_s u, \partial_t u - X_{H_t}(u)) \\ &= \textstyle\int_U \omega(d\psi_t^H \partial_s u', d\psi_t^H \partial_t u') \\ &= \textstyle\int_U \omega(\partial_s u', \partial_t u') \\ &= \textstyle\int_U |\partial_s u'|^2_{J_t'} dt ds = E_{J'}(u';U), \end{aligned}$$

Now let $u^\nu:\mathbb{R}^2\to M$ be a sequence of smooth maps and $z\in\mathbb{R}^2$. If the following limits exist we define the **mass of** $\{u^\nu\}_{\nu\in\mathbb{N}}$ **at** z

$$m_{J.H.}(\{u^\nu\}_{\nu\in\mathbb{N}}, z) := \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E_{J.H.}(u^\nu; B_\epsilon(z)), \tag{22}$$

If it is clear from the context, what J_t and H_t are, we drop these subscripts. Furthermore we define for $z := \pm\infty$ if the limits exist

$$m(\{u^\nu\}_{\nu\in\mathbb{N}}, z) := \begin{cases} \lim_{R\to\infty}\lim_{\nu\to\infty} E_{J.H.}(u^\nu; (R,\infty)\times(0,1)), & z = +\infty, \\ \lim_{R\to\infty}\lim_{\nu\to\infty} E_{J.H.}(u^\nu; (-\infty,-R)\times(0,1)), & z = -\infty. \end{cases}$$

Definition 1.11 (Energy of spheres) *Let* M,ω *be a symplectic manifold,* J *an* ω*-tame almost complex structure on* M *and* $U\subseteq S^2$ *be a measurable subset. The energy on* U *of a smooth map* $u:U\to M$ *is defined as the* L^2*-norm of the 1-form* $du\in\Omega^1(S^2, u^*TM)$*:*

$$E_J(u;U) := \frac{1}{2}\int_{S^2} |du(z)|_J^2 dvol_{S^2}, \tag{23}$$

Here $|.|_J$ *denotes the following norm on* $Hom(T_z S^2, T_p M)$*:*

$$|L|_J := |\zeta|^{-1}\sqrt{|L\zeta|_J^2 + |L(j_{S^2}\zeta)|_J^2}, \tag{24}$$

where $0 \neq \zeta \in T_zS^2$, *is arbitrary, for* $z \in S^2$, $p \in M$. $j_{S^2} := S^*i$ *is the pullback under the stereographic projection* S *of the standard complex structure* i *on* $\mathbb{C} \cup \{\infty\}$ *to the sphere* S^2. *The right hand side is independent of* ζ.

Remark 1.12 If $u : S^2 \to M$ is J-holomorphic then its energy on S^2 is $E_J(u; S^2) = \int_{S^2} u^*\omega dvol_{S^2} =: E(u)$. This is independent of the almost complex structure J.

Remark 1.13 Let $u : \mathbb{C} \to M$ be a smooth function. Then

$$|d(S^*u)|_J^2 dvol_{S^2} = S^*\left((|\partial_s u|_J^2 + |\partial_t u|_J^2) ds \wedge dt\right), \quad \text{and therefore}$$

$$E(S^*u; U) = \frac{1}{2}\int_{S(U)} \left(|\partial_s u|_J^2 + |\partial_t u|_J^2\right) dsdt = E_J(u; S(U)),$$

and therefore the definitions 1.7 and 1.11 agree in the case $J_t = J$ and $H_t = 0$ if we identify $u : \mathbb{R}^2 \to M$ with $S^*u = u \circ S : S^2 \setminus \{\infty\} \to M$.

For an ω-compatible almost complex structure $J \in \mathcal{J}(M, \omega)$, a sequence $u^\nu : S^2 \to M$ of smooth functions and $z \in S^2$ we define, if the limits exist:

$$m_J(\{u^\nu\}_{\nu\in\mathbb{N}}, z) := \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E_J(u^\nu; B_\epsilon(z)) \tag{25}$$

Proposition 1.14 *Assume (H1). Let* $u : [0,\infty) \times \mathbb{R} \to M$ *be a solution of (15), (16). Equivalent are:*

(i) *u has finite energy,*

$$E(u; [0,\infty) \times [0,1]) < \infty.$$

(ii) *There is a* $x_0 \in Fix(\phi \circ \psi_1^H)$ *such that*

$$\lim_{s\to\infty} u(s,t) = \psi_t^H(x_0), \quad \lim_{s\to\infty} \partial_s u(s,t) = 0, \tag{26}$$

uniformly in $t \in [0,1]$.

(iii) $\partial_s u$ *decays exponentially with all derivatives, i.e. there are a* $\delta > 0$ *and* $c_1, c_2, \ldots > 0$ *such that for every* $s \geq 0$ *and every* $k \in \mathbb{N}$

$$||\partial_s u||_{C^k([s,\infty)\times[0,1])} \leq c_k e^{-\delta s} \tag{27}$$

Remark 1.15 Conditions (ii) and (iii) are independent of the choice of the Riemannian metric. Furthermore since $\phi^*g_t = g_{t+1}$ condition (ii) is equivalent to the following

$$d^t(u(s,t), \psi_t^H(x_0)) \to 0, \quad |\partial_s u(s,t)|_t \to 0,$$

uniformly in $t \in \mathbb{R}$.

Our proof of this proposition uses the following result:

Proposition 1.16 *Let M be a smooth manifold, $\{J_t\}_{0\leq t\leq 1}$ a smooth family of almost complex structures on M, and $\mathcal{L}_0 M$ and L_1 be closed (not necessarily compact) transverse submanifolds of M such that L_0 is totally real for J_0 and L_1 is totally real for J_1. Furthermore assume that for each intersection point $p \in L_0 \cap L_1$ there is a nondegenerate skew form*

$$T_pM \times T_pM \to \mathbb{R} : (v,w) \mapsto \omega_p(v,w)$$

that renders each of the subspaces T_pL_0 and T_pL_1 Lagrangian, and such that the form

$$T_pM \times T_pM \to \mathbb{R} : (v,w) \mapsto \omega_p(v, J_t(p)w)$$

is symmetric positive definite for $0 \leq t \leq 1$. Let $u : S = [0,\infty) + i[0,1] \to M$ satisfy

$$\begin{aligned} &\partial_s u + J_t(u)\partial_t u = 0, \\ &u(\partial_0 S = [0,\infty)) \subseteq L_0, \quad u(\partial_1 S = [0,\infty) + i) \subseteq L_1. \end{aligned}$$

Then the following are equivalent.

(I) *u has finite energy:*

$$E(u) := \int_S |\partial_s u|^2 < \infty.$$

(II) *The limits*

$$p := \lim_{s\to\infty} u(s,t), \quad 0 = \lim_{s\to\infty} \partial_s u(s,t)$$

exist uniformly in $t \in [0,1]$.

(III) *$\partial_s u$ decays exponentially in the C^∞ topology, i.e. there are positive constants ϵ and $c_1, c_2, \ldots$ such that, for all s and k,*

$$||\partial_s u||_{C^k([s,\infty)\times[0,1])} \leq c_k e^{-\epsilon s}.$$

Proof: See theorem A of [12]. □

Proof of proposition 1.14: Note that w.l.o.g.we can assume that $H_t = 0$. This is seen as follows: By remark 1.10 condition (i) is equivalent to $E_{J'_\cdot}(u') < \infty$, by remark 1.4 condition (ii)/ (iii) is equivalent to the same statement with u replaced by u' and ϕ replaced by $\phi' := \phi \circ \psi_1^H$. So assume $H_t = 0$.
We define

$$\hat{M} := M \times M,$$

$$\hat{\omega} = \omega \oplus (-\omega) \in \Omega^2(\hat{M}), \quad \hat{\omega}(\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}) := \omega(\xi_1, \eta_1) - \omega(\xi_2, \eta_2),$$

$$\hat{J}_t = J_t \times (-J_t) \in \mathcal{J}(\hat{M}, \hat{\omega}), \;\; \hat{J}_t \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} := \begin{pmatrix} J_t \xi_1 \\ -J_t \xi_2 \end{pmatrix},$$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in T_{\hat{p}} \hat{M}, \hat{p} \in \hat{M},$$

$$\hat{u} : S \to \hat{M}, \;\; s + it \mapsto \begin{pmatrix} u(\frac{s+it}{2}) \\ u(\frac{s-it}{2}) \end{pmatrix},$$

$$L_0 := \{(p,p) | p \in M\}, \;\; L_1 := \{(p, \phi(p)) | p \in M\} \subseteq \hat{M}.$$

Since we assume (H1), L_0 and L_1 intersect transversally.
Now condition (i) is equivalent to (I) with E, u replaced by $E_{\hat{\omega},\hat{J}}, \hat{u}$ since

$$E_{\hat{\omega},\hat{J}}(\hat{u}; [0,\infty) \times [0,1]) = E_{\omega,J}(u; [0,\infty) \times [-\frac{1}{2}, \frac{1}{2}]).$$

Furthermore (ii) is equivalent to (II) and (iii) is equivalent to (III) with u replaced by $\hat{u}$. Therefore applying proposition 1.16 with M replaced by $\hat{M}$, ω replaced by $\hat{\omega}$, J_t replaced by $\hat{J}_t$, u replaced by $\hat{u}$ and with L_0, L_1 as above proves proposition 1.14. □

Theorem 1.17 (Removal of singularities) *Let (M, ω) be a compact symplectic manifold, $J \in \mathcal{J}_\tau(M, \omega)$ be an ω-tame almost complex structure, $r > 0$, $z \in \mathbb{C}$ and $u : B_r(z) \setminus \{z\} \to M$ a J-holomorphic curve with finite energy*

$$E(u; B_r(z)) < \infty.$$

Then u extends to a smooth map $B_r(z) \to M$.

Proof: See theorem 4.1.3 of [9]. □

Proposition 1.18 (Quantization of energy for J-holomorphic spheres) *Let (M, ω) be a compact symplectic manifold and $J \in \mathcal{J}_\tau(M, \omega)$ an ω-tame almost complex structure. Then:*
(A) There exists a constant $c := c(M, \omega, J) > 0$ such that the following holds.

$$\text{If } u : S^2 \to M \text{ is a } J- \text{ holomorphic sphere, then } E(u) \geq c. \tag{28}$$

(B) Furthermore the constant c can be chosen to depend continuously on J and therefore for every compact $K \subseteq \mathbb{R}$ and every family $\{J_t\}_{t\in K} \subseteq \mathcal{J}_\tau(M,\omega)$ there is a constant $c > 0$ such that for every $t \in K$ condition (28) holds with J replaced by J_t.

We define $\hbar_S(M,\omega,J) := \sup\{c > 0 | E(u) \geq c, \forall \text{ const} \neq u \in C^\infty(S^2, M),\ u : J - \text{holomorphic}\}$.

Proof of proposition 1.17: Proposition 1.17 follows from the proof of proposition 4.1.4 in [9]. □

There is an analogous result for Floer connecting cylinders:

Lemma 1.19 *Assume (H1) and let $x^\pm \in \mathcal{P}(H,\phi)$. Then:*

(i) There is an $\epsilon > 0$ and a $c > 0$ such that for every Floer cylinder $u : \mathbb{R}^2 \to M$ connecting x^- with x^+ and $T \in \mathbb{R}$ the following holds.

$$\text{If } E(u;(T,\infty)\times(0,1)) < \epsilon \text{ then}$$
$$|\partial_s u(s,t)| + d(u(s,t), x^+(t)) \leq c\sqrt{E(u,(T,\infty)\times(0,1))},\ \ s \geq T+1, t \in \mathbb{R}.$$

*(ii) **(Quantization of energy for Floer connecting cylinders)** There is an $c > 0$ such that for every solution u of (15), (16) the following holds.*

$$\text{If } E(u) < c \text{ then } E(u) = 0.$$

We define $h_c(M,\omega,\{J_t\},\{H_t\},\phi) := \sup\{c > 0 | E(u) \geq c \text{ for } u : \mathbb{R}^2 \to M \text{ with } (15),(16)\}$.

Since we use Gromov's trick and the local symplectic action in the proof of this lemma, it will be given in section 1.6, after the proof of lemma 1.23.

1.5 Gromov's trick

Let $J_t \in \mathcal{J}(M,\omega), t \in \mathbb{R}$ be a smooth family of ω-compatible almost complex structures and $H_t \in C^\infty(M), t \in \mathbb{R}$ be a smooth family of Hamiltonians. We consider Floer's equation

$$\partial_s u - J_t(u)(\partial_t u - X_{H_t}(u)) = 0. \tag{29}$$

We already know from section 1 that we can eliminate the H_t-term by a trick that uses the flow $\psi_t^H,\ t \in \mathbb{R}$ of $\{H_t\}_{t\in\mathbb{R}}$. But applying this trick, the new J is still t-dependent. Gromov's trick, on the other hand, eliminates the H_t-term and gives a new J that does not depend t. It's idea is to define the new function u such that it takes values in the graph of the old u. It works as follows:

Let $K \subseteq \mathbb{R}^2$ be a compact subset. We define

$$\tilde{M} := K \times M, \ \tilde{\omega} := \omega + ds \wedge dt + dH \wedge dt \in \Omega^2(\tilde{M}), \tag{30}$$

$$\tilde{J} \in \mathrm{End}(T\tilde{M}), \ \tilde{J}(s,t,p) := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ X_{H_t}(p) & -J_t(p)X_{H_t}(p) & J_t(p) \end{pmatrix} \tag{31}$$

Then $\tilde{\omega}$ is a symplectic structure on $\tilde{M}$ and $\tilde{J}$ is almost complex and compatible with $\tilde{\omega}$. Abbreviating $X_t := X_{H_t}$, the metric $g_{\tilde{\omega},\tilde{J}} : \tilde{M} \times \tilde{M} \to \mathbb{R}$ is given by

$$g_{\tilde{\omega},\tilde{J}}(\begin{pmatrix} \sigma \\ \tau \\ \xi \end{pmatrix}, \begin{pmatrix} \sigma' \\ \tau' \\ \xi' \end{pmatrix}) = g_t(\xi - \tau X_t(p), \xi' - \tau' X_t(p)) + \sigma\sigma' + \tau\tau', \tag{32}$$

for $(\sigma,\tau,\xi),(\sigma',\tau',\xi') \in T_{\tilde{p}}\tilde{M}, \ \tilde{p} \in \tilde{M}$. It follows that the norm of a vector $(\sigma,\tau,\xi) \in T_{\tilde{p}}\tilde{M}$ is bounded below by a constant times the norm of ξ, more precisely

$$\left| \begin{pmatrix} \sigma \\ \tau \\ \xi \end{pmatrix} \right|_{\tilde{J}} \geq \frac{|\xi|_t}{1 + \max_{q \in M} |X_t(q)|} \tag{33}$$

For a smooth $u : K \to M$ we define

$$\tilde{u} \in C^\infty(K, \tilde{M}), \ (s,t) \mapsto (s,t,u(s,t)). \tag{34}$$

Then (29) is equivalent to

$$\partial_s \tilde{u} + \tilde{J}(\tilde{u})\partial_t \tilde{u} = 0$$

Furthermore if u solves (29) then for every measurable subset $A \subseteq K$

$$E_{\tilde{\omega},\tilde{J}}(\tilde{u}; A) = E_{\omega,J}(u; A) + |A|.$$

Now let $U \subseteq \mathbb{C} \cong \mathbb{R}^2$ be an open, bounded subset and $u^\nu : \overline{U} \to M$ be solutions of Floer's equations (15), (16). We define $K := \overline{U}$ and $\tilde{M}, \tilde{\omega}, \tilde{J}$ as above and

$$\tilde{u}^\nu : \overline{U} \to \tilde{M}, \ z \mapsto (z, u^\nu(z))$$

We will use the following remarks in the proof of Floer-Gromov-compactness in section 4.

Remark 1.20 Assume that $z_1 = s_1 + it_1 \in U$ is such that the mass $m_{\tilde{J}}(\{\tilde{u}^\nu\}, z_1)$ exists. Then for every small enough $\epsilon > 0$ the limit

$$\lim_{\nu\to\infty} E_{J.H.}(\{u^\nu\}, B_\epsilon(z_1)) = \lim_{\nu\to\infty} E_{\tilde{J}}(\{\tilde{u}^\nu\}, B_\epsilon(z_1)) - \pi\epsilon^2$$

exists and therefore also the mass $m_{J.H.}(\{u^\nu\}, z_1)$ exists and equals the mass $m_{\tilde{J}}(\{\tilde{u}^\nu\}, z_1)$.

The same argument shows that, if the mass $m_{J.H.}(\{u^\nu\}, z_1)$ exists, then also $m_{\tilde{J}}(\{\tilde{u}^\nu\}, z_1)$ and the two masses are the same.

Let now $\psi_1^\nu \in \mathrm{Aut}(\mathbb{C})$ be complex automorphisms that converge to $z_1 = s_1 + it_1$, u.c.s. on $\mathbb{C}$. We define

$$w_1^\nu := \tilde{u}^\nu \circ \psi_1^\nu : (\psi_1^\nu)^{-1}(\overline{U}) \subseteq \mathbb{C} \to \tilde{M}.$$

Assume that $z_2 \in \mathbb{C}$ is such that the mass $m_{\tilde{J}}(\{w_1^\nu\}, z_2)$ exists. Now let $\epsilon > 0$. Since the Lebesque measure $|\psi_1^\nu(B_\epsilon(z_2))|$ converges to 0 for $\nu \to \infty$,

exists and

$$\begin{aligned} \lim_{\nu\to\infty} E_{J.H.}(u^\nu; \psi_1^\nu(B_\epsilon(z_2))) &= \lim_{\nu\to\infty} \big(E_{\tilde{J}}(\tilde{u}^\nu; \psi_1^\nu(B_\epsilon(z_2))) - |\psi_1^\nu(B_\epsilon(z_2))|\big) \\ &= \lim_{\nu\to\infty} E_{\tilde{J}}(\tilde{u}^\nu; \psi_1^\nu(B_\epsilon(z_2))) \\ &= \lim_{\nu\to\infty} E_{\tilde{J}}(\tilde{u}^\nu \circ \psi_1^\nu; B_\epsilon(z_2)). \end{aligned} \tag{35}$$

On the other hand, since ψ_1^ν converges to $s_1 + it_1$, u.c.s. on $\mathbb{C}$, we have

$$\begin{aligned} \lim_{\nu\to\infty} E_{J.H.}(u^\nu; \psi_1^\nu(B_\epsilon(z_2))) &= \lim_{\nu\to\infty} E_{J_{t_1}}(u^\nu; \psi_1^\nu(B_\epsilon(z_2))) \\ &= \lim_{\nu\to\infty} E_{J_{t_1}}(u^\nu \circ \psi_1^\nu; B_\epsilon(z_2)) \end{aligned} \tag{36}$$

Sending $\epsilon \to 0$, it follows from equations (36) and (35) that the mass $m_{J_{t_1}}(\{u^\nu \circ \psi_1^\nu\}, z_2)$ exists and

$$m_{J_{t_1}}(\{u^\nu \circ \psi_1^\nu\}, z_2) = m_{\tilde{J}}(\{w_1^\nu\}, z_2) \tag{37}$$

Remark 1.21 Let $\psi_1^\nu \in \mathrm{Aut}(\mathbb{C})$ be complex automorphisms that converge to some $z_1 = s_1 + it_1 \in U$, define w_1^ν as above, let $V \subseteq \mathbb{C}$ be an open subset and assume that w_1^ν converges to some $\tilde{J}$-holomorphic sphere $w_1 : S^2 \to \tilde{M}$, u.c.s. on V. Then w_1 is of the form

$$w_1(z) = (z_1, v_1(z)), \text{ where } v_1 : S^2 \to S^2,$$

and since w_1 is $\tilde{J}$-holomorphic, the sphere v_1 is J_{t_1}-holomorphic. Furthermore it follows that $u^\nu \circ \psi_1^\nu \to v_1$, u.c.s. on V.

1.6 Local symplectic action

For a metric g on M and a point $p \in M$ we denote by $\epsilon(g,p)$ the radius of injectivity of the exponential map with respect to g at the point p and abbreviate $\epsilon(p,t) := \epsilon(g_t,p)$. Furthermore for every smooth $x_0 : \mathbb{R} \to M$ with $\phi(x_0(t+1)) = x_0(t)$ we define

$$L(\phi,x_0) := L(g.,\phi,x_0) := \{\ x \in C^\infty(\mathbb{R},M) | \phi(x(t+1)) = x(t), \quad d^t(x_0(t),x(t)) < \epsilon(x_0(t),t), \forall t \in \mathbb{R}\}.$$

We now define the local symplectic action $\mathcal{A}_{x_0} : L(\phi,x_0) \to \mathbb{R}$ as follows. For $x \in L(\phi,x_0)$ we define $\xi \in C^\infty(\mathbb{R},x_0^*TM)$ by $exp^t_{x_0(t)}\xi(t) = x(t)$ and $u(s,t) := exp_{x_0(t)}s\xi(t)$. Then the **local symplectic action of** x is given by

$$\mathcal{A}_{x_0}(x) := -\int_{[0,1]\times[0,1]} \omega(\partial_s u, \partial_t u - X_{H_t}(u)).$$

Lemma 1.22 *Let $s_1 < s_2$ be real numbers and $u : [s_1,s_2]\times\mathbb{R} \to M$ be a solution of Floer's equations (15), (16)with $d(u(s,t),x_0(t)) < \epsilon(x_0(t),t),\ (s,t) \in [s_1,s_2]\times\mathbb{R}$. Then*

$$E(u;[s_1,s_2]) = \mathcal{A}_{x_0}(u(s_1,.)) - \mathcal{A}_{x_0}(u(s_2,.)) \tag{38}$$

Proof: W.l.o.g.we assume $H_t = 0,\ t \in \mathbb{R}$. Consider each side of (38) as a function of s_2. Then these functions agree for $s_2 = s_1$. Let now $s_2 > s_1$ be fixed. We have to show that

$$\frac{d}{ds}|_{s=s_2} E(u;[s_1,s]) = -\frac{d}{ds}|_{s=s_2}\mathcal{A}_{x_0}(u(s,.))$$

We abbreviate $\exp := \exp^t_{x_0}$ and for $s \in [s_1,s_2]$ we define $\xi_s : \mathbb{R} \to T_{x_0}M$ by $\exp\xi_s(t) = u(s,t)$ and $u_s(s',t') := \exp s'\xi_s(t')$.

Claim 1 u_{s_2} *satisfies the twist condition*

$$\phi(u_{s_2}(s,t+1)) = u_{s_2}(s,t) \tag{39}$$

Proof: This is seen as follows: Fix $t \in \mathbb{R}$. We abbreviate $u_2 := u_{s_2}$. Since the map $[0,1] \ni s \mapsto u_2(s,t+1)$ is a g_{t+1}-geodesic and ϕ is an isometry from (M,g_{t+1}) to (M,g_t), the map $[0,1] \ni s \mapsto \phi(u_2(s,t+1))$ is a g_t-geodesic. Now $[0,1] \ni s \mapsto \exp s\xi(s_2,t)$ is a g_t-geodesic as well, with the same endpoints

$$\exp 0\xi(s_2,t) = x_0 = \phi(u_2(0,t+1)),$$
$$\exp\xi(s_2,t) = u(s_2,t) = \phi(u(s_2,t+1)) = \phi(u_2(1,t+1))$$

Since the two geodesics stay in an $\epsilon(x_0,t)$-ball around x_0 for $s\in[0,1]$ they must be the same. This proves claim 1.
There is a $\delta>0$ and a smooth map $v:[0,1]\times\mathbb{R}\to M$ with the following properties:

$$\begin{aligned} v(0,t) &= x_0, \\ v(s+1,t) &= u(s+s_2,t), \quad \text{for } -\delta\le s\le 0, \\ \phi(v(s,t+1)) &= v(s,t), \quad t\in\mathbb{R}, \end{aligned} \tag{40}$$

The map $[0,1]\times[0,1]\ni(s',t')\mapsto v((1-s)s',t')$ is homotopic to u_{s_2-s}, with fixed endpoints and respecting the twist condition, $-\delta\le s\le 0$ (41)

Condition (41) is defined in the appendix, subsection A.3. Let $\delta>0$ be as in (41). Then it follows from remark A.2 that for $-\delta\le s\le 0$

$$\begin{aligned} \mathcal{A}_{x_0}(u(s+s_2,.)) &= -\int_{[0,1]\times[0,1]} u_{s+s_2}^*\omega \\ &= -\int_{[0,s+1]\times[0,1]} v^*\omega \qquad \text{and therefore} \\ &= -\int_0^{s+1}\int_0^1 \omega(\partial_s v,\partial_t v)dsdt, \end{aligned}$$

$$\begin{aligned} \frac{d}{ds}|_{s=s_2}\mathcal{A}_{x_0}(u(s,.)) &= -\int_0^1 \omega(\partial_s v(1,t),\partial_t v(1,t)) \\ &= -\int_0^1 \omega(\partial_s u(s_2,t),\partial_t u(s_2,t))dt \\ &= -\frac{d}{ds}|_{s=s_2}E(u;[s_1,s]\times[0,1]) \end{aligned}$$

This proves lemma 1.22. □

Lemma 1.23 (Isoperimetric inequality) *Let $x_0\in\mathcal{P}(H,\phi)$ and $\epsilon>0$ satisfy*

$$\epsilon<\min_{t\in[0,1]}\epsilon(x_0(t),t), \tag{42}$$

$$\epsilon<\min\{d^t(\psi_t^H(x'),x_0(t))|t\in[0,1],x_0(0)\neq x'\in Fix(\phi\circ\psi_1^H)\} \tag{43}$$

Then there is a constant $C:=C(\epsilon)$ such that

$$|\mathcal{A}_{x_0}(x)|\le C\left(\int_0^1|\dot{x}(t)-X_{H_t}(x(t))|\right)^2 \tag{44}$$

for every $x\in L(\phi,x_0)$ with $d^t(x_0(t),x(t))\le\epsilon,\ t\in\mathbb{R}$.

Remark 1.24 In the proof of this lemma $C,C',\ldots>0$ generically denote constants that appear in an estimate. They therefore change their values from estimate to estimate.

Proof of lemma 1.23:
W.l.o.g.$H_t = 0, t \in \mathbb{R}$. Else we define $J' := \psi_t^{H*} J_t$, $\phi' := \phi \circ \psi_1^H$ and $H'_t := 0$. Since ψ_t^H is an isometry from $(M, g^{J'_t})$ to (M, g^t) by remark 1.4 we have for $x'_0, x' \in M$

$$d^t(\psi_t^H(x'), \psi_t^H(x'_0)) = d^{J'_t}(x', x'_0)$$

and the radius of injectivity $\epsilon(g^{J'_t}, p')$ of the exponential map with respect to J'_t at a point $p' \in M$ equals the radius of injectivity of $\exp^t$ at $\psi_t^H(p')$. Now let $x_0 \in \mathcal{P}(H, \phi)$ and let $\epsilon > 0$ satisfy (42) and (43) and let $x \in L(\phi, x_0)$ satisfy $d^t(x_0(t), x(t)) < \epsilon$, $t \in \mathbb{R}$. Then $x'_0 := x_0(0) \in \mathrm{Fix}(\phi')$ and ϵ satisfies

$$\epsilon < \min_{t \in [0,1]} \epsilon(g^{J'_t}, x'_0) \text{ and}$$

$$\epsilon < \min\{d^{J'_t}(x', x'_0) | t \in [0,1],\ x'_0 \neq x' \in \mathrm{Fix}(\phi')\}$$

So ϵ satisfies the hypothesis of the lemma with H_t replaced by 0, J_t replaced by J'_t and ϕ replaced by ϕ', and $x' := (\psi_t^H)^{-1} \circ x$ lies in $L(g^{J'_t}, \phi', x'_0)$ and satisfies $d^{J'_t}(x', x'_0) \leq \epsilon$. Assume now that the lemma has already been proved for the case $H_t = 0$. Denote by $\mathcal{A}(g^{J'_\cdot}, H'_\cdot = 0, x'_0; x')$ the symplectic action of x' with respect to $g^{J'_t}, H'_t = 0$, $t \in \mathbb{R}, x'_0$, and let $\xi : \mathbb{R} \to T_{x_0}M$ and $u : [0,1] \times \mathbb{R}$ be given by

$$\begin{aligned} exp_{x_0}^t \xi(t) &= x(t), \\ u(s,t) &= exp_{x_0}^t s\xi(t). \end{aligned}$$

Furthermore we define $u' := (\psi_t^H)^{-1} \circ u : [0,1] \times \mathbb{R} \to M$. Then there is a $C := C(\epsilon) > 0$ such that

$$\begin{aligned} \left| \int_{[0,1]\times[0,1]} \omega(\partial_s u', \partial_t u') ds dt \right| &= |\mathcal{A}(g^{J'_\cdot}, H'_\cdot = 0, x'_0; x')| \\ &\leq C \left(\int_0^1 |\dot{x}'(s)|_{J'_t} ds \right)^2 \end{aligned} \tag{45}$$

Abbreviating $X_t := X_{H_t}$, we have

$$\partial_t u = \psi_{t*}^H \partial_t u' + X_t(u)$$

and therefore

$$\omega(\partial_s u, \partial_t u - X_t(u)) = \omega(\partial_s u', \partial_t u') \tag{46}$$

On the other hand

$$|\dot{x'}(t)|_{J'_t} = |\dot{x}(t) - X_t(x(t))| \tag{47}$$

Inserting (46) and (47) into the inequality (45) we get inequality (44). This proves the "w.l.o.g.".

So assume that $H_t = 0$ and let $x \in L(\phi, x_0)$ with $d^t(x_0(t), x(t)) < \epsilon,\ \ t \in \mathbb{R}$, abbreviate $exp^t := exp^t_{x_0}$ and let $\xi : \mathbb{R} \to T_{x_0}M$ and $u : [0,1] \times \mathbb{R}$ be given by

$$\begin{aligned} exp^t \xi(t) &= x(t), \\ u(s,t) &= exp^t s\xi(t) \end{aligned}$$

Abbreviating $\mathcal{A} := \mathcal{A}_{x_0}$ and $Q := [0,1] \times [0,1]$ we have

$$\begin{aligned} |\mathcal{A}(x)| &\leq |\int_Q \omega(\partial_s u, \partial_t u) ds dt| \\ &\leq \int_Q |\partial_s u||\partial_t u| ds dt \\ &\leq \int_Q |\xi(t)||\partial_t u(s,t)| ds dt \end{aligned} \tag{48}$$

Now let g be any metric on M. We denote by d^g the distance with respect to this metric, while by d we still mean d^t. Furthermore we denote by $B^t(\epsilon, x_0)$ the ϵ-ball around x_0 with respect to exp^t. Since ϵ satisfies (43) and since we assume (H1) there is a constant $c > 0$ such that

$$\text{if } p \in \bigcup_{t \in [0,1]} \overline{B^t(\epsilon, x_0)} \text{ then } d^g(p, x_0) < c d^g(p, \phi(p)).$$

Therefore

$$\begin{aligned} |\xi(t)| &= d^t(x(t), x_0(t)) < C d^g(x(t), x_0(t)) < C' d^g(x(t), \phi(x(t))) \\ &= C' d^g(x(t), x(t-1)) \leq \int_{t-1}^t |\dot{x}(s)|_g ds \\ &\leq C'' \int_0^1 |\dot{x}(s)|_t ds, t \in [0,1] \end{aligned} \tag{49}$$

We now want to estimate $|\partial_t u(s,t)|$. In order to do this, note that g induces a metric on $T_{x_0}M \times T_{x_0}M$ by identifying tangential vectors $\hat{\xi} \in T_\xi T_{x_0}M$ with elements of $T_{x_0}M$ in the canonical way. We denote the norm on $T_\xi T_{x_0}M$ induced by this metric also by $|.|^g$. Furthermore we denote by $B(\epsilon, T_{x_0}M)$ the ϵ-ball in $T_{x_0}M$ with respect to g, by $\partial_t exp^t \xi$ the partial derivative with respect to t of the function $\mathbb{R} \times \overline{B(\epsilon, T_{x_0}M)} \ni (t, \xi) \mapsto \exp^t \xi \in M$ and by

$$d\exp^t(\xi) : T_\xi \overline{B(\epsilon, T_{x_0}M)} \to T_{\exp^t \xi} M,\ \ \xi \in T_{x_0}M$$

the differential of $\exp^t$ at ξ. Then

$$C_1 := \max\{\frac{|d\exp^t(\xi)\hat{\xi}|_t}{|\hat{\xi}|_g} | 0 \neq \hat{\xi} \in T_\xi T_{x_0}M, \xi \in \overline{B(\epsilon, T_{x_0}M)}, t \in [0,1]\} < \infty$$

Since $\partial_t \exp^t 0 = 0$ there is a constant $C_2 > 0$ such that

$$|\partial_t \exp^t \xi|_g < C_2 |\xi|_g, \ \ \xi \in \overline{B(\epsilon, T_{x_0}M)}, t \in [0,1]$$

Now we have

$$\partial_t u(s,t) = \partial_t \exp^t s\xi(t) + d\exp^t(s\xi(t)) s \partial_t \xi(t) \text{ and therefore} \tag{50}$$

$$|\partial_t u(s,t)|_g \leq C_2 |\xi(t)|_g + C_1 |\partial_t \xi(t)|_g \tag{51}$$

Setting $s = 1$ in equation (50) we see that

$$\partial_t \xi(t) = (d\exp^t \xi(t))^{-1} (\dot{x}(t) - \partial_t \exp^t \xi(t))$$

Since

$c := \min\{\frac{|d\exp^t(\xi)\hat{\xi}|_t}{|\hat{\xi}|_g} | 0 \neq \hat{\xi} \in T_\xi T_{x_0}M, \xi \in \overline{B(\epsilon, T_{x_0}M)}, t \in [0,1]\} > 0$ we have

$$|\partial_t \xi(t)|_g \leq c^{-1}(|\dot{x}(t)|_g + C_2 |\xi(t)|_g) \tag{52}$$

(51) and (52) imply

$$\begin{aligned} |\partial_t u(s,t)|_t &\leq C(|\xi(t)|_g + |\partial_t \xi(t)|_g) \\ &\leq C'(|\xi(t)|_g + |\dot{x}(t)|_g) \\ &\leq C''(|\dot{x}(t)|_t + |\xi(t)|_t) \end{aligned}$$

and by (49) we have

$$|\partial_t u(s,t)|_t \leq C\left(|\dot{x}(t)|_t + \int_0^1 |\dot{x}(t')| dt'\right)$$

Inserting this and the estimate (49) for $|\xi(t)|_t$ into (48) yields

$$\begin{aligned} |\mathcal{A}(x)| &\leq C \int_0^1 |\dot{x}(s)| ds \int_Q \left(\int_0^1 |\dot{x}(t')| dt' + |\dot{x}(t)|\right) ds dt \\ &\leq C \left(\int_0^1 |\dot{x}(s)| ds\right)^2 \end{aligned}$$

This proves lemma 1.23. □

Remark 1.25 Let $\epsilon > 0$ be so small that

$$\epsilon < \min\{d^t(x,y)|t \in [0,1],\ \ x,y \in \mathrm{Fix}(\phi), x \neq y\},$$
$$\epsilon < \min_{t\in[0,1], x\in \mathrm{Fix}(\phi)} \epsilon(x,t),$$

$x, x^-, x^+ \in \mathcal{P}(H,\phi)$ and $u \in \mathcal{M}(x^-, x^+, J., H., \phi)$ be a Floer cylinder connecting x^- with x^+. Then the following holds.

$$\text{If } d(u(s,t),x) \leq \epsilon,\ \ (s,t) \in \mathbb{R}^2 \text{ then } u(s,t) \equiv x(t) = x^-(t) = x^+(t), \forall (s,t) \in \mathbb{R}^2. \tag{53}$$

Using the local symplectic action, this is seen as follows. Assume that $d(u(s,t),x) \leq \epsilon,\ (s,t) \in \mathbb{R}^2$ and let $a < b$ be real numbers. By lemma 1.22 we have

$$E(u;[a,b]) = \mathcal{A}_x(u(a,.)) - \mathcal{A}_x(u(b,.))$$

and by lemma 1.23 it follows that

$$\begin{aligned} E(u;[a,b]) &\leq |\mathcal{A}_x(u(a,.))| + |\mathcal{A}_x(u(b,.))| \\ &\leq C\left(\max_{t\in[0,1]} |\partial_t u(a,t)|\right)^2 + C\left(\max_{t\in[0,1]} |\partial_t u(b,t)|\right)^2. \end{aligned} \tag{54}$$

Since we assume $u \in \mathcal{M}(x^-, x^+, J., H., \phi)$ the right hand side of (54) tends to 0 as $a \to -\infty,\ b \to \infty$ and therefore $E(u) = 0$. This implies $\partial_s u \equiv 0$ and therefore (53).

Proof of lemma 1.19: w.l.o.g.we assume that $H_t = 0,\ t \in \mathbb{R}$. Let $u : \mathbb{R}^2 \to M$ solve Floer's equations (15), (16).

Claim 2 *There are constants ϵ_1 and $c_1 > 0$ such that for $a < b$*

$$\textit{if } E(u;[a-1,b+1]) < \epsilon_1,\ \textit{then} \sup_{[a,b]\times\mathbb{R}} |\partial_s u(s,t)| \leq c_1 \sqrt{E(u;[a-1,b+1])}$$

The proof of claim 2 is based on

Claim 0 There are constants $\epsilon_0, c_0 > 0$ such that for every $a < b$:
If $E(u;[a-1,b+1]) < \epsilon_0$ then $\sup_{[a,b]\times\mathbb{R}} |\partial_s u| \leq c_0$.

Proof:
We define $K := [a-1,b-1] \times [-1,2]$ and $\tilde{M}, \tilde{\omega}, \tilde{J}$ and $\tilde{u}$ as in (30), (31) and (34). Let $z_0 \in [a,b] \times [0,1]$ and apply lemma A.1 with M, ω, J, u replaced by

$\tilde{M}, \tilde{\omega}, \tilde{J}, \tilde{u}$ and with $r := 1$. Then there is a constant $\epsilon_0 = \delta > 0$ such that if $E_{\tilde{J}}(\tilde{u}; B_{\frac{1}{2}}(z_0)) < \epsilon_0$ then

$$
\begin{aligned}
|\partial_s u(z_0)|^2 = |\partial_s \tilde{u}(z_0)|^2_{\tilde{J}} - 1 &\leq \frac{8}{\pi} \int_{B_1(z_0)} |\partial_s \tilde{u}(z_0)|^2 \\
&= \frac{8}{\pi} \left(1 + \int_{B_1(z_0)} |\partial_s u|^2 \right) \\
&\leq \frac{8}{\pi}(1 + \epsilon_0)
\end{aligned}
$$

Setting $c_0 := \frac{8}{\pi}(1 + \epsilon_0)$ claim 0 follows.

Proof of claim 2: We set $\epsilon_1 := \epsilon_0$. Then by Morrey's and Sobolev's inequality:

$$
\begin{aligned}
||\partial_s u||_{C^0([a,b]\times[0,1])} &\leq C_1 ||\partial_s u||_{W^{1,3}([a,b]\times[0,1])} \\
&\leq C_2 ||\partial_s u||_{W^{2,2}([a,b]\times[0,1])}
\end{aligned} \tag{55}
$$

Now assume that $E(u; [a-1, b+1] \times [0,1]) < \epsilon_1$. Then by claim 0 $\sup_{[a,b]\times\mathbb{R}} |\partial_s u| \leq c_0$ and therefore elliptic bootstrapping shows

$$
||\partial_s u||_{W^{2,2}([a,b]\times[0,1])} \leq C_3 ||\partial_s u||_{L^2([a-1,b+1]\times[0,1])} = C_3 \sqrt{E(u; [a-1, b+1])} \tag{56}
$$

Inequalities (55) and (56) imply claim 2.

Claim 3 *Assume that* $u \in \mathcal{M}(x^-, x^+, J., \phi)$. *There are constants* ϵ_2 *and* $c_2 > 0$ *such that for every* $T \in \mathbb{R}$:

$$
\textit{If } E(u; [T, \infty)) \leq \epsilon_2 \textit{ then } d(u(s,t), x^+) \leq c_2 \sqrt{E(u; [T, \infty))},
$$

for all $(s,t) \in [T+1, \infty) \times \mathbb{R}$.

Proof: Let g be any metric on M and denote in the proof of this claim by d_g and by $|.|_g$ the distance function and the norm with respect to this metric. Furthermore we abbreviate

$$
E_T := E(u; [T, \infty))
$$

By remark 1.5 there is a constant $C > 0$ such that the following holds.

$$
\text{If } \min_{x \in \text{Fix}(\phi)} d_g(p, x) < \infty \text{ then } \min_{x \in \text{Fix}(\phi)} d_g(p, x) < C d_g(p, \phi(p)), \; p \in M. \tag{57}
$$

For $(s,t)\in\mathbb{R}^2$ we choose $x_{s,t}\in\mathrm{Fix}(\phi)$ such that

$$d_g(u(s,t),x_{s,t}) = \min_{x\in\mathrm{Fix}(\phi)} d_g(p,x).$$

Setting

$$c' := \max_{0\neq\xi\in TM, t'\in[-1,1]} \frac{|\xi|_g}{|\xi|_{t'}} \text{ we have for } (s,t)\in\mathbb{R}\times[0,1]$$

$$\begin{aligned} d_g(u(s,t),\phi(u(s,t))) &= d_g(u(s,t),u(s,t-1)) \\ &\leq \sup_{t'\in[-1,1]} |\partial_t u(s,t')|_g \\ &\leq c'\sup_{t'\in\mathbb{R}} |\partial_t u(s,t')|_{t'} = c'\sup_{t'\in\mathbb{R}} |\partial_s u(s,t')|_{t'}. \end{aligned} \tag{58}$$

Now assume $E_T<\epsilon_1$, then it follows from (57), (58) and claim 2, that for $s\geq T+1,\ \ t\in[0,1]$

$$\begin{aligned} d_g(u(s,t),x_{s,t}) &< Cd_g(u(s,t),\phi(u(s,t))) \\ &< Cc'\sup_{t'\in\mathbb{R}} |\partial_s u(s,t')|_{t'} \\ &< c_1 Cc'\sqrt{E_T}. \end{aligned} \tag{59}$$

Set $c:=c_1Cc'$, let $\epsilon_2>0$ satisfy

$$\begin{aligned} \epsilon_2 &\leq \epsilon_1 \\ 2c\sqrt{\epsilon_2} &< \min\{d_g(x,y)|x,y\in\mathrm{Fix}(\phi),x\neq y\} \end{aligned} \tag{60}$$

and assume $E_T<\epsilon_2$. It follows from (59) and (60) that for $(s,t)\in[T+1,\infty)\times[0,1]$ the fixed point $x_{s,t}\in\mathrm{Fix}(\phi)$ is unique and that the function

$$[T+1,\infty)\times[0,1]\ni(s,t)\mapsto x_{s,t}$$

is locally constant. For $s\geq T+1$ large enough we have

$$\begin{aligned} d_g(x_{s,0},x^+) &\leq d_g(x_{s,0},u(s,0)) + d_g(u(s,0),x^+) \\ &< c\sqrt{E_T} < \frac{1}{2}\min\{d_g(x,y)|x,y\in\mathrm{Fix}(\phi),x\neq y\} \end{aligned}$$

and therefore $x_{s,0}=x^+$. By (59) we have for every $(s,t)\in[T+1,\infty)\times\mathbb{R}$

$$d^t(u(s,t),x^+) \leq \max_{0\neq\xi\in TM,t'\in[-1,1]} \frac{|\xi|_{t'}}{|\xi|_g} c\sqrt{E_T}.$$

This proves claim 3.

Now assertion (i) of lemma 1.19 follows immediately from claim 2 and 3. Proof of assertion (ii): Choose $\epsilon > 0$ so small that

$$\epsilon \leq \epsilon_2, \tag{61}$$

$$c_2\sqrt{\epsilon} < \min\{d^t(x,y) | t \in [0,1], \ x,y \in \mathrm{Fix}(\phi), x \neq y\}, \tag{62}$$

$$c_2\sqrt{\epsilon} < \min_{t\in[0,1],x\in\mathrm{Fix}(\phi)} \epsilon(x,t), \tag{63}$$

Now let $u \in \mathcal{M}(x^-, x^+, J., \phi)$ with $E(u) < \epsilon$. Then by (61) and claim 3 it follows that

$$d(u(s,t), x^+) \leq c_2\sqrt{\epsilon}, \ \ (s,t) \in \mathbb{R}^2$$

Because of (62) and (63) remark 1.25 shows that $E(u) = 0$. This proves (ii) and therefore the lemma 1.19. □

2 Stable Floer connecting orbits

2.1 Trees

Let (T, E) be a tree, i.e. let T be a finite set and $E \subseteq T \times T$ be an edge relation that is connected and contains no cycles. We denote by $e(T)$ the number of edges of T, i.e.

$$e(T) = |T| - 1.$$

For every pair $\alpha, \beta \in T$ there exist a unique integer $m \geq 0$ and a unique ordered tuple $(\gamma_0, \ldots, \gamma_m)$ of vertices $\gamma_i \in T, \ 0 \leq i \leq m$ such that

$$\gamma_i E \gamma_{i+1}, \quad \gamma_i \neq \gamma_j, \text{ if } i \neq j, \quad \gamma_0 = \alpha, \quad \gamma_m = \beta.$$

We call $(\gamma_0, \ldots, \gamma_m)$ the **directed chain (of edges) running from α to β** and denote it by

$$[\alpha, \beta] := (\gamma_0, \ldots, \gamma_m)$$

In particular $[\alpha, \alpha] = (\alpha)$. Whenever convenient, we will think of $[\alpha, \beta]$ as the set $\{\gamma_0, \ldots, \gamma_m\}$ and for example use the notation $\gamma \in [\alpha, \beta]$, meaning $\gamma \in \{\gamma_0, \ldots, \gamma_m\}$. Cutting (i.e. removing) any edge $\alpha E \beta$ decomposes the tree T into two components. The component containing β will be denoted by $T_{\alpha\beta}$. This is the set of all vertices that can be reached from α by a chain of edges through β. In other words

$$T_{\alpha\beta} := \{\gamma \in T | \beta \in [\alpha, \gamma]\}.$$

This set is called a **branch** of the tree T. For every edge $\alpha E \beta$ the tree T decomposes as a disjoint union of the branches $T_{\alpha\beta}$ and $T_{\beta\alpha}$.

Let now (T, E) be a tree with N vertices and $\alpha_1 \in T$ be a vertex. Then
(i) its other vertices $\alpha_2, \ldots, \alpha_N$ can be ordered so that the subset

$$T_i := \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$$

is a subtree for $1 \leq i \leq N$.
(ii) Given such an ordering then for each $i \geq 2$ there is a unique $j_i < i$ such that $\alpha_{j_i} E \alpha_i$.

For every integer $n \geq 1$ we define an **n-labelling on the tree** (T, E) to be a decomposition $\Lambda := \{\Lambda_\alpha\}_{\alpha \in T}$ of the index set $\{1, \ldots, n\}$ into a disjoint union

$$\{1, \ldots, n\} = \bigcup_{\alpha \in T} \Lambda_\alpha.$$

Every n-labelling Λ of (T, E) can be expressed as a function

$$\{1, \ldots, n\} \to T, i \mapsto \alpha_i$$

defined by $\alpha_i = \alpha$ iff $i \in \Lambda_\alpha$.

We define a **weighted tree** to be a labelled tree with weights $w_\alpha \geq 0,\ \ \alpha \in T$, and set $W(T) := \sum_{\alpha \in T} w_\alpha$. Such a tree is said to be **stable** if all vertices α with $w_\alpha = 0$ satisfy the condition

$$\#\Lambda_\alpha + \#\{\beta \in T | \alpha E \beta\} \geq 3 \tag{64}$$

Remark 2.1 For every stable weighted tree T with $e(T) \geq 1$ the number of edges is bounded by

$$e(T) \leq \frac{2W(T)}{\hbar} + n - 3, \tag{65}$$

where $\hbar$ is the minimum of its nonzero weights.

Definition 2.2 *For every integer $\ell \geq 1$ the **standard chain of length** ℓ is the ordered tuple $T_0^\ell := (1, \ldots, \ell)$ together with the relation $E_0^\ell \subseteq \{1, \ldots, \ell\} \times \{1, \ldots, \ell\}$ defined by*

$$i E_0^\ell i' :\iff i' = i \pm 1, \quad 1 \leq i, i' \leq \ell.$$

Let $\ell \in \mathbb{N}$ be an integer, T be a tree, $T_0 = (\gamma_1, \dots, \gamma_\ell) \subseteq T$ a directed chain of length ℓ in T and let $\gamma_1 \neq \alpha \in T$ be a fixed vertex. If we write $(\gamma_1, \dots, \beta, \alpha) = [\gamma_1, \alpha]$ for the unique chain of vertices from the vertex $\gamma_1 \in T_0$ to α, then the vertex $\beta =: \pi(\alpha)$ is called the **predecessor** of α. The case $\beta = \gamma_1$ is also allowed.
Moreover for $\alpha \in T \setminus T_0$ we define the **root of** α to be the last vertex i in the chain $[\gamma_1, \alpha]$ connecting the two vertices γ_1 and α, such that i still lies in the standard chain T_0. If we write $[\gamma_1, \alpha] = (\gamma_1, \dots, i, \gamma, \dots, \alpha)$, then this means that $i \in T_0$, but $\gamma \in T \setminus T_0$. Here γ may be the same as α and i may be equal to γ_1. We denote the root of α by $\mathrm{root}(\alpha)$ and for $\alpha \in T_0$ we set $\mathrm{root}(\alpha) := \alpha$.
Since we think of T_0 as the standard chain $(1, \dots, \ell)$, we introduce the notation

$$\gamma_i \pm 1 := \gamma_{i \pm 1}, \text{ for } 1 \leq i, i \pm 1 \leq \ell.$$

Whenever convenient, we will think of T_0 as the set $\{\gamma_1, \dots, \gamma_\ell\}$ and for example use the notation $\alpha \in T_0$, meaning $\alpha \in \{\gamma_1, \dots, \gamma_\ell\}$.

Let (T, E) and (T', E') be trees and $f : T \to T'$ be a mapping. Recall that f is called a **tree homomorphism**, if $f^{-1}(\alpha')$ is a tree for every $\alpha' \in T'$ and if $\alpha E \beta$ then either $f(\alpha) = f(\beta)$ or $f(\alpha) E' f(\beta)$. f is called a **tree isomorphism**, if it is bijective and f and f^{-1} are tree homomorphisms. If in addition there are n-labellings Λ on T and Λ' on T' then f is called an **isomorphism from the labelled tree** (T, E, Λ) **to** (T', E', Λ') if it is a tree isomorphism from (T, E) to (T', E') and

$$\Lambda_{f(\alpha)} = \Lambda_\alpha, \ \forall \alpha \in T.$$

Now let $\ell \in \mathbb{N}$ be an integer, (T, E) and (T', E') be trees and $T_0 = (\gamma_1, \dots, \gamma_\ell)$, $T_0' = (\gamma_1', \dots, \gamma_\ell')$ be directed chains of length ℓ in T, T'.

Definition 2.3 *A tree isomorphism f from (T, E) to (T', E') is called an **isomorphism from** (T, T_0, E) **to** (T', T_0', E'), if $f(\gamma_i) = \gamma_i'$, $1 \leq i \leq \ell$.*

If in addition there are labellings Λ and Λ' on the trees T and T' and f is an isomorphism of the n-labelled trees (T, E, Λ) and (T', E', Λ') and an isomorphism from (T, T_0, E) to (T', T_0', E') then it is called an **isomorphism from** (T, T_0, E, Λ) **to** (T', T_0', E', Λ').

2.2 Stable Floer connecting orbits

Let (M, ω) be a compact symplectic manifold and ϕ, J_t and H_t, $t \in \mathbb{R}$ be as in section 1. Let $\ell \geq 1$ and $n \geq 0$ be integers. Let (T, E, Λ) be an n-labelled tree and $T_0 = (\gamma_1, \dots, \gamma_\ell)$ be a directed chain of length ℓ in T. For every $\alpha \in T$ we define the Riemann surface

$$\Sigma_\alpha := \begin{cases} \mathbb{R} \times [0,1), & \alpha \in T_0, \\ S^2, & \text{else.} \end{cases}$$

We denote the subrelation of E on T_0 by

$$E_0 := \{(\alpha, \beta) | \alpha E \beta, \ \ \alpha, \beta \in T_0\}$$

For $(\alpha, \beta) \in E \setminus E_0$ let $z_{\alpha\beta} \in \Sigma_\alpha$ be points on the Riemann surface. We call the points $z_{\alpha\beta}, (\alpha, \beta) \in E \setminus E_0$ the **nodal points on the vertex** α and write $Z_\alpha := \{z_{\alpha\beta} | (\alpha, \beta) \in E \setminus E_0\} \subseteq \Sigma_\alpha$. For $i, i' \in T_0$ with $i' = i \pm 1$ we introduce the notation $z_{ii'} := \pm\infty$.
Furthermore let $\alpha_j \in T, \ 1 \leq j \leq n$ be such that $\Lambda_\alpha := \{j \in \{1, \dots, n\} | \alpha_j = \alpha\}$ for $\alpha \in T$ and let $z_j \in \Sigma_{\alpha_j}, \ 1 \leq j \leq n$. The points z_j with $\alpha_j = \alpha$ are called the **marked points on the vertex** α. Nodal and marked points on α together form the set $Y_\alpha := Z_\alpha \cup \{z_j | \alpha_j = \alpha\}$ of **special points on** α.
We introduce the notation

$$z_{\alpha,j} := \begin{cases} z_j & \text{if } \alpha_j = \alpha, \\ z_{\alpha\beta} & \text{if } \alpha_j \in T_{\alpha\beta} \end{cases} \tag{66}$$

Furthermore let $\alpha \in T \setminus T_0$, write $i := \text{root}(\alpha)$ and let $(i, \beta, \dots, \alpha)$ be the chain of vertices from i to α. Here $\beta = \alpha$ is also allowed. Then we define

$$(s_\alpha, t_\alpha) := z_{i\beta} \in \mathbb{R} \times [0,1).$$

For $\alpha = i \in T_0$ let $u_\alpha : \mathbb{R}^2 \to M$ be a solution of Floer's equations (15),(16) and for $\alpha \in T \setminus T_0$ let $u_\alpha : S^2 \to M$ be a J_{t_α}-holomorphic sphere.

Definition 2.4 *Let (T, E, Λ) be as above. We call the tuple:*

$$(\mathbf{u}, \mathbf{z}) = ((T, T_0, E), \{u_\alpha\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{(\alpha,\beta) \in E \setminus E_0}, \{(\alpha_j, z_j)\}_{1 \leq j \leq n})$$

a ***stable Floer connecting orbit into*** M ***of length*** ℓ ***with*** n ***marked points, modelled over the labelled tree*** (T, T_0, E, Λ)*, if the following conditions hold:*

(i) $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$ *for* $\alpha, \beta \in T$ *with* $(\alpha, \beta) \in E \setminus E_0$.
Moreover $\lim_{s \to \infty} u_i(s,t) = \lim_{s \to -\infty} u_{i+1}(s,t)$*, for* $i \in T_0$.

(ii) *(Special points on one vertex are distinct.) For every* $\alpha \in T$ *the points* $z_{\alpha\beta}$ *for* $\alpha E \beta$ *and* z_j *for* $\alpha = \alpha_j, \ 1 \leq j \leq n$ *are pairwise distinct.*

(iii) *(stability): If* $\alpha \in T_0$ *and* $u_\alpha(s,t) \equiv u_\alpha(t)$ *then* $\#Y_\alpha \geq 1$*. If* $\alpha \in T \setminus T_0$ *and* u_α *is a constant function, then* $\#Y_\alpha \geq 3$.

3 Floer-Gromov-convergence

Let $x^{\pm} \in \mathcal{P}(H,\phi)$, let $(\mathbf{u},\mathbf{z})$ be a stable Floer connecting orbit with n marked points $z_j,\ 1 \le j \le n$ such that

$$\begin{array}{l} \lim_{s\to-\infty} u_{\gamma_1}(s,t) = x^-(t),\ \ t \in \mathbb{R} \\ \lim_{s\to\infty} u_{\gamma_\ell}(s,t) = x^+(t),\ \ t \in \mathbb{R}. \end{array}$$

Furthermore let $u^\nu : \mathbb{R}^2 \to M$ be a solution of Floer's equations (15) and (16) and $z_j^\nu \in \mathbb{R} \times [0,1), 1 \le j \le n,\ \nu \in \mathbb{N}$ be marked points.
For a sequence of smooth maps v^ν and $z \in \Sigma_\alpha$ we write if the limits exist

$$m_\alpha(\{v^\nu\},z) := \begin{cases} m_{J.H.}(\{v^\nu\}_{\nu\in\mathbb{N}},z), \text{ if } \alpha \in T_0, \\ m_{J_{t_\alpha}}(\{v^\nu\}_{\nu\in\mathbb{N}},z), \text{ if } \alpha \in T\setminus T_0. \end{cases}$$

Definition 3.1 *The sequence*

$$(u^\nu,\mathbf{z}^\nu) := (u^\nu, z_1^\nu,\dots,z_n^\nu)$$

is said to ***Floer-Gromov-converge*** *to* $(\mathbf{u},\mathbf{z})$*, written* $(u^\nu,\mathbf{z}^\nu) \to (\mathbf{u},\mathbf{z}), \nu \to \infty$*, if there exists a family of real numbers* $s_i^\nu \in \mathbb{R},\ i \in T_0,\ \nu \in \mathbb{N}$ *and a collection of Möbius-transformations* $\phi_\alpha^\nu \in PSl(2,\mathbb{C}),\ \alpha \in T\setminus T_0,\ \nu \in \mathbb{N}$*, such that the following holds:*

(0) $\phi_\alpha^\nu(z_{\alpha\pi(\alpha)}) = \infty$*, for* $\nu \in \mathbb{N},\ \alpha \in T\setminus T_0$.

(Map) *For* $i \in T_0$ *we define* $\phi_i^\nu : \mathbb{R}^2 \to \mathbb{R}^2,\ (s,t) \mapsto (s+s_i^\nu,t)$*. Furthermore for* $\alpha \in T$ *we define*

$$u_\alpha^\nu := u^\nu \circ \phi_\alpha^\nu : \begin{cases} \mathbb{R}^2 \to M, & \text{if } \alpha \in T_0, \\ S^2 \setminus \{z_{\alpha\pi(\alpha)}\} \to M, & \text{if } \alpha \in T\setminus T_0. \end{cases}$$

Then

$$u_\alpha^\nu \to u_\alpha,\ \ u.c.s.\ on \begin{cases} \mathbb{R}^2 \setminus \bigcup_{z\in Z_\alpha}[z], & \text{if } \alpha \in T_0, \\ S^2 \setminus Z_\alpha, & \text{if } \alpha \in T\setminus T_0. \end{cases}$$

(Energy) *The mass* $m_{J.H.}(\{u_{\gamma_1}^\nu\}_{\nu\in\mathbb{N}};-\infty)$ *is welldefined, i.e. the limits occurring in its definition exist, and it vanishes.*
Furthermore let $\alpha \in T,\ \alpha \ne \gamma_1$ *and write* $\beta := \pi(\alpha)$*. Then the mass* $m_\beta(\{u_\beta^\nu\}_{\nu\in\mathbb{N}}; z_{\beta\alpha})$ *is welldefined and*

$$m_\beta(\{u_\beta^\nu\}_{\nu\in\mathbb{N}}; z_{\beta\alpha}) = E(T_{\beta\alpha}) := \sum_{\gamma\in T_{\beta\alpha}} E(u_\gamma)$$

(Rescaling) *The differences* $s_{i+1}^\nu - s_i^\nu$ *tend to* ∞ *as* $\nu \to \infty$*, for* $i, i+1 \in T_0$*. Furthermore define for* $\alpha E \beta$

$$\phi_{\beta\alpha}^\nu := (\phi_\beta^\nu)^{-1} \circ \phi_\alpha^\nu \in PSl(2,\mathbb{C})$$

Then $\phi^\nu_{\beta\alpha} \to z_{\beta\alpha}$*, u.c.s. on* $S^2 \setminus \{z_{\alpha\beta}\}$ *for* $(\alpha,\beta) \in E \setminus E_0$.

(Marked points) *Let* $1 \le j \le n$ *and denote* $\alpha := \alpha_j$ *and* $\beta := \pi(\alpha_j)$*. If* $\alpha \in T_0$ *then* $[z^\nu_j - s^\nu_\alpha] \to [z_j]$ *in the quotient topology on* $\mathbb{R}^2/\sim$*. If* $\alpha \in T \setminus T_0$ *choose* $\delta > 0$ *such that* $z_j \notin B_\delta(z_{\alpha\pi(\alpha)})$*. Then there is an integer* $\nu_0 \in \mathbb{N}$ *such that* $(\phi^\nu_\alpha)^{-1}([z^\nu_j]) \setminus B_\delta(z_{\alpha\beta})$ *contains precisely one point* ζ^ν_j*, for* $\nu \ge \nu_0$ *and the sequence* ζ^ν_j *converges to* z_j*,* $\nu \to \infty$.

We introduce the notation

$$\lim_{\nu\to\infty} s^\nu + it^\nu = \pm\infty, \text{ if } s^\nu \to \pm\infty.$$

and define for $\alpha \in T$

$$\overline{\Sigma_\alpha} := \begin{cases} \mathbb{R} \times [0,1) \cup \{\pm\infty\}, & \text{if } \alpha \in T_0, \\ S^2, & \text{if } \alpha \in T \setminus T_0. \end{cases}$$

Remark 3.2 Let $\phi^\nu \in PSl(2,\mathbb{C})$ be a sequence that converges to some $x \in S^2$, u.c.s. on $S^2 \setminus \{y\}$ for some $y \in S^2$. Then

$$(\phi^\nu)^{-1} \to y, \text{ u.c.s. on } S^2 \setminus \{x\}. \tag{67}$$

Suppose now that we are given a sequence $(u^\nu, z^\nu_1, \dots, z^\nu_n)$ of Floer connecting cylinders with marked points and have somehow constructed a stable Floer connecting orbit $(\mathbf{u}, \mathbf{z})$ and rescalings ϕ^ν_α, $\nu \in \mathbb{N}, \alpha \in T$ and that we want to prove that $(u^\nu, z^\nu_1, \dots, z^\nu_n)$ Floer-Gromov converges to $(\mathbf{u}, \mathbf{z})$ via ϕ^ν_α, $\nu \in \mathbb{N}, \alpha \in T$. By the above remark, in order to show the (Rescaling) condition it suffices to check for each $\alpha E \beta$ only either that

$$\phi^\nu_{\alpha\beta} \to z_{\alpha\beta}, \text{ u.c.s. on } S^2 \setminus \{z_{\beta\alpha}\} \text{ or}$$

$$\phi^\nu_{\beta\alpha} \to z_{\beta\alpha}, \text{ u.c.s. on } S^2 \setminus \{z_{\alpha\beta}\}.$$

Now let $(u^\nu, z^\nu_1, \dots, z^\nu_n)$ Floer-Gromov-converge to $(\mathbf{u}, \mathbf{z})$.

Remark 3.3 For $\alpha \in T \setminus T_0$ we have, abbreviating $i := \mathrm{root}(\alpha)$

$$(\phi^\nu_i)^{-1} \circ \phi^\nu_\alpha \to s_\alpha + it_\alpha, \text{ u.c.s. on } S^2 \setminus \{z_{\alpha\pi(\alpha)}\}.$$

Remark 3.4 Let $1 \le j \le n$ be such that $\alpha_j \in T \setminus T_0$, abbreviate $\alpha := \alpha_j$ and $i := \mathrm{root}(\alpha_j)$ and let $\delta > 0$ and $\zeta^\nu_j \in (\phi^\nu_\alpha)^{-1}([z^\nu_j]) \setminus B_\delta(z_{\alpha\pi(\alpha)})$ for large ν be as in condition (Marked points). Then by the (Marked points) condition we have, for ν large enough

$$(\phi^\nu_i)^{-1} \circ \phi^\nu_\alpha(\zeta^\nu_j) \in [z^\nu_j - s^\nu_i]. \tag{68}$$

On the other hand, it follows from remark 3.3 that for every $\epsilon > 0$ for ν large enough

$$(\phi_i^\nu)^{-1} \circ \phi_\alpha^\nu(\zeta_j^\nu) \in B_\epsilon(s_\alpha + it_\alpha). \tag{69}$$

(68) and (69) imply, abbreviating $z_j'^\nu := \phi_\alpha^\nu(\zeta_j^\nu) \in [z_j^\nu]$,

$$z_j'^\nu - s_i^\nu \to s_\alpha + it_\alpha, \ \ \nu \to \infty.$$

In the notation given by (66) it follows for every $1 \leq j \leq n$ that, abbreviating $i := \mathrm{root}(\alpha_j)$, there is a sequence $z_j'^\nu \in [z_j^\nu]$ such that

$$z_j'^\nu - s_i^\nu \to z_{i,j} \in \overline{\Sigma_\alpha}$$

Lemma 3.5 (Property of Floer-Gromov-convergent sequences) *Suppose that $(u^\nu, \mathbf{z}^\nu)$ Floer-Gromov-converges to $(\mathbf{u}, \mathbf{z})$ via $\{\phi_\alpha^\nu\}_{\alpha\in T,\ \nu\in\mathbb{N}}$. For $1 \leq j \leq n$ let $i := root(\alpha_j)$ and $z_j'^\nu \in [z_j^\nu]$ be such that $z_j'^\nu - s_i^\nu \to z_{i,j}$. Then:*

$z_{\alpha,j} = \lim_{\nu\to\infty}(\phi_\alpha^\nu)^{-1}(z_j'^\nu)$ *for $\alpha \in T$ and $1 \leq j \leq n$.*

Proof: Recall that $z_{\alpha,j}$ is defined by (66) and that, in particular, $z_{\alpha_j,j} = z_j$. Therefore, for $\alpha = \alpha_j$, the assertion follows from the (Marked point) condition in definition 3.1.
If $\alpha \neq \alpha_j$ then choose a chain of edges $\gamma_0, \ldots, \gamma_m$ running from $\gamma_0 = \alpha_j$ to $\gamma_m = \alpha$. We prove by induction that

$$z_{\gamma_k,j} = \lim_{\nu\to\infty} (\phi_{\gamma_k}^\nu)^{-1}(z_j'^\nu) \tag{70}$$

for $k = 0, \ldots, m$. We have already noted that (70) holds for $k = 0$. Suppose that (70) holds for $k \leq m-1$ and note that

$$z_{\gamma_k,j} = z_{\gamma_k\gamma_{k-1}} \neq z_{\gamma_k\gamma_{k+1}}.$$

If $(\gamma_k, \gamma_{k+1}) \in E \setminus E_0$ then by the (Rescaling) condition in definition 3.1, $(\phi_{\gamma_{k+1}}^\nu)^{-1} \circ \phi_{\gamma_k}^\nu$ converges to $z_{\gamma_{k+1}\gamma_k} = z_{\gamma_{k+1},j}$ u.c.s. on $S^2 \setminus \{z_{\gamma_k\gamma_{k+1}}\}$. Since $z_{\gamma_k,j} \neq z_{\gamma_k\gamma_{k+1}}$, this implies (70) with k replaced by $k+1$. If $(\gamma_k, \gamma_{k+1}) \in E_0$ then abbreviating $i' := \gamma_k$ assume w.l.o.g.$\gamma_{k+1} = i'+1$. Since $z_{i',j} \neq z_{i'i'+1} = \infty$ we have by (70)

$$(\phi_{\gamma_{k+1}}^\nu)^{-1}(z_j'^\nu) = (\phi_{\gamma_k}^\nu)^{-1}(z_j'^\nu) + s_{i'}^\nu - s_{i'+1}^\nu \to -\infty = z_{\gamma_{k+1},j}.$$

This completes the induction. Hence (70) holds for $k = m$, and this proves the lemma.

□

4 Floer-Gromov-Compactness

Theorem 4.1 *(Floer-Gromov-Compactness) Let M, ω, ϕ, J_t and $H_t, t \in \mathbb{R}$ be as in the definition of a stable Floer connecting orbit, $x^\pm \in \mathcal{P}(H,\phi)$ and let $u^\nu : \mathbb{R}^2 \to M$ be a sequence of Floer cylinders connecting x^- with x^+ such that the energies $E(u^\nu)$ are uniformly bounded, i.e.*

$$\sup_{\nu \in \mathbb{N}} E(u^\nu) < \infty$$

and let $z_1^\nu, \dots, z_n^\nu \in \mathbb{R} \times [0,1)$ be marked points. If $n = 0$ then assume that $E(u^\nu) \neq 0$, $\nu \in \mathbb{N}$. Then there is a subsequence of $(u^\nu; z_1^\nu, \dots, z_n^\nu)$ that Floer-Gromov-converges to a stable Floer connecting orbit $(\mathbf{u}, \mathbf{z})$.

First, we will prove this theorem in the case $n = 0$, i.e. when there are no marked points. We will need the following results:

Proposition 4.2 *Let $k \geq 1$, M be a compact manifold, $U \subseteq \mathbb{R}^2$ an open subset, $J^\nu \in \mathcal{J}^k(M)$ be a sequence of almost complex structures of class C^k that converges to $J \in \mathcal{J}^k(M)$ in the C^k-norm, $u^\nu \in W^{1,p}_{loc}(U; M)$ be continuous locally $W^{1,p}$-functions that solve*

$$\partial_s u^\nu + J^\nu(u^\nu)\partial_t u^\nu = 0$$

and g be any metric on M. Assume that there is a real number $p > 2$ such that for every compact $Q \subseteq U$

$$\sup_{\nu \in \mathbb{N}} \int_Q |du^\nu(s,t)|_g^p ds dt < \infty. \tag{71}$$

Then for every compact $Q \subseteq U$ the sequence u^ν is uniformly bounded in $W^{k+1,p}(Q, M)$ and hence has a $C^k(Q)$-convergent subsequence.

Proof: This proposition is the special case of theorem B.4.2 in [9] where the Riemannian surface Σ has no boundary and the Lagrangian submanifold of M is the empty set. □

Lemma 4.3 (Bubbling) *Let (M, ω) be a compact symplectic manifold, $J^\nu \in \mathcal{J}_\tau(M, \omega)$ be a sequence of ω-tame almost complex structures that converges to $J \in \mathcal{J}_\tau(M,\omega)$ in the C^∞-topology, $U \subseteq \mathbb{R}^2$ be an open subset and $u^\nu : U \to M$ be a sequence of J^ν-holomorphic curves such that*

$$\sup_{\nu \in \mathbb{N}} E(u^\nu; U) < \infty.$$

If $|du^\nu(z^\nu)| \to \infty$ for some sequence $z^\nu \to z_0 \in U$, then there exist $\psi^\nu \in Aut(\mathbb{C})$ of the form $\psi^\nu(z) = \epsilon^\nu z + \zeta^\nu$ with $\epsilon^\nu \in \mathbb{R}$ such that $\psi^\nu \to z_0$ u.c.s. on $\mathbb{C}$ and, passing to a subsequence, $v^\nu := u^\nu \circ \psi^\nu$ converges to a nonconstant J-holomorphic sphere $v : S^2 \to M$, u.c.s. on $\mathbb{C}$.

Proof: This lemma follows from the proof of lemma 4.5.5. in [9] for the special case $L = \emptyset$.

Remark 4.4 (Bubbling on a cylinder) Let $U \subseteq \mathbb{R}^2$ be a bounded open subset, $z_1 = s_1 + it_1 \in U$ and $u^\nu : U \to M$ be a sequence of solutions of Floer's equation (15) with bounded energy, i.e.

$$\sup_{\nu \in \mathbb{N}} E(u^\nu; U) < \infty.$$

Furthermore let $z^\nu \in U$ be a sequence that converges to z_1 such that

$$|\partial_s u^\nu(z^\nu)| \to \infty, \;\; \nu \to \infty. \tag{72}$$

We define $K := \overline{U}$ and $\tilde{M}, \tilde{\omega}, \tilde{J}$ as in (30), (31) and

$$\tilde{u}^\nu : U \to \tilde{M}, \;\; z \mapsto (z, u^\nu(z)).$$

Then (72) implies that

$$|\partial_s \tilde{u}^\nu(z^\nu)| \geq |\partial_s u^\nu(z^\nu)| \to \infty, \;\; \nu \to \infty$$

and by lemma 4.3 with

$$M, \omega, J^\nu = J, u^\nu, z_0 \text{ replaced by } \tilde{M}, \tilde{\omega}, \tilde{J}, \tilde{u}^\nu, z_1$$

there is a sequence $\psi^\nu \in \mathrm{Aut}(\mathbb{C})$ that converges to z_1, u.c.s. on $\mathbb{C}$ such that, passing to a subsequence, $w^\nu := \tilde{u}^\nu \circ \psi^\nu$ converges to a nonconstant $\tilde{J}$-holomorphic sphere $w : S^2 \to \tilde{M}$, u.c.s. on $\mathbb{C}$. By remark 1.21 w is of the form $w(z) = (z, v(z))$, where $v : S^2 \to M$ is a J_{t_1}-holomorphic sphere, and $u^\nu \circ \psi^\nu \to v$, u.c.s. on $\mathbb{C}$. Let $R > 0$ and let $\epsilon > 0$ be so small that $B_\epsilon(z_1) \subseteq U$. Since ψ^ν converges to z_1, u.c.s. on $\mathbb{C}$, there is a ν_0 such that for $\nu \geq \nu_0$

$$\psi^\nu(B_R) \subseteq B_\epsilon(z_1).$$

Therefore we have for $\nu \geq \nu_0$

$$\begin{aligned} E_{J_{t_1}}(u^\nu; B_\epsilon(z_1)) &\geq E_{J_{t_1}}(u^\nu; \psi^\nu(R)) \\ &= E_{J_{t_1}}(u^\nu \circ \psi^\nu; B_R) \end{aligned}$$

Sending $\nu \to \infty$ it follows that

$$\liminf_{\nu \to \infty} E_{J_{t_1}}(u^\nu; B_\epsilon(z_1)) \geq E_{J_{t_1}}(v; B_R)$$

and sending $R \to \infty$ and then $\epsilon \to 0$ this implies

$$\lim_{\epsilon\to 0}\liminf_{\nu\to\infty} E_{J_{t_1}}(u^\nu; B_\epsilon(z_1)) \geq E_{J_{t_1}}(v).$$

By proposition 1.18 it follows that

$$\lim_{\epsilon\to 0}\liminf_{\nu\to\infty} E_{J.H.}(u^\nu; B_\epsilon(z_1)) = \lim_{\epsilon\to 0}\liminf_{\nu\to\infty} E_{J_{t_1}}(u^\nu; B_\epsilon(z_1)) \geq \hbar_S(J_{t_1}). \tag{73}$$

Remark 4.5 (Bubbling on a sphere) Let $u^\nu : \mathbb{R}^2 \to M$ be a sequence of solutions of Floer's equation (15), $U \subseteq \mathbb{R}^2$ be a bounded open set and $\psi_1^\nu \in \mathrm{Aut}(\mathbb{C})$ be a sequence of rescalings that converge to some point $z_1 \in U$, u.c.s. on $\mathbb{C}$. Define $K := \overline{U}$ and $\tilde{M}, \tilde{\omega}$ and $\tilde{J}$ as in (30), (31) and

$$\tilde{u}^\nu : K \to \tilde{M},\ \ z \mapsto (z, u^\nu(z)),$$

$$w_1^\nu := \tilde{u} \circ \psi_1^\nu : (\psi_1^\nu)^{-1}(K) \subseteq \mathbb{C} \to \tilde{M}.$$

Furthermore let $V \subseteq \mathbb{C}$ be an open subset, $z_2 \in V$ and $z^\nu \in V$ be a sequence that converges to z_2 and assume that

$$\sup_{\nu\in\mathbb{N}} E_{\tilde{J}}(w_1^\nu; V) < \infty,$$

$$|dw_1^\nu(z^\nu)|_{\tilde{J}} \to \infty,\ \ \nu \to \infty.$$

Then by lemma 4.3 with $M, \omega, J^\nu = J, U, u^\nu$ and z_0 replaced by $\tilde{M}, \tilde{\omega}, \tilde{J}, V, w_1^\nu$ and z_2 there are complex automorphisms $\psi^\nu \in \mathrm{Aut}(\mathbb{C})$ such that $\psi^\nu \to z_2$, u.c.s. on $\mathbb{C}$, and, passing to a subsequence, $w_2^\nu := w_1^\nu \circ \psi^\nu$ converges to a nonconstant $\tilde{J}$-holomorphic sphere $w_2 : S^2 \to \tilde{M}$, u.c.s. on $\mathbb{C}$. By remark 1.21 with ψ_1^ν replaced by $\psi_1^\nu \circ \psi^\nu$ the function w_2 is of the form $w_2(z) = (z_2, v_2(z))$, where $v_2 : S^2 \to M$ is a nonconstant J_{t_1}-holomorphic sphere. Now let $R > 0, \epsilon > 0$. Then there is a $\nu_0 \in \mathbb{N}$ such that for $\nu \geq \nu_0$ we have $\psi^\nu(B_R) \subseteq B_\epsilon(z_2)$. Therefore we have for $\nu \geq \nu_0$

$$\begin{aligned} E_{J_{t_1}}(u^\nu \circ \psi_1^\nu; B_\epsilon(z_2)) &\geq E_{J_{t_1}}(u^\nu \circ \psi_1^\nu; \psi^\nu(B_R)) \\ &= E_{J_{t_1}}(u^\nu \circ \psi_1^\nu \circ \psi^\nu; B_R) \end{aligned}$$

Sending $\nu \to \infty$ it follows that

$$\liminf_{\nu\to\infty} E_{J_{t_1}}(u^\nu \circ \psi_1^\nu; B_\epsilon(z_2)) \geq E_{J_{t_1}}(v_2; B_R)$$

and sending $R \to \infty$ and then $\epsilon \to 0$ this implies

$$\lim_{\epsilon\to 0}\liminf_{\nu\to\infty} E_{J_{t_1}}(u^\nu \circ \psi_1^\nu; B_\epsilon(z_2)) \geq E_{J_{t_1}}(v_2).$$

By proposition 1.18 it follows that

$$\lim_{\epsilon\to 0}\liminf_{\nu\to\infty} E_{J_{t_1}}(u^\nu \circ \psi_1^\nu; B_\epsilon(z_2)) \geq \hbar_S(J_{t_1}). \tag{74}$$

Proposition 4.6 *Let* $x^{\pm} \in \mathcal{P}(H, \phi)$ *and* u^{ν}, $\nu \in \mathbb{N}$ *be a sequence of Floer cylinders connecting* x^- *with* x^+ *and satisfying*

$$\sup_{\nu \in \mathbb{N}} E(u^{\nu}) < \infty.$$

Then there is a finite set $Z \subseteq \mathbb{R} \times [0,1)$ *and a solution* $u : \mathbb{R}^2 \to M$ *of (15),(16), such that, passing to some subsequence,* $u^{\nu} \to u$*, u.c.s. on* $\mathbb{R}^2 \setminus \bigcup_{z \in Z}[z]$*, and for every* $z = (s,t) \in Z$

$$\lim_{\epsilon \to 0} \lim_{\nu \to \infty} E(u^{\nu}; B_{\epsilon}(z)) \geq \hbar_S(M, \omega, J_t) \tag{75}$$

where $\hbar_S(M, \omega, J_t)$ *is the constant defined after proposition 1.18.*

Remark 4.7 If we pass to a further subsequence, such that the limit $E := \lim_{\nu \to \infty} E(u^{\nu})$ and all the limits occurring in the definition of the masses $m(\{u^{\nu}\}; z), z \in Z$ etc. exist then

$$E = E(u) + \sum_{z \in Z} m(\{u^{\nu}\}; z) + m(\{u^{\nu}\}; \infty) + m(\{u^{\nu}\}; -\infty). \tag{76}$$

Proof of proposition 4.6: Our proof follows the one of proposition 3.3. in [13]. The idea is the following. Let $Q \subseteq \mathbb{R}^2$ be a compact subset. If the derivatives of u^{ν} are bounded on Q, uniformly in ν, then it follows from proposition 4.2, using Gromov's trick, that there is a $C^{\infty}(Q)$-convergent subsequence. If the derivatives of u^{ν} are not bounded on Q, uniformly in ν, then there is a point (s,t) in Q, where the sequence u^{ν} bubbles, i.e., passing to some subsequence, the derivatives of u^{ν} tend to ∞ at points $(s^{\nu}, t^{\nu}) \in Q$ that converge to (s,t). By an induction argument we will construct bubble points $(s_1, t_1), (s_2, t_2), \ldots$. By remark 4.4 at each bubble point energy of at least $\hbar_S(J_t)$ gets lost. Since we assume a uniform bound on the energies $E(u^{\nu})$, there can only be finitely many such points. For compact subsets Q that avoid the set Z of bubble points, it will follow by proposition 4.2 that u^{ν} converges to some u in the $C^{\infty}(Q)$-topology. Applying this argument successively for $n = 1, 2, \ldots$ for compact subsets $Q := Q_n$ that exhaust $\mathbb{R}^2 \setminus Z$, we will get a limit function u that is defined on $\mathbb{R}^2 \setminus Z$. Finally the removal of singularities theorem 1.17 will show that u extends smoothly on all of $\mathbb{R}^2$.
The first claim makes precise, what it means to construct the bubble points inductively.

Claim 4 *For every integer* $k \geq 0$ *there is an integer* $N := N(k) \geq 0$ *such that for* $1 \leq j \leq N$ *there exist distinct points* $(s_j, t_j) \in \mathbb{R} \times [0,1)$ *with the following properties.*

Passing to a subsequence, there exist $(s_j^\nu, t_j^\nu) \in \mathbb{R}^2$, *with*

$$(s_j^\nu, t_j^\nu) \to (s_j, t_j), \ \ |\partial_s u^\nu(s_j^\nu, t_j^\nu)| \to \infty, \quad \nu \to \infty, \ 1 \le j \le N \tag{77}$$

If $N < k$ *then for every compact* $K \subseteq \mathbb{R}^2 \setminus \bigcup_{1 \le j \le N} [s_j, t_j]$:

$$\sup_{\nu \in \mathbb{N}} \max\{|\partial_s u^\nu(s,t)| | (s,t) \in K\} < \infty. \tag{78}$$

Proof: The claim holds for $k = 0$. We show $k \to k+1$: Assume, by induction, that $N := N(k) \ge 0$ and $(s_j, t_j) \in \mathbb{R} \times [0,1)$ are distinct points such that conditions (77) and (78) are satisfied. Then there are two cases:

Case 1: For every compact subset $K \subseteq \mathbb{R}^2 \setminus \bigcup_{1 \le j \le N} [s_j, t_j]$:

$$\sup_{\nu \in \mathbb{N}} \max\{|\partial_s u^\nu(s,t)| | (s,t) \in K\} < \infty.$$

Case 2: There is a compact subset $K_0 \subseteq \mathbb{R}^2 \setminus \bigcup_{1 \le j \le N} [s_j, t_j]$ such that

$$\{\max_{(s,t) \in K_0} |\partial_s u^\nu(s,t)|\}_{\nu \in \mathbb{N}} \text{ is unbounded.}$$

If case 1 holds then we set $N(k+1) := N$ and the induction step follows. Assume that case 2 holds. Then $N \ge k$. Let $(s^\nu, t^\nu) \in K_0$ be such that

$$|\partial_s u^\nu(s^\nu, t^\nu)| = \max_{(s,t) \in K_0} |\partial_s u^\nu(s,t)|.$$

Passing to a suitable subsequence, it follows that

$$|\partial_s u^\nu(s^\nu, t^\nu)| \to \infty$$

and (s^ν, t^ν) converges to some $(s,t) \in K_0$ for $\nu \to \infty$. We define $(s_{N+1}, t_{N+1}) \in \mathbb{R} \times [0,1)$ to be the unique representant of $[s^\nu, t^\nu]$ in $\mathbb{R} \times [0,1)$ and

$$s_{N+1}^\nu := s^\nu, \ \ t_{N+1}^\nu := t^\nu + t_{N+1} - t, \quad \nu \in \mathbb{N}.$$

Then

$$|\partial_s u^\nu(s_{N+1}^\nu, t_{N+1}^\nu)| = |\partial_s u^\nu(s^\nu, t^\nu)| \to \infty, \ \ \nu \to \infty.$$

Since $(s^\nu, t^\nu) \in K_0 \subseteq \mathbb{R}^2 \setminus \bigcup_{1 \le j \le N} [s_j, t_j]$ the point (s_{N+1}, t_{N+1}) differs from every (s_j, t_j), $1 \le j \le N$. This proves the induction step and therefore claim 4.

Now let $k \ge 0$, $N := N(k)$ and for $1 \le j \le N$ let $(s_j, t_j) \in \mathbb{R} \times [0,1)$ be distinct points, such that conditions (77) and (78) hold.

We define

$$\hbar_S := \min_{t\in[0,1]} \hbar_S(J_t),$$

where $\hbar_S(J_t)$ is the constant defined after proposition 1.18.

Claim 5 *The number of bubble points N is bounded above, more precisely*

$$N \leq \frac{\sup_{\nu\in\mathbb{N}} E(u^\nu)}{\hbar_S}.$$

Proof: By remark 4.4 we have for every $\epsilon > 0$ and $1 \leq j \leq N$

$$\liminf_{\nu\to\infty} E(u^\nu; B_\epsilon(s_j, t_j)) \geq \hbar_S(J_{t_j}) \geq \hbar_S \tag{79}$$

Now let $0 < \epsilon \leq \frac{1}{2}$ be so small that

$$U_\epsilon([s_j, t_j]) \cap U_\epsilon([s_k, t_k]) = \emptyset, \text{ for } j \neq k.$$

Then

$$\begin{aligned} E(u^\nu; \bigcup_{1\leq j\leq N} U_\epsilon([s_j,t_j]) \cap \mathbb{R}\times(0,1)) &= \sum_{1\leq j\leq N} E(u^\nu; U_\epsilon([s_j,t_j]) \cap \mathbb{R}\times(0,1)) \\ &= \sum_{1\leq j\leq N} E(u^\nu; B_\epsilon(s_j, t_t)). \end{aligned}$$

Taking $\nu \to \infty$ it follows that

$$\begin{aligned} \sup_{\nu\in\mathbb{N}} E(u^\nu) &\geq \liminf_{\nu\to\infty} E(u^\nu; \bigcup_{1\leq j\leq N} U_\epsilon([s_j,t_j]) \cap \mathbb{R}\times(0,1)) \\ &\geq \sum_{1\leq j\leq N} \liminf_{\nu\to\infty} E(u^\nu; B_\epsilon(s_j, t_j)). \end{aligned} \tag{80}$$

Claim (5) now follows from (79) and (80), sending $\epsilon \to 0$.

Now let

$$k > \frac{\sup_{\nu\in\mathbb{N}} E(u^\nu)}{\hbar_S},$$

$N := N(k)$ and for $1 \leq j \leq N$ let $(s_j, t_j) \in \mathbb{R}\times[0,1)$ be distinct points, such that conditions (77) and (78) hold. We define $Z := \{(s_1,t_1),\dots,(s_N,t_N)\}$. We want to use proposition 4.2 to show that, passing to some subsequence, u^ν converges to some solution u of Floer's equations (15), (16), u.c.s. on $\mathbb{R}^2 \setminus \bigcup_{z\in Z}[z]$ and therefore we use Gromov's trick. Let

$$U \subseteq \mathbb{R}^2 \setminus \bigcup_{z\in Z}[z]$$

be an open bounded set. We define $K := \overline{U}$ and $\tilde{M}, \tilde{\omega}, \tilde{J}$, as in (30), (31) and

$$\tilde{u}^\nu : U \to \tilde{M}, \ \ z \mapsto (z, u^\nu(z)).$$

Then for $z \in U$ we have

$$|d\tilde{u}^\nu(z)|^2_{\tilde{J}} = 2|\partial_s \tilde{u}^\nu(z)|^2_{\tilde{J}} = 2|\partial_s u^\nu(z)|^2 + 2$$

and by condition (78) and claim 5 we have for every compact $Q \subseteq U$

$$\sup_{\nu \in \mathbb{N}} \sup_{z \in Q} |d\tilde{u}^\nu(z)|_{\tilde{J}} \leq \sup_{\nu \in \mathbb{N}} \sup_{z \in Q} \sqrt{2}|\partial_s u^\nu(z)| + \sqrt{2} < \infty.$$

It follows that the assumptions of theorem 4.2 with

$$M, \omega, J^\nu = J, u^\nu, g \text{ replaced by } \tilde{M}, \tilde{J}, \tilde{u}^\nu, g_{\tilde{\omega}, \tilde{J}}$$

and with $p := \infty$ are satisfied. Therefore by theorem 4.2 for every compact $Q \subseteq U$ there is a $C^\infty(Q)$-convergent subsequence of $\tilde{u}^\nu$. By remark 1.21 the limit $\tilde{u} : \Sigma \to \tilde{M}$ is of the form $\tilde{u}(z) = (z, u(z))$, where $u : Q \to M$ is a solution of Floer's equations (15), (16). For $n \in \mathbb{N}$ we define

$$U_n := (-n-1, n+1)^2 \setminus \bigcup_{z \in Z} [z], \ \ Q_n := [-n, n]^2 \setminus \bigcup_{z \in Z} U_{\frac{1}{n}}([z]).$$

Applying the above argument successively for $n = 1, 2, \ldots$ with $U := U_n, Q := Q_n$ and passing to a diagonal subsequence, u^ν converges to some solution $u : \mathbb{R}^2 \setminus \bigcup_{z \in Z} [z] \to M$ of Floer's equations (15), (16), u.c.s. on $\mathbb{R}^2 \setminus \bigcup_{z \in Z} [z]$.

Now we want to show that this u smoothly extends to all of $\mathbb{R}^2$. Let therefore $z \in Z$, and define $K := \overline{B_{\frac{1}{2}}(z)}$ and $\tilde{M}, \tilde{\omega}, \tilde{J}$ as in (30), (31) and (34). Then

$$\begin{aligned} E(\tilde{u}; B_{\frac{1}{2}}(z) \setminus \{z\}) &= \int_{B_{\frac{1}{2}}(z) \setminus \{z\}} \lim_{\nu \to \infty} |\partial_s \tilde{u}^\nu|_{\tilde{J}} ds dt \\ &\leq \liminf_{\nu \to \infty} \int_{B_{\frac{1}{2}}(z)} |\partial_s \tilde{u}^\nu|^2_{\tilde{J}} ds dt \\ &\leq \sup_{\nu \in \mathbb{N}} \tilde{E}(\tilde{u}^\nu; B_{\frac{1}{2}}(z)) = \sup_{\nu \in \mathbb{N}} E(u^\nu) + \frac{\pi}{4} < \infty \end{aligned}$$

and therefore the removal of singularities theorem 1.17 applied with

$$M, \omega, J, u \text{ replaced by } \tilde{M}, \tilde{\omega}, \tilde{J}, \tilde{u}$$

shows that $\tilde{u}$ and hence also u smoothly extends to all of $B_{\frac{1}{2}}(z)$. This completes the proof of proposition 4.6. $\square$

Proposition 4.8 (Soft Rescaling) *Let $J^\nu \in \mathcal{J}_\tau(M,\omega)$ be a sequence of ω-tame almost complex structures on M that converges to J in the C^∞-topology. Fix a point $z_0 \in \mathbb{C}$ and a number $r > 0$. Suppose that $u^\nu : B_r(z_0) \to M$ is a sequence of J^ν-holomorphic curves and $u : B_r(z_0) \to M$ is a J-holomorphic curve such that, passing to a subsequence, the following holds.*

(a) *u^ν converges to u in the C^∞-topology on every compact subset of $B_r(z_0) \setminus \{z_0\}$.*
(b) *The limit $m_0 := \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(u^\nu; B_\epsilon(z_0))$ exists and is positive.*

Then there exist a sequence of complex automorphisms $\psi^\nu \in Aut(\mathbb{C})$ of the form $\psi^\nu(z) = \epsilon^\nu z + z^\nu$ with $\epsilon^\nu \in \mathbb{R}$, a J-holomorphic sphere $v : S^2 \to M$, and a finite set $Z \subseteq \overline{B_1} \subseteq \mathbb{C}$, such that the following holds.

(i) *ψ^ν converges to z_0 in the C^∞-topology on every compact subset of $\mathbb{C}$.*

(ii) *The sequence $v^\nu := u^\nu \circ \psi^\nu$ converges to v in the C^∞-topology on every compact subset of $\mathbb{C} \setminus Z$ and for every $z \in Z$ there is a sequence $z^\nu \in \mathbb{C}$, converging to z, such that*

$$|dv^\nu(z^\nu)| \to \infty, \nu \to \infty.$$

Furthermore the limits

$$m(z) := \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(v^\nu; B_\epsilon(z))$$

exist and are positive for $z \in Z$.

(iii) *$E(v) + \sum_{z\in Z} m(z) = m_0$.*

(iv) *If v is constant then $|Z| \geq 2$.*

Proof: This proposition follows from the proof of proposition 4.6.1. in [9].□

Proposition 4.9 *Let $J^\nu \in \mathcal{J}_\tau(M,\omega)$, $z_0 \in \mathbb{C}$, and $u, u^\nu : B_r(z_0) \to M$ be as in the hypotheses of proposition 4.8 and suppose that $\psi^\nu \in PSl(2,\mathbb{C})$, $v : S^2 \to M$, and $z_1, \ldots, z_k, z_\infty \subseteq \mathbb{C}$ are distinct points such that*

(i) ψ^ν converges to z_0 in the C^∞-topology on every compact subset of $S^2 \setminus \{z_\infty\}$.

(ii) The sequence $v^\nu := u^\nu \circ \psi^\nu$ converges to v in the C^∞-topology on every compact subset of $S^2 \setminus \{z_1, \ldots, z_k, z_\infty\}$, and the limits

$$m_j := \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(v^\nu; B_\epsilon(z_j))$$

exist and are positive for $j = 1, \ldots, k$.

(iii) $E(v) + \sum_{j=1}^k m_j = m_0 := \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(u^\nu; B_\epsilon(z_0))$.

Then

$$u(z_0) = v(z_\infty).$$

Moreover, for every $\epsilon > 0$, *there exist constants* $\delta > 0$ *and* $\nu_0 \in \mathbb{N}$ *such that, for every integer* $\nu \geq \nu_0$ *and every* $\zeta \in S^2$

$$d(\zeta, z_0) + d((\psi^\nu)^{-1}(\zeta), z_\infty) < \delta \Rightarrow d(u^\nu(\zeta), u(z_0)) < \epsilon.$$

Proof: See proposition 4.6.2. in [9].

The proof of theorem 4.1 in the case with no marked points is split into three parts: In the first part we construct the numbers s_i^ν, the Floer connecting cylinders u_i and the nodal sets Z_i, $1 \leq i \leq \ell$. In the second part we construct the subtrees $T_{i\alpha}$ for $\alpha E i$, $i \in T_0, \alpha \in T \setminus T_0$, the J_{t_α}-holomorphic spheres u_α, the nodal sets Z_α and the rescalings ψ_α^ν, $\nu \in \mathbb{N}$, for $\alpha \in T \setminus T_0$. In the third part we check that $(\mathbf{u}, \mathbf{z}) := ((T, T_0, E), \{u_\alpha\}_{\alpha\in T}, \{z_{\alpha\beta}\}_{\alpha E\beta})$ indeed is a stable Floer connecting orbit and that u^ν really Floer-Gromov-converges to $(\mathbf{u}, \mathbf{z})$.

Remark 4.10 It suffices to prove the theorem for $H_t = 0$, $t \in \mathbb{R}$. To see this we apply the trick mentioned at page 6 with $K_t := H_t$, i.e. we set $J_t' := \psi_t^{H*} J_t$, $\phi' := \phi \circ \psi_1^H$ and $u'(s,t) := (\psi_t^H)^{-1} \circ u(s,t)$ for a smooth function $u : \mathbb{R}^2 \to M$. Now if $(\mathbf{u}', \mathbf{z}) := ((T, T_0, E), \{u_\alpha'\}_{\alpha\in T}, \{z_{\alpha\beta}\}_{\alpha E\beta})$ is a stable Floer connecting orbit then the same holds for $(\mathbf{u}, \mathbf{z})$ and if $(u'^\nu, \mathbf{z}^\nu)$ Floer-Gromov-converges to $(\mathbf{u}', \mathbf{z})$, then $(u^\nu, \mathbf{z}^\nu) \to (\mathbf{u}, \mathbf{z})$, $\nu \to \infty$. Note in particular that by remark 1.10 the energy of u' with respect to J_t' equals the energy of u with respect to J_t and therefore, if the (Energy) condition in the definition of Floer-Gromov-convergence is satisfied for $(\mathbf{u}'^\nu, \mathbf{z}^\nu)$ and $(\mathbf{u}', \mathbf{z})$, then it holds also for $(u^\nu, \mathbf{z})$ and $(\mathbf{u}, \mathbf{z})$.
Therefore **we assume without loss of generality that** $H_t = 0$, $t \in \mathbb{R}$. So Floer's equations read

$$\partial_s u + J_t(u)\partial_t u = 0 \tag{81}$$
$$\phi(u(s, t+1)) = u(s,t). \tag{82}$$

Note that solutions of (81) and (82) with $E(u) < \infty$ satisfy

$$\lim_{s\to\pm\infty} u(s,t) = x^{\pm}, \quad \forall t \in \mathbb{R}, \tag{83}$$

where $x^{\pm}$ are fixed points of ϕ.

Proof of theorem 4.1 for $n = 0,\ H_t = 0,\ t \in \mathbb{R}$**:**

Part I: construction of $s_i^\nu,\ u_i$ and Z_i:

First of all: No matter how we choose the s_i^ν, proposition 4.6 guarantees that there is a subsequence of $u_i^\nu := u^\nu(s_i^\nu + .,.)$ that converges to a solution u_i of Floer's equations (81),(82), u.c.s. on $\mathbb{R}^2 \setminus \bigcup_{z\in Z_i}[z]$ for some finite subset $Z_i \subseteq \mathbb{R} \times [0,1)$. The idea is now to prove that there are an $\ell \in \mathbb{N}$ and $s_i^\nu \in \mathbb{R},\ \nu \in \mathbb{N},\ 1 \le i \le \ell$ such that, passing to some subsequence, $u_i^\nu := u^\nu(s_i^\nu + .,.)$ and the u_i and Z_i we obtain from proposition 4.6 satisfy the following conditions, to be specified later on:

(i) $u_1^\nu(s,t)$ is close to x^-, for $s < 0$. In part III we will show that this implies $m(\{u_1^\nu\}, -\infty) = 0$.

(ii) If $\ell \ge 2$ then for $2 \le i \le \ell$ the point $u_i^\nu(s,t)$ is close to x_{i-1} for $s \in [s_{i-1}^\nu - s_i^\nu + T, 0]$, where $T \ge 0$ is some fixed constant. This will imply two things: First, that u_i is connected to u_{i-1} in the sense $\lim_{s\to\infty} u_{i-1}(s,t) = \lim_{s\to-\infty} u_i(s,t)$ and secondly that no energy gets lost between the $(i-1)$st and the i-th cylinder in the limit $\nu \to \infty$. More precisely this means that $m(\{u_{i-1}^\nu, \infty) = E(u_i) + \sum_{z\in Z_i} m(\{u_i^\nu\}, z) + m(\{u_i^\nu\}, \infty)$.

(iii) $u_\ell^\nu(s,t)$ is close to x_ℓ for large s. This will imply that $m(\{u_\ell^\nu\}, \infty) = 0$.

(iv) There is a $t^\nu \in [0,1]$ such that $u_i^\nu(0,t^\nu)$ is further away from x_{i-1} than some $\epsilon > 0$. This will imply $E(u_i) > 0$ or that there is bubbling on the i-th cylinder, i.e. $|Z_i| \ge 1$. This condition corresponds to stability.

That there can be only finitely many shifts s_i^ν such that conditions (i),(ii) and (iv) hold will be guaranteed by the quantization of energy assertion (ii) in proposition 1.19 and by inequality (75) in proposition 4.6.
Recall the notation d^t and $exp_x^t : T_xM \to M$ for the distance and exponential map with respect to the metric g_t for $t \in \mathbb{R},\ x \in M$. We denote for $\epsilon > 0$ by $exp_x^{t,\epsilon}$ its restriction to the ball $B_\epsilon(0, T_xM)$. If it is clear from the context what is meant by d and exp_x, we will omit the superscript t. Now let $\epsilon > 0$ satisfy

$$\epsilon < \min_{x \in \mathrm{Fix}(\phi), t \in [0,1]} \epsilon(x,t) \text{ and} \tag{84}$$
$$\epsilon < \min\{d^t(x_0,x_1) | t \in [0,1],\ x_0, x_1 \in \mathrm{Fix}(\phi), x_0 \neq x_1\}. \tag{85}$$

Since we always assume (H1), i.e. that all fixed points of ϕ are nondegenerate, for every fixed point x there is an $\epsilon > 0$ such that $B^t_\epsilon(x) \cap Fix(\phi) = \{x\}$. Since M is assumed to be compact, there are only finitely many fixed points. Therefore condition (85) can be satisfied.

Now let M, ω, ϕ and J be as in the definition of a stable Floer connecting orbit, $x^\pm \in \mathrm{Fix}(\phi)$ be fixed points and let $u^\nu : \mathbb{R}^2 \to M$ solve (81),(82), $\nu \in \mathbb{N}$. Assume that

$$\lim_{s \to \pm} u^\nu(s,t) = x^\pm, \text{ uniformly in } t \in [0,1], \quad \text{for } \nu \in \mathbb{N},$$
$$\lim_{s \to \pm} |\partial_s u^\nu(s,t)| \to 0, \text{ uniformly in } t \in [0,1], \quad \text{for } \nu \in \mathbb{N}$$

and suppose that the energies $E(u^\nu)$ are nonzero and that

$$\sup_{\nu \in \mathbb{N}} E(u^\nu) < \infty$$

We pass to a subsequence such that

$$E := \lim_{\nu \to \infty} E(u^\nu) \in [0, \infty)$$

exists and set $x_0 := x^-$.

Claim 6 *There exists an $\ell \in \mathbb{N}$ such that for $1 \leq i \leq \ell$ there is a fixed point $x_i \in Fix(\phi)$, a Floer cylinder u_i connecting x_{i-1} with x_i, a finite subset $Z_i \subseteq \mathbb{R} \times [0,1)$, a number $T_i \in \mathbb{R}$ and, passing to some subsequence, there are numbers $s^\nu_i \in \mathbb{R}$, $\nu \in \mathbb{N}, 1 \leq i \leq \ell$ such that the following conditions are satisfied:*

$$u^\nu_i := u^\nu(s^\nu_i + .,.) \to u_i, \nu \to \infty, u.c.s. \text{ on } \mathbb{R}^2 \setminus \bigcup_{z \in Z_i} [z], 1 \leq i \leq \ell, \tag{86}$$
$$m(i,z) := m(\{u^\nu_i\}, z) \geq \hbar_S(J_t), \text{ for } z = (s,t) \in Z_i,,\ 1 \leq i \leq \ell, \tag{87}$$
$$d(u^\nu_1(s,t), x^-) < \epsilon, \quad \text{for } s < 0,\ t \in \mathbb{R},\ \nu \in \mathbb{N},$$
$$d(u^\nu_i(s,t), x_{i-1}) < \epsilon, \quad \text{for } (s,t) \in [s^\nu_{i-1} - s^\nu_i + T_{i-1}, 0) \times \mathbb{R}, \quad \nu \in \mathbb{N},\ 2 \leq i \leq \ell, \tag{88}$$
$$d(u^\nu_i(0,t^\nu), x_{i-1}) \geq \epsilon, \quad \text{for some } t^\nu \in [0,1],\ \forall \nu \in \mathbb{N}, 1 \leq i \leq \ell, \tag{89}$$
$$s^\nu_i - s^\nu_{i-1} \to \infty, \quad \nu \to \infty,\ 2 \leq i \leq \ell, \tag{90}$$
$$d(u^\nu_\ell(s,t), x_\ell) < \epsilon, \quad s \geq T_\ell,\ t \in \mathbb{R},\ \nu \in \mathbb{N}. \tag{91}$$

Proof: The proof is split into two parts: In the first part we prove by induction the following slightly modified version of the claim.

Claim 7 *For every $k \in \mathbb{N}$ there is an $\ell := \ell(k)$ such that for $1 \leq i \leq \ell$ there is an $x_i \in Fix(\phi)$, a Floer cylinder u_i connecting x_{i-1} with x_i, a finite subset $Z_i \subseteq \mathbb{R} \times [0,1)$, for $1 \leq i \leq \ell - 1$ there is a number $T_i \in \mathbb{R}$, and, passing to some subsequence, there are numbers $s_i^\nu, \nu \in \mathbb{N}, 1 \leq i \leq \ell$ such that conditions (86-90) hold and*

$$\begin{array}{c}\textit{if } \ell < k \textit{ then there is a } T_\ell \in \mathbb{R}, \textit{ such that,} \\ \textit{passing to some further subsequence, } (91) \textit{ holds.}\end{array} \tag{92}$$

In the second part we will apply claim 7 and use the condition (87), which relies on lemma 1.18 about quantization of energy for J-holomorphic spheres, and lemma 1.19 (ii) about quantization of energy for Floer connecting cylinders to show that $\ell(k)$ is bounded above by some constant. Then claim 6 follows.

Proof of claim 7 For $k = 1$ we see this as follows. We set $\ell(1) := 1$ and fix $\nu \in \mathbb{N}$. By the assumption $u^\nu(s,t) \to x^-,\ s \to -\infty$, uniformly in $t \in [0,1]$ there is an $s' \in \mathbb{R}$ such that $d(u^\nu(s,t), x^-) < \epsilon$ for every $s \leq s',\ t \in \mathbb{R}$. On the other hand, since ϵ satisfies (84) and (85) by remark 1.25 with $x := x^-$ and the assumption $E(u^\nu) \neq 0$ there is a point $(s,t) \in \mathbb{R}^2$ such that $d(u^\nu(s,t), x^-) \geq \epsilon$. We can therefore define the number

$$\mathbb{R} \ni s_1^\nu := \sup\{s' \in \mathbb{R} \mid d(u^\nu(s,t), x^-) < \epsilon,\ \forall s \leq s',\ \forall t \in \mathbb{R}\}$$

Furthermore we define $u_1^\nu : \mathbb{R}^2 \to M,\ (s,t) \mapsto u^\nu(s_1^\nu + s, t)$ and apply proposition 4.6 to the sequence u_1^ν to get a finite subset $Z_1 \subseteq \mathbb{R} \times [0,1)$ and a solution u_1 of Floer's equations (81), (82) such that, passing to some subsequence, the assertions of 4.6 are satisfied. In particular u_1^ν converges to u_1, u.c.s. on $\mathbb{R}^2 \setminus \bigcup_{z \in Z_1} [z]$ and therefore (86) holds. Passing to a subsequence, such that for every $z \in Z_1$ the mass $m(1,z) := m(\{u_1^\nu\}, z)$ exists, condition (87) follows. Conditions (88) and (89) are satisfied by the definition of s_1^ν and conditions (90) and (92) are void. Furthermore

$$E(u_1) \leq E = \lim_{\nu \to \infty} E(u_1^\nu) < \infty$$

and therefore by proposition 1.14 the following limits exist uniformly in $t \in [0,1]$:

$$\begin{array}{c} x_1 := \lim_{s \to \infty} u_1(s,t) \in \mathrm{Fix}(\phi), \\ x := \lim_{s \to -\infty} u_1(s,t) \in \mathrm{Fix}(\phi), \\ \lim_{s \to \pm} |\partial_s u_1(s,t)| = 0. \end{array}$$

Therefore u_1 is indeed a Floer cylinder connecting a fixed point x with a fixed point x_1. We now prove that $x = x^-$. To see this let $t \in [0,1]$ be fixed and choose $\epsilon' > \epsilon$ such that

$$\epsilon' < d^t(x^-, x'), \quad \forall t \in [0,1],\ x' \in \mathrm{Fix}(\phi). \tag{93}$$

Furthermore we define

$$T := \min\left\{ \min_{s'+it' \in Z_1} s', 0 \right\}. \tag{94}$$

Now let $s < T$. Since $u_1^\nu \to u_1$ uniformly on $\{s\} \times [0,1]$ and $d(u_1^\nu(s,t), x^-) \leq \epsilon$, it follows that for ν large enough

$$d(u_1(s,t), x^-) \leq d(u_1(s,t), u_1^\nu(s,t)) + d(u_1^\nu(s,t), x^-) < \epsilon'.$$

Since this holds for every $s < T$ we have

$$d(x, x^-) = \lim_{s \to -\infty} d(u_1(s,t), x^-) \leq \epsilon'$$

and by (93) it follows that $x = x^-$. This proves the assertion of the claim for $k = 1$.

Let now $k \geq 1$ and assume by induction that there exist an $\ell := \ell(k) \in \mathbb{N}$ such that for $1 \leq i \leq \ell$ there is an $x_i \in Fix(\phi)$, a Floer cylinder u_i connecting x_{i-1} with x_i, a finite subset $Z_i \subseteq \mathbb{R} \times [0,1)$, for $1 \leq i \leq \ell - 1$ there is a number $T_i \in \mathbb{R}$, and, passing to some subsequence, there are numbers $s_i^\nu, \nu \in \mathbb{N}, 1 \leq i \leq \ell$ such that conditions (86-90) and (92) hold. There are two cases.

Case 1 There is a $T_\ell \in \mathbb{R}$ such that, passing to some subsequence, condition (91) is satisfied.
Case 2 For every $T_\ell \in \mathbb{R}$ and passing to any subsequence, condition (91) is not satisfied.

If case 1 holds then we set $\ell(k+1) := \ell$ and the induction step holds. If case 2 holds then $\ell \geq k$ and for any $T_\ell \in \mathbb{R}$ for ν large enough there are numbers $s^\nu \geq T_\ell,\ t^\nu \in \mathbb{R}$ such that

$$d(u_\ell^\nu(s^\nu, t^\nu), x_\ell) \geq \epsilon.$$

On the other hand let $T_\ell > \max_{(s,t) \in Z_\ell} s$ be so large that $d(u_\ell(s,t), x_\ell) < \epsilon$ for $s \geq T_\ell,\ t \in \mathbb{R}$. Since the distance $d(u_\ell^\nu(T_\ell, t), u_\ell(T_\ell, t))$ converges to 0 uniformly in $t \in \mathbb{R}$, the distance $d(u_\ell^\nu(T_\ell, t), x_\ell)$ is less than ϵ for $t \in \mathbb{R}$ and ν large enough. It follows that, passing to a subsequence, for $\nu \in \mathbb{N}$ we can define the number

$$\mathbb{R} \ni s^\nu_{\ell+1} := \sup\{s' \geq T_\ell + s^\nu_\ell | d(u^\nu(s,t), x_\ell) < \epsilon,\ \forall (s,t) \in [s^\nu_\ell + T_\ell, s'] \times \mathbb{R}\}$$

By proposition 4.6 applied to $u^\nu_{\ell+1}(s,t) := u^\nu(s + s^\nu_{\ell+1}, t)$ there is a solution $u_{\ell+1}$ of Floer's equations (81), (82) and a finite subset $Z_{\ell+1} \subseteq \mathbb{R} \times [0,1)$ such that the assertions of proposition 4.6 are satisfied. In particular, condition (86) holds with ℓ replaced by $\ell + 1$. Passing to some subsequence, such that for every $z \in Z_{\ell+1}$ the mass $m(\ell+1, z) := m(\{u^\nu_{\ell+1}\}, z)$ exists, condition (87) with ℓ replaced by $\ell+1$ follows. Condition (88) and (89) for $\ell+1$ follow from the definition of $s^\nu_{\ell+1}$. Finally, for every $T'_\ell > T_\ell$ there is a $\nu_0 \in \mathbb{N}$ such that $d(u^\nu_\ell(s,t), x_\ell) < \epsilon$, for $(s,t) \in [T_\ell, T'_\ell] \times \mathbb{R}, \nu \geq \nu_0$. Therefore

$$s^\nu_{\ell+1} - s^\nu_\ell \to \infty, \nu \to \infty$$

and condition (90) is satisfied with ℓ replaced by $\ell + 1$. Since $\ell + 1 \geq k + 1$ condition (92) is void. Since

$$E(u_{\ell+1}) \leq E = \lim_{\nu \to \infty} E(u^\nu_{\ell+1}) < \infty$$

by proposition 1.14 the following limits exist uniformly in $t \in [0,1]$:

$$\begin{aligned} x_{\ell+1} &:= \lim_{s\to\infty} u_{\ell+1}(s,t) \in \mathrm{Fix}(\phi), \\ x &:= \lim_{s\to-\infty} u_{\ell+1}(s,t) \in \mathrm{Fix}(\phi), \\ &\lim_{s\to\pm} |\partial_s u_{\ell+1}(s,t)| = 0. \end{aligned}$$

Therefore $u_{\ell+1}$ is indeed a Floer cylinder connecting a fixed point x with a fixed point $x_{\ell+1}$. As for the case $k := 1$ it follows that $x = x_\ell$. This proves the induction step $k \to k+1$ and therefore claim 7.

Remark 4.11 Using the fact that for a smooth $v : S^2 \to M$ the integral $\int_{S^2} v^*\omega$ is invariant under homotopy, it can be shown, that the sequences $u^\nu_i, 1 \leq i \leq \ell$ do not bubble in $(-\infty, 0) \times \mathbb{R}$, i.e. that $Z_i \subseteq [0,\infty) \times [0,1)$ and therefore we could have set $T := 0$ in formula (94).

We now define

$$\hbar := \min\{\min_{t\in[0,1]} \hbar_S(J_t), \hbar_c(\{J_t\}_{t\in\mathbb{R}})\},$$

where $h_S(J_t)$ is the constant defined after proposition 1.18 and $\hbar_c(\{J_t\}_{t\in\mathbb{R}})$ is the constant defined after lemma 1.19.

Claim 8 *For $k \in \mathbb{N}$ let $\ell := \ell(k)$ be as in claim 7. Then*

$$\ell \leq \frac{m(1,\infty)}{\hbar} \tag{95}$$

Choosing $k > m(1,\infty)/\hbar$ claim 8 implies claim 6.
In order to prove claim 8 we use the same idea as in the proof of claim 5. By the quantization of energy assertion (ii) in lemma 1.19 and condition (87), on each cylinder energy of at least $\hbar$ gets lost. Since on the other hand we assume that the total energies $E(u^\nu)$ are bounded above, there can only be finitely many cylinders.

Proof of claim 8: Let $k \in \mathbb{N}$, $\ell := \ell(k)$ and

$$x_i \in \mathrm{Fix}(\phi),\ \ u_i \in \mathcal{M}(x_{i-1}, x_i, J., \phi),$$
$$Z_i \subseteq \mathbb{R} \times [0,1),\ \ |Z_i| < \infty, 1 \le i \le \ell,\ \ T_i \in \mathbb{R}, 1 \le i \le \ell - 1$$

and, passing to a suitable subsequence, let $s_i^\nu, \nu \in \mathbb{N}, 1 \le i \le N$

be such that conditions (86-90) and (92) hold.

Claim 9

$$m(1,\infty) \ge \sum_{2 \le i \le \ell} \big(E(u_i) + \sum_{z \in Z_i} m(i;z)\big) \tag{96}$$

Claim 10

$$E(u_i) + \sum_{z \in Z_i} m(i,z) \ge \hbar,\ \ 1 \le i \le \ell. \tag{97}$$

We will see that this corresponds to the stability condition in the definition of a stable Floer connecting orbit.
Inserting inequality (97) into inequality (96) proves claim (8) and therefore claim 6.

Proof of claim 9: Note that for $1 \le i \le \ell$ and $R \in \mathbb{R}$ we have

$$E(u_i^\nu; [R,\infty) \times [0,1]) = E(u_{i+1}^\nu; [R - s_{i+1}^\nu + s_i^\nu, \infty) \times [0,1]).$$

By condition (90), passing to a subsequence if necessary, we get for every R and $T \in \mathbb{R}$:

$$\begin{aligned}
\lim_{\nu\to\infty} E(u_i^\nu; [R,\infty)) &= \lim_{\nu\to\infty} E(u_{i+1}^\nu; [R - s_{i+1}^\nu + s_i^\nu, \infty)) \\
&\ge \lim_{\nu\to\infty} E(u_{i+1}^\nu; [T,\infty)) \\
&= \lim_{\nu\to\infty} E(u_{i+1}^\nu) - \lim_{\nu\to\infty} E(u_{i+1}^\nu; (-\infty, T))
\end{aligned}$$

Taking the limit $T \to -\infty$ we see:

$$\begin{aligned}
&\lim\nolimits_{\nu\to\infty} E(u_i^\nu; [R,\infty)) \\
&\quad \ge E - m(i+1; -\infty) \\
&\quad = E(u_{i+1}) + \textstyle\sum_{z \in Z_{i+1}} m(i+1; z) + m(i+1; \infty), \quad \forall R \in \mathbb{R},
\end{aligned}$$

and sending R to infinity it follows that

$$m(i;\infty) \geq E(u_{i+1}) + \sum_{z \in Z_{i+1}} m(i+1;z) + m(i+1;\infty) \qquad (98)$$

Inequality (96) follows by induction. This proves claim 9.

Proof of claim 10: Let $1 \leq i \leq \ell$. If Z_i is nonempty then the claim follows by condition (87). So assume that Z_i is empty. This implies that $u_i^\nu \to u_i$, u.c.s. on the whole of $\mathbb{R}^2$. Now let $t^\nu \in [0,1]$ be a sequence as in (89) such that $d(u_i^\nu(0,t^\nu), x_{i-1}(t^\nu)) \geq \epsilon$. We pass to a subsequence that converges to some $t_0 \in [0,1]$. Since the functions u_i^ν converge C^0 to u_i on the compact set $\{0\} \times [0,1]$ the distance $d(u_i(0,t_0), x_{i-1}(t_0))$ is $\geq \epsilon$. Therefore $\partial_s u$ cannot be constant, so the energy $E(u_i)$ is positive. Now the claim follows from the quantization of energy assertion (ii) in lemma 1.19. This proves claim 10.

Part II: construction of the branches $T_{i\alpha},\ iE\alpha,\ i \in T_0,\ \alpha \in T \setminus T_0$:

Let now $\ell \in \mathbb{N}$,

$$x_i \in \mathrm{Fix}(\phi),\ \ u_i \in \mathcal{M}(x_{i-1}, x_i, J., \phi),$$
$$Z := Z_i \subseteq \mathbb{R} \times [0,1),\ \ |Z_i| < \infty,\ \ T_i \in \mathbb{R},$$

and, passing to a suitable subsequence, let $s_i^\nu \in \mathbb{R},\ \ \nu \in \mathbb{N},\ \ 1 \leq i \leq \ell$

be such that conditions (86-91) hold. It follows from (91) that $x_\ell = x^+$. We define $T_0 := T_0^\ell = (1, \ldots, \ell)$ with the edge relation $E_0 := E_0^\ell$ defined as in section 2.

Let $i \in T_0$ and $z_1 := s_1 + it_1 \in Z_i$. In order to simplify notation, we abbreviate

$$v^\nu := u_i^\nu \text{ and } v := u_i.$$

We define the integer $j_1 := 0$ and for $s + it \in \mathbb{C} \cong \mathbb{R}^2$, if the limits exist, $m_0(s+it) := m_{J.}(\{v^\nu\}, s+it)$.

Claim 11 *There is an integer $N \in \mathbb{N}$, a tuple*

$$\mathbf{v} = (v_1, \ldots, v_N; j_2, \ldots, j_N; z_2, \ldots, z_N),$$

consisting of J_{t_1}-holomorphic spheres $v_i : S^2 \to M,\ 1 \leq i \leq N$, positive integers $j_i < i,\ 2 \leq i \leq N$ and complex numbers $z_i \in \mathbb{C}$ with $|z_i| \leq 1,\ 2 \leq i \leq N$, finite subsets $Z_i^! \subseteq \overline{B}_1, 1 \leq i \leq N$, and, passing to some subsequence, complex automorphisms $\psi_i^\nu \in Aut(\mathbb{C})$ of the form $\psi_i^\nu(z) = \epsilon_i^\nu z + z_i^\nu$ with $\epsilon_i^\nu \in \mathbb{R},\ \nu \in \mathbb{N},\ 1 \leq i \leq N$, such that the following conditions are satisfied:

(i)(Map) $v_i^\nu := v^\nu \circ \psi_i^\nu \to v_i$, *u.c.s. on* $\mathbb{C} \setminus Z_i'$, $1 \le i \le N$.

(ii)(Stability) *If* v_i *is a constant function then* $\# Z_i' \ge 2$, $1 \le i \le N$.

(iii)(Energy) *For* $1 \le i \le N$ *and* $(s,t) = z \in Z_i'$ *the number*

$$m_i(z) := \lim_{\epsilon \to 0} \lim_{\nu \to \infty} E_{J_{t_1}}(v_i^\nu; B_\epsilon(z))$$

exists and is greater than or equal to $\hbar_S(J_{t_1})$. *Furthermore* $z_i \in Z'_{j_i}$ *for* $2 \le i \le N$ *and for* $1 \le i \le N$ *we have*

$$m_{j_i}(z_i) = E(v_i) + \sum_{z \in Z_i'} m_i(z).$$

(iv)(Nodal points distinct) *If* $j_i = j_{i'}$ *for* $i \ne i'$ *then* $z_i \ne z_{i'}$, $2 \le i, i' \le N$.

(v)(Connectedness) $v_1(\infty) = v(s_1, t_1)$
Furthermore $v_{j_i}(z_i) = v_i(\infty)$ *for* $2 \le i \le N$.

(vi)(Rescaling) $\psi_j^\nu \to s_1 + it_1$, *u.c.s. on* $\mathbb{C}$ *for* $1 \le j \le N$. *Furthermore* $\psi_{j_i i}^\nu := (\psi_{j_i}^\nu)^{-1} \circ \psi_i^\nu \to z_i$, *u.c.s. on* $\mathbb{C}$, *for* $2 \le i \le N$.

(vii)(All bubble points resolved) $Z_i' = \{z_k | i < k \le N,\ j_k = i\}$ *for* $1 \le i \le N$.

Proof: As the proof of claim 6 the proof of this claim is split into two parts: In the first part we will show a modified version of claim 11, using induction. In the second part we will show that the induction terminates.

Claim 12 *For every* $k \in \mathbb{N}$ *there is an integer* $N := N(k)$, *a tuple* $\mathbf{v} = (v_1, \ldots, v_N; j_2, \ldots, j_N; z_2, \ldots, z_N)$, *consisting of* J_{t_1}*-holomorphic spheres* $v_i : S^2 \to M$, $1 \le i \le N$, *positive integers* $j_i < i$, $2 \le i \le N$ *and complex numbers* $z_i \in \mathbb{C}$ *with* $|z_i| \le 1$, $2 \le i \le N$, *finite subsets* $Z_i' \subseteq \overline{B}_1, 1 \le i \le N$, *and, passing to some subsequence, complex automorphisms* $\psi_i^\nu \in Aut(\mathbb{C})$ *of the form* $\psi_i^\nu(z) = \epsilon_i^\nu z + z_i^\nu$ *with* $\epsilon_i^\nu \in \mathbb{R}$, $\nu \in \mathbb{N}$, $1 \le i \le N$, *such that (i-vi) are satisfied and such that the following holds.*

$$\text{If } N < k \text{ then (vii) holds, i.e. } Z_i' = \{z_k | i < k \le N,\ j_k = i\},\ 1 \le i \le N. \tag{99}$$

Proof: We prove this claim by induction over k. In each step we will apply the soft rescaling proposition 4.8, using Gromov's trick. Now let $k=1$, set $N := N(1) := 1$ and let $r>0$ be so small that $\overline{B_r(z_1)} \cap \bigcup_{z\in Z}[z] = \{z_1\}$. We define $K := \overline{B_r(z_1)}$, $\tilde{M}, \tilde{\omega}$ and $\tilde{J}$ as in (30) and (31) with $H_t = 0$ and

$$\tilde{v}^\nu : K \to \tilde{M}, \ \ z \mapsto (z, v^\nu(z)),$$
$$\tilde{v} : K \to \tilde{M}, \ \ z \mapsto (z, v(z)).$$

Now we apply proposition 4.8 with

$$M, \omega, J^\nu = J, u^\nu, u, z_0 \text{ replaced by } \tilde{M}, \tilde{\omega}, \tilde{J}, \tilde{v}^\nu, \tilde{v}, z_1$$

to get rescalings $\psi_1^\nu \in \mathrm{Aut}(\mathbb{C})$, a finite subset $Z_1' \subseteq \overline{B_1} \subseteq \mathbb{C}$ and a $\tilde{J}$-holomorphic $w_1 : S^2 \to M$ such that the assertions of proposition 4.8 hold. In particular $w_1^\nu := \tilde{v}^\nu \circ \psi_1^\nu$ converges to w_1, u.c.s. on $\mathbb{C} \setminus Z_1'$. By remark 1.21 w_1 has the form

$$w_1(z) = (z, v_1(z)),$$

where $v_1 : S^2 \to M$ is a J_{t_1}-holomorphic sphere, and $v_1^\nu := v^\nu \circ \psi_1^\nu$ converges to v_1, u.c.s. on $\mathbb{C} \setminus Z_1'$. This proves (i) (with $N=1$). Assertions (ii) and (vi) follow directly from proposition 4.8 and (iv) and (99) are void. By proposition 4.9 we have

$$(z_1, v_1(\infty)) = w_1(\infty) = \tilde{v}(z_1) = (z_1, v(z_1)),$$

and therefore (v) is satisfied. We check the energy assertion (iii): Let $z \in Z_1'$, then it follows by remark 1.20 that the mass

$$m_1(z) := \lim_{\epsilon\to 0} \lim_{\nu\to\infty} E_{J_{t_1}}(v_1^\nu; B_\epsilon(z)) \text{ exists and}$$

$$m_1(z) = m_{\tilde{J}}(\{w_1^\nu\}, z). \tag{100}$$

By proposition 4.8 there is a sequence $z^\nu \in \mathbb{C}$, converging to z, such that

$$|dw_1^\nu(z^\nu)| \to \infty, \nu \to \infty.$$

Therefore by remark 4.5 it follows that $m_1(z) \geq \hbar_S(J_{t_1})$. Furthermore Proposition 4.8 implies that

$$\begin{aligned} m_0(z_1) := m_{J_\cdot}(\{v^\nu\}, z_1) &= m_{\tilde{J}}(\{\tilde{v}^\nu\}, z_1) \\ &= E_{\tilde{J}}(w_1) + \sum_{z\in Z_1'} m_{\tilde{J}}(w_1^\nu, z) \\ &= E_{J_{t_1}}(v_1) + \sum_{z\in Z_1'} m_1(z) \end{aligned}$$

This proves the energy condition (iii) and therefore the statement of claim 12 for $k = 1$.
We show the induction step $k \to k+1$. Let $k \geq 1$ and suppose, by induction, that there is an $N := N(k) \in \mathbb{N}$ and for $i \leq N$ there are v_i, j_i, z_i, Z'_i and, passing to a suitable subsequence, there exist ψ_i^ν as in claim 12, such that conditions (i)-(vi) and (99) are satisfied. There are two cases:

Case 1 (vii) is also satisfied.
Case 2 (vii) is not satisfied.

If case 1 holds then we set $N(k+1) := N$ and the induction step holds. If case 2 holds then $N \geq k$ and there is a $j \in \{1, \dots, N\}$, such that $Z'_j \underset{\neq}{\supset} \{z_i | j < i \leq N, j_i = j\}$. We set $N(k+1) := N+1$ and choose any element

$$z_{N+1} \in Z'_j \setminus \{z_i | j < i \leq N, j_i = j\}.$$

In order to apply proposition 4.8 we again use Gromov's trick. Let therefore $r > 0$ be so small that $B_r(z_{N+1}) \cap Z'_j = \{z_{N+1}\}$. We define $K := \overline{B_r(z_{N+1})}$ and $\tilde{M}, \tilde{\omega}, \tilde{J}$ as in (30), (31) with $H_t = 0$ and $w_j^\nu := \tilde{v}^\nu \circ \psi_j^\nu : K \to \tilde{M}$. By condition (i) and (vi) with $N = N(k)$ we have

$$w_j^\nu \to w_j := (z_1, v_j), \nu \to \infty, \text{ u.c.s. on } B_r(z_{N+1}) \setminus \{z_{N+1}\}$$

and therefore hypothesis (a) of proposition 4.8 with

$$M, \omega, J, u^\nu, u, z_0 \text{ replaced by } \tilde{M}, \tilde{\omega}, \tilde{J}, w_j^\nu, w_j, z_{N+1}$$

is satisfied. Since by remark 1.20

$$m_{\tilde{J}}(w_j^\nu; z_{N+1}) = m_{J_{t_1}}(v^\nu \circ \psi_j^\nu; z_{N+1})$$

hypothesis (b) follows from the energy condition (iii) with $N = N(k)$. Therefore by proposition 4.8 there is a subsequence and there are $\psi^\nu \in \mathrm{Aut}(\mathbb{C})$, a $\tilde{J}$-holomorphic sphere $w_{N+1} : S^2 \to \tilde{M}$ and a finite set Z'_{N+1}, such that the assertions of proposition 4.8 hold. In particular, $w_j^\nu \circ \psi^\nu = \tilde{u}^\nu \circ \psi_j^\nu \circ \psi^\nu$ converges to w_{N+1}, u.c.s. on $\mathbb{C} \setminus Z'_{N+1}$. By remark 1.21 w_{N+1} has the form

$$w_{N+1}(z) = (z, v_{N+1}(z)),$$

where $v_{N+1} : S^2 \to M$ is a J_{t_1}-holomorphic sphere, and $v_{N+1}^\nu := v^\nu \circ \psi_j^\nu \circ \psi^\nu$ converges to v_{N+1}, u.c.s. on $\mathbb{C} \setminus Z'_{N+1}$. Defining $\psi_{N+1}^\nu := \psi_j^\nu \circ \psi^\nu$ this shows condition (i). Furthermore we set $j_{N+1} := j$. Then conditions (ii),(iii) and (vi) with N replaced by $N+1$ follow from proposition 4.8, remark 1.20 and remark 4.5, (iv) is satisfied by the choice of z_{N+1} and (v) follows from 4.9. This completes the induction and therefore proves claim 12.

Now let $k \in \mathbb{N}$, abbreviate $N := N(k)$, for $i \le N$ let v_i, j_i, z_i, Z_i' and, passing to a suitable subsequence, let $\psi_i^\nu,\ \nu \in \mathbb{N}, 1 \le i \le N$, be as in claim 12, such that conditions (i)-(vi) and (99) are satisfied. For $1 \le \ell \le N$ the integers $1, \ldots, \ell$ are the vertices of the tree $T^\ell := \{1, \ldots, \ell\}$ with edge relation E^ℓ given by

$$iE^\ell i' :\Longleftrightarrow\ i = j_{i'} \text{ or } i' = j_i.$$

We want to show that the number N of vertices of T^N is bounded above by some constant. The idea is the same as in the proof of claim 8 and in the proof of claim 7. We will define weights on each vertex $j \in T^N$, using the energy of v_j and masses of bubble points $z \in Z_j$. By the energy condition (iii) for each the mass $m_i(z)$ is bounded below by $\hbar_S(J_{t_1}) > 0$. Since the total weight of the branch T^N is $m(\{v^\nu\}, z_1) < \infty$, it will follow from the stability condition (ii) that N is bounded above by some constant.

Let $1 \le \ell \le N$. On the tree T^ℓ we define a 1-labelling Λ^ℓ by putting 1 marked point on the vertex 1, i.e. $\Lambda_1^\ell := \{1\},\ \Lambda_j^\ell := \emptyset,\ 2 \le j \le \ell$. Furthermore we define

$$Z_{j;\ell} := Z_j' \setminus \{z_i | j < i \le \ell, j_i = j\},$$
$$\text{weights } m(j;\ell) := E(v_j) + \textstyle\sum_{z \in Z_{j;\ell}} m_j(z),\ \ 1 \le j \le \ell,$$
$$\text{the total weight } W(T^\ell) := \textstyle\sum_{j=1}^{\ell} m(j;\ell) \text{ and we abbreviate}$$
$$\hbar_S := \hbar_S(J_{t_1}).$$

Think of the weight $m(j;\ell)$ of the vertex j as the energy of v_j together with the energy of those of its bubble points that have not yet been blown up into a bubble v_i with $j < i \le \ell$. Note that $m(j;\ell) = 0$ only when v_j is constant and all its bubble points have been resolved.

Claim 13 *The number of vertices N of the tree T^N satisfies*

$$N \le \frac{2m(\{v^\nu\}, z_1)}{\hbar_S} - 1.$$

For the proof of this claim we need the following.

Claim 14 (T^N, E^N, Λ^N) *with the above weights is a stable weighted tree.*

Claim 15

$$W(T^N) = m(\{v^\nu\}, z_1) \tag{101}$$

Since by the energy condition (iii) $\hbar_S$ is less than or equal to the minimum of the nonzero weights $m(j;N),\ 1 \le j \le N$, it follows from claim 14 and remark 2.1 that

$$N-1=e(T_N)\leq\frac{2W(T^N)}{\hbar_S}-2 \tag{102}$$

Inserting the expression for $W(T^N)$ in equation (101) into (102) proves claim 13 and therefore claim 11.

Proof of claim 14: We have to show: If $m(j;N)=0$, then $\#\{z_i|iE^Nj\}+\#\Lambda_j^N\geq 3,\ 1\leq j\leq N$. Case $j=1$: Suppose that $m(1;N)=0$, then $Z_{1;N}$ is empty and therefore

$$Z_1'=\{z_i|j_i=1,\ 1<i\leq N\}=\{z_i|iE^N1\}.$$

It follows also from $m(1;N)=0$ that $E(v_1)=0$. By (ii) this implies

$$\#\Lambda_1^N+\#\{z_i|iE^N1\}=1+\#Z_1'\geq 3.$$

Now let $j\geq 2$ and assume that $m(j;N)=0$. Then $Z_{j;N}$ is empty and $Z_j'=\{z_i|j_i=j,\ j<i\leq N\}$. Since

$$\{z_i|iE^Nj\}=\{z_i|j=j_i,\ i<j\leq N\}\dot\cup\{\infty\}=Z_j'\dot\cup\{\infty\}$$

and $E(v_j)=0$ it follows by (ii) that $\#\{z_i|iE^Nj\}\geq 3$. Therefore the labelled weighted tree T^N is indeed stable. This proves the claim.

Proof of claim 15: We show by induction over ℓ, that for $1\leq\ell\leq N$

$$W(T^\ell):=\sum_{j=1}^{\ell}m(j;\ell)=m(\{v^\nu\};z_1) \tag{103}$$

This is satisfied for $\ell=1$, since by (iii) $m(\{v^\nu\};z_1)=E(v_1)+\sum_{z\in Z_1}m_1(z)=m(1;1)$. Now suppose (103) has been proved for some $\ell\in\{1,\dots,N\}$. If $\ell=N$ then we are done, so assume that $\ell<N$. Note that $m(j;\ell+1)=m(j;\ell)$ whenever $j\neq j_{\ell+1}$ and $j\leq\ell$, while

$$m(j_{\ell+1};\ell+1)=m(j_{\ell+1};\ell)-m_{j_{\ell+1}}(z_{\ell+1}),$$

and, by (iii) with $i=\ell+1$,

$$m(\ell+1;\ell+1)=m_{j_{\ell+1}}(z_{\ell+1}).$$

Hence (103) holds with ℓ replaced by $\ell+1$. Thus we have proved (103) for every $\ell\in\{1,\dots,N\}$. This proves claim 15.

Now let $i\in T_0=(1,\dots,\ell),\ z=z_1=s_1+it_1\in Z_i$, where $Z_i\subseteq\mathbb{R}\times[0,1)$ is the finite subset we constructed in part I, and let $j_k,z_k,Z_k,v_k,\ 1\leq k\leq N$ and, passing to a suitable subsequence, $\{\psi_k^\nu\}_{\nu\in\mathbb{N}},1\leq k\leq N$ be such that (i)-(vii) are satisfied. Recall that we have abbreviated $v^\nu := u_i^\nu$. We define the tree $T_{i,z} := \{(i,z)\} \times \{1,\dots,N\} \cup \{i\}$ with the edge relation

$$E_{i,z} := \{(k, j_k), (j_k, k) | 2 \le k \le N\} \cup \{(i, (i,z,1)), ((i,z,1), i).\}$$

Let $i \neq \alpha = (i,z,k) \in T_{i,z}$. We define the predecessor

$$\pi(\alpha) := \begin{cases} (i,z,j_k), & \text{if } k \ge 2, \\ i, & \text{if } k = 1 \end{cases}$$

and the nodal points $z_{\alpha\pi(\alpha)} := \infty$ and $z_{\pi(\alpha)\alpha} := z_k$, and denote the set of nodal points at the vertex α by

$$Z_\alpha := \{z_{\alpha\beta} | \beta E_{i,z} \alpha\}.$$

Furthermore we set $u_\alpha := v_k : S^2 \to M$. Recalling that in part I we have constructed time shifts s_i^ν and defined $\phi_i^\nu(s,t) := (s + s_i^\nu, t)$ we define for $i \neq \alpha \in T_{i,z}$ the rescalings

$$\phi_\alpha^\nu := \phi_i^\nu \circ \psi_k^\nu$$

and extend these functions to all of S^2 by setting $\phi_\alpha^\nu(\infty) := \infty$. Then $\phi_\alpha^\nu \in PSl(2,\mathbb{C})$. In order to prove the stability condition in definition 2.4 we need the following

Remark 4.12 Let $i \neq \alpha = (i,z,k) \in T_{i,z}$. Then $Z_\alpha = Z_k' \dot\cup \{\infty\}$. If $u_\alpha = v_k$ is a constant function then by (ii) it follows that

$$\# Z_\alpha = \# Z_k' + 1 \ge 3.$$

If the limits occurring in its definition exist, we denote for $\alpha \in T_{i,z}$

$$m_\alpha(z) := \begin{cases} m_{J.}(\{u_i^\nu\}, z), & \text{for } z \in \mathbb{R}^2, \text{ if } \alpha = i, \\ m_{J_{t_1}}(\{v_k^\nu\}, z), & \text{for } z \in S^2, \text{ if } \alpha = (i,z,k) \in T_{i,z}. \end{cases}$$

In order to prove the (energy) condition in definition 3.1 we will need the following

Claim 16 *Let $i \neq \alpha = (i,z,k) \in T_{i,z}$, write $\beta := \pi(\alpha)$ and denote by $T_{\beta\alpha}$ the branch of $T_{i,z}$ we obtain by removing the edge $\beta E_{i,z} \alpha$ and which contains α, i.e.*

$$T_{\beta\alpha} := \{\gamma \in T_{i,z} | \alpha \in [\beta, \gamma]\}.$$

Then

$$m_\beta(z_{\beta\alpha}) = \sum_{\gamma \in T_{\beta\alpha}} E(u_\gamma) \tag{104}$$

Proof: We prove this formula by induction over the number of elements of $T_{\beta\alpha}$. If $T_{\beta\alpha}$ contains only the element α then by condition (vii) Z'_k is empty and the claim follows immediately from condition (iii). Assume now that we have proved formula (104) for $\#T_{\beta\alpha} \leq m$. Suppose that $\#T_{\beta\alpha} = m+1$ and let $\gamma_1, \dots, \gamma_s$ be the distinct elements of $\{\gamma \in T_{i,z} | \alpha = \pi(\gamma)\}$. Then $\{\gamma_1, \dots, \gamma_s\} \cup \{\beta\}$ is the set of vertices adjacent to $\alpha = (i, z, k)$. Note that

$$T_{\beta\alpha} = \{\alpha\} \cup \bigcup_{1 \leq r \leq s} T_{\alpha\gamma_r} \tag{105}$$

and therefore $\#T_{\gamma_r} \leq m, \ 1 \leq r \leq s$. Hence, by the induction hypothesis,

$$m_\alpha(z_{\alpha\gamma_r}) = \sum_{\gamma \in T_{\alpha\gamma_r}} E(u_\gamma), \quad 1 \leq r \leq s.$$

Moreover, it follows from (iii) that

$$m_\beta(z_{\beta\alpha}) = E(u_\alpha) + \sum_{r=1}^{s} m_\alpha(z_{\alpha\gamma_r}).$$

Inserting the previous formula and using (105) we obtain (104). This proves the claim.

Now we define the tree

$$\begin{aligned} T &:= \bigcup_{i \in T_0, z \in Z_i} T_{i,z} \text{ with the edge relation} \\ E &:= E_0 \cup \bigcup_{i \in T_0, z \in Z_i} E_{i,z} \end{aligned} \tag{106}$$

and introduce the notation $z_{i,i'} := \pm\infty$, for $i' = i \pm 1, \ i, i' \in T_0$.

Part III: Proof, that $(\mathbf{u}, \mathbf{z})$ is a stable Floer connecting orbit and that, passing to some subsequence, $u^\nu \to (\mathbf{u}, \mathbf{z}), \ \nu \to \infty$:
The property (i) in definition 2.4 follows from (v) in part II and from the construction in part I. Property (ii) follows from (iv) in part II and the construction of the tree T. The assertion for $\alpha \in T_0$ in (iii) follows from the claim 10 and the assertion for $\alpha \in T \setminus T_0$ follows from remark 4.12.
We now verify the conditions of definition 3.1:

(0): By construction $z_{\alpha\pi(\alpha)} = \infty$, and $\phi_\alpha^\nu(z_{\alpha\pi(\alpha)}) = \infty, \ \nu \in \mathbb{N}, \ \alpha \in T \setminus T_0$.

(Map) follows from condition (86) in part I (for $\alpha \in T_0$) and from condition (i) in part II (for $\alpha \in T \setminus T_0$).

The first assertion in the **(Rescaling)** condition is satisfied by (90). The other assertions follow from (vi) in part II.

The **(Marked points)** condition is void.

(Energy) This condition is equivalent to the following conditions:

$$m(1,-\infty) = 0, \tag{107}$$
$$m(\ell,\infty) = 0, \tag{108}$$
$$m(i,\infty) = E(u_{i+1}) + \sum_{z\in Z_{i+1}} m(i+1,z) + m(i+1,\infty), \;\; 1 \le i \le \ell-1, \tag{109}$$
$$m(\{u^\nu_{\pi(\alpha)}\}_{\nu\in\mathbb{N}}, z_{\pi(\alpha)\alpha}) = E(T_{\pi(\alpha)\alpha}) := \sum_{\beta\in T_{\pi(\alpha)\alpha}} E(u_\beta), \quad \alpha \in T\setminus T_0. \tag{110}$$

Proof of (107): Let $a < b < 0$ and $\nu \in \mathbb{N}$. Recall that we have chosen $\epsilon > 0$ such that (84) and (85) are satisfied. Therefore, it follows by lemma 1.22 that the energy of u^ν_1 on $[a,b]\times[0,1]$ is given by

$$E(u^\nu_1;[a,b]) = \mathcal{A}_{x^-}(u^\nu_1(a,.)) - \mathcal{A}_{x^-}(u^\nu_1(b,.)).$$

Abbreviating $\mathcal{A} := \mathcal{A}_{x^-}$, by lemma 1.23 there is a constant $C > 0$ such that

$$|\mathcal{A}(u^\nu_1(a,.)| \le C \max_{t\in\mathbb{R}} |\partial_s u^\nu_1(a,t)|^2$$

Sending $a \to -\infty$ it follows that

$$E(u^\nu_1;(-\infty,b]) = -\mathcal{A}(u^\nu_1(b,.)).$$

Now let

$$b < \min\left\{\min_{s+it\in Z_1} s, 0\right\}, \text{ then}$$

$$|\mathcal{A}(u^\nu_1(b,.))| \le C \max_{t\in\mathbb{R}} |\partial_s u^\nu_1(b,t)|^2 \to C\max_{t\in\mathbb{R}} |\partial_s u_1(b,t)|^2, \;\; \nu\to\infty,$$

and this converges to 0 as $b \to -\infty$. It follows that

$$\begin{aligned} m(\{u^\nu_1\},-\infty) &= \lim_{b\to-\infty}\lim_{\nu\to\infty} E(u^\nu_1;(-\infty,b]) \\ &= \lim_{b\to-\infty}\lim_{\nu\to\infty} -\mathcal{A}(u^\nu_1(b,.)) = 0. \end{aligned}$$

This proves (107).
(108) is shown analogously. We show (109): Let $1 \leq i \leq \ell - 1$. By (98) we already know that the left hand side of (109) is greater or equal to the right hand side. So we have to show that it is also less than or equal. For $i \in T_0$ let $T_i \in \mathbb{R}$ be as in claim 6. Let

$$T > \max\{\max_{s+it \in Z_i} s, T_i\}, \quad -T' < \min\{\min_{s+it \in Z_{i+1}} s, 0\}$$

and choose ν so large that $T \leq s^\nu_{i+1} - s^\nu_i - T'$. By condition (88) the distance $d(u^\nu_i(s,t), x_i)$ is less than or equal to ϵ for $(s,t) \in [T_i, s^\nu_{i+1} - s^\nu_i] \times \mathbb{R}$. Therefore by lemma 1.22 we have

$$\begin{aligned}
&E(u^\nu_i; (T, \infty)) \\
&= E(u^\nu_i; [T, s^\nu_{i+1} - s^\nu_i - T']) + E(u^\nu_i; [s^\nu_{i+1} - s^\nu_i - T', \infty)) \\
&= \mathcal{A}_{x_i}(u^\nu_i(T, .)) - \mathcal{A}_{x_i}(u^\nu_i(s^\nu_{i+1} - s^\nu_i - T', .)) + E(u^\nu_{i+1}; [-T', \infty)),
\end{aligned}$$

and by lemma 1.23, using that $u^\nu_i \to u_i$, u.c.s. on $\{T\} \times [0,1]$ and $u^\nu_{i+1} \to u_{i+1}$, u.c.s. on $\{-T'\} \times [0,1]$ we have

$$\begin{aligned}
&\lim_{\nu\to\infty} E(u^\nu_i; (T, \infty)) \\
&\leq \limsup_{\nu\to\infty} \left(|\mathcal{A}_{x_i}(u^\nu_i(T, .))| + |\mathcal{A}_{x_i}(u^\nu_i(s^\nu_{i+1} - s^\nu_i - T', .))|\right) \\
&\quad + \lim_{\nu\to\infty} E(u^\nu_{i+1}; [-T', \infty)) \\
&\leq C\left(\max_{t\in[0,1]} |\partial_t u_i(T, .)|\right)^2 + C\left(\max_{t\in[0,1]} |\partial_t u_{i+1}(-T', .)|\right)^2 \\
&\quad + \lim_{\nu\to\infty} E(u^\nu_{i+1}; [-T', \infty)).
\end{aligned}$$

By sending $T, T' \to \infty$ we get

$$\begin{aligned}
m(i, \infty) &\leq \lim_{T'\to\infty} \lim_{\nu\to\infty} E(u^\nu_{i+1}; [-T', \infty)) \\
&= E(u_{i+1}) + \sum_{z \in Z_{i+1}} m(i+1, z) + m(i+1, \infty), \ \ 1 \leq i \leq \ell - 1,
\end{aligned}$$

This proves (109). Condition (110) is satisfied by claim 16 and therefore the (Energy) condition of Floer-Gromov-convergence is satisfied. This proves theorem 4.1 in the case $n = 0$, i.e. if there are no marked points.

If there are marked points, we need the following

Lemma 4.13 *Let $x^-, x^+ \in \mathcal{P}(H, \phi)$, $(u^\nu, z^\nu_1, \ldots, z^\nu_n)$ be a sequence of Floer cylinders connecting x^- with x^+ with marked points that Floer-Gromov-converges to a stable Floer connecting orbit $(\mathbf{u}, \mathbf{z})$ via ϕ^ν_α, $\alpha \in T$ and let $\beta \in T \setminus T_0$ be such that, denoting $\alpha := \pi(\beta)$*

(a) *$(\phi^\nu_\alpha)^{-1} \circ \psi^\nu$ converges to $z_{\alpha\beta}$ u.c.s. on $\mathbb{C}$.*
(b) *$(\phi^\nu_\beta)^{-1} \circ \psi^\nu$ converges to $z_{\beta\alpha}$ u.c.s. on $S^2 \setminus \{0\}$.*

Then the following holds:
(i) *Abbreviating* $E := E_{t_\beta}$*, we have for every* $r > 0$

$$\lim_{\nu\to\infty} E(u^\nu \circ \psi^\nu; B_r) = E(T_{\alpha\beta}) = \sum_{\gamma\in T_{\alpha\beta}} E(u_\gamma) \tag{111}$$

(ii) *Passing to some subsequence* $u^\nu \circ \psi^\nu$ *converges to* $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$*, u.c.s. on* $\mathbb{C}\setminus\{0\}$*.*

(iii) *For* $1 \le j \le n$ *abbreviate* $i := root(\alpha_j)$ *and let* $z_j'^\nu \in [z_j^\nu]$ *with* $z_j'^\nu - s_i^\nu \to z_{i,j}, \nu \to \infty$*. Then*

$$(\psi^\nu)^{-1}(z_j'^\nu) \to \begin{cases} 0, & \text{if } \alpha_j \in T_{\alpha\beta} \\ \infty, & \text{if } \alpha_j \in T_{\beta\alpha} \end{cases}$$

Proof: Recall the notation $u_\alpha^\nu := u^\nu \circ \phi_\alpha^\nu$ and $u_\beta^\nu := u^\nu \circ \phi_\beta^\nu$. We write $u_\gamma^\nu := u^\nu \circ \psi^\nu$. We prove (i): Let $r > 0$ be fixed. Then by (a) we have for every $\epsilon > 0$ for ν sufficiently large

$$\psi^\nu(B_r) \subseteq \phi_\alpha^\nu(B_\epsilon(z_{\alpha\beta})) \quad \text{and}$$

$$E(u_\gamma^\nu; B_r) = E(u^\nu; \psi^\nu(B_r)) \le E(u^\nu; \phi_\alpha^\nu(B_\epsilon(z_{\alpha\beta}))) = E(u_\alpha^\nu; B_\epsilon(z_{\alpha\beta})).$$

Letting $\nu \to \infty$ and then $\epsilon \to 0$ we see

$$\limsup_{\nu\to\infty} E(u_\gamma^\nu; B_r) \le m(\{u_\alpha\}, z_{\alpha\beta}) = E(T_{\alpha\beta}) \tag{112}$$

The last identity follows from the (Energy) condition in the definition of Floer-Gromov-convergence.
On the other hand, using an analogous argument, it follows by (b) that for $\epsilon > 0$ and ν sufficiently large

$$E(u_\gamma^\nu; B_r) \ge E(u_\beta^\nu; S^2 \setminus B_\epsilon(z_{\beta\alpha})) \quad \text{and therefore}$$

$$\begin{aligned} \liminf_{\nu\to\infty} E(u_\gamma^\nu; B_r) &\ge \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(u_\beta^\nu; S^2 \setminus B_\epsilon(z_{\beta\alpha})) \\ &= E(T_{\alpha\beta}) \end{aligned} \tag{113}$$

Inequalities (112) and (113) prove the energy identity (111).
We prove (ii): Let $K \subseteq \mathbb{C}\setminus\{0\}$ be a compact subset and let $U \subseteq \mathbb{C}$ be a bounded open subset that contains K. From the energy identity follows that

$$\lim_{\nu\to\infty} E(u_\gamma^\nu; K) = 0. \tag{114}$$

We define $\tilde{M} := \overline{U} \times M$ and

$$\tilde{\omega} \in \Omega^2(\tilde{M}), \tilde{J} \in \mathrm{End}(T\tilde{M}) \text{ and } \tilde{u}^\nu \in C^\infty(U, \tilde{M})$$

as in (30),(31) and (34), with u replaced by u^ν. Furthermore we define $i := \mathrm{root}(\beta)$ and

$$w_\gamma^\nu : \overline{U} \to \tilde{M}, w_\gamma^\nu(z) = ((\phi_i^\nu)^{-1} \circ \psi^\nu(z), u^\nu \circ \psi^\nu(z)).$$

For every measurable subset $A \subseteq \mathbb{C}$ we have

$$\begin{aligned} E_{\tilde{J}}(w_\gamma^\nu; A) &= E_{\tilde{J}}(\tilde{u}^\nu; \psi^\nu(A)) \\ &= E_{J.H.}(u^\nu; \psi^\nu(A)) + |\psi^\nu(A)|. \end{aligned} \tag{115}$$

Therefore, sending $\nu \to \infty$, we see

$$\begin{aligned} \lim_{\nu\to\infty} E_{\tilde{J}}(w_\gamma^\nu; K) &= \lim_{\nu\to\infty} E(u^\nu; \psi^\nu(K)) \\ &= \lim_{\nu\to\infty} E(u^\nu \circ \psi^\nu; K) = 0, \end{aligned} \tag{116}$$

where in the last step we have used (114). We claim that

$$\sup\{|dw_\gamma^\nu(z)||z \in K,\ \nu \in \mathbb{N}\} < \infty \tag{117}$$

This is seen as follows, using lemma 4.3: Assume that the claim were wrong, then, passing to some subsequence, there is a sequence $z^\nu \in K$ that converges to some $z_0 \in K$ such that

$$|dw_\gamma^\nu(z^\nu)| \to \infty,\ \nu \to \infty.$$

By equation (115) we have

$$\sup_{\nu\in\mathbb{N}} E_{\tilde{J}}(w_\gamma^\nu; U) < \infty$$

and therefore the hypotheses of lemma 4.3 with $M, \omega, J^\nu = J$ and u^ν replaced by $\tilde{M}, \tilde{\omega}, \tilde{J}$ and w_γ^ν are satisfied. Therefore there exist complex automorphisms $\chi^\nu \in \mathrm{Aut}(\mathbb{C})$ and a nonconstant $\tilde{J}$-holomorphic sphere $w : S^2 \to \tilde{M}$ such that, passing to a suitable subsequence, the assertions of lemma 4.3 are satisfied. Since $\chi^\nu \to z_0$, uniformly on U, we have for every $\epsilon, R > 0$ and ν large enough

$$\chi^\nu(B_R) \subseteq B_\epsilon(z_0), \text{ and therefore}$$

$$\begin{aligned} E_{\tilde{J}}(w_\gamma^\nu \circ \chi^\nu; B_R) &= E_{\tilde{J}}(w_\gamma^\nu; \chi^\nu(B_R)) \\ &\leq E_{\tilde{J}}(w_\gamma^\nu; B_\epsilon(z_0)) \end{aligned} \tag{118}$$

Choosing $\epsilon > 0$ small enough and sending $\nu \to \infty$ the left hand side converges to $E_{\tilde{J}}(w; B_R)$, while the right hand side converges to 0 by (114). Letting $R \to \infty$ we see

$$E_{\tilde{J}}(w) = 0,$$

a contradiction. This proves (117). Now let $U \subseteq \mathbb{C} \setminus \{0\}$ be a bounded open subset that contains K and define $\tilde{M} := \overline{U} \times M$ and $\tilde{\omega}, \tilde{J}$ as in (30), (31). By proposition 4.2 applied with $M, \omega, J^\nu = J$ and u^ν replaced by $\tilde{M}, \tilde{\omega}, \tilde{J}$ and w_γ^ν there is a subsequence of w_γ^ν that $C^\infty(K)$-converges to a $\tilde{J}$-holomorphic curve $w_\gamma : U \to \tilde{M}$. By assumption (a) and (b), the (Rescaling) condition in the definition of Floer-Gromov-convergence and remark 3.4 we have

$$(\phi_i^\nu)^{-1} \circ \psi^\nu = (\phi_i^\nu)^{-1} \circ \phi_\beta^\nu \circ \phi_{\beta\alpha}^\nu \circ (\phi_\alpha^\nu)^{-1} \circ \psi^\nu \to s_\beta + it_\beta,$$

u.c.s. on $\mathbb{C}$. It follows that w_γ is of the form

$$w_\gamma(z) = (s_\beta, t_\beta, v_\gamma(z)), \text{ where } u_\gamma : U \to M$$

and that $u_\gamma^\nu \to u_\gamma, \nu \to \infty$. By (114) the energy of u_γ vanishes and therefore u_γ is a constant map. Next we prove, using proposition 4.9, that the constant limit is equal to $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$. Let $r > 0$ be so small that

$$\begin{cases} B_r(z_{\alpha\beta}) \cap \bigcup_{z \in Z_\alpha} [z] = \{z_{\alpha\beta}\}, \text{ if } \alpha \in T_0, \\ B_r(z_{\alpha\beta}) \cap Z_\alpha = \{z_{\alpha\beta}\}, \qquad \text{if } \alpha \in T \setminus T_0. \end{cases}$$

Set $z_\infty := z_{\beta\alpha}$, let $z_1, \ldots, z_k$ be the other points of Z_β and define $K := \overline{B_r(z_{\alpha\beta})}$ and $\tilde{M}, \tilde{\omega}, \tilde{J}$ as in (30), (31) and $w_\alpha^\nu, w_\alpha : B_r(z_{\alpha\beta}) \to \tilde{M}$,

$$w_\alpha^\nu(z) := ((\phi_i^\nu)^{-1} \circ \phi_\alpha^\nu(z), u_\alpha^\nu(z)),$$

$$w_\alpha(z) := \begin{cases} (z, u_\alpha(z)), & \text{if } \alpha \in T_0, \\ (s_\alpha, t_\alpha, u_\alpha(z)), & \text{if } \alpha \in T \setminus T_0. \end{cases}$$

Then the assumptions of proposition 4.9 are satisfied with $z_0 := z_{\alpha\beta}$, $v := u_\beta$ and

$$M, \omega, J^\nu = J, u^\nu, u, \psi^\nu \text{ replaced by}$$

$$\tilde{M}, \tilde{\omega}, \tilde{J}, w_\alpha^\nu, w_\alpha, \phi_{\alpha\beta}^\nu$$

Now choose any $z \in \mathbb{C} \setminus \{0\}$ and consider the sequence

$$\zeta^\nu := (\phi_\alpha^\nu)^{-1} \circ \psi^\nu(z).$$

Setting $\zeta := \zeta^\nu$, by proposition 4.9 for every $\epsilon > 0$ there is a $\delta > 0$ and a $\nu_0 \in \mathbb{N}$ such that for $\nu \geq \nu_0$

$$\text{if } d^{S^2}((\phi_\alpha^\nu)^{-1}\circ\psi^\nu(z), z_{\alpha\beta}) + d^{S^2}((\phi_\beta^\nu)^{-1}\circ\psi^\nu(z), z_{\beta\alpha}) < \delta \Rightarrow \tag{119}$$
$$d^{\tilde{J}}(w_\gamma^\nu(z), u(z_{\alpha\beta})) < \epsilon.$$

By hypotheses (a) and (b) condition (119) is satisfied for ν large enough. It follows that

$$d^{\tilde{J}}(w_\gamma^\nu(z), u(z_{\alpha\beta})) \to 0, \nu\to\infty$$

and therefore, using (33),

$$d^t(u_\gamma^\nu(z), u_\alpha(z_{\alpha\beta})) \to 0, \nu\to\infty.$$

This proves (ii).
In order to prove (iii) suppose that $\alpha_j \in T_{\alpha\beta}$. Then $z_{\alpha,j} = z_{\alpha\beta}$ and, by lemma 3.5

$$z_{\beta\alpha} \neq z_{\beta,j} = \lim_{\nu\to\infty}(\phi_\beta^\nu)^{-1}(z_j'^\nu)$$

This implies that $(\psi^\nu)^{-1}(z_j'^\nu)$ converges to 0 since otherwise we would obtain the limit $z_{\beta\alpha}$ after applying the Möbius transformations $(\phi_\beta^\nu)^{-1}\circ\psi^\nu$ to this sequence and passing to a suitable subsequence. A similar argument shows that $(\psi^\nu)^{-1}(z_j'^\nu)$ converges to ∞ whenever $\alpha_j \in T_{\beta\alpha}$. This proves the lemma. □

Recall the notation

$$\overline{\Sigma_\alpha} := \begin{cases} \mathbb{R}\times[0,1)\cup\{\pm\infty\}, & \text{if } \alpha\in T_0,\\ S^2, & \text{if } \alpha\in T\setminus T_0 \text{ and}\end{cases}$$
$$\lim_{\nu\to\infty} s^\nu + it^\nu = \pm\infty, \text{ if } s^\nu\to\pm\infty, \nu\to\infty.$$

Lemma 4.14 *Let $(u^\nu, z_1^\nu, \dots, z_n^\nu)$ Floer-Gromov-converge to a stable Floer connecting orbit $(\mathbf{u}, \mathbf{z})$ via $\phi_\alpha^\nu \in PSl(2,\mathbb{C})$ and $\zeta^\nu \in \mathbb{R}^2 \setminus \bigcup_{1\le j\le n}[z_j^\nu]$ be a sequence such that the limit*

$$\zeta_\alpha := \lim_{\nu\to\infty}(\phi_\alpha^\nu)^{-1}(\zeta^\nu) \in \overline{\Sigma_\alpha}$$

exists for every $\alpha\in T$. Then precisely one of the following conditions holds.

(I) *There exists a (unique) vertex $\alpha \in T$ such that*

$$\zeta_\alpha \in \Sigma_\alpha \setminus (\{z_{\alpha\beta} | (\alpha,\beta)\in E\setminus E_0\} \cup \{z_j | \alpha_j = \alpha,\ 1\le j\le n\})$$

(II) *There exists a (unique) index $j\in\{1,\dots,n\}$ such that $\zeta_{\alpha_j} = z_j$.*

(III) *$\zeta_1 = -\infty$ or $\zeta_\ell = \infty$ or $\ell \geq 2$ and there is a (unique) $i \in \{1, \ldots, \ell-1\}$ such that*

$$\zeta_i = \infty, \quad \textit{and } \zeta_{i+1} = -\infty$$

(IV) *There is a (unique) $\beta \in T \setminus T_0$ such that with $\alpha := \pi(\beta)$ we have $\zeta_\alpha = z_{\alpha\beta}$ and $\zeta_\beta = z_{\beta\alpha}$.*

Proof: We only prove that exactly one of the conditions (I-IV) holds, but not uniqueness. Note that (I),. . . ,(IV) are mutually exclusive. Assume that neither (I) nor (II) nor (III) holds. We have to show that then (IV) is satisfied. For each $\alpha \in T$ there is a $\beta \in T$ with $\beta E \alpha$ and $\zeta_\alpha = z_{\alpha\beta}$. Let us look at $\alpha = 1 \in T_0$. There are two cases

Case 1 $\zeta_1 = \infty$. ($\zeta_1 = -\infty$ is excluded by (III).)
Case 2 $\zeta_1 = z_{1\alpha_1}$ for some $\alpha_1 \in T \setminus T_0$ with $\alpha_1 E 1$.

If case 1 holds we look at $\alpha = 2 \in T_0$. Again there are two cases:

Case 1 $\zeta_2 = \infty$.
Case 2 $\zeta_2 = z_{2\alpha_1}$ for some $\alpha_1 \in T \setminus T_0$ with $\alpha_1 E 2$.

By repeating this argument for $\zeta_3, \ldots$ and noting that ζ_ℓ can not be equal to ∞ by the assumption that (III) does not hold, there is an $i \in T_0$ such that case 2 holds, i.e. such that $\zeta_i = z_{i\alpha_1}$ for some $\alpha_1 \in T \setminus T_0$ with $\alpha_1 E i$. We now look at α_1: By the assumption that (I) and (II) do not hold there is an $\alpha_2 \in T$ with $\alpha_2 E \alpha_1$ and $\zeta_{\alpha_1} = z_{\alpha_1\alpha_2}$. If $\alpha_2 = i$ then case (IV) holds and we are done. So assume that $\alpha_2 \neq i$. Then there is a $\alpha_3 \in T \setminus T_0$ with $\zeta_{\alpha_2} = z_{\alpha_2\alpha_3}$. By repeating this argument we obtain a sequence $\alpha_j \in T \setminus T_0$ with $\alpha_j E \alpha_{j+1}$ and $\zeta_{\alpha_j} = z_{\alpha_j\alpha_{j+1}}$. Since T has no cycles and is a finite set, there must be a j such that $\alpha_j = \alpha_{j-2}$ and therefore $\zeta_{\alpha_j} = z_{\alpha_j\alpha_{j-1}}$. This means that case (IV) holds.
This proves the existence statement of lemma 4.14 □

Proof of theorem 4.1 if $n > 0$:

Let $(u^\nu, z_1^\nu, \ldots, z_n^\nu)$ be a sequence of Floer cylinders connecting x^- with x^+, with n distinct marked points that satisfies the requirements of theorem 4.1. Passing to a subsequence if necessary we assume furthermore that for $1 \leq j \leq n$ there is a sequence $z_j'^\nu := x_j^\nu + y_j^\nu \in [z_j^\nu] \subseteq \mathbb{R}^2$ such that y_j^ν converges to some $y_j \in [0,1)$. We prove, by induction over k, that for $0 \leq k \leq n$ a subsequence of $(u^\nu, z_1^\nu, \ldots, z_k^\nu)$ Floer-Gromov-converges to a stable Floer connecting orbit

$$(\mathbf{u}_k, \mathbf{z}_k) = ((T^k, T_0^k, E^k), \{u_\alpha\}_{\alpha \in T^k}, \{z_{\alpha\beta}\}_{\alpha E^k \beta}, \{z_j\}_{1 \leq j \leq k})$$

with $z_{\alpha\pi(\alpha)} = \infty$ and $z_{\pi(\alpha)\alpha} \in \mathbb{C}$ for every $\alpha \in T \setminus T_0$, via complex automorphisms $\phi_\alpha^\nu \in \mathrm{Aut}(\mathbb{C})$, $\nu \in \mathbb{N}, \alpha \in T^k$. For $k = 0$ this follows from the proof of theorem 4.1 in the case $n = 0$. Let now $k \geq 1$ and assume, by induction, that passing to some subsequence

$$(u^\nu, z_1^\nu, \ldots, z_{k-1}^\nu) \to ((T^{k-1}, T_0^{k-1}, E^{k-1}), \{u_\alpha\}_{\alpha \in T^{k-1}}, \{z_{\alpha\beta}\}_{\alpha E^{k-1}\beta}, \{z_j\}_{1 \leq j \leq k-1})$$

via $\phi_\alpha^\nu \in \mathrm{Aut}(\mathbb{C})$, $\nu \in \mathbb{N}, \alpha \in T^{k-1}$. We abbreviate $T := T^{k-1}$, $T_0 := T_0^{k-1}$, $E := E^{k-1}$. Passing to a further subsequence if necessary, we may assume that the limits

$$z_{\alpha,k} := \lim_{\nu \to \infty} (\phi_\alpha^\nu)^{-1}(z_k'^\nu) \tag{120}$$

exist for all α. We apply lemma 4.14 to $\zeta^\nu := z_k'^\nu$ and $\zeta_\alpha := z_{\alpha,k} := \lim_{\nu\to\infty}(\phi_\alpha^\nu)^{-1}(z_k'^\nu)$.

If (I) holds then there is a unique element $\alpha \in T$ such that

$$z_{\alpha,k} \in \Sigma_\alpha \setminus (\{z_{\alpha\beta} | (\beta, \alpha) \in E \setminus E_0\} \cup \{z_j | \alpha_j = \alpha,\ 1 \leq j \leq k-1\}).$$

In this case we define $T^{new} = T^k := T, E^{new} = E^k := E$ and introduce a new marked point

$$z_k := z_{\alpha,k} \in \begin{cases} \mathbb{R} \times [0,1), & \text{if } \alpha \in T_0, \\ S^2, & \text{if } \alpha \in T \setminus T_0. \end{cases}$$

on the vertex $\alpha_k := \alpha$. Then it follows that

$$(\mathbf{u}_k, \mathbf{z}_k) := (\{u_\alpha\}_{\alpha \in T^{new}}, \{z_{\alpha\beta}\}_{\alpha E^{new}\beta}, \{z_j\}_{1 \leq j \leq k})$$

is a stable Floer connecting orbit and that $(u^\nu, z_1^\nu, \ldots, z_k^\nu)$ Floer-Gromov-converges to $(\mathbf{u}_k, \mathbf{z}_k)$. This proves the induction step in case (I).

If (II) holds then this means that in the limit $\nu \to \infty$ the two marked points with indices j and k lie on the same vertex $\alpha := \alpha_j$ and coincide. In order to satisfy the condition for a Floer connecting orbit, that the special points on each vertex are distinct, we introduce a new bubble vertex γ. This vertex is connected to α and carries the two marked points with indices j and k, which are now separated. The marked point z_j on α becomes the nodal point from α to γ. The new stable Floer connecting orbit is given by

$$T^{new} := T \cup \{\gamma\}, \quad \alpha_k^{new} := \gamma, \quad z_k^{new} := 1, \quad \alpha_j^{new} := \gamma, \quad z_j^{new} := 0,$$
$$z_{\gamma\alpha}^{new} := \infty, \quad z_{\alpha\gamma}^{new} := z_j, \quad u_\gamma : S^2 \to M,\ u_\gamma \equiv u_\alpha(z_j).$$

The new bubble is a "ghost", i.e. the corresponding map $u_\gamma : S^2 \to M$ has zero energy.
We now define rescalings $\phi^\nu_\gamma = \psi^\nu \in \mathrm{Aut}(\mathbb{C})$ by

$$\psi^\nu(z) := \epsilon^\nu_\gamma z + z'^\nu_j, \quad \epsilon^\nu_\gamma := z'^\nu_k - z'^\nu_j.$$

We check the conditions of Floer-Gromov-convergence. We begin with the (Rescaling) condition for the edge $\alpha E^{new}\gamma$. Writing $\phi^\nu_\alpha(z) =: \epsilon^\nu_\alpha z + z^\nu_\alpha$ we have

$$\frac{z'^\nu_j - z^\nu_\alpha}{\epsilon^\nu_\alpha} \to z_j, \ \ \nu \to \infty,$$

and it follows by the assumption $z_j = z_{\alpha,k} := \lim_{\nu\to\infty}(\phi^\nu_\alpha)^{-1}(z'^\nu_k)$ that

$$\frac{z'^\nu_k - z^\nu_\alpha}{\epsilon^\nu_\alpha} \to z_j, \ \ \nu \to \infty \text{ and therefore}$$

$$\frac{\epsilon^\nu_\gamma}{\epsilon^\nu_\alpha} \to 0, \nu \to \infty \text{ and } (\phi^\nu_\alpha)^{-1}\circ\psi^\nu \to z_j = z^{new}_{\alpha\gamma}, \text{ u.c.s. on } \mathbb{C} \cong S^2\backslash\{\infty = z^{new}_{\gamma\alpha}\}. \tag{121}$$

By remark 3.2 we have $(\psi^\nu)^{-1} \circ \phi^\nu_\alpha \to \infty = z^{new}_{\gamma\alpha}$ for $\nu \to \infty$, u.c.s. on $S^2 \setminus \{z_j = z^{new}_{\alpha\gamma}\}$. This proves the (Rescaling) condition.
We continue with the (Map) condition: Since $z_j \notin Z_\alpha$ it follows that

$$u^\nu_\gamma := u^\nu \circ \phi^\nu_\gamma = u^\nu \circ \phi^\nu_\alpha \circ (\phi^\nu_\alpha)^{-1} \circ \psi^\nu \to u_\alpha(z_j), \text{ u.c.s. on } \mathbb{C}.$$

This proves the (Map) condition for γ. Since u_γ is constant,

$$E(u_\gamma) = 0 = m(\{u^\nu_\alpha\}, z_j)$$

and therefore the (Energy) condition is satisfied. The (Marked points) condition is satisfied since

$$(\phi^\nu_\gamma)^{-1}(z'^\nu_k) = 1 \text{ and } (\phi^\nu_\gamma)^{-1}(z'^\nu_j) = 0, \ \forall \nu \in \mathbb{N}.$$

This proves the induction step in the case (II).

In case (III) the new marked point gets lost in the limit $\nu \to \infty$. It either leaves the first cylinder in $-\infty$ direction or the last cylinder in $+\infty$ direction or it gets lost between some cylinder $i \in T_0$ and the next one. If $\zeta_1 = -\infty$ or $\zeta_\ell = \infty$ we extend T_0 by a new vertex γ, which is connected to

$$\begin{cases} 1, \text{ if } \zeta_1 = -\infty, \\ \ell, \text{ if } \zeta_\ell = \infty \end{cases}$$

and carries the new marked point. Recalling that $y_k := \lim_{\nu\to\infty} z_k^\nu \in [0,1)$, the new stable Floer connecting orbit is given by

$$T^{new} := T_0^{new} \cup T \setminus T_0,\ \ T_0^{new} := T_0 \cup \{\gamma\},\quad \alpha_k^{new} := \gamma,\quad z_k^{new} := iy_k,$$
$$u_\gamma : \mathbb{R}^2 \to M,\ \ u_\gamma(s,t) := \begin{cases} x^-(t), & \text{if } \zeta_1 = -\infty,\\ x_\ell(t) = x^+(t), & \text{if } \zeta_\ell = \infty.\end{cases}$$

The new cylinder is a "ghost", i.e. $E(u_\gamma) = 0$.
If $\zeta_i = \infty,\ \zeta_{i+1} = -\infty$ for some $1 \le i \le \ell - 1$ then we extend T_0 by a new vertex γ, which is connected to i and $i+1$ and carries the new marked point. The two vertices i and $i+1$ are no longer adjacent. We define T^{new}, α_k^{new} and z_k^{new} as above and

$$u_\gamma(s,t) := \lim_{s'\to\infty} u_i(s',t)$$

In any of the cases $\zeta_1 = -\infty$, $\zeta_\ell = \infty$ or $\zeta_i = \infty,\ \zeta_{i+1} = -\infty$ for some $1 \le i \le \ell - 1$ we define $\phi_\gamma^\nu \in PSl(2,\mathbb{C})$ by $\phi_\gamma^\nu(s+it) := (s + x_k^\nu, t)$. We check the (Energy) condition for Floer-Gromov-convergence: Since $x_k^\nu - s_i^\nu \to \infty$ and $x_k^\nu - s_{i+1}^\nu \to -\infty, \nu \to \infty$ we have for every $T, T' \in \mathbb{R}$ for ν large enough

$$\begin{aligned} E(u_i^\nu;[T,\infty)) &\ge E(u_i^\nu;[T + x_k^\nu - s_i^\nu,\infty))\\ &= E(u_\gamma^\nu;[T,\infty)) \text{ and}\\ E(u_\gamma^\nu;[T',\infty)) &= E(u_{i+1}^\nu;[T' + x_k^\nu - s_{i+1}^\nu,\infty))\\ &\ge E(u_{i+1}^\nu;[T',\infty))\end{aligned}$$

Sending $\nu \to \infty$ and than $T \to \infty$ and $T' \to -\infty$ we have

$$m(\{u_i^\nu\},\infty) \ge \lim_{T\to\infty} \limsup_{\nu\to\infty} E(u_\gamma^\nu;[T,\infty)), \tag{122}$$

$$\begin{aligned}\lim_{T'\to-\infty} \liminf_{\nu\to\infty} E(u_\gamma^\nu;[T',\infty)) &\ge \lim_{\nu\to\infty} E(u_{i+1}^\nu) - \lim_{T'\to-\infty}\lim_{\nu\to\infty} E(u_{i+1}^\nu;(-\infty,T'])\\ &= E - m(\{u_{i+1}^\nu\},-\infty)\\ &= E(u_{i+1}) + \sum_{z\in Z_{i+1}} m(\{u_{i+1}^\nu\},z) + m(\{u_{i+1}^\nu\},\infty)\\ &= m(\{u_i^\nu\},\infty)\end{aligned} \tag{123}$$

and therefore the mass $m(\{u_\gamma^\nu\},\infty)$ is welldefined and

$$m(\{u_\gamma^\nu\},\infty) = m(\{u_i^\nu\},\infty)$$

This proves the (Energy) condition and the induction step in the case (III).

If case (IV) holds then there is a unique $\beta \in T \setminus T_0$ such that with $\alpha := \pi(\beta)$ we have $\zeta_\alpha = z_{\alpha\beta}$ and $\zeta_\beta = z_{\beta\alpha}$. Then the new marked point gets lost between the bubble vertex β and its predecessor α. We introduce a new bubble γ, which is connected to α and β and carries the new marked point. In the new tree T^{new} the vertices α and β are no longer adjacent. The new stable Floer connecting orbit is given by

$$T^{new} := T \cup \{\gamma\}, \quad \alpha_k^{new} := \gamma, \quad z_k^{new} := 1, \quad z_{\alpha\gamma}^{new} := z_{\alpha\beta},$$
$$z_{\beta\gamma}^{new} := z_{\beta\alpha} = \infty, \quad z_{\gamma\alpha}^{new} := \infty, \quad z_{\gamma\beta}^{new} := 0$$
$$u_\gamma : S^2 \to M, \; u_\gamma(z) \equiv u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha}).$$

γ is a ghost, i.e. the corresponding map u_γ has 0 energy. We write

$$\psi_\alpha^\nu(z) = \epsilon_\alpha^\nu z + z_\alpha^\nu, \quad \psi_\beta^\nu(z) = \epsilon_\beta^\nu z + z_\beta^\nu$$

and define the rescalings $\phi_\gamma^\nu = \psi^\nu \in \mathrm{Aut}(\mathbb{C})$ by

$$\psi^\nu(z) := \epsilon^\nu z + z_\beta^\nu, \quad \epsilon^\nu := z_k'^\nu - z_\beta^\nu.$$

We check the conditions of Floer-Gromov-convergence: For the (Rescaling) condition note that, since we assume case (IV),

$$\frac{z_k'^\nu - z_\alpha^\nu}{\epsilon_\alpha^\nu} \to z_{\alpha\beta} \in \mathbb{C}, \tag{124}$$

$$\frac{z_k'^\nu - z_\beta^\nu}{\epsilon_\beta^\nu} \to z_{\beta\alpha} = \infty. \tag{125}$$

Furthermore by the rescaling condition in the definition of a stable Floer connecting orbit

$$\frac{z_\beta^\nu - z_\alpha^\nu}{\epsilon_\alpha^\nu} \to z_{\alpha\beta}, \;\; \nu \to \infty. \tag{126}$$

It follows from (124) and (126) that

$$(\phi_\alpha^\nu)^{-1} \circ \psi^\nu(z) = \frac{\epsilon^\nu z + z_\beta^\nu - z_\alpha^\nu}{\epsilon_\alpha^\nu} \tag{127}$$

$$= \frac{z_k'^\nu - z_\alpha^\nu}{\epsilon_\alpha^\nu} z + \frac{z_\beta^\nu - z_\alpha^\nu}{\epsilon_\alpha^\nu}(1-z) \to z_{\alpha\beta} = z_{\alpha\gamma}^{new}, \tag{128}$$

u.c.s. on $\mathbb{C} \cong S^2 \setminus \{z_{\gamma\alpha}^{new} = \infty\}$, and (125) implies that

$$(\phi_\beta^\nu)^{-1} \circ \psi^\nu(z) = \frac{z_k'^\nu - z_\beta^\nu}{\epsilon_\beta^\nu} z \to z_{\beta\gamma}^{new} = z_{\beta\alpha} = \infty, \tag{129}$$

u.c.s. on $S^2 \setminus \{z_{\gamma\beta}^{new} = 0\}$. By remark 3.2 it follows from (128) and (129) that

$$\begin{aligned} (\psi^\nu)^{-1} \circ \phi_\alpha^\nu &\to z_{\gamma\alpha}^{new} = \infty, \quad \text{u.c.s. on } S^2 \setminus \{z_{\alpha\gamma}^{new} = z_{\alpha\beta}\}, \\ (\psi^\nu)^{-1} \circ \phi_\beta^\nu &\to z_{\gamma\beta}^{new} = 0, \quad \text{u.c.s. on } S^2 \setminus \{z_{\beta\gamma}^{new} = z_{\beta\alpha}\} \end{aligned}$$

This proves the (Rescaling) condition.
The (Energy) and the (Map) condition follow by lemma 4.13 and the (marked points) condition is satisfied, since $(\psi^\nu)^{-1}(z_k'^\nu) = 1, \ \nu \in \mathbb{N}$. This proves the induction step in case (IV) and therefore theorem 4.1 in the general case.

Remark 4.15 Note that in the cases (III) and (IV) it also follows from lemma 4.13, that

$$(\psi^\nu)^{-1}(z_j'^\nu) \to \begin{cases} 0, & \text{if } \alpha_j \in T_{\alpha\beta} \\ \infty, & \text{if } \alpha_j \in T_{\beta\alpha} \end{cases}$$

and therefore in the induction step $k-1 \to k$ we do not need to pass to a subsequence in order to make sure that the limits in (120) exist.

A Appendix

In this appendix (M, ω) is a compact symplectic manifold.

A.1 Proof of Lemma 1.3

Let $x_0 \in \mathcal{L}_0 M$ be a contractible loop in M and let $\xi \in T_{x_0}\mathcal{L}_0 M$ be a tangent vector to $\mathcal{L}_0 M$ at x_0, i.e.a smooth vector field in M along x_0. We have to prove that

$$\Psi_H(x_0; \xi) = da_H(x_0)\xi,$$

where the circle valued symplectic action $a_H : \mathcal{L}_0 M \to \mathbb{R}/\mathbb{Z}$ and the 1-form $\Psi_H \in \Omega^1(\mathcal{L}_0 M)$ are defined in section 1.2.
Choose a smooth map

$$x : \mathbb{R} \to \mathcal{L}_0 M, \qquad \lambda \mapsto x(\lambda) =: x_\lambda,$$

such that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} x = \xi,$$

and

$$x_\lambda \equiv \begin{cases} x_{-\frac{1}{4}}, & \text{if } \lambda \le -\frac{1}{4}, \\ x_{\frac{1}{4}}, & \text{if } \lambda \ge \frac{1}{4}. \end{cases}$$

Let $u_{-1} : B \to M$ be a smooth map such that

$$u_{-1}(re^{2\pi it}) = x_{-1}(t), \qquad \text{if } \frac{1}{2} \le r \le 1.$$

For $\lambda \le \frac{1}{4}$ we define $u_\lambda : B \to M$ by

$$u_\lambda(re^{2\pi it}) := \begin{cases} u_{-1}(re^{2\pi it}), & \text{for } r \le \frac{1}{2}, \\ x_{\lambda+r-1}(t), & \text{for } \frac{1}{2} \le r \le 1. \end{cases}$$

Then we have $u_\lambda(e^{2\pi it}) = x_\lambda(t)$, for $\lambda \le \frac{1}{4}$. Therefore, denoting the annulus with radii $0 < a < b$ by $A(a,b) := \overline{B}_b \setminus B_a$, we have

$$\begin{aligned} da_H(x_0)\xi &= \frac{d}{d\lambda}\Big|_{\lambda=0} a_H(x_\lambda) \\ &= \frac{d}{d\lambda}\Big|_{\lambda=0} \left(-\int_{B_{\frac{1}{2}}} u_{-1}^*\omega - \int_{A(\frac{1}{2},1)} u_\lambda^*\omega - \int_0^1 H_t(x_\lambda(t))dt \right) \\ &= -\frac{d}{d\lambda}\Big|_{\lambda=0} \int_{A(\frac{1}{2},1)} u_\lambda^*\omega - \frac{d}{d\lambda}\Big|_{\lambda=0} \int_0^1 H_t(x_\lambda(t))dt \\ &=: I + II. \end{aligned}$$

The second term is given by

$$II = -\int_0^1 dH_t(x_0)\xi dt = -\int_0^1 \omega(X_{H_t}(x_0), \xi)dt.$$

For the first term we consider the change of coordinates

$$f : (\frac{1}{2}, 1) \times (0,1) \to A(\frac{1}{2}, 1), \qquad f(r,t) = re^{2\pi it}.$$

Then

$$\begin{aligned} \int_{A(\frac{1}{2},1)} u_\lambda^*\omega &= \int_{(\frac{1}{2},1)\times(0,1)} f^* u_\lambda^*\omega \\ &= \int_{(\frac{1}{2},1)\times(0,1)} (u_\lambda \circ f)^*\omega \\ &= \int_0^1 \int_{\frac{1}{2}}^1 \omega(\partial_r x_{\lambda+r-1}, \partial_t x_{\lambda+r-1})\, dr\, dt. \end{aligned}$$

We define for $\lambda \in \mathbb{R}$

$$\alpha(\lambda) := -\int_0^1 \omega(\partial_\lambda x_\lambda, \partial_t x_\lambda)\, dt.$$

Then

$$\begin{aligned}
I &= -\left.\frac{d}{d\lambda}\right|_{\lambda=0} \int_{\frac{1}{2}}^{1} \int_{0}^{1} \omega(\partial_\lambda x_{\lambda+r-1}, \partial_t x_{\lambda+r-1})\, dt\, dr \\
&= \left.\frac{d}{d\lambda}\right|_{\lambda=0} \int_{\frac{1}{2}}^{1} \alpha(\lambda + r - 1)\, dr \\
&= \left.\int_{\frac{1}{2}}^{1} \alpha'(\lambda + r - 1)\, dr\right|_{\lambda=0} \\
&= \alpha(r-1)|_{r=\frac{1}{2}}^{1} \\
&= \alpha(0) - \alpha(-\frac{1}{2}).
\end{aligned}$$

For $\lambda \leq -\frac{1}{4}$ we have $x_\lambda \equiv x_{-\frac{1}{4}}$, so $\partial_\lambda x_\lambda = 0$ and therefore $\alpha(\lambda) = 0$. This implies

$$\begin{aligned}
I &= \alpha(0) \\
&= -\int_0^1 \omega(\partial_\lambda|_{\lambda=0} x_\lambda, \partial_t x_0)\, dt \\
&= -\int_0^1 \omega(\xi, \partial_t x_0)\, dt
\end{aligned}$$

and therefore

$$da_H(x_0)\xi = I + II = \int_0^1 \omega(\partial_t x_0 - X_{H_t}(x_0), \xi)\, dt = \Psi_H(x_0; \xi).$$

This proves Lemma 1.3.

A.2 An a priori estimate

Lemma A.1 *Let (M, J) be a compact almost complex manifold and g be any Riemannian metric on M. Then there exists a constant δ such that for $r > 0$ and every J-holomorphic curve $u : B_r \to M$ the following holds.*

$$\textit{If } E(u; B_r) < \delta \textit{ then } |du(0)|^2 \leq \frac{8}{\pi r^2} \int_{B_r} |du|^2.$$

Proof: See lemma 4.3.1. in [9]. □

A.3 Homotopy

Let $u, v : [0,1] \times \mathbb{R} \to M$ be smooth functions that satisfy the twist condition (16) and

$$u(0,t) = v(0,t),$$
$$u(1,t) = v(1,t)$$

We call u and v **homotopic with fixed endpoints and respecting the twist condition** if there is a smooth family $v^\lambda : [0,1] \times \mathbb{R} \to M,\ \ \lambda \in [0,1]$ with

$$\begin{aligned}
v^0 &= u & (130)\\
v^1 &= v & (131)\\
v^\lambda(0,t) &= u(0,t) = v(0,t), & (132)\\
v^\lambda(1,t) &= u(1,t) = v(1,t), & (133)\\
\phi(v^\lambda(s,t+1)) &= v^\lambda(s,t), \quad s \in [0,1],\ \ t \in \mathbb{R},\ \ \lambda \in [0,1] & (134)
\end{aligned}$$

Remark A.2 *Let $u, v : [0,1] \times \mathbb{R} \to M$ be homotopic with fixed endpoints and respecting the twist condition. Then, setting $Q := [0,1] \times [0,1]$*

$$\int_Q u^*\omega = \int_Q v^*\omega$$

Proof:
We use the same trick as in the proof of proposition 1.14. We define

$$\begin{aligned}
\hat{M} &:= M \times M,\\
\hat{\omega} &:= \omega \oplus (-\omega) \in \Omega^2(\hat{M}),\\
L_0 &:= \{(p,p) | p \in M\},\\
L_1 &:= \{(p,\phi(p)) | p \in M\} \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\hat{u}, \hat{v}, \hat{v}^\lambda :&\quad [0,1] \times [0,\tfrac{1}{2}] \to \hat{M},\\
\hat{u}(s,t) &:= \begin{pmatrix} u(s,t) \\ u(s,-t) \end{pmatrix},\\
\hat{v}(s,t) &:= \begin{pmatrix} v(s,t) \\ v(s,-t) \end{pmatrix},\\
\hat{v}^\lambda(s,t) &:= \begin{pmatrix} v^\lambda(s,t) \\ v^\lambda(s,-t) \end{pmatrix}
\end{aligned}$$

Then $L_0, L_1 \subseteq \hat{M}$ are Lagrangian submanifolds and

$$\hat{v}^0 = \hat{u}, \ \ \hat{v}^1 = \hat{v}, \tag{135}$$
$$\hat{v}^\lambda(0,t) = \hat{u}(0,t) = \hat{v}(0,t), \ \ \hat{v}^\lambda(1,t) = \hat{u}(1,t) = \hat{v}(1,t), \tag{136}$$
$$\hat{v}^\lambda(s,0) \in L_0, \ \ \hat{v}^\lambda(s,\frac{1}{2}) \in L_1, \tag{137}$$

for $s \in [0,1]$, $t \in \mathbb{R}$, $\lambda \in [0,1]$. Now by a corollary to Cartan's formula the difference $\hat{v}^*\hat{\omega} - \hat{u}^*\hat{\omega}$ is exact, more precisely it is given by $\hat{v}^*\hat{\omega} - \hat{u}^*\hat{\omega} = d\alpha$, where

$$\alpha := \int_0^1 \hat{\omega}(\partial_\lambda \hat{v}^\lambda, d\hat{v}^\lambda.)d\lambda.$$

Therefore by Stokes, setting $Q_{\frac{1}{2}} := [0,1] \times [0, \frac{1}{2}]$

$$\int_Q (v^*\omega - u^*\omega) = \int_{Q_{\frac{1}{2}}} (\hat{v}^*\hat{\omega} - \hat{u}^*\hat{\omega}) = \int_{\partial Q_{\frac{1}{2}}} \alpha$$

and by (136) and (137) this integral vanishes. This proves the remark.

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