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SYMPLECTIC VORTICES ON THE COMPLEX PLANE AND QUANTUM COHOMOLOGY

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Contents

Abstract

Let (M, ω) be a symplectic manifold, G be a compact connected Lie group that acts on M in a Hamiltonian way, with moment map μ, and let P be a principal G-bundle over a Riemann surface Σ. The symplectic vortex equations are nonlinear first order partial differential equations for a G-equivariant map $u : P \to M$ and a connection one form A on P. This dissertation investigates these equations in the case of the trivial bundle over the complex plane $\mathbb{C}$. In this case they become equations for a map $(u, \Phi, \Psi) : \mathbb{C} \to M \times \mathfrak{g} \times \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G. Assume that the moment map μ is proper and that G acts freely on $\mu^{-1}(0)$. The main results are the following:

- Let $w := (u, \Phi, \Psi)$ be a smooth finite energy solution of the vortex equations on $\mathbb{C}$ such that the closure $\overline{u(\mathbb{C})} \subseteq M$ is compact. Then the vertical differential of the vortex equations at a point w, augmented by a gauge fixing operator, is a Fredholm operator between two suitable weighted Sobolev spaces of sections of $u^*TM \oplus \mathfrak{g} \oplus \mathfrak{g}$.
- A bubbling result in the case that M is convex at ∞ and symplectically aspherical. Let $(w_\nu) := ((u_\nu, \Phi_\nu, \Psi_\nu))_{\nu \in \mathbb{N}}$ be a sequence of vortices (i.e. solutions of the vortex equations) on $\mathbb{C}$ for which $\overline{u_\nu(\mathbb{C})}$ is compact. Assume that the energies of the vortices are positive and uniformly bounded above. Then there exists a subsequence of (w_ν) that converges to a new kind of stable map consisting of vortices on $\mathbb{C}$ and pseudo-holomorphic spheres in the symplectic quotient.
- Given a sequence (w_ν) of vortices that converges to some stable map, the equivariant homology class is preserved in the limit.
- Construction of an evaluation map on the set of equivalence classes of stable maps with finitely many marked points. This evaluation map is continuous w.r.t. convergence of vortices against a stable map.

Furthermore, I sketch a proof of an application stating that there exists a vortex on $\mathbb{C}$ with positive and finite energy, provided that at least one of the following conditions holds. 1. There exists a nonzero second equivariant homology class for which the genus 0 symplectic vortex invariants do not vanish. 2. Denoting by $(\bar{M}, \bar{\omega})$ the symplectic quotient, there exists a class $0 \neq \bar{B} \in H_2(\bar{M}, \mathbb{Z})$ such that the 3-point genus 0 Gromov-Witten invariants of $(\bar{M}, \bar{\omega}, \bar{B})$ do not vanish. As an example, once the details of this applications are carried out, it follows that for every positive integer n there exists a positive and finite energy solution of the vortex equations on $\mathbb{C}$, associated to the diagonal action of S^1 on $\mathbb{C}^n$. The application is based on the one hand on the bubbling techniques developed in this dissertation. On

the other hand, it relies on an adiabatic limit argument for the vortex equations, which has been carried out by R. Gaio and D. A. Salamon.

The main difficulty in stating and proving the Fredholm and bubbling results is that the domain of the considered maps is the whole plane $\mathbb{C}$. In the case of the Fredholm theorem one problem is to find the right weights for the Sobolev spaces of sections of $u^*TM \oplus \mathfrak{g} \oplus \mathfrak{g} \to \mathbb{C}$. Unlike the situation of a compact base manifold the definition of these spaces is not canonical. Furthermore, the Kondrachov compactness theorem does not hold on $\mathbb{C}$. This means that the 0-th order terms in the augmented vertical differential are not compact. On the other hand in the case of the bubbling theorem the following problem arises, since $\mathbb{C}$ has infinite measure. If (w_ν) is a sequence of vortices on $\mathbb{C}$ with uniformly bounded energies, then in the limit $\nu \to \infty$, some energy may escape to $\infty \in \mathbb{C} \cup \{\infty\}$.

This dissertation is part of a larger project, whose aim is to define a ring homomorphism from the equivariant cohomology of M to the quantum cohomology of the symplectic quotient, assuming that (M, ω) is symplectically aspherical. This homomorphism should be defined by counting vortices on $\mathbb{C}$.

Zusammenfassung

Sei (M, ω) eine symplektische Mannigfaltigkeit, G eine kompakte zusammenhängende Lie-Gruppe, die auf Hamiltonsche Weise auf M operiere, mit Impuls-Abbildung μ, und sei P ein G-Hauptfaserbündel über einer Riemannschen Fläche Σ. Die symplektischen Vortex-Gleichungen sind nichtlineare partielle Differential-Gleichungen erster Ordnung für eine G-äquivariante Abbildung $u : P \to M$ und eine Zusammenhangs-1-Form A auf P. Diese Doktorarbeit untersucht diese Gleichungen im Fall des trivialen Bündels über der komplexen Ebene $\mathbb{C}$. In diesem Fall werden sie zu Gleichungen für eine Abbildung $(u, \Phi, \Psi) : \mathbb{C} \to M \times \mathfrak{g} \times \mathfrak{g}$, wobei $\mathfrak{g}$ die Lie-Algebra von G bezeichnet. Wir nehmen an, dass μ eigentlich ist und G frei auf $\mu^{-1}(0)$ operiert. Die Hauptresultate sind die folgenden:

- Sei $w := (u, \Phi, \Psi)$ eine glatte Lösung der Vortex-Gleichungen mit endlicher Energie, so dass der Abschluss $\overline{u(\mathbb{C})} \subseteq M$ kompakt ist. Dann ist das Differential der Vortex-Gleichungen am Punkt w, erweitert um einen Eich-Fixierungs-Operator, ein Fredholm-Operator zwischen geeigneten gewichteten Sobolev-Räumen von Schnitten von $u^*TM \oplus \mathfrak{g} \oplus \mathfrak{g}$.
- Ein "Blasen"-Resultat im Fall, dass M konvex bei ∞ und symplektisch asphärisch ist. Sei $(w_\nu) := ((u_\nu, \Phi_\nu, \Psi_\nu))_{\nu \in \mathbb{N}}$ eine Folge von Vortizes (das heisst Lösungen der Vortex-Gleichungen) auf $\mathbb{C}$, für welche $\overline{u(\mathbb{C})}$ kompakt ist. Wir nehmen an, dass die Energien der Vortizes positiv und gleichmässig nach oben beschränkt sind. Dann existiert eine Teilfolge von (w_ν), die gegen eine neue Art von stabiler Abbildung konvergiert. Diese besteht aus Vortizes auf $\mathbb{C}$ und pseudo-holomorphen Sphären im symplektischen Quotienten.
- Die äquivariante Homologie-Klasse jeder Folge (w_ν) von Vortizes, die gegen eine stabile Abbildung konvergiert, bleibt im Limes erhalten.
- Konstruktion einer Auswertungs-Abbildung auf der Menge der Äquivalenz-Klassen von stabilen Abbildungen mit endlich vielen markierten Punkten. Diese Auswertungs-Abbildung ist stetig bezüglich der Konvergenz von Vortizes gegen eine stabile Abbildung.

Darüber hinaus skizziere ich einen Beweis einer Anwendung, die besagt, dass ein Vortex auf $\mathbb{C}$ mit positiver und endlicher Energie existiert, falls mindestens eine der folgenden Bedingungen erfüllt ist. 1. Es gibt eine zweite äquivariante Homologie-Klasse ungleich 0, für welche die symplektischen Vortex-Invarianten für das Geschlecht 0 nicht verschwinden. 2. Wenn wir den symplektischen Quotienten mit $(\bar{M}, \bar{\omega})$ bezeichnen, dann existiert eine Klasse $0 \neq \bar{B} \in H_2(\bar{M}, \mathbb{Z})$

so, dass die Geschlecht-0-Gromov-Witten-Invarianten von $(\bar{M}, \bar{\omega}, \bar{B})$ nicht verschwinden. Als ein Beispiel folgt, sobald die Details dieser Anwendung ausgeführt sind, dass es Vortizes auf $\mathbb{C}$ mit positiver und endlicher Energie gibt, die zur diagonalen Operation von S^1 auf $\mathbb{C}^n$ gehören. Die Anwendung beruht auf den "Blasen"-Techniken, die ich in dieser Doktorarbeit entwickle, und auf einem Argument von R. Gaio und D. A. Salamon über einen adiabatischen Limes in den Vortex-Gleichungen.

Die Hauptschwierigkeit in der Formulierung und im Beweis der Fredholm- und "Blasen"-Resultate ist, dass der Definitions-Bereich der betrachteten Abbildungen die ganze Ebene $\mathbb{C}$ ist. Im Fall des Fredholm-Theorems ergibt sich unter anderem das Problem, die richtigen Gewichte für die Sobolev-Räume von Schnitten von $u^*TM \oplus \mathfrak{g} \oplus \mathfrak{g} \to \mathbb{C}$ zu finden. Anders als im Fall einer kompakten Basis-Mannigfaltigkeit, ist die Definition dieser Räume nicht kanonisch. Überdies gilt der Kondrachovsche Kompaktheitssatz auf $\mathbb{C}$ nicht. Das führt dazu, dass die Terme nullter Ordnung im erweiterten vertikalen Differential nicht kompakt sind. Da $\mathbb{C}$ unendliches Mass hat, tritt auf der anderen Seite im Fall des "Blasen"-Theorems das folgende Problem auf: Wenn (w_ν) eine Folge von Vortizes auf $\mathbb{C}$ mit gleichmässig beschränkten Energien ist, dann kann im Limes $\nu \to \infty$ Energie nach $\infty \in \mathbb{C} \cup \{\infty\}$ entweichen.

Diese Doktorarbeit ist Teil eines grösseren Projektes, in dem es darum geht, im symplektisch asphärischen Fall mittels Zählen von Vortizes auf $\mathbb{C}$ einen Ring-Homomorphismus von der äquivarianten Kohomologie von M zur Quanten-Kohomologie des symplektischen Quotienten zu konstruieren.

0 Introduction

0.1 Main results

Let (M, ω) be a symplectic manifold (without boundary) and G be a compact connected Lie group that acts on M in a Hamiltonian way. We denote by $\mathfrak{g}$ the Lie algebra of G and fix an inner product $\langle .,. \rangle$ on $\mathfrak{g}$ that is invariant under the adjoint action of G. We also fix an (equivariant) moment map $\mu : M \to \mathfrak{g}$ and a smooth G-invariant ω-compatible almost complex structure J on M. The *symplectic vortex equations* on the complex plane are the equations for a map $(u, \Phi, \Psi) : \mathbb{C} \to M \times \mathfrak{g} \times \mathfrak{g}$ given by

$$\partial_s u + X_\Phi(u) + J(u)(\partial_t u + X_\Psi(u)) = 0, \tag{0.1}$$

$$\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \mu(u) = 0. \tag{0.2}$$

Here we write an element of $z \in \mathbb{C}$ as $z = s + it$. Furthermore, for every $x \in M$ and $\xi \in \mathfrak{g}$ the vector $X_\xi(x) \in T_xM$ denotes the infinitesimal action of ξ at the point x. These equations were discovered, independently, on the one hand by K. Cieliebak, R. Gaio and D. A. Salamon [CGS] and on the other hand by I. Mundet i Riera [Mu1], [Mu2]. They arise from the energy functional $E : C^\infty(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g}) \to [0, \infty]$ defined by

$$\begin{aligned} E(u, \Phi, \Psi) \;:=\; & \frac{1}{2} \int_{\mathbb{C}} \big(|\partial_s u + X_\Phi(u)|^2 + |\partial_t u + X_\Psi(u)|^2 + \\ & |\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi]|^2 + |\mu(u)|^2 \big)\, ds\, dt \end{aligned} \tag{0.3}$$

as follows. We denote by $\widetilde{\mathcal{M}}$ the set of all smooth solutions $w := (u, \Phi, \Psi)$ of the vortex equations on $\mathbb{C}$ (0.1), (0.2) (vortices) such that $E(w) < \infty$ and the closure $\overline{u(\mathbb{C})} \subseteq M$ is compact. We impose the following hypothesis.

(H1) G *acts freely on* $\mu^{-1}(0)$ *and the moment map* μ *is proper.*

It follows that every vortex $w \in \widetilde{\mathcal{M}}$ determines a second equivariant homology class $[w]_{\mathrm{G}} \in H_2^{\mathrm{G}}(M, \mathbb{Z})$. We fix a class $B \in H_2^{\mathrm{G}}(M, \mathbb{Z})$ and assume that there exists a vortex $w \in \widetilde{\mathcal{M}}$ such that $[w]_{\mathrm{G}} = B$. Then the vortices $w \in \widetilde{\mathcal{M}}$ such that $[w]_{\mathrm{G}} = B$ are precisely the minimizers of E among all smooth maps behaving suitably at ∞ and representing B.

From now on, throughout this dissertation, unless otherwise mentioned, we assume that hypothesis (H1) is satisfied.

This hypothesis has the following consequences. That G acts freely on $\mu^{-1}(0)$ implies that 0 is a regular value of μ and therefore $\mu^{-1}(0) \subseteq M$ is a smooth submanifold of dimension $\dim M - \dim \mathrm{G}$. Since G is compact, the action of G on $\mu^{-1}(0)$ is proper. It follows that the quotient $\bar{M} := \mu^{-1}(0)/\mathrm{G}$ carries a canonical manifold structure of dimension $\dim M - 2\dim \mathrm{G}$ and a canonical symplectic form $\bar{\omega}$. That μ is proper implies that $\mu^{-1}(0)$ and $\bar{M}$ are compact.

The first main result is a Fredholm theorem. We use a complexified notation. This makes the notation simpler and, more importantly, reveals a symmetry in the augmented vertical differential of the vortex equations. Fix a point $x \in M$ and consider the linearized action $L_x : \mathfrak{g} \to T_xM$ defined by $L_x\xi := X_\xi(x)$. Denoting by $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the complexified Lie algebra, the map L_x extends to a complex linear map $L_x^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \to T_xM$. The "complex action bundle" is the complex subbundle $\mathrm{im}L^{\mathbb{C}} \subseteq TM$ whose fibre over a point $x \in M$ is the subspace $\mathrm{im}L_x^{\mathbb{C}} = \mathrm{im}L_x \oplus J\mathrm{im}L_x$. We denote by $\widehat{P} : TM \to \mathrm{im}L^{\mathbb{C}}$ the (real) orthogonal projection to this subbundle, w.r.t. the Riemannian metric $g_{\omega,J} := \omega(\cdot, J\cdot)$. Furthermore, we define

$$P := \widehat{P} \oplus \mathrm{id} : TM \oplus \mathfrak{g}^{\mathbb{C}} \to \mathrm{im}L^{\mathbb{C}} \oplus \mathfrak{g}^{\mathbb{C}}.$$

The Lie bracket $[\cdot,\cdot]$ and the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, and the Riemannian metric $g_{\omega,J}$ on M extend uniquely to a complex linear Lie bracket $[\cdot,\cdot]$ on $\mathfrak{g}^{\mathbb{C}}$, to a hermitian inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$ and to a hermitian metric $g^{\mathbb{C}}$ on TM. We fix real numbers $p > 2$ and $\lambda > 1 - 2/p$ and a linear G-invariant torsionfree connection ∇ on M. We also fix a vortex $w := (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}$. We identify $\mathfrak{g}^{\mathbb{C}}$ with the trivial bundle $\mathbb{C} \times \mathfrak{g}^{\mathbb{C}}$ and fix a locally $W^{1,p}$-section $\zeta := (v, \beta) : \mathbb{C} \to u^*TM \oplus \mathfrak{g}^{\mathbb{C}}$. We denote by $\nabla_s v$ and $\nabla_t v$ the covariant derivatives of v in the s- and t-direction, and define

$$\begin{aligned}
\nabla_s^\Phi(v,\beta) &:= (\nabla_s^\Phi v, \partial_s^\Phi \beta) := (\nabla_s v + \nabla_v X_\Phi, \partial_s\beta + [\Phi,\beta]),\\
\nabla_t^\Psi(v,\beta) &:= (\nabla_t^\Psi v, \partial_t^\Psi \beta) := (\nabla_t v + \nabla_v X_\Psi, \partial_t\beta + [\Psi,\beta]).
\end{aligned}$$

We define the (possibly infinite) (p,λ,w)-norm of $\zeta := (v,\beta)$ to be

$$||\zeta||_w^{p,\lambda} := ||\zeta||_{L^\infty(\mathbb{C})} + \big|\big|\big(|P\zeta| + |(\nabla_s^\Phi\zeta, \nabla_t^\Psi\zeta)|\big)(1+|\cdot|^2)^{\frac{\lambda}{2}}\big|\big|_{L^p(\mathbb{C})},$$

where for each $x \in M$ the norm on $T_xM \oplus \mathfrak{g}^{\mathbb{C}}$ is taken w.r.t. $g_{\omega,J}$ and $\langle\cdot,\cdot\rangle_{\mathbb{C}}$. We define $\mathcal{X}_w^{p,\lambda}$ to be the vector space

$$\mathcal{X}_w^{p,\lambda} := \big\{\zeta \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, u^*TM \oplus \mathfrak{g}^{\mathbb{C}}) \,\big|\, ||\zeta||_w^{p,\lambda} < \infty\big\}$$

with norm $||\cdot||_w^{p,\lambda}$. Furthermore, we define the normed vector space

$$\mathcal{Y}_w^{p,\lambda} := \big\{\zeta \in L^p_{\mathrm{loc}}(\mathbb{C}, u^*TM \oplus \mathfrak{g}^{\mathbb{C}}) \,\big|\, ||\zeta||_{p,\lambda} < \infty\big\},$$

where the norm

$$||\zeta||_{p,\lambda} := ||\zeta(1+|\cdot|^2)^{\frac{\lambda}{2}}||_{L^p(\mathbb{C})}$$

is taken w.r.t. $g_{\omega,J}$ and $\langle\cdot,\cdot\rangle_{\mathbb{C}}$. For every locally $W^{1,p}$-section $(v,\beta):\mathbb{C}\to u^*TM\oplus\mathfrak{g}^{\mathbb{C}}$ we denote

$$\nabla_{\bar z}^{\Phi+i\Psi}v := (\nabla_s^{\Phi}+J\nabla_t^{\Psi})v, \qquad \partial_z^{\Phi-i\Psi}\beta := (\partial_s^{\Phi}-i\partial_t^{\Psi})\beta.$$

We also abbreviate $\partial_t^{\Psi}u := \partial_t u + X_{\Psi}(u)$ and denote by $(\nabla_{.}J):TM\oplus TM\to TM$ the map taking $(v,w)\in T_xM\times T_xM$ to $(\nabla_v J)w\in T_xM$, for each point $x\in M$. For $x\in M$ we denote by $(L_x^{\mathbb{C}})^*:T_xM\to\mathfrak{g}^{\mathbb{C}}$ the adjoint map of $L_x^{\mathbb{C}}$ w.r.t. $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ and $g^{\mathbb{C}}$. The vertical differential of the vortex equations at the point $w=(u,\Phi+i\Psi)$, augmented by a gauge fixing operator, is the map

$$\mathcal{D}_w^{p,\lambda} = \begin{pmatrix} 2\nabla_{\bar z}^{\Phi+i\Psi}+(\nabla_{.}J)\partial_t^{\Psi}u & L_u^{\mathbb{C}} \\ (L_u^{\mathbb{C}})^* & 2\partial_z^{\Phi-i\Psi}\end{pmatrix} : \mathcal{X}_w^{p,\lambda}\to\mathcal{Y}_w^{p,\lambda}. \tag{0.4}$$

Note that $\mathcal{D}_w^{p,\lambda}$ is in some sense symmetrical in the two components (v,β) of its argument $\zeta\in\mathcal{X}_w^{p,\lambda}$. The vortex w carries a Maslov index $m(w)\in\mathbb{Z}$, see Definition 2.6. The first main result is the following.

Theorem 1 *Assume that* $\dim M > 2\dim \mathrm{G}$. *Then there exists a real number* $p_0>2$ *such that the following holds. Let* $2<p<p_0$, $1-2/p<\lambda<1-2/p_0$ *and* $w:=(u,\Phi,\Psi)\in\widetilde{\mathcal{M}}$ *be a vortex. Then the spaces* $\mathcal{X}_w^{p,\lambda}$ *and* $\mathcal{Y}_w^{p,\lambda}$ *are complete. Furthermore, the operator* $\mathcal{D}_w^{p,\lambda}:\mathcal{X}_w^{p,\lambda}\to\mathcal{Y}_w^{p,\lambda}$ *is well-defined and Fredholm of real index given by*

$$\mathrm{ind}\mathcal{D}_w^{p,\lambda} = \dim M - 2\dim\mathrm{G} + 2m(w).$$

The proof of Theorem 1 has two main ingredients. The first one is a suitable complex trivialization of the bundle $u^*TM\oplus\mathfrak{g}^{\mathbb{C}}$. For $|z|$ large this trivialization respects the splitting $T_{u(z)}M = (\mathrm{im}L^{\mathbb{C}})^{\perp}\oplus\mathrm{im}L_{u(z)}^{\mathbb{C}}$ of the tangent space at $u(z)$ into the "horizontal" and "complex action" parts. It induces an isomorphism of normed vector spaces from $\mathcal{X}_w^{p,\lambda}$ to some weighted Sobolev space on $\mathbb{C}$, and similarly for $\mathcal{Y}_w^{p,\lambda}$. The proof of this relies on a Hardy-type inequality. Since the weighted Sobolev spaces are complete, the same holds for $\mathcal{X}_w^{p,\lambda}$ and $\mathcal{Y}_w^{p,\lambda}$. The second ingredient of the proof of Theorem 1 are two propositions stating that the $\partial_{\bar z}$-operator and a related matrix differential operator are Fredholm maps between suitable weighted Sobolev spaces.

We come now to the next result, the bubbling theorem. We fix real numbers $p>2$ and $E>0$ and define $\widetilde{\mathcal{M}}_E^{1,p}$ to be the set of solutions $w=(u,\Phi,\Psi)\in W^{1,p}_{\mathrm{loc}}(\mathbb{C},M\times\mathfrak{g}\times\mathfrak{g})$ of the vortex equations (0.1), (0.2) with energy $E(w)=E$ such that $\overline{u(\mathbb{C})}$ is compact. We endow $\widetilde{\mathcal{M}}_E^{1,p}$ with the compact open topology. The gauge group $\mathcal{G}^{2,p}:=W^{2,p}_{\mathrm{loc}}(\mathbb{C},\mathrm{G})$ acts on $W^{1,p}_{\mathrm{loc}}(\mathbb{C},M\times\mathfrak{g}\times\mathfrak{g})$ by

$$g^*(u,\Phi,\Psi) := (g^{-1}u, g^{-1}(\Phi g+\partial_s g), g^{-1}(\Psi g+\partial_t g)). \tag{0.5}$$

This action restricts to an action on $\widetilde{\mathcal{M}}^{1,p}_E$.

Question Is the set $\mathcal{M}_E := \widetilde{\mathcal{M}}^{1,p}_E/\mathcal{G}^{2,p}$ endowed with the quotient topology compact?

In general, the answer to this question is **no**. Let for example M be the sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$, ω be the standard volume form on S^2, $J := i$, and $\mathrm{G} := \{\mathbf{1}\}$ be the trivial Lie group. Consider the sequence of holomorphic maps $u_\nu : \mathbb{C} \to S^2$ defined by $u_\nu(z) := z + \nu$, and let $w_\nu := (u_\nu, 0, 0)$. Then $w_\nu \in \widetilde{\mathcal{M}}^{1,p}_{4\pi}$ for every ν, and the sequence (w_ν) converges to the constant map $w \equiv (\infty, 0, 0) \in S^2 \times \{0\} \times \{0\}$, uniformly on every compact subset of $\mathbb{C}$. However, $w \notin \widetilde{\mathcal{M}}^{1,p}_{4\pi}$, since $E(w) = 0$. So in the limit $\nu \to \infty$, all the energy of the sequence w_ν escapes to the point $\infty \in S^2 = \mathbb{C} \cup \{\infty\}$. It follows that $\mathcal{M}^{1,p}_{4\pi} \cong \widetilde{\mathcal{M}}^{1,p}_{4\pi}$ is noncompact. The bubbling theorem remedies this noncompactness, by allowing for some new kind of limit object, namely a stable map of vortices on $\mathbb{C}$ and $\bar{J}$-holomorphic maps from S^2 to the symplectic quotient $\bar{M} := \mu^{-1}(0)/\mathrm{G}$. Here the $\bar{\omega}$-compatible almost complex structure $\bar{J}$ on $\bar{M}$ is induced by the almost complex structure J. Such a stable map is a tuple

$$\big(\bar{T}, V, E, (u_\alpha, \Phi_\alpha, \Psi_\alpha)_{\alpha \in V}, (\bar{u}_\alpha)_{\alpha \in \bar{T}}, (z_{\alpha\beta})_{\alpha E \beta}, (\alpha_i, z_i)_{i=0,\dots,k-1}\big),$$

where $\bar{T}$ and V are finite sets, E is a tree relation on the disjoint union $T := \bar{T} \sqcup V$, $w_\alpha := (u_\alpha, \Phi_\alpha, \Psi_\alpha) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ is a finite energy vortex such that $\overline{u_\alpha(\mathbb{C})}$ is compact, for every $\alpha \in V$, $\bar{u}_\alpha : S^2 \to \bar{M}$ is a $\bar{J}$-holomorphic map, for every $\alpha \in \bar{T}$, $z_{\alpha\beta} \in S^2 \cong \mathbb{C} \cup \{\infty\}$ is a nodal point, for adjacent α and $\beta \in T$ and the pair $(\alpha_i, z_i) \in T \times S^2$ is a marked point, for every $i = 0, \dots, k-1$. Every component $\alpha \in V$ (corresponding to a vortex) is a leaf of the tree $T = \bar{T} \sqcup V$. For each two adjacent components α and β of T the maps w_α (or $\bar{u}_\alpha$) and w_β (or $\bar{u}_\beta$) are connected at the nodal points $z_{\alpha\beta}$ and $z_{\beta\alpha}$. If $\alpha \in V$, corresponding to a vortex, and $\beta \in T$ is a component adjacent to α, then $\beta \in \bar{T}$, corresponding to a sphere, and $z_{\alpha\beta}$ equals the point $\infty \in S^2 \cong \mathbb{C} \cup \{\infty\}$. The special (i.e. marked or nodal) points lying on a fixed component $\alpha \in T$ are all distinct. Furthermore, the stability condition says that every component α of the tree T for which $E(w_\alpha) = 0$, carries at least three special points. The notion of a stable map of vortices on $\mathbb{C}$ and $\bar{J}$-holomorphic spheres in $\bar{M}$ is modelled on the notion of a genus 0 stable map of pseudo-holomorphic spheres, as introduced by M. Kontsevich in [Ko]. In order to state the bubbling theorem, we impose the following hypotheses.

(H2) (Convexity at ∞) *There exists a proper* G*-invariant function* $f : M \to [0, \infty)$ *and a constant* $c > 0$ *such that*

$$f(x) \geq c \Longrightarrow \omega(\nabla_v \nabla f(x), Jv) - \omega(\nabla_{Jv} \nabla f(x), v) \geq 0 \qquad (0.6)$$

for every nonzero vector $v \in T_xM$ and

$$f(x) \geq c \Longrightarrow df(x)JX_{\mu(x)}(x) \geq 0. \tag{0.7}$$

This hypothesis implies that there exists a compact subset $K_0 \subseteq M$ such that the following holds. If $(u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ is a finite energy vortex such that $\overline{u(\mathbb{C})}$ is compact then $u(\mathbb{C}) \subseteq K_0$ (see Proposition D.6(B)).

(H3) (M symplectically aspherical)

$$\int_{S^2} v^*\omega = 0$$

for every smooth map $v : S^2 \to M$.

The bubbling theorem includes also the case of marked points. Note that for some technical reason we add the marked point $z_0^\nu := \infty$ to each member of the sequence of vortices (w_ν).

Theorem 2 (Bubbling) *Assume that the hypotheses (H2), (H3) (and (H1)) hold. Let $w^\nu = (u^\nu, \Phi^\nu, \Psi^\nu) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ be a sequence of positive energy solutions of the vortex equations (0.1), (0.2) such that $\overline{u^\nu(\mathbb{C})}$ is compact for every ν, let $k \geq 0$ be an integer, and let $z_1^\nu, \dots, z_k^\nu \in \mathbb{C}$ be sequences of points. Assume that*

$$\sup_{\nu \in \mathbb{N}} E(w^\nu) < \infty$$

and that

$$\limsup_{\nu \to \infty} |z_i^\nu - z_j^\nu| > 0,$$

for $i \neq j$. Then there exists a subsequence of $(w^\nu, z_0^\nu := \infty, z_1^\nu, \dots, z_k^\nu)$ that converges to some stable map of vortices on $\mathbb{C}$ and $\bar{J}$-holomorphic maps from S^2 to $\bar{M}$ with $k+1$ marked points.

This Theorem generalizes the Gromov compactness theorem, which corresponds to the case $\mathrm{G} := \{\mathbf{1}\}$. Its intuitive meaning is that the space of (equivalence classes of) stable maps (of vortices on $\mathbb{C}$ and $\bar{J}$-holomorphic spheres in $\bar{M}$) compactifies the space of vortices on $\mathbb{C}$. An important feature of convergence of a sequence of vortices w_ν against a stable map is that the limit of the energies $E(w_\nu)$ exists and equals the sum of the energies of the components of the stable map.

To understand why the limit object should consist of vortices on $\mathbb{C}$ and $\bar{J}$-holomorphic spheres, we fix a sequence (w_ν) of vortices on $\mathbb{C}$. Furthermore, we fix a sequence of positive numbers R_ν and points $z_\nu \in \mathbb{C}$, and consider the rescaled sequence

$$(\widetilde{u}_\nu, \widetilde{\Phi}_\nu, \widetilde{\Psi}_\nu) := (u_\nu, R_\nu \Phi_\nu, R_\nu \Psi_\nu)(R_\nu \cdot + z_\nu) : \mathbb{C} \to M \times \mathfrak{g} \times \mathfrak{g}.$$

Then $\widetilde{w}_\nu$ solves the first vortex equation (0.1) and a modified version of the second vortex equation, namely

$$\partial_s \widetilde{\Psi}_\nu - \partial_t \widetilde{\Psi}_\nu + [\widetilde{\Phi}_\nu, \widetilde{\Psi}_\nu] + R_\nu^2 \mu \circ \widetilde{u}_\nu = 0. \tag{0.8}$$

By passing to some subsequence, we may assume w.l.o.g. that R_ν converges to some value $R_0 \in [0, \infty]$. There are three cases.

Case $R_0 = 0$. Then from looking at (0.1) and (0.8), we may expect that in the limit $\nu \to \infty$ we get a solution (u, Φ, Ψ) of (0.1) such that $\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] = 0$. By gauge transforming this solution we may assume w.l.o.g. that $(\Phi, \Psi) = 0$, so $u : \mathbb{C} \to M$ is a J-holomorphic map. By removal of singularities it extends to a J-holomorphic sphere in M. The hypothesis (H3) that (M, ω) is symplectically aspherical implies that u is constant.

Case $0 < R_0 < \infty$. In the limit we get a solution of (0.1) and of

$$\partial_s \Psi - \partial_t \Psi + [\Phi, \Psi] + R_0^2 \mu \circ u = 0.$$

By rescaling we get a vortex.

Case $R_0 = \infty$. Rewriting (0.8) as

$$R_\nu^{-2}(\partial_s \widetilde{\Psi}_\nu - \partial_t \widetilde{\Phi}_\nu + [\widetilde{\Phi}_\nu, \widetilde{\Psi}_\nu]) + \mu \circ \widetilde{u}_\nu = 0,$$

in the limit we expect to get a solution of (0.1) such that $\mu \circ u = 0$. It follows that the map $\mathrm{G}u : \mathbb{C} \to \bar{M} := \mu^{-1}(0)/\mathrm{G}$ is $\bar{J}$-holomorphic, and by removal of singularities it extends to a $\bar{J}$-holomorphic sphere in $\bar{M}$.

The proof of Theorem 2 combines Gromov compactness for pseudo-holomorphic spheres with Uhlenbeck compactness. It involves versions for vortices on $\mathbb{C}$ of all the ingredients of the proof of Gromov compactness for pseudoholomorphic spheres: quantization of energy, an a priori Lemma, compactness modulo bubbling, compactness with bounded energy density, hard rescaling, soft rescaling, an annulus lemma.

The next result guarantees that for a sequence w_ν of vortices on $\mathbb{C}$ converging to a stable map, in the limit $\nu \to \infty$ not only the energy but even the equivariant homology class is preserved.

Proposition 3 (Conservation of homology class) *Assume that (H2) holds. Let* $w^\nu := (u^\nu, \Phi^\nu, \Psi^\nu)$ *be a sequence of finite energy solutions of (0.1), (0.2) such that* $\overline{u^\nu(\mathbb{C})}$ *is compact, and let* $z_1^\nu, \dots, z_k^\nu \in \mathbb{C}$ *be sequences of points. Assume that* $(w^\nu, z_0^\nu := \infty, z_1^\nu, \dots, z_k^\nu)$ *converges to some stable map* $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$. *Then for large enough* ν *the equivariant homology class of* w^ν *equals the equivariant homology class of* $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$.

The following Proposition states that for every positive integer there is a continuous evaluation map from the set $\overline{\mathcal{M}}_k$ of equivalence classes of stable maps with k marked points to $((M \times \mathrm{EG})/\mathrm{G})^k$. Here EG denotes a contractible topological space on which G acts continuously and freely. We denote by $\pi : M_{\mathrm{G}} := (M \times \mathrm{EG})/\mathrm{G} \to M/\mathrm{G}$ the canonical projection to the topological space M/G, and write $\mathrm{G}x \in M/\mathrm{G}$ for the orbit of any point $x \in M$.

Proposition 4 (Existence of a continuous evaluation map) *Assume that hypothesis (H2) (Convexity at* ∞*) is satisfied. Then for every* $k \in \mathbb{N}$ *there exists a map* $\mathrm{ev} : \overline{\mathcal{M}}_k \to M_{\mathrm{G}}^k$ *with the following properties.*

(i) **(Evaluation)** *For every equivalence class of stable maps*

$$[\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}] = [\bar{T}, V, E, (u_\alpha, \Phi_\alpha, \Psi_\alpha)_{\alpha \in V}, (\bar{u}_\alpha)_{\alpha \in \bar{T}}, (z_{\alpha\beta})_{\alpha E \beta}, (\alpha_i, z_i)_{i=0,\dots,k-1}] \in \overline{\mathcal{M}}_k$$

and every index $i \in \{0, \dots, k-1\}$ *we have*

$$\pi(\mathrm{ev}^i([\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}])) = \mathrm{G}u_{\alpha_i}(z_i) \in M/\mathrm{G}.$$

(ii) **(Continuity)** *Let* $(w^\nu, z_0^\nu, \dots, z_{k-1}^\nu)$ *be a sequence of vortices on* $\mathbb{C}$ *and marked points that converges to some stable map* $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ *of vortices on* $\mathbb{C}$ *and pseudoholomorphic spheres in* $\bar{M}$. *Then*

$$\mathrm{ev}([w^\nu, z_0^\nu, \dots, z_{k-1}^\nu]) \to \mathrm{ev}([\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}]).$$

We come now to the application. There always exist solutions of the symplectic vortex equations on $\mathbb{C}$, (0.1), (0.2), namely the constant solutions $(u, \Phi, \Psi) := (x_0, 0, 0)$, where $x_0 \in \mu^{-1}(0)$ is a point, and the solutions gauge equivalent to the constant solutions. These solutions have zero energy. Solutions with finite and positive energy are known to exist in the case of linear actions of the torus $\mathrm{G} := \mathbb{T}^n$ on $M := \mathbb{C}^n$, see the books [Ya] by Y. Yang, and [JT] by A. Jaffe and C. Taubes for the case $n = 1$. The following conjecture gives sufficient criteria for existence of finite and positive energy vortices in the general setting. Assume that the symplectic quotient $(\bar{M}, \bar{\omega})$ is semipositive. Then for every integer $k \geq 0$ and every spherical homology class $\bar{B} \in H_2(\bar{M}, \mathbb{Z})$ the genus 0 Gromov-Witten invariants of $(\bar{M}, \bar{\omega})$ with k fixed marked points are well-defined, see for example the book [MS3] by D. McDuff and D. A. Salamon. They are given by a k-linear map

$$\mathrm{GW}^{\bar{M},\bar{\omega}}_{k,\bar{B}} : H^*(\bar{M})^k \to \mathbb{Z},$$

where $H^*(\bar{M})$ denotes the quotient of $H^*(\bar{M}, \mathbb{Z})$ over its torsion subgroup. Furthermore, under the hypotheses (H1), (H2) and (H3) for every equivariant homology class $B \in H_2^{\mathrm{G}}(M, \mathbb{Z})$ the genus 0 symplectic vortex invariants of (M, ω, μ, B) are well-defined. They are given by a linear map

$$\Phi^{M,\omega,\mu}_B : H^*_{\mathrm{G}}(M) \to \mathbb{Q},$$

see the paper by K. Cieliebak et al. [CGMS].

Conjecture 5 (Existence of vortices on $\mathbb{C}$ with positive energy) *Assume that the hypotheses (H1), (H2) and (H3) are satisfied, that the symplectic quotient $(\bar{M}, \bar{\omega})$ is semipositive, and that $\bar{J}$ is regular. If at least one of the following conditions is satisfied, then there exists a vortex on $\mathbb{C}$ with positive energy.*

(i) There exists an equivariant homology class $0 \neq B \in H_2^{\mathrm{G}}(M, \mathbb{Z})$ such that

$$\Phi^{M,\omega,\mu}_B \not\equiv 0 \tag{0.9}$$

and the following condition holds. If there exists a spherical homology class $\bar{B} \in H_2(\bar{M}, \mathbb{Z})$ with $\kappa_. \bar{B} = B$ then $\bar{B}$ is not a nontrivial multiple of a homology class with first Chern number 0, and every $\bar{J}$-holomorphic sphere representing $\bar{B}$ is simple. Here $\kappa_. : H_*(\bar{M}, \mathbb{Z}) \to H_*(M_{\mathrm{G}}, \mathbb{Z})$ denotes the Kirwan homomorphism.

(ii) There exists a homology class $0 \neq \bar{B} \in H_2(\bar{M}, \mathbb{Z})$ that is not a nontrivial multiple of a homology class with first Chern number 0, such that

$$\mathrm{GW}^{\bar{M},\bar{\omega}}_{3,\bar{B}} \not\equiv 0$$

and every $\bar{J}$-holomorphic map $\bar{u} : S^2 \to \bar{M}$ that represents $\bar{B}$ is simple.

As an example, if the conjecture is true, then there exists a finite and positive energy vortex on $\mathbb{C}$ for the diagonal action of S^1 on $\mathbb{C}^n$, for every integer $n \geq 1$. (As mentioned above, in the case $n = 1$ this is already known.) The idea to prove the conjecture is to use the bubbling techniques developed in this dissertation and the adiabatic limit analysis of R. Gaio and D. A. Salamon [GS], see section 7. No other techniques are needed, in particular no transversality result for vortices on $\mathbb{C}$.

0.2 Relation with other work and motivation for the results

The bubbling phenomenon was discovered by J. Sacks and K. Uhlenbeck in [SU] in the context of minimal surfaces. Later, Gromov [Gr] realized its meaning for symplectic geometry.

Let (Σ, j) be a Riemann surface, dvol be a volume form on Σ that is compatible with j and P be a principal G-bundle over Σ. Then the symplectic vortex equations on P (see the paper [CGS]) are the equations for a pair (u, A), where $u : P \to M$ is a G-invariant map and A is a connection 1-form on P:

$$\bar{\partial}_{J,A}(u) = 0, \qquad (0.10)$$

$$*F_A + \mu \circ u = 0. \qquad (0.11)$$

Here $\bar{\partial}_{J,A}(u)$ denotes the complex antilinear part of $d_A u := du + X_A(u)$, which we think of as a one form on Σ with values in the complex vector bundle $u^*TM/\mathrm{G} \to \Sigma$. Furthermore, we view the curvature F_A of A as a 2-form on Σ with values in the adjoint bundle $\mathfrak{g}_P := (P \times \mathfrak{g})/\mathrm{G}$ and we view $\mu \circ u$ as a map from Σ to $\mathfrak{g}_P$. Moreover, $*$ denotes the Hodge-*-operator w.r.t. the Riemann metric $\mathrm{dvol}(\cdot, j\cdot)$ on Σ. The symplectic vortex equations on $\mathbb{C}$, (0.1), (0.2) correspond to the case $\Sigma := \mathbb{C}$, $j := i$, $\mathrm{dvol} := ds \wedge dt$ and the trivial principal G-bundle $P := \mathbb{C} \times \mathrm{G}$.

Under hypothesis (H1), (H2) and (H3), for a general principal G-bundle P over a *compact* Riemann surface Σ, K. Cieliebak, R. Gaio, I. Mundet i Riera and D. A. Salamon proved in [CGMS] that the space of gauge equivalence classes of vortices on P is compact. Furthermore, without the assumption of symplectic asphericity (H3), in the case $\mathrm{G} := S^1$ and P being a principal S^1-bundle over a *compact* Riemann surface, I. Mundet i Riera and G. Tian compactified the space of gauge equivalence classes of vortices on P in the dissertation [Mu1] and the papers [Mu2] and [MT].

Furthermore, in the case of a *compact* Riemann surface Σ, in [CGMS] K. Cieliebak et al. proved that the augmented vertical differential of the vortex equations is Fredholm. I would also like to mention work related to this dissertation by S. Bradlow [Br], O. Garcia-Prada [G-P] and U. Frauenfelder [Fr1], [Fr2], [Fr3], [Fr4]. In the article [Fr3], U. Frauenfelder gives a new proof for existence of finite and positive energy vortices on the cylinder. His approach is based on Floer theoretical methods.

The main motivation for the results presented above is Conjecture 6 below. It states that there is a ring homomorphism from the equivariant cohomology of M $H^*_{\mathrm{G}}(M)$ to the quantum cohomology of $(\bar{M}, \bar{\omega})$ with coefficients in the equivariant Novikov ring. This homomorphism should intertwine the genus 0 symplectic vortex invariants of (M, ω, μ) with the genus 0 Gromov-Witten invariants of $(\bar{M}, \bar{\omega})$. Under some rather strong additional hypotheses, R. Gaio and D. A. Salamon proved that there is a ring homomorphism from $H^*_{\mathrm{G}}(M)$ to the quantum cohomology with coefficients in the usual Novikov ring, see the dissertation [Ga] and the paper [GS].

The *equivariant Novikov ring* $\Lambda^{\mathrm{G}}_\omega$ is defined as follows. As a set, it consists of all maps $\lambda : H_2(M_{\mathrm{G}}) := H_2(M_{\mathrm{G}}, \mathbb{Z})/\text{torsion} \to \mathbb{Z}$ such that for every real number c the set

$$\big\{B \in H_2(M_{\mathrm{G}}) \,\big|\, \lambda(B) \neq 0, \langle[\omega - \mu]^{\mathrm{G}}, B\rangle \leq c\big\}$$

is finite. Here $[\omega - \mu]^{\mathrm{G}} \in H^2_{\mathrm{G}}(M)$ is the equivariant cohomology class of $\omega - \mu$, see the paper by Cieliebak et al. [CGS]. We define addition in $\Lambda^{\mathrm{G}}_\omega$ in the usual way and multiplication to be convolution. This means that for each two elements $\lambda, \nu \in \Lambda^{\mathrm{G}}_\omega$ we set

$$\lambda \cdot \nu(B) := \sum_{B' \in H_2(M_{\mathrm{G}})} \lambda(B')\nu(B - B'),$$

for every homology class $B \in H_2(M_{\mathrm{G}})$. In the case of the trivial Lie group $\mathrm{G} = \{\mathbf{1}\}$ the equivariant Novikov ring is the usual Novikov ring Λ_ω, which consists of all maps $B : H_2(M) \to \mathbb{Z}$ such that for every real number c the set

$$\big\{B \in H_2(M) \,\big|\, \lambda(B) \neq 0,\ \langle[\omega], B\rangle \leq c\big\}$$

is finite. In the general case, since the Kirwan homomorphism $\kappa_. : H_*(\bar{M}) \to H^{\mathrm{G}}_*(M)$ is injective (see the book by F. Kirwan [Ki]), there is an injective ring homomorphism

$$\Lambda_{\bar{\omega}} \to \Lambda^{\mathrm{G}}_\omega, \quad \bar{\lambda} \mapsto \lambda,$$

where $\lambda : H^{\mathrm{G}}_2(M) \to \mathbb{Z}$ is defined by

$$\lambda(B) := \begin{cases} \bar{\lambda}(\bar{B}), & \text{if } \kappa_. \bar{B} = B, \\ 0, & \text{if } B \notin \mathrm{im}(\kappa_.). \end{cases}$$

Therefore, we can view $\Lambda^{\mathrm{G}}_\omega$ as an extension of $\Lambda_{\bar{\omega}}$. We define the quantum cohomology ring of $(\bar{M}, \bar{\omega})$ with coefficients in $\Lambda^{\mathrm{G}}_\omega$ to be the tensor product

$$\mathrm{QH}^*(\bar{M}, \bar{\omega}, \Lambda^{\mathrm{G}}_\omega) := H^*(M) \otimes_{\mathbb{Z}} \Lambda^{\mathrm{G}}_\omega,$$

with multiplication

$$* : \mathrm{QH}^*(\bar{M}, \bar{\omega}, \Lambda^{\mathrm{G}}_\omega) \times \mathrm{QH}^*(\bar{M}, \bar{\omega}, \Lambda^{\mathrm{G}}_\omega) \to \mathrm{QH}^*(\bar{M}, \bar{\omega}, \Lambda^{\mathrm{G}}_\omega)$$

defined to be the $\Lambda^{\mathrm{G}}_\omega$-bilinear extension of the quantum cup product on ordinary cohomology classes in $H^*(\bar{M})$. (For this see the book by D. McDuff and D. A. Salamon [MS3].) The ring $QH^*(\bar{M}, \bar{\omega}, \Lambda^{\mathrm{G}}_\omega)$ is a $\Lambda^{\mathrm{G}}_\omega$-module and a $\Lambda_{\bar{\omega}}$-module. The quantum cohomology ring of $(\bar{M}, \bar{\omega})$ with coefficients in the usual Novikov ring $\Lambda_{\bar{\omega}}$ can be regarded as a $\Lambda_{\bar{\omega}}$-submodule of $\mathrm{QH}^*(\bar{M}, \bar{\omega}, \Lambda^{\mathrm{G}}_\omega)$.

For each nonnegative integer k and equivariant homology class $B \in H_2(M_{\mathrm{G}})$ the genus 0 Gromov-Witten invariants of $(\bar{M}, \bar{\omega})$ with k fixed marked points can naturally be extended to a $\mathbb{Z}$-multi-linear map

$$\mathrm{GW}_{k,B}^{\bar{M},\bar{\omega}} : (\mathrm{QH}^*(\bar{M}, \bar{\omega}, \Lambda_\omega^{\mathrm{G}}))^k \to \mathbb{Z}.$$

We denote by $\smile: H_{\mathrm{G}}^*(M) \times H_{\mathrm{G}}^*(M) \to H_{\mathrm{G}}^*(M)$ the cup product.

Conjecture 6 *Assume that the hypotheses (H1), (H2) and (H3) are satisfied and that the symplectic quotient* $(\bar{M}, \bar{\omega})$ *is semipositive. Then there exists a ring homomorphism*

$$\varphi : (H_{\mathrm{G}}^*(M), \smile) \to (\mathrm{QH}^*(\bar{M}, \bar{\omega}, \Lambda_\omega^{\mathrm{G}}), *),$$

such that for every nonnegative integer k*, every equivariant homology class* $B \in H_2(M_{\mathrm{G}})$ *and every* k*-tuple of equivariant cohomology classes* $\alpha_1, \ldots, \alpha_k \in H_{\mathrm{G}}^*(M)$ *we have*

$$\mathrm{GW}_B^{\bar{M}}(\varphi(\alpha_1), \cdots, \varphi(\alpha_k)) = \Phi_{k,B}^{M,\omega,\mu}(\alpha_1 \smile \ldots \smile \alpha_k). \tag{0.12}$$

Assuming that the hypotheses (H2) and (H3) (and (H1)) hold, that the action of G on M is monotone (hypothesis (H3) in the paper [GS]) and that $H^*(M_{\mathrm{G}})$ is generated by classes of degree less than twice the minimal Maslov number, R. Gaio and D. A. Salamon [Ga, GS] have defined a ring homomorphism from $H_{\mathrm{G}}^*(M)$ to the quantum cohomomology of $(\bar{M}, \bar{\omega})$ with coefficients in the usual Novikov ring. Their additional assumptions are however rather strong. Monotonicity of the action for example implies that the symplectic quotient is monotone. In their proof, Gaio and Salamon use an adiabatic limit argument. This method fails in the more general situation that is considered here.

The idea for the proof of Conjecture 6 is more geometric. It is to define the map φ by counting solutions of the vortex equations on $\mathbb{C}$. Note that in general, we have to use the *equivariant* Novikov ring $\Lambda_\omega^{\mathrm{G}}$ instead of the usual Novikov ring of the symplectic quotient. Otherwise, the map φ whose construction is indicated below, will not be a ring homomorphism.

The definition of the map φ of Conjecture 6 is based on the symplectic vortex equations on $\mathbb{C}$, (0.1), (0.2). Let $B \in H_2^{\mathrm{G}}(M)$ be an equivariant homology class. We denote by $\mathcal{M}_B$ the set of gauge equivalence classes of vortices that represent B, see section 5. There is an evaluation map at infinity, $\overline{\mathrm{ev}} : \mathcal{M}_B \to \bar{M} = \mu^{-1}(0)/\mathrm{G}$ defined by

$$\overline{\mathrm{ev}}([u, \Phi, \Psi]) := \lim_{r\to\infty} \mathrm{G}u(r) \in \mu^{-1}(0)/\mathrm{G} \subseteq M/\mathrm{G}.$$

The idea is to define

$$\varphi(\alpha) := \sum_{i,B} \varphi_B(\alpha, \bar{e}_i)\bar{e}_i^* e^B := \sum_{i,B} \left(\int_{\mathcal{M}_B} \mathrm{ev}_0^* \alpha \smile \overline{\mathrm{ev}}^* \bar{e}_i \right) \bar{e}_i^* e^B. \tag{0.13}$$

Here $\bar{e}_i$, $i = 1, \dots, N$ and $\bar{e}_i^*$, $i = 1, \dots, N$ are bases of $H^*(\bar{M})$ dual to each other, in the sense that

$$\int_{\bar{M}} \bar{e}_i \smile \bar{e}_j^* = \delta_{ij}.$$

For integral classes $\alpha \in H^*_{\mathrm{G}}(M, \mathbb{Z})$ and $\bar{\beta} \in H^*(\bar{M}, \mathbb{Z})$ the idea is that $\varphi_B(\alpha, \bar{\beta})$ equals the number of gauge equivalence classes of vortices $[w] \in \mathcal{M}_B$ such that w "passes through" X at the point $0 \in \mathbb{C}$ and through $\bar{Y}$ at ∞, where $X \subseteq M_{\mathrm{G}}$ is a "closed oriented submanifold Poincaré dual to α" and $\bar{Y} \subseteq \bar{M}$ is a closed oriented submanifold Poincaré dual to $\bar{\beta}$. (To make this statement precise we have to replace the contractible topological space EG on which G acts continuously and freely by a finite dimensional approximation $\widetilde{\mathrm{EG}} \subseteq \mathrm{EG}$ and M by a compact submanifold with boundary $M' \subseteq M$.)

The idea of proof that φ is a ring homomorphism, is the following. Let $B \in H_2^{\mathrm{G}}(M, \mathbb{Z})$, $\alpha_1, \alpha_2 \in H^*_{\mathrm{G}}(M, \mathbb{Z})$ and $\bar{a} \in H_*(\bar{M}, \mathbb{Z})$. One has to prove that

$$\langle \varphi_B(\alpha_1 \smile \alpha_2), \bar{a} \rangle = \langle (\varphi(\alpha_1) * \varphi(\alpha_2))_B, \bar{a} \rangle. \tag{0.14}$$

Consider the sequences of points $z_1^\nu := -\nu$, $z_2^\nu := \nu \in \mathbb{C}$ and consider a sequence w_ν of vortices on $\mathbb{C}$, representing B and passing through "submanifolds" $X_1, X_2 \subseteq M_{\mathrm{G}}$ at z_1^ν, z_2^ν and through a submanifold $\bar{X} \subseteq \bar{M}$ at ∞, where X_1, X_2 are "Poincaré dual" to α_1, α_2 and $\bar{X}$ represents $\bar{a}$. In the transversal case, it follows from Theorem 2 (Bubbling) that in the limit $\nu \to \infty$ we get a stable map consisting of two vortices on $\mathbb{C}$ and a pseudo-holomorphic sphere in $\bar{M}$, such that certain conditions hold. The number of such stable maps equals the right hand side of (0.14). By the gluing argument described below this number should equal the number of vortices representing B and passing through $X_1, X_2, \bar{X}$ at $-1, 1$ and ∞.

On the other hand, consider the sequences $z_1'^\nu := -\frac{1}{\nu}$, $z_2'^\nu := \frac{1}{\nu}$ and a sequence w_ν of vortices representing B and passing through X_1, X_2 and $\bar{X}$ at $z_1'^\nu, z_2'^\nu$ and ∞ respectively. In the limit $\nu \to \infty$, in the transversal case, we get a single vortex on $\mathbb{C}$ passing through $X_1 \cap X_2$ at 0 and $\bar{X}$ at ∞. The number of such vortices equals the left hand side of (0.14). By a gluing argument this number should equal the number of vortices representing B and passing through $X_1, X_2, \bar{X}$ at $-1, 1$ and ∞. Together with the argument above this would conclude the proof of (0.14).

In order to make sense of the formula (0.13), one has to prove that the maps ev_0 and $\overline{\mathrm{ev}}$ are pseudo-cycles, as defined in the book [MS3]. This involves the bubbling theorem 2, the conservation of the equivariant homology class (Proposition 3) existence of a continuous evaluation map (Proposition 4) and the Fredholm theorem 1. It remains to prove transversality for vortices on $\mathbb{C}$ and for the edge evaluation maps for collections of vortices on $\mathbb{C}$

and $\bar{J}$-holomorphic spheres. In order to show that the map φ intertwines the cup product with the quantum product, in addition, one has to prove the following gluing result. If $\bar{u} : S^2 \cong \mathbb{C} \cup \{\infty\} \to \bar{M}$ is a pseudo-holomorphic sphere and w_1 and w_2 are two vortices on $\mathbb{C}$ such that $\overline{\mathrm{ev}}(w_1) = \bar{u}(1)$ and $\overline{\mathrm{ev}}(w_2) = \bar{u}(2)$, then $\bar{u}$, w_1 and w_2 can be glued together to a new vortex on $\mathbb{C}$ that represents the sum of the homology classes of $\bar{u}$, w_1 and w_2. The proof of the gluing theorem will be similar to the gluing theorem for pseudo-holomorphic spheres in [MS2, MS3]. Finally, in order to prove that φ intertwines $\Phi_{k,B}^{M,\omega,\mu}$ and $\mathrm{GW}_{k,\bar{B}}^{\bar{M},\bar{\omega}}$, one has to prove a bubbling result similar to theorem 2. It says that given a sequence (u_ν, A_ν) of vortices on a principal G-bundle P over S^2 with volume form $\lambda_\nu \mathrm{dvol}$, where dvol is a fixed volume form on S^2, and the numbers $\lambda_\nu > 0$ converge to ∞, there exists a subsequence of (u_ν, A_ν) that converges in a suitable sense to a stable map of vortices on $\mathbb{C}$ and pseudo-holomorphic spheres in $\bar{M}$. The techniques for the proof of this are all developed in this dissertation. Moreover, one has to prove that a $\bar{J}$-holomorphic sphere $\bar{u}$ and k vortices $w_1, \ldots, w_k$ on $\mathbb{C}$ can be glued together to a vortex (u, A) on a principal G-bundle P over S^2 with volume form $\lambda \mathrm{dvol}$, for a large number $\lambda > 0$.

Instead of establishing transversality, one could also try to use abstract perturbation theory, as developed by F. Fukaya and K. Ono [FO], H. Hofer, K. Wysocki, E. Zehnder [HWZ] and others.

0.3 Organization of the dissertation

Section 1 briefly reviews some standard definitions from symplectic geometry that are used throughout this dissertation. It also contains some background information about the symplectic vortex equations on $\mathbb{C}$ and about trees. Trees are the underlying structures of stable maps. Theorem 1 (Fredholm) is proved in section 2 (Corollary 2.9). In section 3 I introduce the notion of a stable map of vortices on $\mathbb{C}$ and of pseudo-holomorphic spheres in the quotient. Section 4 contains the proof of the Bubbling theorem 2 (see Theorem 4.1), section 5 contains the proof of Proposition 3 (Conservation of the equivariant homology class, see Proposition 5.4), and section 6 the proof of Proposition 4 (Existence of a continuous evaluation map, see Proposition 6.1). The application is explained in section 7 (Conjecture 7.1).

1 Preliminaries

1.1 Symplectic manifolds and symplectic quotients

In this subsection we recall some classical concepts referring to the books by D. McDuff and D. A. Salamon [MS1] and by F. Scheck [Sch] for details.

A symplectic structure on a manifold M is a 2-form $\omega \in \Omega^2(M)$ that is closed, i.e. $d\omega = 0$, and nondegenerate, i.e. for every point $x \in M$ and every vector $0 \neq v \in T_xM$ there is a vector $w \in T_xM$ such that $\omega(v, w) \neq 0$. As an example, consider the manifold $M := \mathbb{R}^{2n}$ with the *standard symplectic form* ω_0 defined as follows. For every point $z \in \mathbb{R}^{2n}$ and each two vectors $(\xi^1, \eta_1, \ldots, \xi^n, \eta_n)$, $(\xi'^1, \eta'_1, \ldots, \xi'^n, \eta'_n) \in T_z\mathbb{R}^{2n}$ we define

$$\omega_0\big((\xi^1, \eta_1, \ldots, \xi^n, \eta_n), (\xi'^1, \eta'_1, \ldots, \xi'^n, \eta'_n)\big) := \sum_{i=1}^{n} \xi^i \eta'_i - \xi'^i \eta_i. \tag{1.1}$$

Symplectic structures originated from mechanics. There the configuration space of a system (for example of a particle) is a manifold L, the phase space is the cotangent bundle $M := T^*L$ and there is a canonical symplectic form $\omega_{\mathrm{can}} = -d\lambda_{\mathrm{can}} \in \Omega^2(T^*L)$, where $\lambda_{\mathrm{can}} = pdq$ is the Liouville form. The dimension of every symplectic manifold is an even integer.

An action $\rho : \mathrm{G} \times M \to M$ of a Lie group G on a manifold M is a smooth map satisfying

$$\rho(\mathrm{id}, x) = x, \qquad \rho(gh, x) = \rho(g, \rho(h, x))$$

for every $g, h \in \mathrm{G}$ and $x \in M$. Given an action of G on M then for every fixed $g \in \mathrm{G}$ the map

$$\rho_g : M \to M, \qquad \rho_g(x) := \rho(g, x)$$

is a diffeomorphism. For every element $g \in \mathrm{G}$, every point $x \in M$ and every vector $v \in T_xM$ we abbreviate

$$\begin{aligned} gx &:= \rho(g, x) \in M, \\ gv &:= d\rho_g(x)v. \end{aligned}$$

The action of G on M is called proper iff the map

$$\mathrm{G} \times M \to M \times M, \quad (g, x) \mapsto (x, gx)$$

is proper, in the sense that the preimage of each compact subset of $M \times M$ is compact. We denote by $\mathfrak{g}$ the Lie algebra of G and for each $\xi \in \mathfrak{g}$ we define the vector field X_ξ by

$$X_\xi(x) := \frac{d}{dt}\bigg|_{t=0} \exp(t\xi)x.$$

We choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ that is invariant under the adjoint action of G, i.e.

$$\langle g\xi g^{-1}, g\eta g^{-1} \rangle = \langle \xi, \eta \rangle$$

for every $\xi, \eta \in \mathfrak{g}$ and $g \in \mathrm{G}$.

Assume now that M is equipped with a symplectic structure $\omega \in \Omega^2(M)$. The action of G on M is said to be *Hamiltonian*, iff there exists a smooth map $\mu : M \to \mathfrak{g}$ such that the following conditions are satisfied. The map μ is equivariant, i.e.

$$\mu(gx) = g\mu(x)g^{-1},$$

for every $g \in \mathrm{G}$, $x \in M$. Furthermore, μ generates the action, i.e.

$$\langle d\mu(x)v, \xi \rangle = \omega(X_\xi(x), v),$$

for every $x \in M$, $v \in T_xM$ and $\xi \in \mathfrak{g}$.

Such a map μ is called a moment map for the action of G on M. We assume that a moment map μ exists. Since μ is equivariant, for every point $x \in \mu^{-1}(0)$ and every $g \in \mathrm{G}$ the point gx lies again in $\mu^{-1}(0)$. Therefore, the action of G on M restricts to an action of G on $\mu^{-1}(0)$. We denote the quotient by

$$\bar{M} := \mu^{-1}(0)/\mathrm{G}$$

and the canonical projection to the quotient by $\pi : \mu^{-1}(0) \to \bar{M}$. We assume that this restricted action is free, i.e. if $gx = x$ then $g = \mathrm{id}$, for every $g \in \mathrm{G}$, $x \in \mu^{-1}(0)$. We also assume that G acts properly on $\mu^{-1(0)}$. This condition is satisfied if G is compact. It follows that the quotient $\bar{M}$ is a manifold and carries the symplectic structure $\bar{\omega} \in \Omega^2(\bar{M})$ defined as follows. Given a point $\bar{x} \in \bar{M}$ and vectors $\bar{v}, \bar{w} \in T_{\bar{x}}\bar{M}$, we choose an arbitrary point x in the G-orbit $\bar{x} \subseteq \mu^{-1}(0)$ and two arbitrary vectors $v, w \in T_x\mu^{-1}(0)$ such that $d\pi(x)v = \bar{v}$, $d\pi(x)w = \bar{w}$ and define

$$\bar{\omega}(\bar{v}, \bar{w}) := \omega(v, w). \tag{1.2}$$

This definition does not depend on the choice of the point x in the orbit $\bar{x}$ and of the vectors $v, w \in T_x\mu^{-1}(0)$. The pair $(\bar{M}, \bar{\omega})$ is called the symplectic quotient or the Marsden-Weinstein quotient of the action of G on M. In mechanics, the action of G on M plays the rôle of a symmetry of the system and going to the symplectic quotient corresponds to the reduction of the number of degrees of the system.

1.2 Almost complex structures and pseudo-holomorphic spheres

A good reference for this subsection is the recent book by D. McDuff and D. A. Salamon [MS3].

For every manifold M an endomorphism of the tangent bundle TM is a smooth map $J : TM \to TM$ such that for every $x \in M$ there is a linear map $J_x : T_xM \to T_xM$ such that $J(x, v) = (x, J_x v)$. An almost complex structure on M is an endomorphism $J : TM \to TM$ such that $J^2 = -\mathrm{id}$. As an example, the *standard complex structure* $J := i$ on $M := S^2$ is given by

$$iv := x \times v,$$

for $x \in S^2 \subseteq \mathbb{R}^3$ and $v \in T_xS^2$, where $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ denotes the vector product. We fix a manifold M and an almost complex structure J on M and Riemann surface (Σ, j), i.e. a real two dimensional manifold Σ and an almost complex structure j on Σ. Since the real dimension of Σ is two, the structure j is integrable, i.e. it arises from a holomorphic atlas of charts on Σ. A smooth map $u : \Sigma \to M$ is called *J-holomorphic* iff it solves the Cauchy-Riemann equations

$$du(z)j = Jdu(z) : T_z\Sigma \to T_{u(z)}M, \tag{1.3}$$

for every $z \in \Sigma$. An almost complex structure J on a symplectic manifold (M, ω) is called ω-compatible iff $g_{\omega,J} := \omega(\cdot, J\cdot) : TM \times TM \to \mathbb{R}$ is a Riemannian metric. Furthermore, if M is a manifold, G is a Lie group, and $\rho : \mathrm{G} \times M \to M$ is a smooth action then J is called invariant under ρ iff

$$d\rho_g(x)Jv = Jd\rho_g(x)v,$$

for every $x \in M$, $v \in T_xM$, $g \in \mathrm{G}$.

1.3 The symplectic vortex equations

Let (M, ω) be a symplectic manifold, G be a compact connected Lie group, $\rho : \mathrm{G} \times M \to M$ be a Hamiltonian action and J be an ω-compatible G-invariant almost complex structure on M. There is always such a structure, see the book by D. McDuff and D. A. Salamon [MS1]. We fix a number $p > 2$. The *symplectic vortex equations* on $\mathbb{C}$ are the system of first order nonlinear partial differential equations for a triple $(u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ given by

$$\begin{aligned} \partial_s u + X_\Phi(u) + J(u)(\partial_t u + X_\Psi(u)) &= 0, & (1.4)\\ \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \mu(u) &= 0, & (1.5) \end{aligned}$$

where we write an element of $z \in \mathbb{C}$ as $z = s + it$. These equations were discovered, independently, by K. Cieliebak, R. Gaio and D. A. Salamon [CGS] and I. Mundet i Riera [Mu1], [Mu2]. For the case $M := \mathbb{C}^n$ and G a closed subgroup of $\mathrm{U}(n)$, they were already known in physics as *gauged* *sigma models*, starting with the work of E. Witten [Wi]. Before that, in the case $\mathrm{G} := S^1 \subseteq \mathbb{C}$ acting on $M := \mathbb{C}$ by multiplication, the equations were introduced by V. L. Ginzburg and L. D. Landau in [GL] as a model for superconductivity. Note that in the case that $\mathrm{G} := \{\mathbf{1}\}$ is the trivial Lie group, the vortex equations are equivalent to the Cauchy-Riemann equations given by (1.3). Besides this, other important equations such as anti-self-dual Yang-Mills instantons, Bradlow pairs and the Seiberg-Witten equations are special cases of the symplectic vortex equations, see [CGS].

Consider the set $\mathcal{G}^{2,p} := W^{2,p}_{\mathrm{loc}}(\mathbb{C}, \mathrm{G})$ of *gauge transformations*. It is a group, with neutral element the constant map $\mathbf{1} : \mathbb{C} \to \mathrm{G}$. There is an action

$$\begin{aligned} \mathcal{G}^{2,p}(\mathbb{C}, \mathrm{G}) \times W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g}) \to W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g}), \\ (g; u, \Phi, \Psi) \mapsto g^*(u, \Phi, \Psi) := (g^{-1}u, g^{-1}(\Phi g + \partial_s g), g^{-1}(\Psi g + \partial_t g)). \quad (1.6) \end{aligned}$$

Equations (1.4), (1.5) are invariant under this action, i.e. if $(u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ solves (1.4), (1.5) and $g \in \mathcal{G}^{2,p}$ is a gauge transformation then $g^*(u, \Phi, \Psi)$ also solves (1.4), (1.5). The solutions of (1.4), (1.5) with zero energy are precisely the maps $g^*(w_0, 0, 0)$, where $x_0 \in \mu^{-1}(0)$ is a point and $g \in \mathcal{G}^{2,p}$ is a gauge transformation, see Lemma D.14. In the case of $\mathrm{G} := \mathbb{T}^n$ acting on $M := \mathbb{C}^n$ linearly and diagonally, there exist solutions of (1.4) and (1.5) on $\mathbb{C}$ with finite positive energy, and they can be classified up to gauge equivalence. This follows from Theorem 4.1.1. in the book by Y. Yang, and in the case $n := 1$ from Theorem 1.1 in the book by A. Jaffe and C. Taubes. Furthermore, for S^1 acting on $\mathbb{C}$ by multiplication, in the article [Fr3] U. Frauenfelder gives a new proof that vortices on the cylinder with positive and finite energy exist. These vortices lift to vortices $\mathbb{C}$ of positive and infinite energy. Another existence result is sketched in section 7 of this dissertation. The approach in this article is based on Floer theoretical methods.

We define the energy functional $E : W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g}) \to [0, \infty]$ by

$$\begin{aligned} E(u, \Phi, \Psi) \ := \ & \frac{1}{2} \int_{\mathbb{C}} \big(|\partial_s u + X_\Phi(u)|^2 + |\partial_t u + X_\Psi(u)|^2 + \\ & |\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi]|^2 + |\mu(u)|^2\big)\, ds\, dt. \quad (1.7) \end{aligned}$$

It is invariant under the action of the gauge group $\mathcal{G}^{2,p}$ on $W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$. Assume that $\mu : M \to \mathfrak{g}$ is proper, i.e. the preimage $\mu^{-1}(K) \subseteq M$ is compact, for every compact subset $K \subseteq \mathfrak{g}$. Suppose also that G acts freely on $\mu^{-1}(0)$. Then the equations (1.4),(1.5) arise from E given by (1.7) in the following way. We choose a contractible topological space EG on which G acts continuously and freely. Such a space always exists, see for example the book by V. W. Guillemin and S. Sternberg [GSt]. Let $w := (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ be a solution of (1.4),(1.5) with finite energy $E(w)$ such that $\overline{u(\mathbb{C})}$ is compact. By Lemma 5.2 there exists a continuous map $f : S^2 \cong$

$\mathbb{C} \cup \{\infty\} \to (M \times \mathrm{EG})/\mathrm{G}$ for which a continuous map $\mathsf{e} : \mathbb{C} \to \mathrm{EG}$ exists such that

$$f(z) = [u(z), \mathsf{e}(z)], \qquad \forall z \in \mathbb{C}.$$

We define the equivariant homology class $[w]_{\mathrm{G}} \in H_2(M_{\mathrm{G}}, \mathbb{Z})$ to be the push-forward of the fundamental class $[S^2]$ under f. Again by Lemma 5.2 this definition does not depend on the choice of f. Furthermore, the class $[w]_{\mathrm{G}}$ depends only on the gauge equivalence class of w. Fix a second homology class $B \in H_2((M \times \mathrm{EG})/\mathrm{G}, \mathbb{Z})$. We denote by $\widetilde{\mathcal{M}}_B^{1,p}$ the set of all solutions $w := (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ of (1.4), (1.5) such that $\overline{u(\mathbb{C})}$ is compact and $[w]_{\mathrm{G}} = B$. If $\widetilde{\mathcal{M}}_B$ is nonempty than it can be characterized as the set of global minimizers of the functional E among all maps $w := (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ for which there exists a number $\delta > 0$ such that

$$\sup_{z \in \mathbb{C} \setminus B_1} e_w(z)|z|^{-2-\delta} < \infty.$$

This follows from the proof of the energy identity, Proposition 3.1 in the paper by K. Cieliebak, R. Gaio and D. A. Salamon [CGS], and from Proposition D.6(C).

1.4 Trees

Let T be a finite set. A tree relation on T is a subset $E \subseteq T \times T$ with the following properties.

(i) **(Symmetry)** If $(\alpha, \beta) \in E$ then $(\beta, \alpha) \in E$.

(ii) **(Antireflexivity)** If $(\alpha, \beta) \in E$ then $\alpha \neq \beta$.

(iii) **(No cycles)** If $m \geq 2$ and $\alpha_0, \dots, \alpha_m \in T$ are such that $(\alpha_i, \alpha_{i+1}) \in E$ for $i = 0, \dots, m-1$ and $\alpha_{i+2} \neq \alpha_i$ for $i = 0, \dots, m-2$ then $\alpha_0 \neq \alpha_m$.

(iv) **(Connectedness)** For each two distinct elements $\alpha, \beta \in T$ there are a natural number m and elements $\alpha_1, \dots, \alpha_{m-1} \in T$ such that setting $\alpha_0 := \alpha$ and $\alpha_m := \beta$ we have $(\alpha_i, \alpha_{i+1}) \in E$ for $i = 0, \dots, m-1$.

An element of T is called a *vertex* and each pair $(\alpha, \beta) \in E$ is called an *oriented edge*. Furthermore, the set $\{(\alpha, \beta), (\beta, \alpha)\} \subseteq E$ is called an *unoriented edge*. Let $T := \{0, \dots, m\}$. An example of a tree relation on T is

$$E_1 := \{(i, i+1) \,|\, i = 1, \dots, m-1\} \cup \{(i+1, i) \,|\, i = 1, \dots, m-1\}.$$

Another example is

$$E_2 := \{(0, i) \,|\, i = 1, \dots, m\} \cup \{(i, 0) \,|\, i = 1, \dots, m\}.$$

We can represent a tree (T, E) graphically by drawing a point x_α on the plane for each vertex $\alpha \in T$ and the straight line segment from x_α to x_β for each pair $(\alpha, \beta) \in E$. The tree $(\{0, \ldots, m\}, E_2)$ corresponds to a "flower" whose center corresponds to the vertex $0 \in T$.

Let E be a tree relation on T. We write $\alpha E \beta$ for the statement $(\alpha, \beta) \in E$ and say that α and β are adjacent iff this holds. For $\alpha, \beta \in T$ we define the *(oriented) chain of edges running from α to β* to be

$$[\alpha, \beta] := (\alpha_0, \alpha_1, \ldots, \alpha_m),$$

where $\alpha_i \in T$, $i = 0, \ldots, m$ are the unique vertices such that $\alpha_0 = \alpha$, $\alpha_m = \beta$ and $\alpha_{i-1} E \alpha_i$ for $i = 1, \ldots, m$.
Assume now that $\alpha E \beta$ and consider the forest obtained from T by deleting the edge $\{(\alpha, \beta), (\beta, \alpha)\}$. We define the subtree $T_{\alpha\beta} \subseteq T$ to be the connected component of this forest containing the vertex β.
A vertex $\alpha \in T$ is called an *leaf of* T iff there is at most one vertex $\beta \in T$ such that $\alpha E \beta$.

Let (T, E) and (T', E') be trees and $f : T \to T'$ be a map. Then f is called a *tree isomorphism* iff it is bijective and if $\alpha E \beta$ implies $f(\alpha) E' f(\beta)$. Note that if f is a tree isomorphism and $\alpha' E' \beta'$ then $f^{-1}(\alpha') E f^{-1}(\beta')$. This follows from the fact

$$|E'| = 2|T'| - 2 = 2|T| - 2 = |E|.$$

1.5 Notation

Our convention for the natural numbers is $\mathbb{N} := \{1, 2, \ldots\}$. Let $N \in \mathbb{N}$ be a number, $x \in \mathbb{R}^N$ a point and $r > 0$. We denote by $B_r(x)$ and $\bar{B}_r(x)$ the open and the closed ball and by $S_r(x)$ the sphere with radius r around x. Furthermore,

$$\langle x \rangle := \sqrt{|x|^2 + 1}.$$

We denote the space of Schwartz functions on $\mathbb{R}^N$ by $\mathcal{S}$ and the space of temperate distributions by $\mathcal{S}'$. Furthermore, by $\widehat{\ } : \mathcal{S}' \to \mathcal{S}'$ we denote the Fourier transform, defined by

$$\widehat{u}(\varphi) := u(\widehat{\varphi}),$$

for $\varphi \in \mathcal{S}$, where

$$\widehat{\varphi}(\xi) := (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \varphi(x) e^{-i(\xi, x)} \, d^N x.$$

Moreover, we denote by $^\vee : \mathcal{S}' \to \mathcal{S}'$ the inverse Fourier transform. For $\psi \in \mathcal{S}$ it is given by

$$\overset{\vee}{\psi}(x) := (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \psi(\xi) e^{i(\xi,x)}\, d^N\xi.$$

We denote the closed unit disk in $\mathbb{C}$ by

$$\mathbb{D} := \{z \in \mathbb{C} \,|\, |z| \le 1\}$$

and the closed annulus around $z \in \mathbb{C}$ with radii $0 \le r \le r' \le \infty$ by

$$A(z, r, r') := \bar{B}_{r'}(z) \setminus B_r(z_0).$$

Here if $r = 0$ then $B_r(z) := \emptyset$ and if $r = \infty$ then $\bar{B}_r(z) := \mathbb{C}$. We identify $S^1 \cong \mathbb{R}/\mathbb{Z}$. The sphere $S^2 \subseteq \mathbb{R}^3$ carries the metric g_{S^2} and the volume form dvol_{S^2} induced from $\mathbb{R}^3$. We identify $S^2 \cong \mathbb{C} \cup \{\infty\}$, using stereographic projection. Under this identification we have

$$g_{S^2} = \frac{4}{(1+|z|^2)^2}(\cdot,\cdot), \qquad \mathrm{dvol} = \frac{4}{(1+|z|^2)^2} ds \wedge dt,$$

where $(\cdot,\cdot)$ is the standard inner product on $\mathbb{R}^2$. For every $z \in S^2$ and every real linear map $\alpha : T_zS^2 \to \mathbb{C}$ we denote by $|\alpha|$ the norm of the differential w.r.t. to the metrics g_{S^2} and $(\cdot,\cdot)$. This means that

$$|\alpha| := \sqrt{|\alpha(e_1)|^2 + |\alpha(e_2)|^2},$$

where e_1, e_2 is an g_{S^2}-orthonormal basis of T_zS^2.

For every topological group G we denote by EG an arbitrary contractible topological space on which G acts continuously and freely and by BG := EG /G the classifying space of G. That such an EG exists is proven in the book by Husemoller [Hu]. We denote $M_{\mathrm{G}} := M \times_{\mathrm{G}} \mathrm{EG} := (M \times \mathrm{EG})/\mathrm{G}$.

Let $\rho : \mathrm{G} \times M \to M$ be an action of a Lie group G on a manifold M, $g \in \mathrm{G}$ and $x \in M$. We abbreviate $gx := \rho(g, x)$. Furthermore, for every $v \in T_xM$ we denote by $gv := d\rho_g(x)v \in T_{gx}M$ the differential of the action of g. We denote the projection to the quotient by $pi : M \to M/\mathrm{G}$, and the orbit of a point $x \in M$ by $\mathrm{G}x = \pi(x)$. Recall that for every $\xi \in \mathfrak{g}$ the infinitesimal action of ξ is the vector field X_ξ on M defined by

$$X_\xi(x) := \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)x.$$

For every $x \in M$ we define $L_x : \mathfrak{g} \to T_xM$ by

$$L_x\xi := X_\xi(x). \tag{1.8}$$

The map L_x is linear. The notation $M^* \subseteq M$ means the subset of points $x \in M$ on which G acts freely.

We fix a symplectic manifold (M,ω). For every ω-compatible almost complex structure J on M we denote by $g_{\omega,J} := \omega(\cdot, J\cdot)$ the Riemannian metric induced by ω and J. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ that is invariant under the adjoint action of G. Assume that G acts on M in a Hamiltonian way with moment map $\mu : M \to \mathfrak{g}$. Then for every $c \geq 0$ we denote

$$M_c := \{x \in M \,|\, |\mu(x)| \leq c\}.$$

Recall that an almost complex structure J on M is called G-invariant iff for every $g \in \mathrm{G}$ and $x \in M$ we have

$$J d\rho_g(x) = d\rho_g(x) J.$$

We fix a structure G-invariant ω-compatible almost complex structure J on M. Unless otherwise mentioned, all norms of vectors in TM are w.r.t. $g_{\omega,J}$. For every point $x \in M$ and every linear subspace $V \subseteq T_xM$ we denote by $V^\perp \subseteq T_xM$ the orthogonal complement of V w.r.t. $g_{\omega,J}$. For every $x \in M$ we define the *horizontal subspace* to be

$$H_x := \ker d\mu(x) \cap (\mathrm{im} L_x)^\perp \subseteq T_xM.$$

We define the almost complex structure $\bar{J}$ on the symplectic quotient $\bar{M} := \mu^{-1}(0)/\mathrm{G}$ as follows. For every point $\bar{x} \in \bar{M}$ and every vector $\bar{v} \in T_{\bar{x}}\bar{M}$ we define

$$\bar{J}\bar{v} := d\pi(x)Jv, \tag{1.9}$$

where $x \in \mu^{-1}(0)$ is a point in the orbit $\bar{x}$ and $v \in H_x$ is the unique horizontal vector such that $d\pi(x)v = \bar{v}$. We define the metric $\bar{g} := \bar{g}_{\omega,J}$ on M^*/G as follows. For every $\bar{x} \in M^*/\mathrm{G}$, $\bar{v}, \bar{w} \in T_{\bar{x}}(M^*/\mathrm{G})$ we define

$$\bar{g}(\bar{v}, \bar{w}) := g_{\omega,J}(v, w), \tag{1.10}$$

where $x \in M^*$ is any point such that $\pi(x) = \bar{x}$ and $v, w \in \mathrm{Im}\, L_x^\perp$ are the unique vectors such that $d\pi(x)v = \bar{v}$, $d\pi(x)w = \bar{w}$. Note that the restriction of $\bar{g}$ to the submanifold $\bar{M} \subseteq M^*/\mathrm{G}$ coincides with the metric induced by $\bar{\omega}$ and $\bar{J}$.

Fix $p > 2$, $R \in [0,\infty]$ and $w := (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$. We abbreviate

$$\kappa := \partial_s\Psi - \partial_t\Phi + [\Phi, \Psi].$$

If $0 < R < \infty$ we define the *R-energy density of w* by

$$e^R_w := \frac{1}{2}\big(|\partial_s u + X_\Phi \circ u|^2 + |\partial_t u + X_\Psi \circ u|^2 + R^{-2}|\kappa|^2 + R^2|\mu \circ u|^2\big). \tag{1.11}$$

If $R = 0$ and $\kappa = 0$ or if $R = \infty$ and $\mu \circ u = 0$ we define

$$e^R_w := \frac{1}{2}\big(|\partial_s u + X_\Phi \circ u|^2 + |\partial_t u + X_\Psi \circ u|^2\big). \tag{1.12}$$

For every measurable subset $X \subseteq \mathbb{C}$ we define the *R-energy of w on X* by

$$E^R(w, X) := \int_X e_w^R \, ds\, dt \in [0, \infty]. \tag{1.13}$$

In the case $R = 1$ we abbreviate $e_w := e_w^1$ and $E(w, X) := E^1(w, X)$ and call this the *energy density* of w and the *energy* of w on X for short. In the case $X = \mathbb{C}$ we write $E^R(w) := E^R(w, \mathbb{C})$. Moreover, we denote the energy of a map $\bar{u} \in W^{1,p}(X, \bar{M})$ on a measurable subset $X \subseteq S^2$ by

$$E(\bar{u}, X) := \frac{1}{2} \int_X |d\bar{u}|^2 \mathrm{dvol}_{S^2}. \tag{1.14}$$

For every integer $k \geq 1$ and every real number $p > 2$ we define

$$\begin{aligned} \widetilde{\mathcal{M}}_J^{k,p} := \big\{ & (u, \Phi, \Psi) \in W^{k,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g}) \mid \\ & (0.1), (0.2), E(u, \Phi, \Psi) < \infty, \overline{u(\mathbb{C})} \text{ compact} \big\}. \end{aligned} \tag{1.15}$$

Furthermore, we define

$$\begin{aligned} \widetilde{\mathcal{M}}_J := \big\{ & (u, \Phi, \Psi) \in C^\infty(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g}) \mid \\ & (0.1), (0.2),\ E(u, \Phi, \Psi) < \infty,\ \overline{u(\mathbb{C})} \text{ compact} \big\}. \end{aligned} \tag{1.16}$$

If it is clear which almost complex structure J is meant, we will drop it from the notation.

Remark 1.1 By Proposition D.6(B) there is a compact subset $K_0 \subseteq M$ such that for every $w \in \widetilde{\mathcal{M}}^{1,p}$ we have $\overline{u(\mathbb{C}) \subseteq K_0}$. Therefore, in the definition of $\widetilde{\mathcal{M}}^{1,p}$, we could replace the condition that $\overline{u(\mathbb{C})}$ is compact by the condition that $u(\mathbb{C})$ is contained in the fixed compact set K_0.

The gauge group

$$\mathcal{G}^{k+1,p} := W^{k+1,p}_{\mathrm{loc}}(\mathbb{C}, \mathrm{G}). \tag{1.17}$$

acts on $\widetilde{\mathcal{M}}^{k,p}$ by

$$g^*(u, \Phi, \Psi) := (g^{-1}u, g^{-1}\partial_s g + g^{-1}\Phi g, g^{-1}\partial_t g + g^{-1}\Psi g). \tag{1.18}$$

We call two vortices $w, w' \in \widetilde{\mathcal{M}}^{1,p}$ gauge equivalent iff there is a gauge transformation $g \in \mathcal{G}^{2,p}$ such that $g^* w = w'$. We denote the gauge equivalence class of $w \in \widetilde{\mathcal{M}}^{1,p}$ by $[w] := (\mathcal{G}^{2,p})^* w$ and the set of all such equivalence classes by

$$\mathcal{M} := \widetilde{\mathcal{M}}^{1,p} / \mathcal{G}^{2,p}.$$

The group of smooth gauge transformations

$$\mathcal{G} := C^\infty(\mathbb{C}, \mathrm{G}) \tag{1.19}$$

acts on $\widetilde{\mathcal{M}}$ by formula (1.18). It follows from the proof of Proposition D.3 that the inclusion $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}^{1,p}$ induces a bijection

$$\widetilde{\mathcal{M}} / \mathcal{G} \to \widetilde{\mathcal{M}}^{1,p} / \mathcal{G}^{2,p} = \mathcal{M}.$$

The energy E is invariant under gauge transformation and therefore we can define $E([w]) := E(w)$ for every $[w] \in \mathcal{M}$.

2 The Fredholm property

Let (M, ω) be a symplectic manifold, G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ that is invariant under the adjoint action of G. Let G act on M in a Hamiltonian way, with moment map $\mu : M \to \mathfrak{g}$. Here we identify the dual space $\mathfrak{g}^*$ with $\mathfrak{g}$, using the inner product $\langle\cdot,\cdot\rangle$. We fix an ω-compatible G-invariant almost complex structure J on M. Let $w := (u, \Phi, \Psi)$ be a finite energy solution of the symplectic vortex equations (0.1), (0.2) on $\mathbb{C}$ such that the closure $\overline{u(\mathbb{C})}$ is compact. The main result of this section states that the vertical differential of the vortex equations at w, augmented by some gauge fixing operator, is a Fredholm map between suitable weighted Sobolev spaces. This means that it is bounded, its kernel has finite dimension, and its image is closed and has finite codimension. Its index, i.e. the dimension of its kernel minus the codimension of its image, equals the dimension of the symplectic quotient plus twice the Maslov index of w. The situation here differs heavily from the case of the vortex equations on some principal G-bundle over a *compact* surface. In that case K. Cieliebak et al. already proved a Fredholm result using ordinary Sobolev spaces, see [CGMS], Proposition 4.6. In the present situation, the augmented vertical differential $\mathcal{D}_w$ is an operator from a space of sections of the bundle $u^*TM \oplus \mathfrak{g} \oplus \mathfrak{g}$ with noncompact base $\mathbb{C}$ to another space of sections of this bundle. Fix a nonnegative integer k and a real number $p > 2$. Since $\mathbb{C}$ is noncompact, the definition of the space of $W^{k,p}$-vector fields along w is not canonical, but it depends on a choice of weights (or of a metric). Furthermore, the operator $\mathcal{D}_w$ will only be Fredholm for a suitable choice of the weights, and its index will depend on them. The weighted Sobolev spaces we will use are natural in the sense that the weights correspond to the asymptotical behaviour of vortices on $\mathbb{C}$.

Since the Kondrachov compactness theorem does not hold on $\mathbb{C}$, the 0-th order terms in the operator $\mathcal{D}_w$ are no longer compact. This means that we can not ignore them. It forces us to incorporate the splitting $TM = H \oplus (\mathrm{im}L \oplus J\mathrm{im}L)$ into the definition of the domain of $\mathcal{D}_w$. Here H denotes the "horizontal bundle" defined by $H_x := \ker d\mu(x) \cap \mathrm{im}L_x^\perp$, for $x \in M$. We denote by $\mathfrak{g}^\mathbb{C} := \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ the complexified Lie algebra. Then for each vector field $\zeta := (v, \beta) : \mathbb{C} \to u^*TM \oplus \mathfrak{g}^\mathbb{C}$ along $w := (u, \Phi + i\Psi)$ the orthogonal projection of $v : \mathbb{C} \to u^*TM$ to the "complex action sub-bundle" $\mathrm{im}L \oplus J\mathrm{im}L \subseteq TM$ plays the rôle of a counter part of the "complex gauge part" $\beta : \mathbb{C} \to \mathfrak{g}^\mathbb{C}$. This correspondence is incorporated in the operator $\mathcal{D}_w$. Namely, it is in some sense symmetrical w.r.t. the orthogonal projection of v to the "complex action bundle" and the component β, see formula (2.31).

Throughout this section, we will use a complexified notation.

2.1 An abstract setting

In this subsection I set up the notation, in order to state the main result of the section, Theorem 2.8. Furthermore, I motivate it by explaining the abstract setting it fits into. Namely, intuitively, the vortex equations over $\mathbb{C}$ can be viewed as a section $\mathcal{S}$ of an infinite dimensional vector bundle $\mathcal{E}$ over an infinite dimensional manifold $\mathcal{B}$. The abstract framework will not be used any further after this subsection. Readers who want to get to the core of the matter may directly turn to subsection 2.2.

We denote the action of an element $g \in \mathrm{G}$ on a point $x \in M$ by $gx := \rho_g(x)$. Furthermore, if $v \in T_xM$ is a tangent vector, we write $gv := d\rho_g(x)v \in T_{gx}M$. We choose a linear torsionfree connection ∇ on TM that is invariant under the action of G, i.e.

$$\nabla_{gv} gX = g\nabla_v X, \tag{2.1}$$

for every vector field X on M, every tangent vector $v \in T_xM$ with $x \in M$, and every element $g \in \mathrm{G}$. For instance we may choose ∇ to be the Levi-Civita connection of the Riemannian metric $g_{\omega,J} := \omega(\cdot, J\cdot)$, since the action of G on M preserves ω and J, and hence $g_{\omega,J}$. We fix a map $(u, \Phi, \Psi) \in C^\infty(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$. We use the notation

$$\partial_s^\Phi u := \partial_s u + X_\Phi(u), \qquad \partial_t^\Psi u := \partial_t u + X_\Psi(u), \qquad d^{\Phi,\Psi} u := \partial_s^\Phi u\, ds + \partial_t^\Psi u\, dt.$$

We denote by $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the complexified Lie algebra, and we write an element of $\mathfrak{g}^{\mathbb{C}}$ as $\Phi + i\Psi$, where $(\Phi, \Psi) \in \mathfrak{g} \times \mathfrak{g}$. We fix a positive integer N, two real numbers λ and $1 < p < \infty$ and a measurable subset $X \subseteq \mathbb{R}^N$. For every vector $x \in R^N$ we denote

$$\langle x \rangle := \sqrt{|x|^2 + 1}.$$

We define the (possibly infinite) (p, λ)-norm of a measurable function $u : X \to \mathbb{C}$ to be

$$||u||_{L^p_\lambda(X)} := \left(\int_X \left(|u| \langle \cdot \rangle^\lambda \right)^p \right)^{\frac{1}{p}}. \tag{2.2}$$

If $X = \mathbb{R}^N$, we abbreviate

$$||u||_{p,\lambda} := ||u||_{L^p_\lambda(\mathbb{R}^N)}, \qquad L^p_\lambda := L^p_\lambda(\mathbb{R}^N).$$

We define

$$L^p_\lambda(X) := \{ u : X \to \mathbb{C} \mid u \text{ measurable}, ||u||_{L^p_\lambda(X)} < \infty \}. \tag{2.3}$$

For every pair of real numbers $p > 2$ and $\lambda > -2/p + 1$ we define the set

$$\begin{aligned} \mathcal{B}^p_\lambda := \big\{ & (u, \Phi + i\Psi) \in C^\infty(\mathbb{C}, M \times \mathfrak{g}^{\mathbb{C}}) \,\big|\, \overline{u(\mathbb{C})} \text{ is compact}, \\ & |\mu \circ u(re^{i\varphi})| \to 0,\, r \to \infty,\, \forall \varphi,\, ||d^{\Phi,\Psi} u|| \in L^p_\lambda, \\ & \exists g \in C^\infty(\mathbb{C} \setminus B_1, \mathrm{G}) :\ g^*(\Phi + i\Psi) \in L^p_\lambda(\mathbb{C} \setminus B_1, \mathfrak{g}^{\mathbb{C}}) \big\}. \end{aligned} \tag{2.4}$$

Recall the definition (0.3) of the energy functional. We define

$$\widetilde{\mathcal{M}} := \big\{ w := (u, \Phi + i\Psi) \in C^\infty(\mathbb{C}, M \times \mathfrak{g}^\mathbb{C}) \mid \\ w \text{ solves (0.1), (0.2), } E(w) < \infty, \overline{u(\mathbb{C})} \text{ is compact} \big\}. \tag{2.5}$$

We will see below that this set is the zero set of a section of an "infinite dimensional vector bundle" over $\mathcal{B}^p_\lambda$. For this to be true it is necessary that the set $\widetilde{\mathcal{M}}$ is contained in $\mathcal{B}^p_\lambda$. That this is the case for a good choice of the parameters p and λ is the content of Proposition D.11 in the appendix. It states that there exists a number $p_0 > 2$ such that for every $2 < p < p_0$ and every $-2/p+1 < \lambda < -2/p_0+1$ we have $\widetilde{\mathcal{M}} \subseteq \mathcal{B}^p_\lambda$. By Remark D.12, if $q > 2$ and $\mu > -2/q+1$ are such that $\widetilde{\mathcal{M}} \subseteq \mathcal{B}^q_\mu$, then $\widetilde{\mathcal{M}}$ is also contained in $\mathcal{B}^p_\lambda$ for every pair (p, λ) such that $2 < p \leq q$ and $\lambda + 2/p < \mu + 2/q$. Furthermore, using the Uhlenbeck gauge theorem, it should be possible to prove a stronger version of Proposition D.11 stating that there exists a number $\varepsilon > 0$ such that for every $p > 2$ and every $-2/p+1 < \lambda < -2/p+1+\varepsilon$ we have $\widetilde{\mathcal{M}} \subseteq \mathcal{B}^p_\lambda$. (Note that setting $p_0 := 2/(1-\varepsilon)$ this would imply Proposition D.11.)

We fix $p > 2$ and $\lambda > -2/p+1$ and abbreviate $\mathcal{B} := \mathcal{B}^p_\lambda$. The gauge group $\mathcal{G} := C^\infty(\mathbb{C}, \mathrm{G})$ acts on $\mathcal{B}$ by the formula

$$g^*(u, \Phi + i\Psi) := \Big(g^{-1}u, g^{-1}\big(\partial_s + \Phi + i(\partial_t + \Psi)\big)g \Big). \tag{2.6}$$

We define

$$\widehat{\mathcal{B}} := \mathcal{B}/\mathcal{G}.$$

Formally, we may think of $\widehat{\mathcal{B}}$ as a Banach manifold. For each $W \in \widehat{\mathcal{B}}$ we define a Banach space $\mathcal{X}_W$, of which we think of as the "tangent space" of $\widehat{\mathcal{B}}$ at W, in the following way. We first fix a map $w := (u, \Phi + i\Psi) \in \mathcal{B}$ and define the space $\mathcal{X}_w$ as follows.

We extend the definition of the Lie bracket on $\mathfrak{g}$ to the complexified Lie algebra $\mathfrak{g}^\mathbb{C}$ by setting

$$[\varphi + i\psi, \varphi' + i\psi'] := [\varphi, \varphi'] - [\psi, \psi'] + i\big([\psi, \varphi'] + [\varphi, \psi']\big).$$

Furthermore, we identify $\mathfrak{g}^\mathbb{C}$ with the trivial bundle $\mathbb{C} \times \mathfrak{g}^\mathbb{C}$ and abbreviate

$$u^*TM \oplus \mathfrak{g}^\mathbb{C} := u^*TM \oplus (\mathbb{C} \times \mathfrak{g}^\mathbb{C}).$$

This is a smooth complex vector bundle over $\mathbb{C}$. We denote by $W^{1,p}_{\mathrm{loc}}(\mathbb{C}, u^*TM \oplus \mathfrak{g}^\mathbb{C})$ the space of locally $W^{1,p}$-sections of $u^*TM \oplus \mathfrak{g}^\mathbb{C}$ and fix a section $(v, \beta) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, u^*TM \oplus \mathfrak{g}^\mathbb{C})$. The covariant derivative of v in the s-direction $\nabla_s v$ is the section of the pullback bundle $u^*TM \to \mathbb{C}$ defined as follows. Let $s_0 + it_0 \in \mathbb{C}$ be a point. We choose a local frame around the point $u(s_0+it_0) \in M$, i.e. a collection of vector fields $e_1, \dots e_{2n}$ defined on some neighbourhood $U \subseteq M$ of $u(s_0 + it_0)$ that span the tangent space of M at every point in U.

Writing $v =: \sum_{i=1}^{2n} v^i e_i \circ u$ on the neighbourhood $u^{-1}(U) \subseteq \mathbb{C}$ of $s_0 + it_0$, we define

$$\nabla_s v(s_0 + it_0) := \sum_{i=1}^{2n} \big(v^i \nabla_{\partial_s u} e_i + (\partial_s v^i) e_i \circ u\big)(s_0 + it_0).$$

We define $\nabla_t v$ analogously, and we denote

$$\nabla_s^\Phi v := \nabla_s v + \nabla_v X_\Phi, \qquad \nabla_t^\Psi v := \nabla_t v + \nabla_v X_\Psi, \tag{2.7}$$

$$\partial_s^\Phi \beta := \partial_s \beta + [\Phi, \beta], \qquad \partial_t^\Psi \beta := \partial_t \beta + [\Psi, \beta], \tag{2.8}$$

$$\nabla_s^\Phi (v, \beta) := (\nabla_s^\Phi v, \partial_s^\Phi \beta), \qquad \nabla_t^\Psi (v, \beta) := (\nabla_t^\Psi v, \partial_t^\Psi \beta). \tag{2.9}$$

Moreover, we define $\nabla^{\Phi,\Psi} v$ to be the real linear one form on $\mathbb{C}$ with values in u^*TM given by

$$(\nabla^{\Phi,\Psi} v)_z(\sigma + i\tau) := \big(\sigma \nabla_s^\Phi v + \tau \nabla_t^\Psi v\big)(z),$$

where $z \in \mathbb{C}$ and $\sigma + i\tau \in \mathbb{C} \cong T_z\mathbb{C}$. Moreover, we define $d^{\Phi,\Psi}\beta$ to be the real linear one form on $\mathbb{C}$ with values in $\mathfrak{g}^\mathbb{C}$ given by

$$(d^{\Phi,\Psi}\beta)_z(\sigma + i\tau) := \big(\sigma \partial_s^\Phi + \tau \partial_t^\Psi\big)\beta,$$

and we denote

$$\nabla^{\Phi,\Psi}(v, \beta) := \big(\nabla^{\Phi,\Psi} v, d^{\Phi,\Psi}\beta\big). \tag{2.10}$$

Furthermore, we define the hermitian metric $g^\mathbb{C}$ on TM

$$g^\mathbb{C}(v, v') := g_{\omega,J}(v, v') + i\omega(v, v'), \tag{2.11}$$

and the hermitian inner product $\langle \cdot, \cdot \rangle_\mathbb{C}$ on $\mathfrak{g}^\mathbb{C}$ by

$$\langle \varphi + i\psi, \varphi' + i\psi' \rangle_\mathbb{C} := \langle \varphi, \varphi' \rangle + \langle \psi, \psi' \rangle + i\big(\langle \varphi, \psi' \rangle - \langle \psi, \varphi' \rangle\big). \tag{2.12}$$

Both the metric $g^\mathbb{C}$ and the inner product $\langle \cdot, \cdot \rangle_\mathbb{C}$ are complex antilinear in the first argument. We denote by $\widehat{P} : TM \to TM$ the orthogonal projection to the "complex action subbundle" $\mathrm{im}L \oplus J\mathrm{im}L \subseteq TM$, and we set

$$P := \widehat{P} \oplus \mathrm{id} : TM \oplus \mathfrak{g}^\mathbb{C} \to TM \oplus \mathfrak{g}^\mathbb{C}. \tag{2.13}$$

We define the (possibly infinite) (p, λ, w)-norm of a section

$$\zeta := (v, \beta) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, u^*TM \oplus \mathfrak{g}^\mathbb{C})$$

to be

$$||\zeta||_w := ||\zeta||_w^{p,\lambda} := ||\zeta||_{L^\infty(\mathbb{C})} + ||\,|P\zeta| + |\nabla^{\Phi,\Psi}\zeta|\,||_{L^p_\lambda(\mathbb{C})}. \tag{2.14}$$

Here the norms on each tangent space of M and on $\mathfrak{g}^\mathbb{C}$ are taken w.r.t. $g_{\omega,J}$ and $\langle \cdot, \cdot \rangle_\mathbb{C}$. We define now the space $\mathcal{X}_w$ to be

$$\mathcal{X}_w := \mathcal{X}_w^{p,\lambda} := \big\{\zeta \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, u^*TM \oplus \mathfrak{g}^\mathbb{C}) \,\big|\, ||\zeta||_w < \infty\big\}. \tag{2.15}$$

By Corollary 2.13 below this space is complete. Consider the map

$$L_w : C_0^\infty(\mathbb{C}, \mathfrak{g}) \to \mathcal{X}_w, \qquad L_w\xi := \big(L_u\xi, -\partial_s^\Phi\xi - i\partial_t^\Psi\xi\big).$$

Formally, this map corresponds to the infinitesimal action of the Lie algebra of the gauge group $\mathcal{G}$ on $\mathcal{B}$. We denote by $C_0^\infty(\mathbb{C}, \mathfrak{g})$ and $C_0^\infty(\mathbb{C}, u^*TM \oplus \mathfrak{g}^{\mathbb{C}})$ the spaces of C^∞-maps and C^∞-sections with compact support. Consider the real L^2-inner product on $C_0^\infty(\mathbb{C}, \mathfrak{g})$ given by

$$\langle \xi, \eta\rangle_{L^2} := \int_{\mathbb{C}} \langle \xi, \eta\rangle \, ds\, dt,$$

and the real L^2-inner product on $C_0^\infty(\mathbb{C}, u^*TM \oplus \mathfrak{g}^{\mathbb{C}})$ given by

$$\langle (v, \varphi + i\psi), (v', \varphi' + i\psi')\rangle_{L^2} := \int_{\mathbb{C}} \big(g_{\omega,J}(v, v') + \langle \varphi, \varphi'\rangle + \langle \psi, \psi'\rangle\big)\, ds\, dt.$$

The formal adjoint $L_w^* : C_0^\infty(\mathbb{C}, u^*TM \oplus \mathfrak{g}^{\mathbb{C}}) \to C_0^\infty(\mathbb{C}, \mathfrak{g})$ of L_w w.r.t. these inner products is given by

$$L_w^*(v, \varphi + i\psi) = L_u^* v + \partial_s^\Phi \varphi + \partial_t^\Psi \psi.$$

It satisfies

$$\langle L_w^*(v, \beta), \xi\rangle_{L^2} = \langle (v, \beta), L_w\xi\rangle_{L^2},$$

for every $(v, \beta) \in C_0^\infty(\mathbb{C}, u^*TM \oplus \mathfrak{g}^{\mathbb{C}})$ and every $\xi \in C_0^\infty(\mathbb{C}, \mathfrak{g})$. This map extends continuously to a map

$$L_w^* : \mathcal{X}_w \to L_\lambda^p(\mathbb{C}, \mathfrak{g}).$$

We fix a gauge equivalence class $W = [w] = [u, \Phi, \Psi] \in \widehat{\mathcal{B}}$. Heuristically, we should have

$$T_W\widehat{\mathcal{B}} \cong \mathcal{X}_w/\mathrm{im} L_w \cong \ker L_w^*.$$

Fix a gauge transformation $g \in \mathcal{G} = C^\infty(\mathbb{C}, \mathrm{G})$ and a section $\zeta := (v, \beta) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, u^*TM \oplus \mathfrak{g}^{\mathbb{C}})$. Then g acts on ζ by

$$g^*(v, \beta) := (g^{-1}v, g^{-1}\beta g). \tag{2.16}$$

Heuristically, this corresponds to the "differential" of the action of g on $\mathcal{B}$ at the point w, applied to ζ. By Lemma A.8 and a density argument we have

$$\nabla^{g^*(\Phi,\Psi)} g^*\zeta = (\nabla^{g^*(\Phi,\Psi)} v, d^{g^*(\Phi,\Psi)}\beta) = g^*\nabla^{\Phi,\Psi}\zeta.$$

It follows that the map

$$\mathcal{X}_w \ni (v, \beta) \mapsto g^*(v, \beta) := (g^{-1}v, g^{-1}\beta g) \in \mathcal{X}_{g^*w} \tag{2.17}$$

is well-defined and an isometric isomorphism. Note also that for every $x \in M$, $v \in T_xM$ and $g \in \mathrm{G}$ we have

$$L^*_{g^{-1}x} g^{-1} v = g^{-1}(L^*_u v) g \in \mathfrak{g}.$$

Hence for every $g \in \mathcal{G}$ and every $(v, \beta) \in \mathcal{X}_w$

$$L^*_{g^* w}(g^{-1} v, g^{-1} \beta g) = g^{-1}(L^*_w(v, \beta)) g.$$

Therefore, the map (2.17) identifies $\ker L^*_w \subseteq \mathcal{X}_w$ with $\ker L^*_{g^* w}$. We fix an equivalence class $W \in \widehat{\mathcal{B}} = \mathcal{B}/\mathcal{G}$ and define the normed vector space $\widehat{\mathcal{X}}_W$ to be the quotient

$$\widehat{\mathcal{X}}_W := \left(\bigsqcup_{w \in W} \ker L^*_w \right) / \mathcal{G}, \tag{2.18}$$

together with the norm

$$||Z||_W := ||\zeta||_w, \tag{2.19}$$

for $Z \in \widehat{\mathcal{X}}_W$, where $(w, \zeta) \in \bigsqcup_{w \in W} \ker L^*_w$ is an arbitrary representative of (W, Z). Since the map (2.17) is an isometric isomorphism, this norm is well-defined, and $\mathcal{X}_W$ is isometrically isomorphic to $\ker L^*_w$, for every $w \in W$. Since the map $L^*_w : \mathcal{X}_w \to L^p_\lambda(\mathbb{C}, \mathfrak{g})$ is continuous, the subspace $\ker L^*_w \subseteq \mathcal{X}_w$ is closed. Since $\mathcal{X}_w$ is a Banach space, it follows that $\ker L^*_w$ and hence $\widehat{X}_W$ are Banach spaces.

Consider now the "Banach space bundle"

$$\mathcal{E} := \bigsqcup_{w \in \mathcal{B}} \mathcal{E}_w \to \mathcal{B},$$

where for each point $w := (u, \Phi + i\Psi) \in \mathcal{B}$ the fibre $\mathcal{E}_w$ is defined by

$$\mathcal{E}_w := \left\{ (v, \varphi) \in L^p_{\mathrm{loc}}(\mathbb{C}, u^*TM \oplus \mathfrak{g}) \,\middle|\, ||(v, \varphi)||_{L^p_\lambda(\mathbb{C})} < \infty \right\},$$

where the norm $||\cdot||_{L^p_\lambda(\mathbb{C})}$ is induced by the metric $g_{\omega,J}$ on M and the real inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. The gauge group $\mathcal{G}$ acts on $\mathcal{E}$ by isometric isomorphisms, by the formula

$$g^*(v, \varphi) := (g^{-1} v, g^{-1} \varphi g), \tag{2.20}$$

lifting the action of $\mathcal{G}$ on $\mathcal{B}$. Hence we can view the quotient

$$\widehat{\mathcal{E}} := \mathcal{E}/\mathcal{G}$$

as a "Banach space bundle" over $\widehat{\mathcal{B}} = \mathcal{B}/\mathcal{G}$. We can now interpret the vortex equations (0.1), (0.2) as the section $\mathcal{S} : \mathcal{B} \to \mathcal{E}$ given by

$$\mathcal{S}(w) := (w, \sigma(w)),$$

where

$$\sigma(u, \Phi + i\Psi) := \begin{pmatrix} \partial_s u + X_\Phi(u) + J(\partial_t u + X_\Psi(u)) \\ \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \mu \circ u \end{pmatrix}. \tag{2.21}$$

By a straight-forward calculation, using that μ is equivariant, we have $\mathcal{S}(g^* w) = g^* \mathcal{S}(w)$, hence the section $\mathcal{S}$ descends to a section

$$\widehat{\mathcal{S}} : \widehat{\mathcal{B}} \to \widehat{\mathcal{E}}.$$

We fix a zero of $\mathcal{S}$, i.e. a point $W \in \widehat{\mathcal{B}}$ such that $\widehat{\mathcal{S}}(W) = (W, 0)$. Then the "tangent space" of $\widehat{\mathcal{E}}$ at $(W, 0)$ splits canonically as

$$T_{(W,0)} \widehat{\mathcal{E}} \cong T_W \widehat{\mathcal{B}} \oplus \widehat{\mathcal{E}}_W.$$

We denote by $\widehat{P}_W : T_{(W,0)} \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}_W$ the projection to $\widehat{\mathcal{E}}_W$ along $T_W \widehat{\mathcal{B}}$ and consider the "vertical differential"

$$\widehat{\mathcal{D}}_W := \widehat{P}_W d\widehat{\mathcal{S}}(W) : T_W \widehat{\mathcal{B}} = \widehat{\mathcal{X}}_W \to \widehat{\mathcal{E}}_W.$$

The main result of this section, Theorem 2.8 below, is the main step in proving that this map is a Fredholm map. To understand this, fix a vector $Z \in T_W \widehat{\mathcal{B}}$. We choose representatives $w \in \mathcal{B}$ of W and $\zeta \in T_w \mathcal{B}$ of Z. Since $\mathcal{S}(w) = 0$, the "tangent space" of $\mathcal{E}$ at w splits canonically as

$$T_{(w,0)} \mathcal{E} \cong T_w \mathcal{B} \oplus \mathcal{E}_w.$$

We denote by $P_w : T_{(w,0)} \mathcal{E} \to \mathcal{E}_w$ the projection to $\mathcal{E}_w$ along $T_w \mathcal{B}$ and consider the "vertical differential"

$$\mathcal{D}_w := P_w d\mathcal{S}(w) : T_w \mathcal{B} = \mathcal{X}_w \to \mathcal{E}_w.$$

We derive an explicit formula for $\mathcal{D}_w$. Formally, the connection ∇ on M induces a "covariant derivative" $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ in a pointwise way. Namely, for every "smooth" path $\mathbb{R} \ni a \mapsto w_a \in \mathcal{B}$ and every "smooth" section of $\mathcal{E}$ along w, $\mathbb{R} \ni a \mapsto (v_a, \varphi_a) \in \mathcal{E}_{w_a}$, the "covariant derivative" at $a := 0$, $\nabla^{\mathcal{E}}_a|_{a=0}(v_a, \varphi_a) \in \mathcal{E}_{w_a}$, is given by

$$\big(\nabla^{\mathcal{E}}_a|_{a=0}(v_a, \varphi_a)\big)(z) := \Big(\nabla_a|_{a=0}(v_a(z)), \left.\frac{d}{da}\right|_{a=0} \varphi_a(z)\Big).$$

Here for every $z \in \mathbb{C}$ and every $a \in \mathbb{R}$ the vector $\nabla_a(v_a(z)) \in T_{u_a(z)}M$ is the covariant derivative of the vector field $v_\cdot(z)$ in M along the curve $u_\cdot(z)$, at the point a. If $g \in \mathcal{G}$ is a gauge transformation, then

$$\nabla^{\mathcal{E}}_a(g^*(v_a, \varphi_a)) = g^* \nabla^{\mathcal{E}}_a(v_a, \varphi_a),$$

i.e. $\nabla^{\mathcal{E}}$ is invariant under the action of $\mathcal{G}$. For $w \in \mathcal{B}$ such that $\mathcal{S}(w) = (w, 0)$ and $\zeta \in T_w\mathcal{B}$ the "vertical differential" of $\mathcal{S}$ at w in the direction ζ is now given by

$$\mathcal{D}_w\zeta = \nabla^{\mathcal{E}}_a|_{a=0}\mathcal{S}(w_a), \tag{2.22}$$

where $\mathbb{R} \ni a \mapsto w_a \in \mathcal{B}$ is such that $w_0 = w$ and $d/da|_{a=0}w_a = \zeta$. The following lemma gives a formula for the right hand side of (2.22), which is also valid in the case that $\mathcal{S}(w) \neq (w, 0)$.

Lemma 2.1 ("Vertical differential" of the vortex equations on $\mathbb{C}$) *Let ∇ be a torsionfree connection on M and*

$$w := (u, \Phi + i\Psi) \in C^\infty(\mathbb{C}, M \times \mathfrak{g}^{\mathbb{C}}), \qquad \zeta := (v, \varphi) \in C^\infty(\mathbb{C}, M \oplus \mathfrak{g}).$$

Furthermore, let

$$\mathbb{R} \times \mathbb{C} \ni (a, z) \mapsto \big(u_a(z), \Phi_a(z) + i\Psi_a(z)\big) \in M \times \mathfrak{g}^{\mathbb{C}}$$

be a smooth map such that

$$(w_0, \partial_a|_{a=0}w_a) = (w; \zeta).$$

Then

$$\nabla_a|_{a=0}\sigma(w_a) =$$

$$\begin{pmatrix} (\nabla^\Phi_s + J\nabla^\Psi_t)v + L_u\varphi + JL_u\psi + (\nabla_v J)(\partial_t u + X_\Psi(u)) \\ \partial^\Phi_s\psi - \partial^\Psi_t\varphi + d\mu(u)v \end{pmatrix}, \tag{2.23}$$

where σ is defined as in (2.21).

Proof of Lemma 2.1: That ∇ is torsionfree implies that

$$\nabla_a|_{a=0}\partial_s u_a = \nabla_s\partial_a|_{a=0}u_a = \nabla_s v, \tag{2.24}$$

$$\nabla_a|_{a=0}\partial_t u_a = \nabla_t\partial_a|_{a=0}u_a = \nabla_t v. \tag{2.25}$$

Furthermore, we claim that

$$\nabla_a|_{a=0}(X_{\Phi_a}(u_a)) = L_u\varphi + \nabla_v X_\Phi \tag{2.26}$$

$$\nabla_a|_{a=0}(X_{\Psi_a}(u_a)) = L_u\psi + \nabla_v X_\Psi. \tag{2.27}$$

To see this, we fix a point $z_0 \in \mathbb{C}$, and consider the map $f : \mathbb{R}^2 \to M$, $f(a, b) := u_b(z_0)$ and the vector field X along f given by

$$X(a, b) := X_{\Phi_a(z_0)}(u_b(z_0)).$$

We have

$$\begin{aligned}\nabla_a|_{a=0}\big(X_{\Phi_a(z_0)}(u_a(z_0))\big) &= \nabla_a|_{a=0}X(a,0)+\nabla_b|_{b=0}X(0,b)\\ &= X_{\partial_a|_{a=0}\Phi_a(z_0)}(u(z_0))+\nabla_{\partial_b|_{b=0}u_b(z_0)}X_\Phi\\ &= L_{u(z_0)}\varphi(z_0)+\nabla_{v(z_0)}X_\Phi.\end{aligned}$$

This proves (2.26). Equality (2.27) follows analogously. It follows that

$$\begin{aligned}\nabla_{a=0}\big(J(\partial_t u_a+X_{\Phi_a}(u_a))\big) &= (\nabla_{\partial_a|_{a=0}u_a}J)(\partial_t u+X_\Phi(u))\\ &\quad + J\nabla_a|_{a=0}(\partial_t u_a+X_{\Psi_a}(u_a))\\ &= (\nabla_v J)(\partial_t u+X_\Phi(u))+J(\nabla_t v+L_u\psi+\nabla_v X_\Psi).\end{aligned}$$

Therefore, using (2.26)

$$\nabla_a|_{a=0}\big(\partial_s u_a+X_{\Phi_a}(u_a)+J(\partial_t u_a+X_{\Psi_a}(u_a))\big)=$$
$$\nabla_s v+L_u\varphi+\nabla_v X_\Phi+(\nabla_v J)(\partial_t u+X_\Psi(u))+J(\nabla_t v+L_u\psi+\nabla_v X_\Psi(u)).$$

This shows that the first component of $\nabla_a|_{a=0}\mathcal{S}(w_a)$ is as claimed. That the second component is as claimed follows from a straight-forward calculation. This proves Lemma 2.1. □

The "vertical differential" $\mathcal{D}_w : T_w\mathcal{B}\to\mathcal{E}(w)$ at a zero w of the section $\mathcal{S}:\mathcal{B}\to\mathcal{E}$ is *not* a Fredholm operator. The reason is that the image of $C_0^\infty(\mathbb{C},\mathfrak{g})$ under the infinitesimal gauge action L_w is an infinite dimensional subspace of the kernel of $\mathcal{D}_w$. However, if we augment $\mathcal{D}_w$ by the gauge fixing operator $L_w^*: T_w\mathcal{B}=\mathcal{X}_w\to L^p_\lambda(\mathbb{C},\mathfrak{g})$, then the resulting operator $\mathcal{D}_w:\mathcal{X}_w\to\mathcal{E}_w\oplus L^p_\lambda(\mathbb{C},\mathfrak{g})$ is Fredholm. This is the content of the main result 2.8. More precisely, for every $w=(u,\Phi+i\Psi)\in\mathcal{B}$ we define

$$\mathcal{Y}_w:=\mathcal{Y}_w^{p,\lambda}:=\big\{\zeta\in L^p_{\mathrm{loc}}(\mathbb{C},u^*TM\oplus\mathfrak{g}^{\mathbb{C}})\,\big|\,||\zeta||_{p,\lambda}<\infty\big\} \tag{2.28}$$

where the norm

$$||\zeta||_{p,\lambda}:=||\zeta||_{L^p_\lambda(\mathbb{C})} \tag{2.29}$$

is taken w.r.t. the Riemannian metric $g_{\omega,J}$ on M and the hermitian inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}^{\mathbb{C}}}$ on $\mathfrak{g}^{\mathbb{C}}$ defined in (2.12). Then $\mathcal{Y}_w$ is isomorphic as a normed vector space to the space $\mathcal{E}_w\oplus L^p_\lambda(\mathbb{C},\mathfrak{g})$. By Corollary 2.13 below it is complete. We denote $\partial_t^\Psi u:=\partial_t u+X_\Psi(u)$ and define the augmented "vertical differential" to be the linear operator

$$\mathcal{D}_w:=\mathcal{D}_w^{p,\lambda}:\mathcal{X}_w^{p,\lambda}\to\mathcal{Y}_w^{p,\lambda},$$

$$\mathcal{D}_w^{p,\lambda}(v,\varphi+i\psi):=\begin{pmatrix}(\nabla_s^\Phi+J\nabla_t^\Psi)v+L_u\varphi+JL_u\psi+(\nabla_v J)\partial_t^\Psi u\\ L_u^*v+\partial_s^\Phi\varphi+\partial_t^\Psi\psi\\ d\mu(u)v+\partial_s^\Phi\psi-\partial_t^\Psi\varphi\end{pmatrix}. \tag{2.30}$$

The operator $\mathcal{D}_w$ is well-defined and bounded. We can rewrite it in a compact form. Namely, for every $x \in M$ we extend $L_x : \mathfrak{g} \to T_xM$ to a complex linear map

$$L_x^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \to T_xM, \qquad L_x^{\mathbb{C}}(\varphi + i\psi) := L_x\varphi + JL_x\psi.$$

The adjoint map of $L_x^{\mathbb{C}}$ w.r.t. $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ and $g^{\mathbb{C}}$ (defined in (2.11) is the complex linear map given by

$$(L_x^{\mathbb{C}})^* = L_x^* - iL_x^*J = L_x^* + id\mu(x) : T_xM \to \mathfrak{g}^{\mathbb{C}}.$$

We also introduce the notations for $(v,\beta) \in \mathcal{X}_w$

$$\begin{aligned} \nabla_{\bar{z}}^{\Phi+i\Psi} v &:= \frac{1}{2}(\nabla_s^{\Phi} + J\nabla_t^{\Psi})v \\ \partial_z^{\Phi-i\Psi}\beta &:= \frac{1}{2}(\partial_s^{\Phi} - i\partial_t^{\Psi})\beta. \end{aligned}$$

By $\nabla_\cdot J$ we mean the map taking $(x,v) \in TM$ to the endomorphism $\nabla_v J(x) : T_xM \to T_xM$. Then the operator $\mathcal{D}_w$ is given by

$$\mathcal{D}_w = \begin{pmatrix} 2\nabla_{\bar{z}}^{\Phi+i\Psi} + (\nabla_\cdot J)\partial_t^{\Psi} u & L_u^{\mathbb{C}} \\ (L_u^{\mathbb{C}})^* & 2\partial_z^{\Phi-i\Psi} \end{pmatrix} : \mathcal{X}_w \to \mathcal{Y}_w. \tag{2.31}$$

Remark 2.2 This formula reveals some symmetry of $\mathcal{D}_w$ with respect to the components v and β of the vector $(v,\beta) \in \mathcal{X}_w$. Assume that we are in the Kähler case, i.e. that J is integrable, and let ∇ be the Levi-Civita connection of $g_{\omega,J}$. Then $\nabla J = 0$, see for example Lemma C.7.1 in the book by D. McDuff and D. A. Salamon [MS3]. Therefore, the term $(\nabla_\cdot J)\partial_t^{\Psi} u$ in (2.31) vanishes, and hence the symmetry of the operator becomes even more apparent.

Remark 2.3 Note that the off-diagonal terms in the formula (2.31) are not compact operators. This is due to the fact that the Morrey embedding of $W^{1,p}(\mathbb{C})$ into the space of bounded continuous functions on $\mathbb{C}$ is not compact, in contrast to the case of $W^{1,p}(\Omega)$, where $\Omega \subseteq \mathbb{C}$ is a *bounded* domain.

The next lemma shows that if W is a zero of $\widehat{\mathcal{S}} : \widehat{\mathcal{B}} \to \widehat{\mathcal{E}}$, $w \in W$ is a representative, the map $\mathcal{D}_w : \mathcal{X}_w \to \mathcal{Y}_w$ is Fredholm and the gauge fixing operator $L_w^* : \mathcal{X}_w \to L_\lambda^p(\mathbb{C},\mathfrak{g})$ is surjective, then the "vertical differential" of $\widehat{\mathcal{S}}$ at the point W, $\widehat{\mathcal{D}}_W : \widehat{\mathcal{X}}_W \to \widehat{\mathcal{E}}_W \cong \mathcal{E}_w$ is also a Fredholm operator. Moreover, under the identification $\widehat{\mathcal{X}}_W \cong \ker L_w^* \subseteq \mathcal{X}_w$, the kernels of $\widehat{\mathcal{D}}_W$ and $\mathcal{D}_w$ agree, and we can identify the cokernels. This is the motivation for the Fredholm theorem 2.8.

Lemma 2.4 *Let X,Y,Z be vector spaces and $D : X \to Y$ and $T : X \to Z$ be linear maps. We define $\widehat{D} : \ker T \to Y$ to be the restriction of D to $\ker T$. Then the following statements hold.*

(i)

$$\ker \widehat{D} = \ker(D, T).$$

(ii) Assume that $T : X \to Z$ is surjective. Then the map

$$\begin{aligned} \operatorname{coker} \widehat{D} := Y/\operatorname{im}\widehat{D} &\to \operatorname{coker}(D, T) := (Y \oplus Z)/\operatorname{im}(D, T), \\ y + \operatorname{im}\widehat{D} &\mapsto (y, 0) + \operatorname{im}(D, T) \end{aligned} \tag{2.32}$$

is well-defined, i.e. independent of the choice of the representative y of $y + \operatorname{im}\widehat{D}$. Furthermore, it is an isomorphism.

(iii) Let $||\cdot||_Y$, $||\cdot||_Z$ be norms on Y and Z and assume that the image $\operatorname{im}(D, T)$ is closed in $Y \oplus Z$. Then the image $\operatorname{im}\widehat{D}$ is closed in Y.

Proof of Lemma 2.4: Statement (i) and linearity and well-definedness of the map (2.32) in (ii) follow immediately from the definitions. To see that (2.32) is surjective, let $(y, z) \in Y \oplus Z$ be a vector. By our assumption that $T : X \to Z$ is surjective, there exists a vector $x \in X$ such that $Tx = z$. It follows that under the map (2.32)

$$\begin{aligned} y - Dx + \operatorname{im}\widehat{D} &\mapsto (y - Dx, 0) + \operatorname{im}(D, T) \\ &= (y - Dx, 0) + (Dx, Tx) + \operatorname{im}(D, T) \\ &= (y, z) + \operatorname{im}(D, T). \end{aligned}$$

Hence the map (2.32) is surjective.

We prove statement (iii). Let $y_\nu \in \operatorname{im}\widehat{D}$ be a sequence that converges in Y, and let y be its limit. Then $(y_\nu, 0) \in \operatorname{im}(D, T)$. By our assumption that $\operatorname{im}(D, T) \subseteq Y \oplus Z$ is closed, the limit $(y, 0) = \lim_{\nu \to \infty}(y_\nu, 0)$ lies in $\operatorname{im}(D, T)$. It follows that there exists a vector $x \in \ker T$ such that $\widehat{D}x = Dx = y$. Therefore, $\operatorname{im}\widehat{D}$ is closed in Y. This proves Lemma 2.4. □

2.2 The Fredholm Theorem

We come now to the main result of the section. It is Theorem 2.8 below, which says that the augmented vertical differential of the symplectic vortex equations at every point $w \in \mathcal{B}^p_\lambda$ is Fredholm. Its index is the dimension of the symplectic quotient plus twice the Maslov index of w. We fix a real numbers $p > 2$ and $\lambda > -2/p + 1$ and define $\mathcal{B} := \mathcal{B}^p_\lambda$ as in (2.4). The definition of the Maslov index is based on the following lemma.

Lemma 2.5 *Let $w := (u, \Phi, \Psi) \in \mathcal{B}$ be a map. Then there exists a gauge transformation $g \in \mathcal{G} = C^\infty(\mathbb{C}, \mathrm{G})$, a map $g_\infty \in C^\infty(S^1, \mathrm{G})$ and a point $x_\infty \in \mu^{-1}(0)$ such that $(g^{-1}u)(re^{i\varphi})$ converges to $g_\infty(e^{i\varphi})x_\infty$, as $r \to \infty$, uniformly in $\varphi \in \mathbb{R}$.*

Proof of Lemma 2.5: Let $w := (u, \Phi + i\Psi) \in \mathcal{B}$ be a map. By the definition of $\mathcal{B}$ there exists a gauge transformation $g_0 \in C^\infty(\mathbb{C} \setminus B_1, \mathrm{G})$ such that $w' := (u', \Phi' + i\Psi') := g_0^* w$ satisfies $\Phi' + i\Psi' \in L^p_\lambda(\mathbb{C} \setminus B_1, \mathfrak{g}^\mathbb{C})$. We set $N := 2 \dim M + 1$ and choose an embedding $\iota : M \to \mathbb{R}^N$, as in Whitney's embedding theorem, see Theorem 3.2.14 p. 55 in the book by M. W. Hirsch [Hi]. Furthermore, we fix a smooth map $\rho : \mathbb{C} \to [0, 1]$ such that $\rho(z) = 0$ if $|z| \leq 1$ and $\rho(z) = 1$ if $|z| \geq 2$. By the definition of $\mathcal{B}$ the closure of the image of u in M is compact. Since $\overline{u'(\mathbb{C})} = \overline{g_0^{-1} u(\mathbb{C})}$ is a closed subset of the compact set $\mathrm{G} \cdot \overline{u(\mathbb{C})}$, it is also compact. Hence

$$a := \sup \big\{ ||\iota \circ u'||_{L^\infty(\mathbb{C} \setminus B_1)}, ||d\iota(u')||_{L^\infty(\mathbb{C} \setminus B_1)} \big\} < \infty.$$

It follows that

$$\begin{aligned}
||d(\rho\iota \circ u')||_{L^p_\lambda(\mathbb{C})} &\leq ||(d\rho)\iota \circ u'||_{L^p_\lambda(A(1,2))} + ||\rho\, d\iota(u') du'||_{L^p_\lambda(\mathbb{C} \setminus B_1)} \\
&\leq C||\iota \circ u'||_{L^\infty(A(1,2))} + ||d\iota(u')||_{L^\infty(\mathbb{C} \setminus B_1)} ||du'||_{L^p_\lambda(\mathbb{C} \setminus B_1)} \\
&\leq Ca + a||du'||_{L^p_\lambda(\mathbb{C} \setminus B_1)} \\
&\leq Ca\big(1 + ||d^{\Phi' + i\Psi'} u'||_{L^p_\lambda(\mathbb{C} \setminus B_1)} + ||\Phi' + i\Psi'||_{L^p_\lambda(\mathbb{C} \setminus B_1)}\big) \\
&=: b < \infty.
\end{aligned}$$

Here C is a constant depending only on p, λ and ρ. Therefore, by Proposition E.4 (Hardy-type inequality), applied with u replaced by $\rho\iota \circ u' : \mathbb{C} \to \mathbb{R}^N$, the points $\iota \circ u'(re^{i\varphi})$ converge to some point $y_\infty \in \mathbb{R}^N$, for $r \to \infty$, uniformly in φ. Since $\iota(\overline{u'(\mathbb{C})})$ is compact, we have $y_\infty \in \iota(\overline{u'(\mathbb{C})})$, hence there exists a point $x_\infty \in M$ such that $\iota(x_\infty) = y_\infty$. Furthermore, by the definition of $\mathcal{B}$ we have $|\mu \circ u'(re^{i\varphi})| = |\mu \circ u(re^{i\varphi})| \to 0$ for $r \to \infty$, for every $\varphi \in \mathbb{R}$, and therefore $x_\infty \in \mu^{-1}(0)$. We define the map $\widetilde{g} : \mathbb{C} \to \mathrm{G}$ by

$$\widetilde{g}(z) := \begin{cases} g_0(z) g_0\big(\frac{2z}{|z|}\big)^{-1}, & \text{if } |z| \geq 2, \\ \mathbf{1}, & \text{if } |z| < 2. \end{cases}$$

Smoothing the map $\widetilde{g}$ in the ball B_3 yields a smooth map $g : \mathbb{C} \to \mathrm{G}$. Furthermore, we define $g_\infty : S^1 \to \mathrm{G}$ by $g_\infty(z) := g_0(2z)$. The triple g, g_∞, x_∞ has the required properties. This proves Lemma 2.5. □

We define the continuous map $\psi_0 : B_1 \to \mathbb{C}$ by

$$\psi_0(z) := \begin{cases} \frac{z}{|z|} \tan\left(\frac{\pi|z|}{2}\right), & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases} \tag{2.33}$$

Definition 2.6 (Maslov index) *Let $w := (u, \Phi + i\Psi) \in \mathcal{B}$ be a map. Let $g : \mathbb{C} \to \mathrm{G}$ and $g_\infty : S^1 \to \mathrm{G}$ be continuous maps, and $x_\infty \in \mu^{-1}(0)$ be a point, such that $(g^{-1}u)(re^{i\varphi})$ converges to $g_\infty(e^{i\varphi})x_\infty$, as $r \to \infty$, uniformly* *in* $\varphi \in \mathbb{R}$, *as in Lemma 2.5. Let* $v : \mathbb{D} \to M$ *be the unique continuous extension of the map* $(g^{-1}u) \circ \psi_0 : B_1 \to M$, *and let*

$$F : \mathbb{D} \times \mathbb{C}^n \to v^*TM, \qquad (z, v) \mapsto F(z, v) =: F_z v$$

be a continuous complex trivialization of the pullback bundle $v^*TM \to \mathbb{D}$. *We define the* Maslov index of w *to be*

$$m(w) := \deg\Big(S^1 \ni z \mapsto \frac{\det\big(F_z^{-1} g_\infty(z) \cdot F_1\big)}{\big|\det\big(F_z^{-1} g_\infty(z) \cdot F_1\big)\big|} \in S^1\Big). \tag{2.34}$$

Remark 2.7 The Maslov index $m(w)$ does not depend on the choice of g, g_∞, x_∞ and F, since if g', g'_∞, x'_∞ and F' are other choices, then the resulting map

$$S^1 \ni \mapsto F'^{-1}_z g'_\infty(z) \cdot F'_1 \in \mathrm{Aut}(\mathbb{C}^n)$$

is homotopic to

$$S^1 \ni \mapsto F_z^{-1} g_\infty(z) \cdot F_1 \in \mathrm{Aut}(\mathbb{C}^n).$$

Furthermore, the Maslov index depends only on the gauge equivalence class $\mathcal{G}^*w$ of w.

We come now to the main result of this section. Recall that we always assume hypothesis (H1), i.e. that μ is proper and that G acts freely on $\mu^{-1}(0)$.

Theorem 2.8 (Fredholm property) *Assume that* $\dim M > 2 \dim \mathrm{G}$. *Let* $p > 2$ *and* $\lambda > -2/p + 1$ *be real numbers. Let* $w := (u, \Phi + i\Psi) \in \mathcal{B}^p_\lambda$ *be a map, where* $\mathcal{B}^p_\lambda$ *is defined as in (2.4). Then the following statements hold.*

(i) The normed spaces $(\mathcal{X}^{p,\lambda}_w, ||\cdot||^{p,\lambda}_w)$ *and* $(\mathcal{Y}^{p,\lambda}_w, ||\cdot||_{p,\lambda})$, *defined in (2.15), (2.14), (2.28) and (2.29), are complete.*

(ii) Assume that $-2/p + 1 < \lambda < -2/p + 2$. *Let the Maslov index* $m(w)$ *be defined as in (2.34). Then the operator* $\mathcal{D}^{p,\lambda}_w : \mathcal{X}^{p,\lambda}_w \to \mathcal{Y}^{p,\lambda}_w$ *defined in (2.30) is Fredholm of real index*

$$\mathrm{ind}\mathcal{D}^{p,\lambda}_w = \dim M - 2\dim \mathrm{G} + 2m(w). \tag{2.35}$$

Recall that $\widetilde{\mathcal{M}}$ denotes the set of all smooth finite energy solutions of the vortex equations (0.1), (0.2) such that $\overline{u(\mathbb{C})}$ is compact.

Corollary 2.9 *Assume that* $\dim M > 2 \dim \mathrm{G}$. *Then there exists a real number* $p_0 > 2$ *such that the following holds. Let* $2 < p < p_0$, $-2/p + 1 < \lambda < -2/p_0 + 1$ *and* $w \in \widetilde{\mathcal{M}}$ *be a vortex. Then the spaces* $\mathcal{X}^{p,\lambda}_w$ *and* $\mathcal{Y}^{p,\lambda}_w$ *are complete, and the operator* $\mathcal{D}^{p,\lambda}_w : \mathcal{X}^{p,\lambda}_w \to \mathcal{Y}^{p,\lambda}_w$ *given by (2.30) is well-defined and Fredholm of real index given by (2.35).*

Proof of Corollary 2.9: By Proposition D.11 there exists a real number $p_0 > 2$ such that if $2 < p < p_0$ and $-2/p + 1 < \lambda < -2/p_0 + 1$ then

$$\widetilde{\mathcal{M}} \subseteq \mathcal{B}^p_\lambda.$$

Hence the assertion follows from Theorem 2.8. □

The proof of Theorem 2.8 is given on page 60. It contains two main ingredients. The first one is a suitable complex trivialization of the bundle $u^*TM \oplus \mathfrak{g}^{\mathbb{C}}$. For $|z|$ large, this trivialization respects the splitting $T_{u(z)}M = H_{u(z)} \oplus \mathrm{im} L^{\mathbb{C}}_{u(z)}$ of the tangent space at $u(z)$ into the "horizontal" and the "complex action" parts. It induces an isomorphism of normed vector spaces from $\mathcal{X}_w$ to some weighted Sobolev space on $\mathbb{C}$, and similarly for $\mathcal{Y}_w$. In the proof of this we will use a Hardy-type inequality. It says that if $\lambda > -N/p$ then there exists a constant C such that for every weakly differentiable map $u : \mathbb{R}^N \to \mathbb{C}$ there exists a point $x_\infty \in \mathbb{C}$ such that the L^p_λ-norm of $u - x_\infty$ is bounded above by C times the $L^p_{\lambda+1}$-norm of the derivative of u. The proofs of this result, and of more results about weighted Sobolev spaces, are given in appendix E.2. Since the weighted Sobolev spaces are complete, the same holds for $\mathcal{X}_w$ and $\mathcal{Y}_w$. The second ingredient of the proof of Theorem 2.8 are two propositions stating that the $\partial_{\bar{z}}$-operator and a related matrix differential operator are Fredholm maps between suitable weighted Sobolev spaces.

The trivialization we will use is of the following kind. We assume that

$$\bar{n} := \dim M/2 - \dim \mathrm{G} \geq 1.$$

Fix a map $(u, \Phi + i\Psi) \in C^\infty(\mathbb{C}, M \times \mathfrak{g}^{\mathbb{C}})$. We abbreviate

$$V := \mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}.$$

By a complex trivialization of $u^*TM \oplus \mathfrak{g}^{\mathbb{C}}$ we mean a complex vector bundle isomorphism

$$F : \mathbb{C} \times V \to u^*TM \oplus \mathfrak{g}^{\mathbb{C}}.$$

Here the domain of F is the trivial complex vector bundle with base $\mathbb{C}$ and fibre V. We define $\nabla^{\Phi,\Psi} F$ to be the one form on $\mathbb{C}$ with values in the vector bundle of real vector bundle homomorphisms from $\mathbb{C} \times V$ to $u^*TM \times \mathfrak{g}^{\mathbb{C}}$ by

$$(\nabla^{\Phi,\Psi} F)(\sigma + i\tau)\zeta := \sigma \nabla^\Phi_s (F\zeta) + \tau \nabla^\Psi_t (F\zeta), \tag{2.36}$$

for $\sigma + i\tau \in T_z\mathbb{C} = \mathbb{C}$, $z \in \mathbb{C}$ and $\zeta \in V$. Here we use the notation of (2.9). For $d \in \mathbb{Z}$ we denote by $p_d : \mathbb{C} \to \mathbb{C}$ the monomial $p_d(z) := z^d$, and

$$p_d \cdot \oplus \mathrm{id} : \mathbb{C} \times V \to V,$$
$$(z; v^1, \ldots, v^{\bar{n}}, \alpha, \beta) \mapsto (p_d(z)v^1, v^2, \ldots, v^{\bar{n}}, \alpha, \beta).$$

For $x \in M$ recall the definition of the "horizontal" part of T_xM,

$$H_x := \ker d\mu(x) \cap (\mathrm{im} L_x)^{\perp} = (\mathrm{im} L_x^{\mathbb{C}})^{\perp}.$$

We denote by $\mathrm{Ad} : \mathrm{G} \to \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$ the adjoint representation, i.e.

$$\mathrm{Ad}_g(\varphi + i\psi) := g(\varphi + i\psi)g^{-1},$$

for every $g \in \mathrm{G}$ and $\varphi + i\psi \in \mathfrak{g}^{\mathbb{C}}$.

Definition 2.10 *Let $p > 2$, $\lambda > -2/p + 1$ and $w := (u, \Phi + i\Psi) \in \mathcal{B}$ be a map. We call a smooth complex trivialization*

$$F : \mathbb{C} \times V \to u^*TM \oplus \mathfrak{g}^{\mathbb{C}}$$

good, *if the following properties are satisfied.*

(i) **(Splitting)** *For every $z \in \mathbb{C}$ we have*

$$F_z(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}} \times \{0\}) = T_{u(z)}M \times \{0\}, \tag{2.37}$$
$$F_z(\{0\} \times \{0\} \times \mathfrak{g}^{\mathbb{C}}) = \{0\} \times \mathfrak{g}^{\mathbb{C}}. \tag{2.38}$$

Furthermore, there exists a number $R > 0$ and a gauge transformation $g_0 \in C^{\infty}(\mathbb{C} \setminus B_R, \mathrm{G})$ such that the following conditions are satisfied. For every $z \in \mathbb{C} \setminus B_R$ we have

$$F_z(\mathbb{C}^{\bar{n}} \times \{0\} \times \{0\}) = H_{u(z)}. \tag{2.39}$$

Moreover, $g_0^*(\Phi + i\Psi) \in L^p_{\lambda}(\mathbb{C} \setminus B_R, \mathfrak{g}^{\mathbb{C}})$, and for every $z \in \mathbb{C} \setminus B_R$

$$F_z(0, \alpha, \beta) = \big(L^{\mathbb{C}}_{u(z)} \mathrm{Ad}_{g_0(z)} \alpha, \mathrm{Ad}_{g_0(z)} \beta\big), \quad \forall (\alpha, \beta) \in \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}. \tag{2.40}$$

(ii) There exists a number $c > 0$ such that for every $(z, \zeta) \in \mathbb{C} \times V$

$$c^{-1}|\zeta| \leq \big|F_z(\langle z \rangle^{m(w)} \cdot \oplus \mathrm{id})\zeta\big| \leq c|\zeta|, \tag{2.41}$$

where $m(w)$ denotes the Maslov index of w (Definition 2.6).

(iii)

$$\nabla^{\Phi,\Psi}\big(F(p_{m(w)} \cdot \oplus \mathrm{id})\big) \in L^p_{\lambda}(\mathbb{C} \setminus B_1). \tag{2.42}$$

Proposition 2.11 *If $p > 2$ and $\lambda > -2/p + 1$ and $w := (u, \Phi + i\Psi) \in \mathcal{B}^p_{\lambda}$ is a map, then there exists a good complex trivialization of u^*TM.*

Proof of Proposition 2.11: By the definition of $\mathcal{B}$ there exists a gauge transformation $g_0 \in C^\infty(\mathbb{C} \setminus B_1, \mathrm{G})$ such that $w' := (u', \Phi' + i\Psi') := g_0^* w$ satisfies $\Phi' + i\Psi' \in L^p_\lambda$. Therefore, as in the proof of Lemma 2.5 there exists a point $x_\infty \in \mu^{-1}(0)$ such that $u'(re^{i\varphi})$ converges to x_∞, for $r \to \infty$, uniformly in $\varphi \in \mathbb{R}$. By the local slice theorem A.5 there exists a G-invariant neighbourhood $U \subseteq M^*$ of x_∞ and a smooth equivariant parametrization $\psi : \mathbb{R}^{2n-\dim \mathrm{G}} \times \mathrm{G} \to U$. We choose a smooth complex trivialization

$$\widehat{F} : (\mathbb{R}^{2n-\dim \mathrm{G}} \times \{\mathbf{1}\}) \times \mathbb{C}^{\bar{n}} \to \psi^* H|_{\mathbb{R}^{2n-\dim \mathrm{G}} \times \{\mathbf{1}\}}.$$

We define the smooth complex trivialization $F^U : U \times \mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}} \to TM|_U$ by

$$F^U_x(v_0, \alpha) := g\widehat{F}_{(\widehat{x},\mathbf{1})} v_0 + L^{\mathbb{C}}_x \alpha,$$

for $x \in U$, $(v_0, \alpha) \in \mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}$, where $(\widehat{x}, g) := \psi^{-1}(x) \in \mathbb{R}^{2n-\dim \mathrm{G}} \times \mathrm{G}$.

Furthermore, we choose a number $R > 1$ so large that $u(z) \in U$ if $|z| \geq R$. We also choose a complex trivialization

$$F^0 : \bar{B}_R \times \mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}} \to u^* TM|_{\bar{B}_R}.$$

We define $d \in \mathbb{Z}$ to be the degree of the map

$$S^1_R \ni z \mapsto \frac{\det\left((F^0_z)^{-1} F^U_{u(z)}(\cdot, \mathrm{Ad}_{g_0(z)} \cdot)\right)}{\left|\det\left((F^0_z)^{-1} F^U_{u(z)}(\cdot, \mathrm{Ad}_{g_0(z)} \cdot)\right)\right|} \in S^1.$$

Consider the loop

$$S^1_R \ni z \mapsto A(z) := (F^0_z)^{-1} F^U_{u(z)}(\cdot, \mathrm{Ad}_{g_0(z)} \cdot) \cdot (z^{-d} \cdot \oplus \mathrm{id}) \in \mathrm{Aut}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}),$$

where by $\mathrm{Aut}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}})$ we denote the group of complex automorphisms of V. The map $\det \circ A / |\det \circ A| : S^1_R \to S^1$ has degree 0. Therefore, it is homotopic to a constant map.

We denote by $\mathrm{U}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}})$ the group of unitary automorphisms of $\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}$ w.r.t. the standard hermitian product on $\mathbb{C}^{\bar{n}}$ and $\langle \cdot, \rangle_{\mathbb{C}}$. Recall that the space $\mathrm{Aut}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}})$ strongly deformation retracts to $\mathrm{U}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}})$. (This follows from the Gram-Schmidt orthonormalization procedure.) Furthermore, the determinant map $\det : \mathrm{U}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}) \to S^1$ and hence the map $\det : \mathrm{Aut}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}) \to S^1$ induce isomorphisms of fundamental groups, see e.g. Proposition 2.23 in the book by D. McDuff and D. A. Salamon [MS2]. For two topological spaces X and Y we denote by $[X, Y]$ the set of all (free) homotopy classes of continuous maps from X to Y. Since $\pi_1(\mathrm{Aut}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}})) \cong \mathbb{Z}$ is abelian, by Satz 16.3.13 in the book by R. Stöcker and H. Zieschang [SZ] the map $\pi_1(\mathrm{Aut}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}})) \to [S^1, \mathrm{Aut}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}})]$ that forgets the base point, is a bijection. Similarly, the "forgetful map" $\mathbb{Z} \cong \pi_1(S^1) \to [S^1, S^1]$ is a bijection. It follows that the map $[S^1, \mathrm{Aut}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}})] \to [S^1, S^1]$ induced by $\det/|\det|$: $\mathrm{Aut}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}) \to S^1$ is bijective.

Returning to our setting, since $\det \circ A/|\det \circ A| : S^1_R \to S^1$ is homotopic to a constant map, it follows that the same holds for the map $A : S^1_R \to \mathrm{Aut}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}})$. Hence there exists a continuous map $h : \bar{B}_R \setminus B_1 \to \mathrm{Aut}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}})$ such that

$$h(z) = \begin{cases} \mathbf{1}, & \text{if } z \in S^1_1, \\ A(z) & \text{if } z \in S^1_R. \end{cases}$$

We define $\widetilde{F} : \mathbb{C} \times (\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}) \to u^*TM$ by

$$\widetilde{F}_z := \begin{cases} F^0_z h(z), & \text{for } z \in B_R, \\ F^U_{u(z)}(\cdot, \mathrm{Ad}_{g_0(z)} \cdot) \cdot (z^{-d} \cdot \oplus \mathrm{id}), & \text{for } z \in \mathbb{C} \setminus B_R. \end{cases}$$

This is a continuous complex vector bundle isomorphism. Smoothing $\widetilde{F}$ out on the ball of radius $R + 1$, we get a smooth complex vector bundle isomorphism

$$\widetilde{\widetilde{F}} : \mathbb{C} \times (\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}) \to u^*TM.$$

This is a trivialization of the "manifold component" u^*TM of the bundle $u^*TM \oplus \mathfrak{g}^{\mathbb{C}}$. In order to extend this to a trivialization $F : \mathbb{C} \times V \to u^*TM \oplus \mathfrak{g}^{\mathbb{C}}$, we look now at the "complex gauge component" $\mathfrak{g}^{\mathbb{C}}$.

Claim 1 *The loop*

$$S^1_1 \ni z \mapsto \mathrm{Ad}_{g_0(z)} \in \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$$

is homotopic to a constant.

Proof of Claim 1: We claim that $\det(\mathrm{Ad}_g) \in \mathbb{R}$, for every $g \in \mathrm{G}$. To see this we fix $g \in \mathrm{G}$ and choose a basis $e_1, \ldots, e_k$ of $\mathfrak{g}$. Then $\mathrm{Ad}_g e_i \in \mathfrak{g}$ and $\mathrm{Ad}_g i e_j = i\mathrm{Ad}_g e_j \in i\mathfrak{g}$ for every i, so the matrix of Ad_g w.r.t. the basis $e_1, \ldots, e_k, ie_1, \ldots, ie_k$ of $\mathfrak{g}^{\mathbb{C}}$ has real entries. Hence $\det(\mathrm{Ad}_g) \in \mathbb{R}$, as claimed. It follows that

$$f(z) := \frac{\det(\mathrm{Ad}_{g_0(z)})}{|\det(\mathrm{Ad}_{g_0(z)})|} = 1$$

for every $z \in \mathbb{C} \setminus B_1$ or it equals -1 for every $z \in \mathbb{C} \setminus B_1$. Hence the degree of the map $S^1_1 \ni z \mapsto f(z) \in S^1$ is zero. Since the map $[S^1, \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})] \to [S^1, S^1]$ induced by $\det/|\det|$ is a bijection, Claim 1 follows. □

By Claim 1 we may choose a continuous map $\widetilde{k} : \bar{B}_1 \to \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$ such that $\widetilde{k}(z) = \mathrm{Ad}_{g_0(z)}$ for $z \in S^1$. Smoothing the connected sum $\widetilde{k} \# g_0$ on the ball B_2, we get a map $k \in C^\infty(\mathbb{C}, \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}}))$. We now define the smooth complex trivialization

$$F : \mathbb{C} \times V = \mathbb{C} \times \left(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}\right) \to u^*TM \oplus \mathfrak{g}^{\mathbb{C}}, \quad F_z := (\widetilde{\widetilde{F}}_z, k(z)).$$

We check that F satisfies conditions (i), (ii) and (iii) of Definition 2.10. The first condition follows from the construction of F. We check (ii).

Claim 2 *The number d equals the Maslov index of w,*

$$d = m(w). \tag{2.43}$$

Proof of Claim 2: In order to calculate $m(w)$, we gauge transform w by a map $g \in C^\infty(\mathbb{C}, \mathrm{G})$ defined as follows. We first define the continuous map $\widetilde{g} : \mathbb{C} \to \mathrm{G}$ by

$$\widetilde{g}(z) := \begin{cases} g_0(z) g_0\left(\frac{z}{|z|}\right)^{-1}, & \text{for } z \in \mathbb{C} \setminus B_1, \\ \mathbf{1}, & \text{for } z \in B_1. \end{cases}$$

Then by smoothing this map on B_2, we get our map $g \in C^\infty(\mathbb{C}, \mathrm{G})$. We define $w'' := (u'', \Phi'', \Psi'') := g^* w$ and

$$g_\infty : S^1 \to \mathrm{G}, \quad g_\infty(z) := g_0(z) g_0(1)^{-1}, \qquad y_\infty := g_0(1) x_\infty.$$

Recall the definition (2.33) of the map $\psi_0 : B_1 \to \mathbb{C}$. We define $v : \mathbb{D} \to M$ by

$$v(z) := \begin{cases} u'' \circ \psi_0(z), & \text{for } z \in B_1, \\ g_\infty(z) y_\infty, & \text{for } z \in S^1. \end{cases}$$

Since $(g_0^{-1} u)(re^{i\varphi})$ converges to x_∞ for $r \to \infty$, uniformly in $\varphi \in \mathbb{R}$, it follows that the map v is continuous. We define the complex trivialization

$$F' : \mathbb{D} \times (\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}) \to v^* TM,$$

$$F'_z := F'(z, \cdot) := \begin{cases} (g \circ \psi_0(z))^{-1} \cdot \widetilde{\widetilde{F}}_{\psi_0(z)} \circ (|\psi_0(z)|^d \cdot \oplus \mathrm{id}), & \text{for } z \in B_1, \\ g_\infty(z) \cdot F^U_{y_\infty}(\cdot, \mathrm{Ad}_{g_0(1)} \cdot) \circ (z^{-d} \cdot \oplus \mathrm{id}), & \text{for } z \in S^1. \end{cases}$$

Claim 3 *The map F' is continuous.*

Proof of Claim 3: By a short calculation we have for every $x \in M$, $h \in \mathrm{G}$, $v_0 \in \mathbb{C}^{\bar{n}}$ and $\alpha \in \mathfrak{g}^{\mathbb{C}}$

$$F^U_{hx}(v_0, \mathrm{Ad}_h \alpha) = h F^U_x(v_0, \alpha). \tag{2.44}$$

Therefore, for $r < 1$ close enough to 1 we have for every $\varphi \in \mathbb{R}$, abbreviating $z := \psi_0(re^{i\varphi})$,

$$\begin{aligned} F'_{re^{i\varphi}} &= g(z)^{-1} F^U_{u(z)}\big(\cdot, \mathrm{Ad}_{g_0(z)} \cdot\big) \cdot (e^{-id\varphi} \cdot \oplus \mathrm{id}) \\ &= F^U_{u''(z)}\big(\cdot, \mathrm{Ad}_{g(z)^{-1}} \mathrm{Ad}_{g_0(z)} \cdot\big)(e^{-id\varphi} \cdot \oplus \mathrm{id}) \\ &= F^U_{v(re^{i\varphi})}\big(\cdot, \mathrm{Ad}_{g_\infty(e^{i\varphi}) g_0(1)} \cdot\big)(e^{-id\varphi} \cdot \oplus \mathrm{id}), \end{aligned}$$

and this converges for $r \to \infty$ to

$$\begin{aligned} &F^U_{g_\infty(e^{i\varphi})y_\infty}\big(\cdot, \mathrm{Ad}_{g_\infty(e^{i\varphi})g_0(1)}\cdot\big)(e^{-id\varphi}\cdot \oplus \mathrm{id}) \\ &= g_\infty(e^{i\varphi})F^U_{y_\infty}\big(\cdot, \mathrm{Ad}_{g_0(1)}\cdot\big)(e^{-id\varphi}\cdot \oplus \mathrm{id}), \end{aligned}$$

uniformly in $\varphi \in \mathbb{R}$. This proves Claim 3. □

By Claim 3 the tuple $(g, g_\infty, y_\infty, F')$ satisfies the condition of Definition 2.6, and therefore the Maslov index $m(w)$ is given by the degree of the map

$$S^1 \ni \mapsto F_z'^{-1} g_\infty(z) \cdot F_1' \in \mathrm{Aut}(\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}). \tag{2.45}$$

Since $g_\infty(1) = \mathbf{1}$, we have for $z \in S^1$

$$\begin{aligned} F_z'^{-1} g_\infty(z)\cdot F_1' &= (z^d \cdot \oplus \mathrm{id}) F^U_{y_\infty}\big(\cdot, \mathrm{Ad}_{g_0(1)}\cdot\big)^{-1} g_\infty(z)^{-1} g_\infty(z)\cdot F^U_{y_\infty}\big(\cdot, \mathrm{Ad}_{g_0(1)}\cdot\big) \\ &= z^d \cdot \oplus \mathrm{id}. \end{aligned}$$

Hence the degree of the map (2.45) is d. This proves Claim 2. □

Condition (ii) follows from Claim 2 and the construction of F, since for $|z| \geq R+1$ we have

$$F_z(z^{m(w)} \cdot \oplus \mathrm{id}) = F^U_{u(z)}(\cdot, \mathrm{Ad}_{g_0(z)}\cdot) \oplus \mathrm{Ad}_{g_0(z)}.$$

We check condition (iii). We fix a vector

$$\zeta := (v_0, \alpha, \beta) \in V = \mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}.$$

Then by Lemma A.8 we have for $z \in \mathbb{C} \setminus B_{R+1}$, dropping the z,

$$\begin{aligned} \nabla_s^\Phi\big(F(p_d \cdot \oplus \mathrm{id})\zeta\big) &= (g_0)_*\Big(\nabla_s^{g_0^*\Phi}\big(g_0^{-1}F^U_u(v_0, \mathrm{Ad}_{g_0}\alpha)\big), \partial_s^{g_0^*\Phi}\beta\Big) \\ &= (g_0)_*\Big(\nabla_s^{\Phi'}\big(F^U_{u'}(v_0,\alpha)\big), [\Phi', \beta]\Big) \\ &= (g_0)_*\big(\nabla^{\Phi'}_{\partial_s u'}(F^U(v_0,\alpha)), [\Phi', \beta]\big). \end{aligned}$$

It follows that

$$\begin{aligned} &\big|\big|\nabla_s^\Phi\big(F(p_d\cdot\oplus\mathrm{id})\zeta\big)\big|\big|_{L^p_\lambda(\mathbb{C}\setminus B_{R+1})} \\ &= \big|\big|\big(\nabla^{\Phi'}_{\partial_s u'}(F^U(v_0,\alpha)), [\Phi',\beta]\big)\big|\big|_{L^p_\lambda(\mathbb{C}\setminus B_{R+1})} \\ &\leq C\big(||\partial_s u'||_{L^p_\lambda(\mathbb{C}\setminus B_{R+1})} + ||\Phi'||_{L^p_\lambda(\mathbb{C}\setminus B_{R+1})}\big)|\zeta| \\ &\leq C'\Big(||\partial_s u' + X_{\Phi'}(u')||_{L^p_\lambda(\mathbb{C}\setminus B_{R+1})} + ||\Phi'||_{L^p_\lambda(\mathbb{C}\setminus B_{R+1})}\Big)|\zeta| \\ &= \Big(||\partial_s u + X_\Phi(u)||_{L^p_\lambda(\mathbb{C}\setminus B_{R+1})} + ||\Phi'||_{L^p_\lambda(\mathbb{C}\setminus B_{R+1})}\Big)|\zeta| \\ &< \infty, \end{aligned}$$

where $C, C' > 0$ are constants depending on the local trivialization F^U, but not on the point $z \in \mathbb{C} \setminus B_{R+1}$ nor on ζ. This implies condition (iii) of Definition 2.10, and completes the proof of Proposition 2.11. $\square$

The purpose of a good complex trivialization of $u^*TM \oplus \mathfrak{g}^{\mathbb{C}}$ is to translate the operator $\mathcal{D}_w$ into a simpler operator defined on maps from $\mathbb{C}$ to $\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$. This translation is the content of the next result, Proposition 2.12 below. In order to state it, we now introduce the two types of weighted Sobolev-spaces mentioned in the introduction of section 2.

We fix a value $1 \leq p \leq \infty$, two integers $N \geq 1$, $k \geq 0$, a real number λ, an open subset $\Omega \subseteq \mathbb{R}^N$ and a real or complex vector space V. For every multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{N} \sqcup \{0\})^N$ we denote its length by

$$|\alpha| := \sum_{i=1}^{N} \alpha_i$$

and the multiple partial derivative operator corresponding to α by

$$\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}.$$

We define the *strongly λ-weighted (k,p)-Sobolev norm* of a map $u \in W^{k,p}_{\mathrm{loc}}(\Omega, V)$ by

$$||u||_{L^{k,p}_\lambda(\Omega,V)} := \sum_{|\alpha|=0}^{k} ||\langle\cdot\rangle^{\lambda+|\alpha|}\partial^\alpha u||_{L^p(\Omega,V)} \in [0,\infty],$$

and the *small λ-weighted (k,p)-Sobolev space over Ω* by

$$L^{k,p}_\lambda(\Omega, V) := \big\{u \in W^{k,p}_{\mathrm{loc}}(\Omega, V) \,|\, ||u||_{L^{k,p}_\lambda(\Omega,V)} < \infty\big\}.$$

Furthermore, we define the *weakly λ-weighted (k,p)-Sobolev norm* of u by

$$||u||_{W^{k,p}_\lambda(\Omega,V)} := \sum_{|\alpha|=0}^{k} ||\langle\cdot\rangle^{\lambda}\partial^\alpha u||_{L^p(\Omega,V)} \in [0,\infty],$$

and the *big λ-weighted (k,p)-Sobolev space over Ω* by

$$W^{k,p}_\lambda(\Omega, V) := \big\{u \in W^{k,p}_{\mathrm{loc}}(\Omega, V) \,|\, ||u||_{W^{k,p}_\lambda(\Omega,V)} < \infty\big\}.$$

Note that

$$||u||_{W^{k,p}_\lambda(\Omega,V)} \leq ||u||_{L^{k,p}_\lambda(\Omega,V)} \leq ||u||_{W^{k,p}_{\lambda+k}(\Omega,V)},$$

and therefore $W^{k,p}_{\lambda+k}(\Omega, V) \subseteq L^{k,p}_\lambda(\Omega, V) \subseteq W^{k,p}_\lambda(\Omega, V)$. Furthermore the space $L^{0,p}_\lambda(\Omega, V) = W^{0,p}_\lambda(\Omega, V)$ agrees with $L^p_\lambda(\Omega, V)$ defined in (2.3). In the papers [McO1], [McO2], [McO3], [Lo2], [LM1], [LM2], [Lo3] R. B. Lockhart and R. C. McOwen established Fredholm theorems for elliptic partial differential operators (for example for the Laplacian on $\mathbb{R}^N$) between small weighted Sobolev spaces. (They use different notations.)

Assume that $\bar{n} \geq 1$. Let d be an integer and $p > 2$ and $\lambda > -2/p + 1$ be real numbers. We choose a smooth function $\rho_0 : \mathbb{C} \to [0,1]$ such that $\rho_0(z) = 0$ for $|z| \leq 1/2$ and $\rho_0(z) = 1$ for $|z| \geq 1$. Recall that $p_d : \mathbb{C} \to \mathbb{C}$ denotes the monomial $p_d(z) := z^d$. We define the vector space

$$\begin{aligned} \mathcal{X}_d := \mathcal{X}_d^{p,\lambda} :=& \left(\mathbb{C}\rho_0 p_d + L^{1,p}_{\lambda-1-d}(\mathbb{C},\mathbb{C})\right) \times \left(\mathbb{C}^{\bar{n}-1} + L^{1,p}_{\lambda-1}(\mathbb{C},\mathbb{C}^{\bar{n}-1})\right) \\ &\times W^{1,p}_\lambda(\mathbb{C}, \mathfrak{g}^\mathbb{C} \times \mathfrak{g}^\mathbb{C}). \end{aligned} \tag{2.46}$$

Here in the first factor $\mathbb{C}\rho_0 p_d$ denotes the vector space of complex multiples of the function $\rho_0 p_d : \mathbb{C} \to \mathbb{C}$. Analogously, in the second factor we identify $\mathbb{C}^{\bar{n}-1}$ with the complex vector space of constant functions from $\mathbb{C}$ to $\mathbb{C}^{\bar{n}-1}$. We fix a vector

$$\zeta := \begin{pmatrix} v_0 \\ \alpha \\ \beta \end{pmatrix} := \begin{pmatrix} v^1_\infty \rho_0 p_d + v^1 \\ v^2_\infty + v^2 \\ \vdots \\ v^{\bar{n}}_\infty + v^{\bar{n}} \\ \alpha \\ \beta \end{pmatrix} \in \mathcal{X}_d,$$

where $(v^1_\infty, \ldots, v^{\bar{n}}_\infty) \in \mathbb{C}^{\bar{n}}$. By Lemma E.3 the vector $(v^1_\infty, \ldots, v^{\bar{n}}_\infty)$ is uniquely determined. We define the norm of ζ by

$$\begin{aligned} &||\zeta||_{\mathcal{X}_d} := \\ |(v^1_\infty, \ldots, v^{\bar{n}}_\infty)| + ||v^1||_{L^{1,p}_{\lambda-1-d}(\mathbb{C})} &+ ||(v^2, \ldots, v^{\bar{n}})||_{L^{1,p}_{\lambda-1}(\mathbb{C})} + ||(\alpha,\beta)||_{W^{1,p}_\lambda(\mathbb{C})}. \end{aligned} \tag{2.47}$$

Furthermore, we define the normed vector space

$$\mathcal{Y}_d := \mathcal{Y}_d^{p,\lambda} := L^p_{\lambda-d}(\mathbb{C},\mathbb{C}) \times L^p_\lambda(\mathbb{C}, \mathbb{C}^{\bar{n}-1} \times \mathfrak{g}^\mathbb{C} \times \mathfrak{g}^\mathbb{C}). \tag{2.48}$$

We use super- or subscripts to indicate the target of functions on which an operator acts. For example, the operator $\partial_{\bar{z}}^{\mathbb{C}^{\bar{n}}}$ acts on functions from $\mathbb{C}$ to $\mathbb{C}^{\bar{n}}$. Likewise, $\mathrm{id}_{\mathfrak{g}^\mathbb{C}}$ is the identity on a space of functions from $\mathbb{C}$ to $\mathfrak{g}^\mathbb{C}$. For every finite dimensional hermitian vector space $(V, \langle\cdot,\cdot\rangle)$ we call a linear map $A : V \to V$ strictly positive iff for every $0 \neq v \in V$ we have

$$\langle Av, v\rangle > 0.$$

Note that such a map A is self-adjoint, see for example Satz V.5.6 in the book by D. Werner [Wer].

Proposition 2.12 *Let $2 < p < \infty$ and $\lambda > -2/p + 1$ be real numbers. Assume that $\bar{n} = \dim M/2 - \dim \mathrm{G} \geq 1$. Let $w := (u, \Phi + i\Psi) \in \mathcal{B}^p_\lambda$ be a map and $F : \mathbb{C} \times V \to u^*TM \oplus \mathfrak{g}^\mathbb{C}$ be a good complex trivialization. Let $m(w) \in \mathbb{Z}$ denote the Maslov index of w, as in Definition 2.6, let $\mathcal{X}_w := \mathcal{X}^{p,\lambda}_w$ and $\mathcal{Y}_w := \mathcal{Y}^{p,\lambda}_w$ be defined as in (2.15) and (2.28), and let $\mathcal{X}_{m(w)} := \mathcal{X}^{p,\lambda}_{m(w)}$ and $\mathcal{Y}_{m(w)} := \mathcal{Y}^{p,\lambda}_{m(w)}$ be defined as in (2.46) and (2.48). Then the following statements hold.*

(i) The maps

$$\mathcal{X}_{m(w)} \ni \zeta \mapsto F\zeta \in \mathcal{X}_w, \tag{2.49}$$
$$\mathcal{Y}_{m(w)} \ni \zeta \mapsto F\zeta \in \mathcal{Y}_w \tag{2.50}$$

are well-defined and isomorphisms of normed vector spaces, i.e. there exists a constant $C > 0$ such that

$$C^{-1}||\zeta||_{\mathcal{X}_{m(w)}} \leq ||F\zeta||_w \leq C||\zeta||_{\mathcal{X}_{m(w)}}, \tag{2.51}$$
$$C^{-1}||\zeta||_{\mathcal{Y}_{m(w)}} \leq ||F\zeta||_{p,\lambda} \leq C||\zeta||_{\mathcal{Y}_{m(w)}}. \tag{2.52}$$

(ii) The operator

$$\mathcal{D}^{\mathrm{triv}}_w := F^{-1}\mathcal{D}_w F : \mathcal{X}_{m(w)} \to \mathcal{Y}_{m(w)}$$

is a compact perturbation of the operator

$$2\partial^{\mathbb{C}^{\bar{n}}}_{\bar{z}} \oplus \begin{pmatrix} 2\partial^{\mathfrak{g}^\mathbb{C}}_{\bar{z}} & \mathrm{id} \\ A & 2\partial^{\mathfrak{g}^\mathbb{C}}_{z} \end{pmatrix} : \mathcal{X}_{m(w)} \to \mathcal{Y}_{m(w)}, \tag{2.53}$$

where $A : \mathfrak{g}^\mathbb{C} \to \mathfrak{g}^\mathbb{C}$ is some positive $\mathbb{C}$-linear map.

Corollary 2.13 *For each two numbers $p > 2$ and $\lambda > -2/p + 1$ and every map $w := (u, \Phi + i\Psi) \in \mathcal{B}^p_\lambda$ the normed vector spaces $\mathcal{X}^{p,\lambda}_w$ and $\mathcal{Y}^{p,\lambda}_w$ are complete.*

Proof of Corollary 2.13: Recall that if $(X_1, ||\cdot||_1), \ldots, (X_k, ||\cdot||_k)$ are normed vector spaces then the product norm on $X_1 \times \cdots \times X_k$ is defined by $||(x_1, \ldots, x_k)|| := ||x_1||_1 + \cdots + ||x_k||_k$. We abbreviate $d := m(w)$. By definition of the norm on $\mathcal{X}_d$ this space is isometrically isomorphic to the vector space

$$\mathbb{C}^{\bar{n}} \times L^{1,p}_{\lambda-1-d}(\mathbb{C}, \mathbb{C}) \times L^{1,p}_{\lambda-1}(\mathbb{C}, \mathbb{C}^{\bar{n}-1}) \times W^{1,p}_\lambda(\mathbb{C}, \mathfrak{g}^\mathbb{C}), \tag{2.54}$$

with the product norm. By Proposition E.6(iv) the normed space $W^{1,p}_\lambda(\mathbb{C}, \mathfrak{g}^\mathbb{C})$ is isomorphic to $W^{1,p}(\mathbb{C}, \mathfrak{g}^\mathbb{C})$. Furthermore, as R. B. Lockhart proved in his PhD thesis [Lo1], the space $L^{1,q}_\mu(\mathbb{C}, \mathbb{C})$ is complete, for every two real numbers μ and $1 \leq q < \infty$. It follows that the space $\mathcal{X}_d$ is complete. The same holds for $\mathcal{Y}_d$. Since by Proposition 2.12 the spaces $\mathcal{X}_d$ and $\mathcal{Y}_d$ are isometrically isomorphic to $\mathcal{X}_w$ and $\mathcal{Y}_w$, these spaces are also complete. This proves Corollary 2.13. □

Proof of Proposition 2.12: Let $F : \mathbb{C} \times V \to u^*TM \oplus \mathfrak{g}^{\mathbb{C}}$ be a good complex trivialization. We choose a number $R > 0$ and a gauge transformation $g_0 \in C^\infty(\mathbb{C} \setminus B_R, \mathrm{G})$ such that the conditions in Definition 2.10(i) are satisfied. We abbreviate $d := m(w)$.

We prove (i). We choose a smooth function $\rho_0 : \mathbb{C} \to [0,1]$ such that $\rho_0(z) = 0$ for $z \in B_{1/2}$ and $\rho_0(z) = 1$ for $z \in \mathbb{C} \setminus B_1$. Let $\zeta \in W^{1,1}_{\mathrm{loc}}(\mathbb{C}, u^*TM \oplus \mathfrak{g}^{\mathbb{C}})$. Then by the definitions (2.10) and (2.36) of $\nabla^{\Phi,\Psi}\zeta$ and $\nabla^{\Phi,\Psi}F$ we have on $\mathbb{C} \setminus \{0\}$

$$\begin{aligned} \nabla^{\Phi,\Psi}(F\zeta) &= \nabla^{\Phi,\Psi}\Big(F(p_d \cdot \oplus \mathrm{id})(p_{-d} \cdot \oplus \mathrm{id})\zeta\Big) \\ &= \nabla^{\Phi,\Psi}\big(F(p_d \cdot \oplus \mathrm{id})\big)(p_{-d} \cdot \oplus \mathrm{id})\zeta \\ &\quad + F(p_d \cdot \oplus \mathrm{id})d\big((p_{-d} \cdot \oplus \mathrm{id})\zeta\big). \end{aligned} \tag{2.55}$$

Claim 1 *The map (2.49) is well-defined and bounded.*

Proof of Claim 1: Let

$$\zeta := (v_0, \alpha, \beta) := \begin{pmatrix} v^1_\infty \rho_0 p_d + v^1 \\ v^2_\infty + v^2 \\ \vdots \\ v^{\bar{n}}_\infty + v^{\bar{n}} \\ \alpha \\ \beta \end{pmatrix} \in \mathcal{X}_d.$$

In the following $C > 0$ will denote a constant that may depend on p, λ, d, the map w, the number R and the trivialization F, but not on the map ζ. It may change from estimate to estimate. We abbreviate

$$||\cdot||_{1,p,\lambda} := ||\cdot||_{L^{1,p}_\lambda(\mathbb{C})}, \qquad ||\cdot||_{p,\lambda} := ||\cdot||_{L^p_\lambda(\mathbb{C})}, \qquad ||\cdot||_\infty := ||\cdot||_{L^\infty(\mathbb{C})}.$$

We have

$$\begin{aligned} ||F\zeta||_\infty &\le ||F(\langle\cdot\rangle^d \cdot \oplus \mathrm{id})(\langle\cdot\rangle^{-d} \cdot \oplus \mathrm{id})\zeta||_\infty \\ &\le C||(\langle\cdot\rangle^{-d} \cdot \oplus \mathrm{id})\zeta||_\infty \\ &\le C\Big(|(v^1_\infty, \dots, v^{\bar{n}}_\infty)| + ||v^1\langle\cdot\rangle^{-d}||_\infty + \big|\big|\big(v^2, \dots, v^{\bar{n}}, \alpha, \beta\big)\big|\big|_\infty\Big) \\ &\le C\Big(|(v^1_\infty, \dots, v^{\bar{n}}_\infty)| + ||v^1||_{1,p,-\frac{2}{p}-d} + \big|\big|\big(v^2, \dots, v^{\bar{n}}, \alpha, \beta\big)\big|\big|_{1,p,-\frac{2}{p}}\Big) \\ &\le C\Big(|(v^1_\infty, \dots, v^{\bar{n}}_\infty)| + ||v^1||_{1,p,\lambda-1-d} \\ &\qquad + ||(v^2, \dots, v^{\bar{n}})||_{1,p,\lambda-1} + ||(\alpha,\beta)||_{W^{1,p}_\lambda(\mathbb{C})}\Big) \\ &= C||\zeta||_{\mathcal{X}_d}. \end{aligned} \tag{2.56}$$

Here in the second line we have used inequality (2.41), in the forth line we have used Proposition E.6(iii), and in the fifth line we have used that $\lambda > -2/p+1$ and therefore $||v^1||_{1,p,-2/p-d} \leq ||v^1||_{1,p,\lambda-1-d}$, and we have used that

$$||(\alpha,\beta)||_{1,p,\lambda-1} \leq ||(\alpha,\beta)||_{W^{1,p}_\lambda(\mathbb{C})}.$$

Furthermore, recalling the definition (2.13) of P, we have on $\mathbb{C}\setminus B_R$

$$PF\zeta = (L_u^{\mathbb{C}}\mathrm{Ad}_{g_0}\alpha, \mathrm{Ad}_{g_0}\beta),$$

and therefore

$$\begin{aligned}||PF\zeta||_{p,\lambda} &\leq ||PF\zeta||_{L^p_\lambda(\mathbb{C}\setminus B_R)} + C||PF\zeta||_{L^p(B_R)}\\ &\leq C\Big(||\mathrm{Ad}_{g_0}(\alpha,\beta)||_{L^p_\lambda(\mathbb{C}\setminus B_R)} + ||\zeta||_{L^p(B_R)}\Big)\\ &\leq C||\zeta||_{\mathcal{X}_w}. \end{aligned} \tag{2.57}$$

Here in the last line we have used that $|\mathrm{Ad}_{g_0}(\alpha,\beta)| = |(\alpha,\beta)|$.

Furthermore, by (2.55), we have

$$\begin{aligned}&||\nabla^{\Phi,\Psi}(F\zeta)||_{L^p_\lambda(\mathbb{C}\setminus B_1)}\\ &\leq \big|\big|\nabla^{\Phi,\Psi}\big(F(p_d\cdot\oplus\mathrm{id})\big)\big|\big|_{L^p_\lambda(\mathbb{C}\setminus B_1)}||(p_{-d}\cdot\oplus\mathrm{id})\zeta||_{L^\infty(\mathbb{C}\setminus B_1)}\\ &+ C\big|\big|d\big((p_{-d}\cdot\oplus\mathrm{id})\zeta\big)\big|\big|_{L^p_\lambda(\mathbb{C}\setminus B_1)}\\ &\leq C\Big(|(v^1_\infty,\ldots,v^{\bar n}_\infty)| + ||\langle\cdot\rangle^{-d}v^1||_\infty + ||(v^2,\ldots,v^{\bar n},\alpha,\beta)||_\infty\\ &+ ||p_{-d}v^1||_{L^{1,p}_{\lambda-1}(\mathbb{C}\setminus B_1)} + \big|\big|\big(v^2,\ldots,v^{\bar n},\alpha,\beta\big)\big|\big|_{1,p,\lambda-1}\Big)\\ &\leq C\Big(|(v^1_\infty,\ldots,v^{\bar n}_\infty)| + ||v^1||_{1,p,-\frac{2}{p}-d} + \big|\big|\big(v^2,\ldots,v^{\bar n},\alpha,\beta\big)\big|\big|_{1,p,-\frac{2}{p}}\\ &+ ||v^1||_{1,p,\lambda-1-d} + \big|\big|\big(v^2,\ldots,v^{\bar n},\alpha,\beta\big)\big|\big|_{1,p,\lambda-1}\Big)\\ &\leq C||\zeta||_{\mathcal{X}_d}. \end{aligned} \tag{2.58}$$

On the other hand, since

$$\nabla^{\Phi,\Psi}(F\zeta) = (\nabla^{\Phi,\Psi}F)\zeta + Fd\zeta, \tag{2.59}$$

it follows that

$$\begin{aligned}||\nabla^{\Phi,\Psi}(F\zeta)||_{L^p(B_1)} &\leq C||\nabla^{\Phi,\Psi}F||_{L^\infty(B_1)}||\zeta||_{L^p(B_1)} + ||F||_{L^\infty(B_1)}||d\zeta||_{L^p(B_1)}\\ &\leq C||\zeta||_{\mathcal{X}_d}. \end{aligned} \tag{2.60}$$

Here in the second inequality we have used condition (iii) of Definition 2.10 and Proposition E.6(iii). Combining estimates (2.56), (2.57), (2.58) and (2.60) we get

$$||F\zeta||_w \leq C||\zeta||_{\mathcal{X}_d}.$$

This proves Claim 1. □

Claim 2 *The map*

$$\mathcal{X}_w \ni \zeta' \mapsto F^{-1}\zeta' \in \mathcal{X}_d \tag{2.61}$$

is well-defined and bounded.

Proof of Claim 2: We fix $\zeta' := (v', \beta') \in \mathcal{X}_w$ and define $\zeta := (v_0, \alpha, \beta) := F^{-1}\zeta' : \mathbb{C} \to V$. Choosing $R > 0$ as in condition (i) of Definition 2.10, it follows that on $\mathbb{C} \setminus B_R$

$$|(\alpha, \beta)| \leq |P\zeta'|.$$

Therefore,

$$||(\alpha, \beta)||_{L^p_\lambda(\mathbb{C}\setminus B_R)} \leq C||P\zeta'||_{L^p_\lambda(\mathbb{C}\setminus B_R)} \leq C||\zeta'||_w. \tag{2.62}$$

Claim 3 *We have*

$$\big|\big|d\big((\rho_0 p_{-d} \cdot \oplus \mathrm{id})F^{-1}\zeta'\big)\big|\big|_{L^p_\lambda(\mathbb{C})} \leq C||\zeta'||_w. \tag{2.63}$$

Proof of Claim 3: By equality (2.55) we have on $\mathbb{C} \setminus \{0\}$

$$\begin{aligned} d\big((p_{-d} \cdot \oplus \mathrm{id})F^{-1}\zeta'\big) &= (p_{-d} \cdot \oplus \mathrm{id})F^{-1} \cdot \\ &\quad \Big(\nabla^{\Phi,\Psi}\zeta' - \nabla^{\Phi,\Psi}\big(F(p_d \cdot \oplus \mathrm{id})\big)(p_{-d} \cdot \oplus \mathrm{id})F^{-1}\zeta'\Big). \end{aligned}$$

Hence using conditions (ii) and (iii) of Definition 2.10, we have

$$\begin{aligned} \big|\big|d\big((p_{-d} \cdot \oplus \mathrm{id})F^{-1}\zeta'\big)\big|\big|_{L^p_\lambda(\mathbb{C}\setminus B_1)} &\leq C\Big(||\nabla^{\Phi,\Psi}\zeta'||_{p,\lambda} \\ &\quad + \big|\big|\nabla^{\Phi,\Psi}\big(F(p_d \cdot \oplus \mathrm{id})\big)\big|\big|_{p,\lambda}||\zeta'||_\infty\Big) \\ &\leq C||\zeta'||_w < \infty. \end{aligned} \tag{2.64}$$

On the other hand, (2.59) with $\zeta := F^{-1}\zeta'$ implies that

$$d(F^{-1}\zeta') = F^{-1}\big(\nabla^{\Phi,\Psi}\zeta' - (\nabla^{\Phi,\Psi}F)F^{-1}\zeta'\big). \tag{2.65}$$

Therefore,

$$\begin{aligned} \big|\big|d\big((\rho_0 p_{-d} \cdot \oplus \mathrm{id})F^{-1}\zeta'\big)\big|\big|_{L^p(B_1)} &\leq \big|\big|\big(d(\rho_0 p_{-d} \cdot \oplus \mathrm{id})\big)F^{-1}\zeta'\big|\big|_{L^p(B_1)} \\ &\quad + \big|\big|(\rho_0 p_{-d} \cdot \oplus \mathrm{id})d(F^{-1}\zeta')\big|\big|_{L^p(B_1)} \\ &\leq C\big(||\zeta'||_{L^\infty(B_1)} + ||\nabla^{\Phi,\Psi}\zeta'||_{L^p(B_1)}\big) \\ &\leq C||\zeta'||_w. \end{aligned}$$

Combining this with (2.64) proves Claim 3. □

By Claim 3, the hypotheses of Proposition E.4 (Hardy-type inequality) with u replaced by $(\rho_0 p_{-d} \cdot \oplus \mathrm{id})F^{-1}\zeta'$ are satisfied. Hence, applying that Proposition, there exists a vector

$$\zeta_\infty := \big(v^1_\infty, \ldots, v^{\bar{n}}_\infty, \alpha_\infty, \beta_\infty\big) \in V = \mathbb{C}^{\bar{n}} \times \mathfrak{g}^\mathbb{C} \times \mathfrak{g}^\mathbb{C},$$

such that

$$(\rho_0 p_{-d}\cdot \oplus \mathrm{id})F^{-1}\zeta'(re^{i\varphi}) \to \zeta_\infty, \tag{2.66}$$

for $r \to \infty$, uniformly in $\varphi \in \mathbb{R}$. Furthermore, we have

$$\begin{aligned}
&\big|\big|\big((\rho_0 p_{-d}\cdot \oplus \mathrm{id})F^{-1}\zeta' - \zeta_\infty\big)|\cdot|^{\lambda-1}\big|\big|_{L^p(\mathbb{C})} \\
&\le C\big|\big|d\big((\rho_0 p_{-d}\cdot \oplus \mathrm{id})F^{-1}\zeta'\big)|\cdot|^{\lambda}\big|\big|_{L^p(\mathbb{C})} \\
&\le C||\zeta'||_w,
\end{aligned} \tag{2.67}$$

where in the second inequality we have used Claim 3. Since $\lambda > -2/p+1$, we have

$$\int_{\mathbb{C}\setminus B_R} \langle\cdot\rangle^{p\lambda} = \infty.$$

Hence estimate (2.62) implies that $(\alpha_\infty, \beta_\infty) = (0,0)$.

The convergence (2.66) implies that

$$\begin{aligned}
|(v^1_\infty,\dots,v^{\bar{n}}_\infty)| &\le ||(\rho_0 p_{-d}\cdot \oplus \mathrm{id})F^{-1}\zeta'||_\infty \\
&\le C||\zeta'||_\infty \\
&\le C||\zeta'||_w,
\end{aligned} \tag{2.68}$$

where in the second inequality we have used condition (ii) of Definition 2.10. We define

$$\big(v^1,\dots,v^{\bar{n}},\alpha,\beta\big) := F^{-1}\zeta' - \big(\rho_0 p_d v^1_\infty, v^2_\infty,\dots,v^{\bar{n}}_\infty,0,0\big).$$

Using Proposition E.6(ii) and inequalities (2.67) and (2.63), we have

$$\begin{aligned}
||v^1||_{L^{1,p}_{\lambda-1-d}(\mathbb{C}\setminus B_1)} + ||(v^2,\dots,v^{\bar{n}})||_{L^{1,p}_{\lambda-1}(\mathbb{C}\setminus B_1)} &\le C\big(||p_{-d}v^1||_{L^{1,p}_{\lambda-1}(\mathbb{C}\setminus B_1)} \\
&\quad +||(v^2,\dots,v^{\bar{n}})||_{L^{1,p}_{\lambda-1}(\mathbb{C}\setminus B_1)}\big) \\
&\le C||\zeta'||_w.
\end{aligned} \tag{2.69}$$

Finally, on the ball B_R, we estimate $F^{-1}\zeta'$ by

$$\begin{aligned}
||F^{-1}\zeta'||_{L^{1,p}(B_R)} &\le C||\zeta'||_{L^{1,p}(B_R)} \\
&\le C||\zeta'||_w.
\end{aligned} \tag{2.70}$$

Inequalities (2.62), (2.68), (2.69) and (2.70) imply that

$$||F^{-1}\zeta'||_{\mathcal{X}_d} \le C||\zeta'||_w,$$

i.e. the map (2.61) is well-defined and bounded. This proves Claim 2. □
So we have proved that the map (2.49) is an isomorphism of normed vector spaces. In order to see that the map (2.50) is an isomorphism of normed

vector spaces, observe that by condition (ii) of Definition 2.10, we have for every $\zeta \in \mathcal{Y}_d$

$$\begin{aligned} ||F\zeta||_{p,\lambda} &= \left|\left|F(\langle\cdot\rangle^d \cdot \oplus \mathrm{id})(\langle\cdot\rangle^{-d} \cdot \oplus \mathrm{id})\zeta\right|\right|_{p,\lambda} \\ &\leq C||(\langle\cdot\rangle^{-d} \cdot \oplus \mathrm{id})\zeta||_{p,\lambda} \\ &\leq C||\zeta||_{\mathcal{Y}_d}. \end{aligned}$$

Analogously, we have

$$||\zeta||_{\mathcal{Y}_d} \leq C||F\zeta||_{p,\lambda}.$$

This proves that the map (2.50) is an isomorphism of normed vector spaces, and completes the proof of **(i)**.

We prove statement (ii). Recall that we have chosen $R > 0$ and g_0 as in Definition 2.10(i). It follows as in the proof of Lemma 2.5, that the point $u'(re^{i\varphi}) = g_0^{-1}(re^{i\varphi})u(re^{i\varphi}) \in M$ converges to some point $x_\infty \in \mu^{-1}(0)$, for $r \to \infty$, uniformly in $\varphi \in \mathbb{R}$. By our hypothesis (H1) the Lie group G acts freely on $\mu^{-1}(0)$. Hence Lemma A.4 implies that $L_x : \mathfrak{g} \to T_xM$ is injective for every $x \in \mu^{-1}(0)$. It follows that there exists a number $R' \geq R$ such that for every $z \in \mathbb{C} \setminus B_{R'}$ the map $L_{u(z)} : \mathfrak{g} \to T_{u(z)}M$ is injective. Thus the complexified infinitesimal action $L^{\mathbb{C}}_{u(z)} : \mathfrak{g}^{\mathbb{C}} \to T_{u(z)}M$ is injective, for $z \in \mathbb{C} \setminus B_{R'}$. We define the linear map

$$A := (L^{\mathbb{C}}_{x_\infty})^* L^{\mathbb{C}}_{x_\infty} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}.$$

This map is strictly positive complex linear. By the splitting property (2.37) and (2.38) in Definition 2.10(i), the trivialization $F : \mathbb{C} \times V \to u^*TM \oplus \mathfrak{g}^{\mathbb{C}}$ is the direct sum of two complex vector bundle isomorphisms

$$F_1 : \mathbb{C} \times (\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}) \to u^*TM, \qquad F_2 : \mathbb{C} \times \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}.$$

Hence by (2.31), the operator $\mathcal{D}^{\mathrm{triv}}_w : \mathcal{X}_d \to \mathcal{Y}_d$ is given by

$$\begin{aligned} \mathcal{D}^{\mathrm{triv}}_w &= (F_1 \oplus F_2)^{-1}\mathcal{D}_w(F_1 \oplus F_2) \\ &= (F_1 \oplus F_2)^{-1} \begin{pmatrix} 2\nabla^{\Phi+i\Psi}_{\bar{z}} + (\nabla_{\cdot} J)\partial^{\Phi}_t u & L^{\mathbb{C}}_u \\ (L^{\mathbb{C}}_u)^* & 2\partial^{\Phi-i\Psi}_z \end{pmatrix} F_1 \oplus F_2 \\ &= (F_1 \oplus F_2)^{-1} \cdot \\ &\quad \cdot \begin{pmatrix} (2\nabla^{\Phi+i\Psi}_{\bar{z}} F_1) + 2F_1\partial_{\bar{z}} + \nabla_{F_1}\partial^{\Psi}_t u & L^{\mathbb{C}}_u F_2 \\ (L^{\mathbb{C}}_u)^* F_1 & 2(\partial^{\Phi-i\Psi}_z F_2) + 2F_2\partial_z \end{pmatrix} \\ &= 2\partial^{\mathbb{C}^{\bar{n}}}_{\bar{z}} \oplus \begin{pmatrix} 2\partial^{\mathfrak{g}^{\mathbb{C}}}_{\bar{z}} & \mathrm{id}_{\mathfrak{g}^{\mathbb{C}}} \\ A & 2\partial^{\mathfrak{g}^{\mathbb{C}}}_z \end{pmatrix} + S. \end{aligned} \tag{2.71}$$

Here $S : \mathcal{X}_d \to \mathcal{Y}_d$ is the 0-th order perturbation defined by

$$S := \begin{pmatrix} F_1^{-1}\big((2\nabla^{\Phi+i\Psi}_{\bar{z}} F_1) + (\nabla_{F_1} J)\partial^{\Psi}_t u\big) & F_1^{-1} L^{\mathbb{C}}_u F_2 - \iota \\ F_2^{-1}(L^{\mathbb{C}}_u)^* F_1 - A\,\mathrm{pr} & 2F_2^{-1}(\partial^{\Phi-i\Psi}_z F_2) \end{pmatrix}, \tag{2.72}$$

where $\iota : \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}$ is the inclusion to the second factor and pr : $\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is the projection to the second factor. By (2.71) the statement (ii) follows, once we have proved that the operator S is compact. In order to prove compactness of this operator, observe that we can write it as a sum

$$S = S_0 + S_1 : \mathcal{X}_d = \mathcal{X}_d^0 \oplus \mathcal{X}_d^1 \to \mathcal{Y}_d,$$

where

$$\begin{aligned}\mathcal{X}_d^0 &:= \mathbb{C}\rho_0 p_d \times \mathbb{C}^{\bar{n}-1} \times \{(0,0)\},\\ \mathcal{X}_d^1 &:= L^{1,p}_{\lambda-1-d}(\mathbb{C},\mathbb{C}) \times L^{1,p}_{\lambda-1}(\mathbb{C},\mathbb{C}^{\bar{n}-1}) \times W^{1,p}_{\lambda}(\mathbb{C},\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}).\end{aligned}$$

The operators S_0 and S_1 are given by the same formula as S. Since $\mathcal{X}_d^0$ is finite dimensional, S_0 is compact. We prove that S_1 is compact as well. In order to see that S_1 is compact, we also write it as a sum

$$S_1 = \chi_{B_{R'}} S_1 + \chi_{\mathbb{C}\setminus B_{R'}} S_1.$$

Since $B_{R'}$ is bounded and has smooth boundary, the operator $\chi_{B_{R'}} S_1 : \mathcal{X}_d^1 \to \mathcal{Y}_d$ is compact. We prove that $\chi_{\mathbb{C}\setminus B_{R'}} S_1 : \mathcal{X}_d^1 \to \mathcal{Y}_d$ is compact.

Claim 4 *The map*

$$\begin{aligned}\chi_{\mathbb{C}\setminus B_{R'}} F_1^{-1}(\nabla_{\bar{z}}^{\Phi+i\Psi} F_1) \quad &: \quad L^{1,p}_{\lambda-1-d}(\mathbb{C},\mathbb{C}) \times L^{1,p}_{\lambda-1}(\mathbb{C},\mathbb{C}^{\bar{n}-1}) \times W^{1,p}_{\lambda}(\mathbb{C},\mathfrak{g}^{\mathbb{C}})\\ &\to \quad L^{p}_{\lambda-d}(\mathbb{C},\mathbb{C}) \times L^{p}_{\lambda}(\mathbb{C},\mathbb{C}^{\bar{n}-1} \times \mathfrak{g}^{\mathbb{C}}) \qquad (2.73)\end{aligned}$$

is compact.

Proof of Claim 4: On $\mathbb{C} \setminus \{0\}$, we have

$$\begin{aligned}F_1^{-1}(\nabla_{\bar{z}}^{\Phi+i\Psi} F_1) &= F_1^{-1}\big(\nabla_{\bar{z}}^{\Phi+i\Psi}(F_1(p_d \cdot \oplus \mathrm{id}))\big)(p_{-d} \cdot \oplus \mathrm{id})\\ &\quad +(p_d \cdot \oplus \mathrm{id})\big(\partial_{\bar{z}}(p_{-d} \cdot \oplus \mathrm{id})\big)\\ &= F_1^{-1}\big(\nabla_{\bar{z}}^{\Phi+i\Psi}(F_1(p_d \cdot \oplus \mathrm{id}))\big)(p_{-d} \cdot \oplus \mathrm{id}). \qquad (2.74)\end{aligned}$$

Here we have used that p_{-d} is holomorphic. By Proposition E.6(ii), the map

$$\chi_{\mathbb{C}\setminus B_{R'}}(p_{-d}) \cdot : L^{1,p}_{\lambda-1-d}(\mathbb{C},\mathbb{C}) \to L^{1,p}_{\lambda-1}(\mathbb{C},\mathbb{C}) \qquad (2.75)$$

is bounded. By the same Lemma, statement (iii), there exists a compact inclusion

$$L^{1,p}_{\lambda-1}(\mathbb{C},\mathbb{C}^{\bar{n}}) \times W^{1,p}_{\lambda}(\mathbb{C},\mathfrak{g}^{\mathbb{C}}) \subseteq C_b(\mathbb{C},\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}). \qquad (2.76)$$

Here we have used that the inclusion $W^{1,p}_{\lambda} \subseteq L^{1,p}_{\lambda-1}$ is bounded. By condition (iii) of Definition 2.10, the map

$$\chi_{\mathbb{C}\setminus B_{R'}}\big(\nabla_{\bar{z}}^{\Phi+i\Psi}(F_1(p_d \cdot \oplus \mathrm{id}))\big) : C_b(\mathbb{C},\mathbb{C}^{\bar{n}} \times \mathfrak{g}^{\mathbb{C}}) \to L^p_{\lambda}(\mathbb{C},u^*TM) \qquad (2.77)$$

is bounded. Finally, it follows from condition (ii) of Definition 2.10 that the map

$$F_1^{-1} : L^p_\lambda(\mathbb{C}, u^*TM) \to L^p_{\lambda-d}(\mathbb{C},\mathbb{C}) \times L^p_\lambda(\mathbb{C}, \mathbb{C}^{\bar n-1} \times \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}) \tag{2.78}$$

is bounded. Combining equality (2.74), the boundedness of the map (2.75), the compactness of the inclusion (2.76), and the boundedness of (2.77) and (2.78), it follows that the map (2.73) is compact. This proves Claim 4. □

In the same manner as in the proof of Claim 4, Proposition E.6(iii) and conditions (ii) and (iii) of Definition 2.10 imply that the map

$$\chi_{\mathbb{C}\setminus B_{R'}} F_2^{-1}(\partial_z^{\Phi - i\Psi} F_2) : W^{1,p}_\lambda(\mathbb{C}, \mathfrak{g}^{\mathbb{C}}) \to L^p_\lambda(\mathbb{C}, \mathfrak{g}^{\mathbb{C}}) \tag{2.79}$$

is compact. Furthermore, by the definition of $\mathcal{B}$, we have $\partial_t^\Psi u \in L^p_\lambda(\mathbb{C})$. This together with statements (ii) and (iii) of Proposition E.6 and condition (ii) of Definition 2.10 implies that the map

$$\begin{aligned} \chi_{\mathbb{C}\setminus B_{R'}} F_1^{-1}(\nabla_{F_1} J)\partial_t^\Psi u \quad : \quad & L^{1,p}_{\lambda-1-d}(\mathbb{C},\mathbb{C}) \times L^{1,p}_{\lambda-1}(\mathbb{C}, \mathbb{C}^{\bar n-1}) \times W^{1,p}_\lambda(\mathbb{C}, \mathfrak{g}^{\mathbb{C}}) \\ \to \quad & L^p_{\lambda-d}(\mathbb{C},\mathbb{C}) \times L^p_\lambda(\mathbb{C}, \mathbb{C}^{\bar n-1} \times \mathfrak{g}^{\mathbb{C}}) \end{aligned} \tag{2.80}$$

is compact. By the splitting condition (i) of Definition 2.10 we have for $z \in \mathbb{C} \setminus B_{R'}$

$$F_1^{-1} L_u^{\mathbb{C}} F_2|_z = \iota.$$

Thus

$$\chi_{\mathbb{C}\setminus B_{R'}} \big(F_1^{-1} L_u^{\mathbb{C}} F_2 - \iota\big) = 0 : W^{1,p}_\lambda(\mathbb{C}, \mathfrak{g}^{\mathbb{C}}) \to L^p_\lambda(\mathbb{C}, \mathbb{C}^{\bar n} \times \mathfrak{g}^{\mathbb{C}}). \tag{2.81}$$

Moreover, by short calculations, we have for every $x \in M$, $g \in \mathrm{G}$, $\alpha \in \mathfrak{g}^{\mathbb{C}}$ and $v \in T_xM$

$$L^{\mathbb{C}}_{gx} \mathrm{Ad}_g \alpha = g L^{\mathbb{C}}_x \alpha, \tag{2.82}$$

$$(L^{\mathbb{C}}_{gx})^* gv = \mathrm{Ad}_g (L^{\mathbb{C}}_x)^* v. \tag{2.83}$$

We fix a point $z \in \mathbb{C} \setminus B_{R'}$. Recall that $u' = g_0^{-1}u$. We define

$$f(z) := (L^{\mathbb{C}}_{u'(z)})^* L^{\mathbb{C}}_{u'(z)} - (L^{\mathbb{C}}_{x_\infty})^* L^{\mathbb{C}}_{x_\infty} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}.$$

By the splitting condition 2.10(i) we have

$$F_1^z(\mathbb{C}^{\bar n} \times \{0\}) = H_{u(z)} = \ker(L^{\mathbb{C}}_{u(z)})^*.$$

It follows that

$$\begin{aligned} F_2^{-1}(L_u^{\mathbb{C}})^* F_1|_z - A\,\mathrm{pr} &= \big(F_2^{-1}(L_u^{\mathbb{C}})^* F_1|_z - A\big)\,\mathrm{pr} \\ &= \big((L^{\mathbb{C}}_{u'})^* L^{\mathbb{C}}_{u'}|_z - (L^{\mathbb{C}}_{x_\infty})^* L^{\mathbb{C}}_{x_\infty}\big)\,\mathrm{pr} \\ &= f(z)\,\mathrm{pr} : \mathbb{C}^{\bar n} \times \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}. \end{aligned} \tag{2.84}$$

Here we have used equalities (2.82) and (2.83). Since $u'(re^{i\varphi})$ converges to x_∞, for $r \to \infty$, uniformly in φ, the linear map $f(re^{i\varphi})$ converges to 0, for $r \to \infty$, uniformly in φ. Hence by statement (v) of Proposition E.6, the map

$$W^{1,p}_\lambda(\mathbb{C}, \mathfrak{g}^\mathbb{C}) \ni \alpha \mapsto f\alpha \in L^p_\lambda(\mathbb{C}, \mathfrak{g}^\mathbb{C})$$

is compact. Equality (2.84) implies that the map

$$\chi_{\mathbb{C}\setminus B_{R'}}\big(F_2^{-1}(L_u^\mathbb{C})^* F_1 - A\,\mathrm{pr}\big) : W^{1,p}_\lambda(\mathbb{C}, \mathbb{C}^{\bar{n}} \times \mathfrak{g}^\mathbb{C}) \to L^p_\lambda(\mathbb{C}, \mathfrak{g}^\mathbb{C}) \tag{2.85}$$

is compact. Combining the compactness of the maps (2.73), (2.79), (2.80), (2.81) and (2.85) we conclude that the map

$$\chi_{\mathbb{C}\setminus B_{R'}} S_1 : \mathcal{X}^1_d := L^{1,p}_{\lambda-1-d}(\mathbb{C}, \mathbb{C}) \times L^{1,p}_{\lambda-1}(\mathbb{C}, \mathbb{C}^{\bar{n}-1}) \times W^{1,p}_\lambda(\mathbb{C}, \mathfrak{g}^\mathbb{C} \times \mathfrak{g}^\mathbb{C}) \to \mathcal{Y}_d$$

is compact. So we have proved that the perturbation $S : \mathcal{X}_d \to \mathcal{Y}_d$, defined in (2.72), is compact. This proves statement **(ii)** and concludes the proof of Proposition 2.12. □

Remark 2.14 To have identity (2.74) in the proof of Proposition 2.12 was the main reason for the precise form of condition (iii) in Definition 2.10 (good trivialization), involving the function p_d rather than for example the function $\langle \cdot \rangle^d$.

For the proof of the main result, Theorem 2.8, we need two more ingredients, namely that the two summands in the direct sum (2.53) in part (ii) of Proposition 2.12 are Fredholm. These are the statements of Corollary 2.17 and Proposition 2.18 below. Fix an integer d. We define P_d to be the space of polynomials in $z \in \mathbb{C}$ with complex coefficients, of degree *less than* d. So it has complex dimension d. Our convention is that if $d \leq 0$ then $P_d = \{0\}$. Likewise, we denote by $\bar{P}_d$ the space of polynomials in $\bar{z} \in \mathbb{C}$ with complex coefficients, of degree less than d. We abbreviate

$$L^{1,p}_\lambda := L^{1,p}_\lambda(\mathbb{C}, \mathbb{C}), \qquad L^p_\lambda := L^p_\lambda(\mathbb{C}, \mathbb{C}).$$

For a normed vector space X we denote by X^* its dual space.

Proposition 2.15 *Let d be an integer and $1 < p < \infty$ and $-2/p + 1 < \lambda < -2/p + 2$ be real numbers. Then the operator*

$$T := \partial_{\bar{z}} : L^{1,p}_{\lambda-1-d} \to L^p_{\lambda-d} \tag{2.86}$$

is Fredholm. Furthermore, its kernel is

$$\ker T = P_d. \tag{2.87}$$

Moreover, the map

$$\bar{P}_{-d} \to \big(L^p_{\lambda-d}/\mathrm{im}T\big)^*, \quad u \mapsto \Big(v + \mathrm{im}T \mapsto \int_{\mathbb{C}} uv\,ds\,dt\Big) \tag{2.88}$$

is well-defined and an isometric isomorphism. Here we equip $L^p_{\lambda-d}/\mathrm{im}T$ *with the quotient norm.*

Remark 2.16 It follows that $\partial_{\bar{z}} : L^{1,p}_{\lambda-1-d} \to L^p_{\lambda-d}$ is injective if $d \leq 0$ and surjective if $d \geq 0$. In particular, for $d = 0$ it is an isomorphism. In general, its real Fredholm index is $2d$.

We denote by $\mathcal{S}$ the space of Schwartz functions on $\mathbb{C}$ and by $\mathcal{S}'$ the space of temperate distributions. By $\widehat{\ }: \mathcal{S}' \to \mathcal{S}'$ we denote the Fourier transform, defined by

$$\widehat{u}(\varphi) := u(\widehat{\varphi}),$$

for $\varphi \in \mathcal{S}$, where

$$\widehat{\varphi}(\zeta_1 + i\zeta_2) := \frac{1}{2\pi}\int_{\mathbb{C}} \varphi(s+it)e^{-i(s\zeta_1 + t\zeta_2)}\,ds\,dt.$$

Furthermore, we denote by $^\vee : \mathcal{S}' \to \mathcal{S}'$ the inverse transformation. For $\psi \in \mathcal{S}$ it is given by

$$\overset{\vee}{\psi}(s+it) = \frac{1}{2\pi}\int_{\mathbb{C}} \psi(\zeta_1 + i\zeta_2)e^{i(s\zeta_1 + t\zeta_2)}\,d\zeta_1\,d\zeta_2.$$

Proof of Proposition 2.15:

Claim 1 *The kernels of T and T^* are*

$$\ker T = P_d, \tag{2.89}$$

$$\ker T^* = \bar{P}_{-d}. \tag{2.90}$$

Proof of Claim 1: We prove (2.89). A calculation in polar coordinates shows that for every polynomial u in the variable z we have

$$u \in L^{1,p}_{\lambda-1-d} \iff \deg u < d - \lambda + 1 - \frac{2}{p}. \tag{2.91}$$

We prove that $\ker T \supseteq P_d$**.** Fix a polynomial $u \in P_d$. Then $Tu = \partial_{\bar{z}}u = 0$. Since by our assumption $\lambda < -2/p + 2$ and $\deg u$ is an integer less than d, it follows that $\deg u < d - \lambda + 1 - 2/p$. Hence by the equivalence (2.91) we have $u \in L^{1,p}_{\lambda-1-d}$, and therefore $u \in \ker T$. This proves that $\ker T \supseteq P_d$.

We prove that $\ker T \subseteq P_d$**.** We fix a function $u \in \ker T$. Then $\partial_{\bar{z}}u = 0$ and therefore in the sense of temperate distributions

$$0 = \widehat{\partial_{\bar{z}}u}(\zeta) = i\zeta\widehat{u}.$$

It follows that the support of $\widehat{u}$ is either empty or consists of the point $0 \in \mathbb{C}$. Hence the Paley-Wiener theorem implies that u is real analytic in the variables s and t, and there exists an integer N such that

$$\sup_{z\in\mathbb{C}} \left(|u(z)|\langle z\rangle^N\right) < \infty,$$

see for example Theorem IX.12 in Vol. I of the book by M. Reed and B. Simon [RS]. Since u is holomorphic, Liouville's Theorem implies that it is a polynomial in the variable z. Since by our assumption $\lambda > -2/p+1$, it follows from the equivalence (2.91) that $\deg u < d$, hence $u \in P_d$. This proves that $\ker T \subseteq P_d$ and thus equality (2.89).

We prove (2.90). We define $p' := p/(p-1)$. For every $\mu \in \mathbb{R}$ the map

$$L^{p'}_{-\mu} \to (L^p_\mu)^*, \quad u \mapsto \left(v \mapsto \int_{\mathbb{C}} uv\right)$$

is an isometric isomorphism. Hence we may identify $(L^p_{\lambda-d})^*$ with $L^{p'}_{-\lambda+d}$. The adjoint operator T^* is given by

$$T^* = \partial_z = \frac{1}{2}(\partial_s - i\partial_t) : L^{p'}_{-\lambda+d} \cong (L^p_{\lambda-d})^* \to (L^{1,p}_{\lambda-1-d})^*,$$

where the derivatives are taken in the sense of distributions. Analogously to (2.91), for every polynomial u in the variable $\bar{z}$ we have

$$u \in (L^p_{\lambda-d})^* = L^{p'}_{-\lambda+d} \iff \deg u < -d + \lambda - \frac{2}{p'} = -d + \lambda - 2 + \frac{2}{p}. \quad (2.92)$$

We prove that $\ker T^* \supseteq \bar{P}_{-d}$**.** Fix a polynomial $u \in \bar{P}_{-d}$ in the variable $\bar{z}$. Since by our assumption $\lambda > -2/p+1$ and $\deg u$ is an integer less than $-d$, it follows that $\deg u < -d+\lambda-2+2/p$. Therefore by the equivalence (2.92) $u \in (L^p_{\lambda-d})^*$, and hence $u \in \ker T^*$. This proves that $\ker T^* \supseteq \bar{P}_{-d}$.

We prove that $\ker T^* \subseteq \bar{P}_{-d}$**.** We fix a function $u \in \ker T^*$. Then $\partial_z u = 0$ in the sense of temperate distributions, and therefore

$$0 = \widehat{\partial_z u} = i\bar{\zeta}\widehat{u}.$$

Hence as above, the Paley-Wiener theorem and Liouville's Theorem (for anti-holomorphic functions) imply that u is a polynomial in the variable $\bar{z}$. Since by our assumption $\lambda < -2/p+2$ it follows from (2.92) that $\deg u < -d$, hence $u \in \bar{P}_{-d}$. This proves that $\ker T^* \subseteq \bar{P}_{-d}$, and therefore (2.90).

This completes the proof of Claim 1. □

We apply now Theorem 4.3 in the paper by R. B. Lockhart [Lo2]. The hypotheses of that theorem are satisfied, since by our assumption $-2/p+1 <$

$\lambda < -2/p + 2$, and since the operator $T = \partial_{\bar{z}}$ has constant coefficients and is elliptic, in the sense that its principal symbol

$$\sigma_T : \mathbb{C} \to \mathbb{C}, \quad \sigma_T(\zeta) = \frac{1}{2}(\zeta_1 + i\zeta_2)$$

does not vanish on $S^1 \subseteq \mathbb{C}$. Hence that theorem implies that in the case $d \leq 0$ the operator $T : L^{1,p}_{\lambda-1} \to L^p_\lambda$ is Fredholm, and that in the case $d \geq 0$ the operator $T^* : (L^p_\lambda)^* \to (L^{1,p}_{\lambda-1})^*$ is Fredholm. Consider the case $d \geq 0$. Since T^* is Fredholm, it has a closed image. Therefore, Proposition E.2 implies that the image of T is also closed. It follows that $\mathrm{im}T$ is always closed. Thus we may apply Lemma E.1 with $X := L^p_{\lambda-d}$ and $V := \mathrm{im}T$, to deduce that the map

$$\ker T^* = (\mathrm{im}T)^\perp \to (L^p_{\lambda-d}/\mathrm{im}T)^*, \quad u \mapsto \big(v + \mathrm{im}T \mapsto \int_{\mathbb{C}} uv \, ds \, dt\big)$$

is well-defined and an isometric isomorphism. Here $(\mathrm{im}T)^\perp \subseteq (L^p_\lambda)^*$ denotes the annihilator of $\mathrm{im}T$. The statements (2.87) and (2.88) follow now from Claim 1. So $\ker T$ and $\operatorname{coker} T$ are always finite dimensional, hence T is Fredholm also in the case $d \geq 0$. This proves Proposition 2.15. □

Let d be an integer and $1 < p < \infty$ and $-2/p + 1 < \lambda < -2/p + 2$ be real numbers. Let $\rho_0 : \mathbb{C} \to [0, 1]$ be a smooth function that vanishes on $B_{1/2}$ and equals 1 on $\mathbb{C} \setminus B_1$. By Lemma E.3 the map

$$\mathbb{C} \times L^{1,p}_{\lambda-1-d} \to \mathbb{C}\rho_0 p_d + L^{1,p}_{\lambda-1-d}, \qquad (x_\infty, u) \mapsto x_\infty \rho_0 p_d + u \tag{2.93}$$

is an isomorphism of vector spaces. We may therefore define the norm of a vector $v \in \mathbb{C}\rho_0 p_d + L^{1,p}_{\lambda-1-d}$ by setting

$$||v||_d := |x_\infty| + ||u||_{L^p_{\lambda-1-d}},$$

where $(x_\infty, u) \in \mathbb{C} \times L^{1,p}_{\lambda-1-d}$ is the inverse image of v under the map (2.93). The vector space $\mathbb{C}\rho_0 p_d + L^{1,p}_{\lambda-1-d}$ equipped with this norm is complete, since $\mathbb{C} \times L^{1,p}_{\lambda-1-d}$ is complete, see the PhD thesis by R. B. Lockhart [Lo1].

Corollary 2.17 *Let d, p, λ and ρ_0 be as above. Then the operator*

$$\partial_{\bar{z}} : \mathbb{C}\rho_0 p_d + L^{1,p}_{\lambda-1-d} \to L^p_{\lambda-d} \tag{2.94}$$

is Fredholm, and its real index equals $2 + 2d$.

Proof of Corollary 2.17: A short calculation using the Leibnitz product rule shows that the composition of the maps (2.93) and (2.94) is given by

$$T + S : \mathbb{C} \times L^{1,p}_{\lambda-1-d} \to L^p_{\lambda-d}, \tag{2.95}$$

where

$$\begin{aligned} T(x_\infty, u) &:= \partial_{\bar z} u, \\ S(x_\infty, u) &:= x_\infty (\partial_{\bar z}\rho_0) p_d. \end{aligned}$$

The map T is the composition of the projection to the second factor

$$\mathrm{pr} : \mathbb{C} \times L^{1,p}_{\lambda-1-d} \to L^{1,p}_{\lambda-1-d}$$

with the operator

$$\partial_{\bar z} : L^{1,p}_{\lambda-1-d} \to L^p_{\lambda-d}. \tag{2.96}$$

By Proposition 2.15 the operator (2.96) is Fredholm of real index $2d$. Since the projection pr is Fredholm of real index 2, it follows that T is Fredholm of real index $2+2d$. Furthermore, the operator S is compact, since it is the composition of the two compact operators

$$\begin{aligned} \mathbb{C} \times L^{1,p}_{\lambda-1-d} \ni (x_\infty, u) &\mapsto x_\infty \in \mathbb{C}, \\ \mathbb{C} \ni x_\infty \mapsto x_\infty (\partial_{\bar z}\rho_0) p_d \in L^p_{\lambda-d}. \end{aligned}$$

It follows that the operator $T + S : \mathbb{C} \times L^{1,p}_{\lambda-1-d} \to L^p_{\lambda-d}$ is Fredholm of real index $2+2d$, and hence the operator (2.94) is Fredholm with the same index. This proves Corollary 2.17. □

Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional hermitian vector space, and let $A, B : V \to V$ be strictly positive complex linear maps. Recall that this means that

$$\langle Au, u\rangle > 0, \qquad \langle Bu, u\rangle > 0,$$

for every $0 \neq u \in V$. (It follows that A and B are self-adjoint, see for example Satz V.5.6 in the book by D. Werner [Wer].) We fix real numbers λ and $1 < p < \infty$ and define the operator

$$T := \begin{pmatrix} 2\partial_{\bar z} & A \\ B & 2\partial_z \end{pmatrix} : W^{1,p}_\lambda(\mathbb{C}, V \times V) \to L^p_\lambda(\mathbb{C}, V \times V).$$

Proposition 2.18 *The operator T is Fredholm of index 0.*

For the proof of Proposition 2.18 we need the following.

Theorem 2.19 (A. P. Calderón) *Let N be a positive integer, $1 < p < \infty$ be a real number, $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional hermitian vector space and $A : V \to V$ be a strictly positive complex linear map. Then the map*

$$-\Delta + A : W^{2,p}(\mathbb{R}^N, V \times V) \to L^p(\mathbb{R}^N, V \times V) \tag{2.97}$$

is an isomorphism of Banach spaces.

Proof of Theorem 2.19: We denote the complex dimension of V by k and the eigenvalues of A by $\lambda_1, \ldots, \lambda_k$ and choose a unitary basis $e_1, \ldots, e_k$ of eigenvectors of A such that $Ae_i = \lambda_i e_i$ for every i. We define the linear map

$$T : \mathbb{C}^k \to V, \quad Tu := \sum_{i=1}^{k} u^i e_i.$$

Since T is invertible, it induces isomorphisms of Banach spaces

$$T\cdot : W^{2,p}(\mathbb{R}^N, \mathbb{C}^k) \to W^{2,p}(\mathbb{R}^N, V), \tag{2.98}$$

$$T^{-1}\cdot : L^p(\mathbb{R}^N, V) \to L^p(\mathbb{R}^N, \mathbb{C}^k). \tag{2.99}$$

It follows that

$$T^{-1}\cdot(-\Delta+A)T\cdot = -\Delta + \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_k \end{pmatrix} : W^{2,p}(\mathbb{R}^N, \mathbb{C}^k) \to L^p(\mathbb{R}^N, \mathbb{C}^k). \tag{2.100}$$

We define the temperate distribution

$$G := (2\pi)^{\frac{N}{2}} \left(\langle\cdot\rangle^{-2}\right)^{\vee} \in \mathcal{S}'.$$

Then the map

$$\mathcal{S} \ni u \mapsto G * u \in \mathcal{S}$$

is well-defined. By the usual Calderón Theorem this map extends uniquely to an isomorphism of Banach spaces

$$L^p(\mathbb{R}^N, \mathbb{C}) \ni u \mapsto G * u \in W^{2,p}(\mathbb{R}^N, \mathbb{C}), \tag{2.101}$$

see Theorem 1.2.3. in the book by D. R. Adams [Ad] or Theorem V.3 p.135 in the book by E. M. Stein [1]. Note that for every Schwartz function u we have

$$(-\Delta + 1)(G * u) = \left(\langle\cdot\rangle^2 (G * u)^{\wedge}\right)^{\vee} = u.$$

It follows that the operator

$$-\Delta + 1 : W^{2,p}(\mathbb{R}^N, \mathbb{C}) \to L^p(\mathbb{R}^N, \mathbb{C}) \tag{2.102}$$

is the inverse of (2.101), and hence it is an isomorphism of Banach spaces. Since $\lambda_i > 0$ for every i, it follows from a scaling and direct sum argument that the operator (2.100) is an isomorphism of Banach spaces. Since the transformations (2.98) and (2.99) are isomorphisms of Banach spaces, by equality (2.100) the same holds for the operator

$$-\Delta + A : W^{2,p}(\mathbb{R}^N, V) \to L^p(\mathbb{R}^N, V).$$

This proves Theorem 2.19. □

Proof of Proposition 2.18: We abbreviate

$$L^p := L^p(\mathbb{C}, V \times V), \text{ etc.}$$

Claim 1 *We may assume w.l.o.g. that* $\lambda = 0$.

Proof of Claim 1: Assume that we have already proved Proposition 2.18 in the case $\lambda = 0$. The map

$$L^p_\lambda \ni (u,v) \mapsto \langle\cdot\rangle^\lambda (u,v) \in L^p \tag{2.103}$$

is an isometric isomorphism. Furthermore, by Proposition E.6(iv) the map

$$W^{1,p} \ni (u,v) \mapsto \langle\cdot\rangle^{-\lambda}(u,v) \in W^{1,p}_\lambda \tag{2.104}$$

is well-defined and an isomorphism of Banach spaces. Moreover, we have

$$\langle\cdot\rangle^\lambda T \langle\cdot\rangle^{-\lambda} = T_0 + S : W^{1,p} \to L^p, \tag{2.105}$$

where

$$T_0 := \begin{pmatrix} 2\partial_{\bar z} & A \\ B & 2\partial_z \end{pmatrix}, \qquad S := \begin{pmatrix} 2\langle\cdot\rangle^\lambda(\partial_{\bar z}\langle\cdot\rangle^{-\lambda}) & 0 \\ 0 & 2\langle\cdot\rangle^\lambda(\partial_z\langle\cdot\rangle^{-\lambda}) \end{pmatrix}.$$

By our assumption the operator T_0 is Fredholm of index 0. Furthermore, a calculation using the identity $|z|^2 = z\bar z$ shows that

$$|\partial_{\bar z}\langle\cdot\rangle^{-\lambda}| \le \frac{|\lambda|}{2}\langle\cdot\rangle^{-\lambda-1}, \qquad |\partial_z\langle\cdot\rangle^{-\lambda}| \le \frac{|\lambda|}{2}\langle\cdot\rangle^{-\lambda-1}.$$

Therefore, Proposition E.6(v) implies that the operator S is compact. This together with equality (2.105) and the fact that the maps (2.103) and (2.104) are isomorphisms of Banach spaces, implies that T is a Fredholm map of index 0. This proves Claim 1. □

So we assume from now on that $\lambda = 0$. We show that in this case the operator $T : W^{1,p} \to L^p$ is an isomorphism of Banach spaces. We define $A^{1/2}, B^{1/2} : V \to V$ to be the unique strictly positive linear maps such that

$$(A^{\frac12})^2 = A, \qquad (B^{\frac12})^2 = B.$$

For existence and uniqueness of these maps see Satz VI.3.4 in the book by D. Werner [Wer]. We define the operator

$$S := \begin{pmatrix} 2\partial_{\bar z} & A^{\frac12}B^{\frac12} \\ B^{\frac12}A^{\frac12} & 2\partial_z \end{pmatrix} : W^{1,p} \to L^p.$$

Consider the automorphisms of Banach spaces

$$\begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{pmatrix} \; : \; W^{1,p} \to W^{1,p}, \tag{2.106}$$

$$\begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & B^{-\frac{1}{2}} \end{pmatrix} \; : \; L^p \to L^p. \tag{2.107}$$

A short calculation shows that

$$\begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & B^{-\frac{1}{2}} \end{pmatrix} T \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{pmatrix} = S. \tag{2.108}$$

Claim 2 *The operator $S : W^{1,p} \to L^p$ is an isomorphism of Banach spaces.*

Proof of Claim 2: We define

$$S' := \begin{pmatrix} -2\partial_z & A^{\frac{1}{2}}B^{\frac{1}{2}} \\ B^{\frac{1}{2}}A^{\frac{1}{2}} & -2\partial_{\bar{z}} \end{pmatrix} : W^{2,p} \to W^{1,p}.$$

Note that S' is the formal adjoint of S w.r.t. the L^2 hermitian product on C_0^∞ given by

$$\langle (u,v),(u',v')\rangle_{L^2} := \int_{\mathbb{C}} \big(\langle u,v\rangle + \langle u',v'\rangle\big) d^N x.$$

This means that

$$\langle S(u,v),(u',v')\rangle = \langle (u,v), S'(u',v')\rangle,$$

for every pair $(u,v),(u',v') \in C_0^\infty$. By a short calculation we have

$$SS' = \begin{pmatrix} -\Delta + A^{\frac{1}{2}}BA^{\frac{1}{2}} & 0 \\ 0 & -\Delta + B^{\frac{1}{2}}AB^{\frac{1}{2}} \end{pmatrix} : W^{2,p} \to L^p. \tag{2.109}$$

Since the linear maps

$$A^{\frac{1}{2}}BA^{\frac{1}{2}},\ B^{\frac{1}{2}}AB^{\frac{1}{2}} : V \to V$$

are positive, Theorem 2.19 (Calderón) implies that SS' is an isomorphism of Banach spaces. We denote by $(SS')^{-1} : L^p \to W^{2,p}$ its inverse and define the bounded operator

$$R := S'(SS')^{-1} : L^p \to W^{1,p}.$$

It follows that

$$SR = \mathrm{id}_{L^p}.$$

We show that

$$RS = \mathrm{id}_{W^{1,p}}. \tag{2.110}$$

By a short calculation, we have for every Schwartz function $(u,v)\in\mathcal{S}$

$$SS'(u,v)=S'S(u,v).$$

This implies that $(SS')^{-1}S|_{\mathcal{S}}=S(SS')^{-1}|_{\mathcal{S}}$, and therefore

$$RS|_{\mathcal{S}}=\mathrm{id}_{\mathcal{S}}.$$

Since $RS:W^{1,p}\to W^{1,p}$ is continuous and $\mathcal{S}\subseteq W^{1,p}$ is dense, it follows that $RS=\mathrm{id}_{W^{1,p}}$. This shows (2.110) and proves Claim 2. □

Since the maps (2.106) and (2.107) are automorphisms of Banach spaces and by Claim 2 the map S is an isomorphism of Banach spaces, it follows from (2.108) that T is an isomorphism of Banach spaces, assuming that $\lambda=0$. Therefore, for a general number $\lambda\in\mathbb{R}$ it follows from Claim 1 that T is a Fredholm operator of index 0. This proves Proposition 2.18. □

We are now ready for the proof of the main result of this section.

Proof of Theorem 2.8 (Fredholm): Let $p>2$, $\lambda>-2/p+1$, and let $w:=(u,\Phi+i\Psi)\in\mathcal{B}^p_\lambda$ be a map. That the normed vector spaces $\mathcal{X}^{p,\lambda}_w$ and $\mathcal{Y}^{p,\lambda}_w$ are complete is the statement of Corollary 2.13. Assume now that $-2/+1<\lambda<-2/p+2$. Recall the Definition 2.6 of the Maslov index $d:=m(w)$. We prove that the operator $\mathcal{D}^{p,\lambda}_w:\mathcal{X}^{p,\lambda}_w\to\mathcal{Y}^{p,\lambda}_w$ is Fredholm of real index

$$\mathrm{ind}\mathcal{D}_w=\dim M-2\dim\mathrm{G}+2d.$$

By Proposition 2.11 there exists a good complex trivialization

$$F:\mathbb{C}\times\left(\mathbb{C}^{\bar{n}}\times\mathfrak{g}^{\mathbb{C}}\times\mathfrak{g}^{\mathbb{C}}\right)\to u^*TM\oplus\mathfrak{g}^{\mathbb{C}},$$

as defined in Definition 2.10. We recall the definitions (2.46) and (2.48)

$$\begin{aligned}\mathcal{X}_d &:= \left(\mathbb{C}\rho_0p_d+L^{1,p}_{\lambda-1-d}(\mathbb{C},\mathbb{C})\right)\times\left(\mathbb{C}^{\bar{n}-1}+L^{1,p}_{\lambda-1}(\mathbb{C},\mathbb{C}^{\bar{n}-1})\right)\times W^{1,p}_\lambda(\mathbb{C},\mathfrak{g}^{\mathbb{C}}\times\mathfrak{g}^{\mathbb{C}}),\\ \mathcal{Y}_d &:= L^p_{\lambda-d}(\mathbb{C},\mathbb{C})\times L^p_\lambda(\mathbb{C},\mathbb{C}^{\bar{n}-1}\times\mathfrak{g}^{\mathbb{C}}\times\mathfrak{g}^{\mathbb{C}}),\end{aligned}$$

where $\rho_0:\mathbb{C}\to[0,1]$ is a smooth function such that $\rho_0(z)=0$ for $z\in B_{1/2}$ and $\rho_0(z)=1$ for $z\in\mathbb{C}\setminus B_1$, and $p_d(z):=z^d$. We recall also the definition (2.47) of the norm on $\mathcal{X}_d$. Namely, for every map

$$\zeta:=\begin{pmatrix}v^1_\infty\rho_0p_d+v^1\\ v^2_\infty+v^2\\ \vdots\\ v^{\bar{n}}_\infty+v^{\bar{n}}\\ \alpha\\ \beta\end{pmatrix}\in\mathcal{X}_d,$$

with $(v^1_\infty, \dots, v^{\bar{n}}_\infty) \in \mathbb{C}^{\bar{n}}$, we defined

$$||\zeta||_{\mathcal{X}_d} := |(v^1_\infty, \dots, v^{\bar{n}}_\infty)| + ||v^1||_{L^{1,p}_{\lambda-1-d}(\mathbb{C})} + ||(v^2, \dots, v^{\bar{n}})||_{L^{1,p}_{\lambda-1}(\mathbb{C})} + ||(\alpha,\beta)||_{W^{1,p}_\lambda(\mathbb{C})}.$$

By Proposition 2.12 the linear operator

$$\mathcal{D}^{\mathrm{triv}}_w := F^{-1}\mathcal{D}_w F : \mathcal{X}_d \to \mathcal{Y}_d$$

is a compact perturbation of the operator

$$2\partial^{\mathbb{C}^{\bar{n}}}_{\bar{z}} \oplus \begin{pmatrix} 2\partial^{\mathfrak{g}^{\mathbb{C}}}_{\bar{z}} & \mathrm{id} \\ A & 2\partial^{\mathfrak{g}^{\mathbb{C}}}_{z} \end{pmatrix} : \mathcal{X}_d \to \mathcal{Y}_d, \tag{2.111}$$

where $A : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is a strictly positive complex linear map. By Corollary 2.17 and a direct sum argument the operator

$$\begin{aligned} 2\partial^{\mathbb{C}^{\bar{n}}}_{\bar{z}} &: \big(\mathbb{C}\rho_0 p_d + L^{1,p}_{\lambda-1-d}(\mathbb{C},\mathbb{C})\big) \times \big(\mathbb{C}^{\bar{n}-1} + L^{1,p}_{\lambda-1}(\mathbb{C},\mathbb{C}^{\bar{n}-1})\big) \\ &\to L^p_{\lambda-d}(\mathbb{C},\mathbb{C}) \times L^p_\lambda(\mathbb{C},\mathbb{C}^{\bar{n}-1}) \end{aligned}$$

is Fredholm of real index $2d + 2 + 2\bar{n} - 2 = 2\bar{n} + 2d$. Moreover, since $A, \mathrm{id} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ are strictly positive linear maps, by Proposition 2.18 the operator

$$\begin{pmatrix} 2\partial^{\mathfrak{g}^{\mathbb{C}}}_{\bar{z}} & \mathrm{id} \\ A & 2\partial^{\mathfrak{g}^{\mathbb{C}}}_{z} \end{pmatrix} : W^{1,p}_\lambda(\mathbb{C}, \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}) \to L^p_\lambda(\mathbb{C}, \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}})$$

is Fredholm of index 0. It follows that the operator (2.111) is Fredholm of real index $2\bar{n} + 2d$. This proves Theorem 2.8. □

3 Stable maps of vortices on $\mathbb{C}$ and bubbles in $\bar{M}$

3.1 Stable maps and convergence

Let (M, ω) be a symplectic manifold, let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$, and let $\langle\cdot,\cdot\rangle$ be a G-invariant inner product on $\mathfrak{g}$. Assume that G acts on M in a Hamiltonian way, with moment map $\mu : M \to \mathfrak{g}$. Let J be an ω-compatible G-invariant almost complex structure. Recall from (1.9) that J induces an almost complex structure $\bar{J}$ on the symplectic quotient $\bar{M} = \mu^{-1}(0)/\mathrm{G}$, defined by

$$\bar{J}\bar{v} := d\pi(x)Jv,$$

for $\bar{v} \in T_{\bar{x}}\bar{M}$, $\bar{x} \in \bar{M}$, where $v \in H_x = \ker d\mu(x) \cap (\mathrm{im}L_x)^\perp$ and $x \in M$ are chosen such that $\pi(x) := \mathrm{G}x = \bar{x}$, $d\pi(x)v = \bar{v}$. The structure $\bar{J}$ is $\bar{\omega}$-compatible, i.e. $\bar{\omega}(\cdot, \bar{J}\cdot)$ is a Riemannian metric on $\bar{M}$. In this section I introduce stable maps of vortices on $\mathbb{C}$ and $\bar{J}$-holomorphic spheres in $\bar{M}$. The idea is that they compactify the space of finite energy vortices (u, Φ, Ψ) on $\mathbb{C}$ for which $\overline{u(\mathbb{C})}$ is compact, see Theorem 4.1 in section 4. The definition of a stable map of vortices on $\mathbb{C}$ and $\bar{J}$-holomorphic spheres is modelled on the notion of a genus 0 stable map of pseudo-holomorphic spheres, as introduced by Kontsevich in [Ko]. For an exhaustive exposition of those stable maps see the book by D. McDuff and D. A. Salamon [MS3].

As always, we assume that the hypothesis (H1) is satisfied, i.e. that G acts freely on $\mu^{-1}(0)$ and that μ is proper. We fix a real number $p > 2$ and define the energy functional $E : W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g}) \to [0, \infty]$ by

$$\begin{aligned} E(u, \Phi, \Psi) \;:=\; & \frac{1}{2}\int_{\mathbb{C}} \big(|\partial_s u + X_\Phi(u)|^2 + |\partial_t u + X_\Psi(u)|^2 + \\ & |\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi]|^2 + |\mu(u)|^2\big)\, ds\, dt. \end{aligned} \tag{3.1}$$

Furthermore, we define $\widetilde{\mathcal{M}}^{1,p}$ to be the set of finite energy vortices (u, Φ, Ψ) on $\mathbb{C}$ for which the closure of the image of u in M is compact,

$$\begin{aligned} \widetilde{\mathcal{M}}^{1,p} := \big\{ & (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g}) \,\big| \\ & (0.1),\ (0.2),\ E(u, \Phi, \Psi) < \infty\, \overline{u(\mathbb{C})} \text{ compact}\big\}. \end{aligned} \tag{3.2}$$

The gauge group $\mathcal{G}^{2,p} := W^{2,p}_{\mathrm{loc}}(\mathbb{C}, \mathrm{G})$ acts on $\widetilde{\mathcal{M}}^{1,p}$ by the formula

$$g^*(u, \Phi, \Psi) := \big(g^{-1}, g^{-1}(\partial_s + \Phi)g, g^{-1}(\partial_t + \Psi)g\big). \tag{3.3}$$

We denote the quotient by

$$\mathcal{M} := \widetilde{\mathcal{M}}^{1,p}/\mathcal{G}^{2,p}. \tag{3.4}$$

Definition 3.1 *We define the* evaluation map

$$\overline{\mathrm{ev}} : (C^0(S^2, M/\mathrm{G}) \sqcup \widetilde{\mathcal{M}}^{1,p}) \times S^2 \to M/\mathrm{G}$$

as follows. For $(\bar{u}, z) \in C^0(S^2, M/\mathrm{G}) \times S^2$ *we define*

$$\overline{\mathrm{ev}}_z(\bar{u}) := \overline{\mathrm{ev}}(\bar{u}, z) := \bar{u}(z). \tag{3.5}$$

Furthermore, for $(w, z) := (u, \Phi, \Psi; z) \in \widetilde{\mathcal{M}}^{1,p} \times S^2$ *we define*

$$\overline{\mathrm{ev}}_z(w) := \overline{\mathrm{ev}}(w, z) := \begin{cases} \mathrm{G}u(z), & \textit{if } z \neq \infty, \\ \lim_{r\to\infty} \mathrm{G}u(r), & \textit{if } z = \infty. \end{cases} \tag{3.6}$$

That the limit in (3.6) exists, follows from Proposition D.6. Furthermore, for every $w \in \widetilde{\mathcal{M}}^{1,p}$ and $g \in \mathcal{G}^{2,p}$ we have $\overline{\mathrm{ev}}_\infty(g^*w) = \overline{\mathrm{ev}}_\infty(w)$. Therefore, we can define by abuse of notation $\overline{\mathrm{ev}}_\infty : \mathcal{M} := \widetilde{\mathcal{M}}^{1,p}/\mathcal{G}^{2,p} \to M/\mathrm{G}$ by $\overline{\mathrm{ev}}_\infty([w]) := \overline{\mathrm{ev}}_\infty(w)$.

Definition 3.2 *For every nonnegative integer k a* stable map of vortices on $\mathbb{C}$ and pseudo-holomorphic spheres in $\bar{M}$ with $k+1$ marked points *is a tuple*

$$(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}) := \big(V, \bar{T}, E, (w_\alpha)_{\alpha\in V}, (\bar{u}_\alpha)_{\alpha\in\bar{T}}, (z_{\alpha\beta})_{\alpha E\beta}, (\alpha_i, z_i)_{i=0,\ldots,k}\big),$$

where V and $\bar{T}$ are finite sets, E is a tree relation on $T := V \sqcup \bar{T}$, $w_\alpha := (u_\alpha, \Phi_\alpha, \Psi_\alpha) \in \widetilde{\mathcal{M}}^{1,p}$ is a vortex for $\alpha \in V$, $\bar{u}_\alpha : S^2 \to \bar{M} = \mu^{-1}(0)/\mathrm{G}$ is a $\bar{J}$-holomorphic map for $\alpha \in \bar{T}$, $z_{\alpha\beta} \in S^2 \cong \mathbb{C} \cup \{\infty\}$ is a point for each adjacent pair $\alpha E\beta$, $\alpha_i \in T$ is a vertex and $z_i \in S^2$ is a point, for $i = 0, \ldots, k$, such that the following conditions hold.

(i) **(Special points)**

- *If $\alpha_0 \in V$ then $z_0 = \infty$.*
- *Fix $\alpha \in T$. Then the points $z_{\alpha\beta}$ with $\beta \in T$ such that $\alpha E\beta$ and the points z_i with $i = 0, \ldots, k$ such that $\alpha_i = \alpha$ are all distinct.*
- *If $\alpha \in V$, $\beta \in T$ are such that $\alpha E\beta$ then $z_{\alpha\beta} = \infty$.*

(ii) **(Connectedness)** *Let $\alpha, \beta \in T$ be such that $\alpha E\beta$. Then*

$$\overline{\mathrm{ev}}_{z_{\alpha\beta}}(w_\alpha) = \overline{\mathrm{ev}}_{z_{\beta\alpha}}(w_\beta).$$

Here $\overline{\mathrm{ev}}$ is defined as in (3.5) and (3.6) and by abuse of notation we set $w_\alpha := \bar{u}_\alpha$ if $\alpha \in \bar{T}$.

(iii) **(Stability)** *If $\alpha \in T$ is such that $E(w_\alpha) = 0$ then*

$$\#\{\beta \in T \,|\, \alpha E\beta\} + \#\{i \in \{0, \ldots, k\} \,|\, \alpha_i = \alpha\} \geq 3.$$

Here we set $w_\alpha := \bar{u}_\alpha$ if $\alpha \in \bar{T}$.

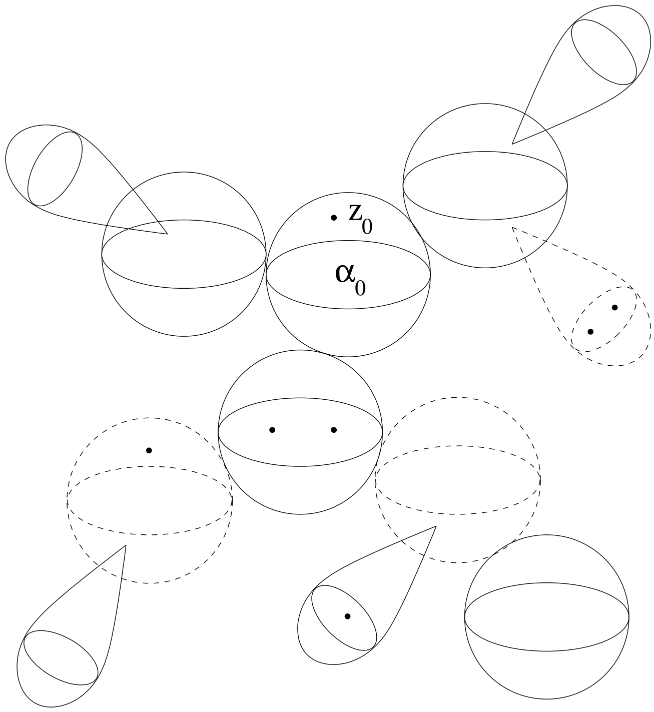

Figure 1: Stable map. The "raindrops" correspond to vortices on $\mathbb{C}$ and the spheres to pseudo-holomorphic spheres in $\bar{M}$. The seven dots are marked points. The dashed objects are "ghosts", i.e. they carry no energy.

Examples 3.3 The easiest example of a stable map consists of the tree with one vertex $T = V = \{\alpha_0\}$, a vortex $w \in \widetilde{\mathcal{M}}^{1,p}$, the point $z_0 := \infty$ and a finite number of distinct points $z_i \in \mathbb{C}$, $i = 1, \ldots, k$, where $k \geq 2$ if $E(w) = 0$.

As a second example we set $V := \emptyset$. Then a stable map of vortices on $\mathbb{C}$ and pseudo-holomorphic spheres in $\bar{M}$ is the same as a genus 0 stable map of $\bar{J}$-holomorphic spheres in $\bar{M}$, modelled over the tree $\bar{T}$.

As another example we set $k := 0$, choose an integer $\ell \geq 0$, and define $V := \{1, \ldots, \ell\}$, $\bar{T} := \{0\}$, $\alpha_0 := 0$, with the tree relation such that j is adjacent to 0 for $j = 1, \ldots, \ell$ and to no other vertex of T. We define $z_{i0} := \infty$ for $i = 1, \ldots, \ell$. Furthermore, let z_0 and $z_{0i} \in S^2$, $i = 1, \ldots, \ell$ be distinct points, let $w_i \in \widetilde{\mathcal{M}}^{1,p}$ be such that $E(w_i) > 0$, for $i = 1, \ldots, \ell$, and let $\bar{u}_0$ be a $\bar{J}$-holomorphic sphere. If $\ell \leq 1$ then assume that $\bar{u}_0$ is nonconstant. Then the tuple

$$\big(\{1, \ldots, \ell\}, \{0\}, (w_i)_{i \in \{1,\ldots,\ell\}}, \bar{u}_0, (z_{ij})_{iEj}, (0, z_0)\big)$$

is a stable map.

Remark 3.4 Note that it follows from condition (i) that if $\alpha \in V$ then there is at most one $\beta \in T$ such that $\alpha E \beta$. This means that every vortex is a *leaf* of the tree T. Furthermore, if $\alpha_0 \in V$ then it follows that $T = V$ consists only of α_0. It follows that if T has at least two elements, then $\alpha_0 \in \bar{T}$, and hence $\bar{T} \neq \emptyset$. Furthermore, if $\alpha \in V$ and $\beta \in T$ are such that $\alpha E \beta$ then $\beta \in \bar{T}$.

Remark 3.5 If $i \geq 1$ and $\alpha_i \in V$ then $z_i \neq \infty$. To see this assume first that $\alpha_0 = \alpha_i$. Then by condition (i) we have $z_i \neq z_0 = \infty$. Assume now that $\alpha_0 \neq \alpha_i$. Let $\beta \in \bar{T}$ be the unique vertex such that $\alpha_i E \beta$. Then by condition (i) we have $z_i \neq z_{\alpha_i \beta} = \infty$.

We fix a stable map $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ as in Definition 3.2. For every $\alpha \in T$ we define the *set of nodal points* to be

$$Z_\alpha := \{z_{\alpha\beta} \,|\, \beta \in T, \alpha E \beta\} \subseteq S^2.$$

For $\alpha \in T$ the set of *marked points on* α is defined to be the set $\{z_i \,|\, \alpha_i = \alpha, i = 1, \ldots, k\}$. We define the *set* Y_α *of special points* to be the union of Z_α and the set of marked points at α. The stability condition (iii) says that if $\alpha \in T$ is such that $E(w_\alpha) = 0$ (or $E(\bar{u}_\alpha) = 0$) then there are at least three special points on α. The point of this condition is the following. We fix two finite sets $\bar{T}, V$ and a tree relation on the disjoint union $T := \bar{T} \sqcup V$ such that every element of V is a leaf. There is an associated group $\mathrm{G}_{\bar{T},V}$ that acts on the set of stable maps modelled over $(\bar{T}, V, E)$. Condition (iii) implies that every isotropy group of this action is finite. For the next definition we need the following notation. Let $\alpha \in T$ and $i = 0, \ldots, k$. We define $z_{\alpha,i} \in S^2$ as follows. If $\alpha = \alpha_i$ then we set

$$z_{\alpha,i} := z_i. \tag{3.7}$$

Otherwise let $\beta \in \bar{T}$ be the unique vertex such that $[\alpha, \alpha_i] = (\alpha, \beta, \ldots, \alpha_i)$, where $[\alpha, \alpha_i]$ is the chain of vertices of T running from α to α_i. ($\beta = \alpha_i$ is also allowed.) We define

$$z_{\alpha,i} := z_{\alpha\beta}. \tag{3.8}$$

Given integers $k \geq 0$, $\ell \geq 1$, a compact subset $Q \subseteq \mathbb{R}^\ell$, a manifold X and maps $u_\nu, u_0 \in C^k(Q, X)$, we say that the sequence u_ν converges to u_0 in $C^k(Q, X)$ iff $\iota \circ u_\nu$ converges in $C^k(Q, \mathbb{R}^N)$, where $\iota : X \to \mathbb{R}^N$ is an embedding. That such a ι exists is guaranteed by the Whitney embedding theorem, see [Wh] Theorem II.1 or [Hi], Theorem 3.2.14 p. 55.

Definition 3.6 (Convergence) *Let* $w^\nu = (u^\nu, \Phi^\nu, \Psi^\nu) \in \widetilde{\mathcal{M}}^{1,p}$ *be a sequence of finite energy vortices,* $k \geq 0$ *be an integer,* $z_1^\nu, \ldots, z_k^\nu \in \mathbb{C}$ *be sequences of points, and let*

$$(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}) := \big(V, \bar{T}, E, (z_{\alpha\beta})_{\alpha E \beta}, (\alpha_i, z_i)_{i=0,\ldots,k}, (w_\alpha)_{\alpha \in V}, (\bar{u}_\alpha)_{\alpha \in \bar{T}}\big)$$

be a stable map. The sequence $(w^\nu, z_0^\nu := \infty, z_1^\nu, \dots, z_k^\nu)$ *is said to converge to* $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ *as* $\nu \to \infty$ *iff the limit* $E := \lim_{\nu\to\infty} E(w^\nu)$ *exists and*

$$E = \sum_{\alpha\in V} E(w_\alpha) + \sum_{\alpha\in\bar{T}} E(\bar{u}_\alpha), \tag{3.9}$$

and there exist Möbius transformations $\varphi_\alpha^\nu : S^2 \to S^2$*, for* $\alpha \in T := V \sqcup \bar{T}$*,* $\nu \in \mathbb{N}$*, such that the following conditions hold.*

(i)
- *If* $\alpha \in V$ *then* φ_α^ν *is a translation on the plane* $\mathbb{C}$*, for every* $\nu \in \mathbb{N}$*.*
- *For every* $\alpha \in \bar{T}$ *we have* $\varphi_\alpha^\nu(z_{\alpha,0}) = \infty$*, where* $z_{\alpha,0}$ *is defined as in (3.7), (3.8).*
- *We fix a vertex* $\alpha \in \bar{T}$ *and choose a Möbius transformation* ψ_α *such that* $\psi_\alpha(\infty) = z_{\alpha,0}$*. Then the derivatives* $(\varphi_\alpha^\nu \circ \psi_\alpha)'(z)$ *converge to* ∞*, for every* $z \in \mathbb{C}$*.*

(ii) If $\alpha, \beta \in T$ *are such that* $\alpha E \beta$ *then* $(\varphi_\alpha^\nu)^{-1} \circ \varphi_\beta^\nu \to z_{\alpha\beta}$*, uniformly on compact subsets of* $S^2 \setminus \{z_{\beta\alpha}\}$*.*

(iii)
- *For every* $\alpha \in V$ *there exist gauge transformations* $g_\alpha^\nu \in \mathcal{G}^{2,p}$*, such that for every compact subset* $Q \subseteq \mathbb{C}$ *the sequence* $(g_\alpha^\nu)^{-1}(u^\nu \circ \varphi_\alpha^\nu)$ *converges to* u_α *in* $C^1(Q)$ *and the sequence* $(g_\alpha^\nu)^*\big((\Phi_\nu, \Psi_\nu) \circ \varphi_\alpha^\nu\big)$ *converges to* $(\Phi_\alpha, \Psi_\alpha)$ *in* $C^0(Q)$*.*
- *Fix* $\alpha \in \bar{T}$*. If* $\alpha \neq \alpha_0$ *let* Q *be a compact subset of* $S^2 \setminus Z_\alpha$ *and if* $\alpha = \alpha_0$ *let* Q *be a compact subset of* $S^2 \setminus (Z_{\alpha_0} \cup \{z_0\})$*. Then for* ν *large enough*

$$u^\nu \circ \varphi_\alpha^\nu(Q) \subseteq M^* := \{x \in M \,|\, \textit{if } gx = x \Rightarrow g = \mathbf{1}\}.$$

Furthermore, $\mathrm{G}u^\nu \circ \varphi_\alpha^\nu$ *converges to* $\bar{u}_\alpha$ *in* $C^1(Q, M^*/\mathrm{G})$*. Here we denote the orbit of a point* $x \in M$ *by* $\mathrm{G}x \in M/\mathrm{G}$*.*

(iv) $(\varphi_{\alpha_i}^\nu)^{-1}(z_i^\nu) \to z_i$ *for every* $i = 1, \dots, k$*.*

Remark 3.7 This definition is based on the notion of convergence of a sequence of pseudo-holomorphic spheres to a genus 0 stable map of pseudo-holomorphic spheres. (For that notion see for example the book by D. McDuff and D. A. Salamon [MS3]). The meaning of Definition 3.6 is the following. Assume that a sequence $w^\nu \in \widetilde{\mathcal{M}}^{1,p}$ of vortices converges to some stable map $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$. Fix a vertex $\alpha \in V$ and an index $\nu \in \mathbb{N}$. We define by $w_\alpha^\nu := (g_\alpha^\nu)^*(w^\nu \circ \varphi_\alpha^\nu)$ the reparametrized and gauge transformed map. Since $\alpha \in V$, the Möbius transformation φ_α^ν is a translation, and therefore the map w_α^ν is again a vortex. The first part of condition (iii) says that w_α^ν converges to the vortex w_α. Fix now a vertex $\alpha \in \bar{T}$, and recall the

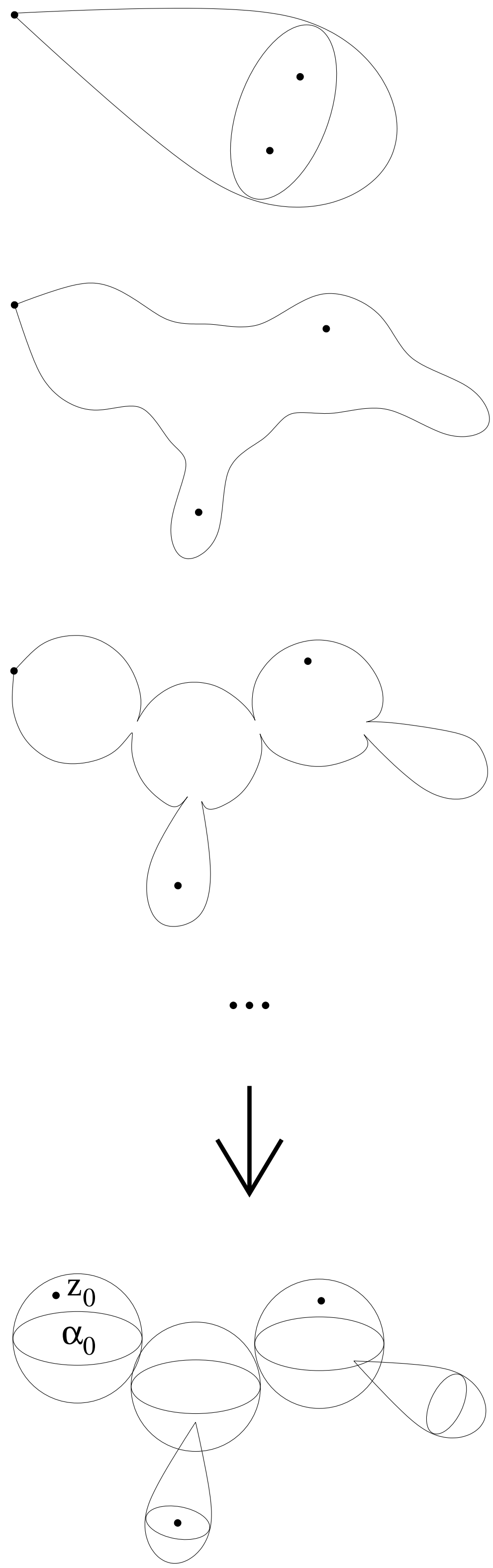

Figure 2: Convergence of a sequence of vortices on $\mathbb{C}$ against a stable map.

definition $Z_\alpha := \{z_{\alpha\beta} \,|\, \beta E \alpha\}$ of the set of nodal points at α, and the definition (3.7) and (3.8) of $z_{\alpha,0}$. Then the second part of condition (iii) says that the composition $\bar{u}^\nu_\alpha : S^2 \setminus \{z_{\alpha,0}\} \to M/\mathrm{G}$ of the rescaled map $u^\nu \circ \varphi^\nu_\alpha$ with the projection to the quotient $M \to M/\mathrm{G}$ converges the $\bar{J}$-holomorphic map $\bar{u}_\alpha : S^2 \to \bar{M} \subseteq M/\mathrm{G}$, away from $Z_\alpha \cup \{z_{\alpha,0}\}$. The motivation for this condition is the following. In the case $\alpha \in \bar{T}$, the last part of condition (i) implies that we "zoom out". This means that the image of a fixed compact ball $B \subseteq S^2 \setminus \{z_{\alpha,0}\}$ under φ^ν_α becomes a larger and larger subset of $\mathbb{C}$. If $z_{\alpha,0} = \infty$, then there exist numbers $\lambda^\nu_\alpha \in \mathbb{C} \setminus \{0\}$ and $z^\nu_\alpha \in \mathbb{C}$ such that $\varphi^\nu_\alpha(z) = \lambda^\nu_\alpha z + z^\nu_\alpha$. In this case the condition (i) says that λ^ν_α converges to ∞, for $\nu \to \infty$. Moreover, we identify $\mathfrak{g} \times \mathfrak{g}$ with the complexification $\mathfrak{g}^\mathbb{C}$. Then the rescaled maps

$$w^\nu_\alpha := (u^\nu_\alpha, \Phi^\nu_\alpha + i\Psi^\nu_\alpha) := (u^\nu, \bar{\lambda}^\nu_\alpha \cdot (\Phi^\nu + i\Psi^\nu))(\lambda^\nu_\alpha \cdot) : \mathbb{C} \to M \times \mathfrak{g}^\mathbb{C}$$

satisfy the $|\lambda^\nu_\alpha|$-vortex equations. These are the equations (0.1) and

$$|\lambda^\nu_\alpha|^{-2}\big(\partial_s \Psi^\nu_\alpha - \partial_t \Phi^\nu_\alpha + [\Phi^\nu_\alpha, \Psi^\nu_\alpha]\big) + \mu \circ u^\nu_\alpha = 0,$$

see section B.2. Taking the limit $\nu \to \infty$, the second equation becomes the equation $\mu \circ u = 0$ for a map $(u, \Phi + i\Psi) : \mathbb{C} \to M \times \mathfrak{g}^\mathbb{C}$. Together with the first vortex equation (0.1), this corresponds to the equation for a $\bar{J}$-holomorphic map from $\mathbb{C}$ to $\bar{M}$, see (E.11). This motivates the second part of condition (iii).

For every vertex $\alpha \in \bar{T}$ we set

$$w^\nu_\alpha := (u^\nu_\alpha, \Phi^\nu_\alpha, \Psi^\nu_\alpha) := w^\nu \circ \varphi^\nu_\alpha.$$

We fix a vertex $\alpha \in T$ and a compact subset $Q \subseteq S^2 \setminus (Z_\alpha \cup \{z_{\alpha,0}\})$. The φ^ν_α-energy of w^ν_α on Q is given by

$$E^{\varphi^\nu_\alpha}(w^\nu_\alpha, Q) = \int_Q \big(|\partial_s u^\nu_\alpha + X_{\Phi^\nu_\alpha}(u^\nu_\alpha)| + |(\varphi^\nu_\alpha)'|^2 |\mu \circ u^\nu_\alpha|^2\big)\, ds\, dt,$$

see section B.2. It follows from (iii) that $E^{\varphi^\nu_\alpha}(w^\nu_\alpha, Q)$ converges to $E(w_\alpha, Q)$ (or $E(\bar{u}_\alpha, Q)$) for $\nu \to \infty$. An important feature of Definition 3.2 is that in the limit $\nu \to \infty$ no energy gets lost. This is the content of equality (3.9) and of the fact that in the limit $\nu \to \infty$ we count the energy only one time. The latter means the following. For each vertex $\alpha \in T$ we fix a large compact subset $Q_\alpha \subseteq S^2 \setminus (Z_\alpha \cup \{z_{\alpha,0}\})$. Then for ν large enough the compact sets $\varphi^\nu_\alpha(Q_\alpha)$ and $\varphi^\nu_\beta(Q_\alpha)$ for different α and β do not overlap. It follows that

$$\sum_{\alpha \in T} E^{\varphi^\nu_\alpha}(w^\nu_\alpha, Q_\alpha) = E\Big(w^\nu, \bigcup_{\alpha \in T} \varphi^\nu_\alpha(Q_\alpha)\Big).$$

Example 3.8 Consider the action of $\mathrm{G} := S^1$ on $M := \mathbb{C}$ by multiplication, and let $J := i$ be the standard complex structure. We define a moment map $\mu : \mathbb{C} \to \mathrm{Lie}(S^1) = i\mathbb{R}$ for this action by $\mu(z) := \frac{i}{2}(1 - |z|^2)$. Then $\bar{M} = \{\mathrm{pt}\}$ and therefore there are only "ghost bubbles" in $\bar{M}$, i.e. constant $\bar{J}$-holomorphic spheres. Fix a vortex $w := (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$. By Proposition 2.2 in the book [JT] by A. Jaffe and C. Taubes, the set $Z_u \subseteq \mathbb{C}$ of zeros of u is finite. We define the *local degree map* $\deg_w : \mathbb{C} \to \mathbb{Z}$ by

$$\deg_w(z) := \deg\left(\frac{u}{|u|} : S_\varepsilon(z) \to S^1\right),$$

where $\varepsilon > 0$ is a number smaller than the distance of each two points in Z_u. Again by Proposition 2.2 in the book [JT] $\deg_w(z) \geq 0$ for every $z \in \mathbb{C}$, and if $u(z) = 0$ then $\deg_w(z) > 0$. Furthermore, the local degree only depends on the gauge equivalence class of w. We define the degree of w to be

$$\deg(w) := \sum_{z \in Z_u} \deg_w(z).$$

Let d be a nonnegative integer. The symmetric group S_d acts on $\mathbb{C}^d$ by

$$\sigma \cdot (z_1, \ldots, z_d) := (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(d)}).$$

The d-fold symmetric product of $\mathbb{C}$ is the quotient $\mathrm{Sym}^d(\mathbb{C}) := \mathbb{C}^d/S_d$, endowed with the quotient topology. It can be identified with the set of all maps $m : \mathbb{C} \to \mathbb{N} \cup \{0\}$ such that $m(z) \neq 0$ only for finitely many $z \in \mathbb{C}$ and $\sum_{z \in \mathbb{C}} m(z) = d$. By Proposition D.22 the map

$$\mathcal{M}_d := \{w \in \widetilde{\mathcal{M}}^{1,p} \mid \deg(w) = d\}/\mathcal{G}^{2,p} \to \mathrm{Sym}^d(\mathbb{C}), \qquad [w] \mapsto \deg_w$$

is a bijection.

Let $k := 0$ and for $\nu \in \mathbb{N}$ let $W^\nu \in \mathcal{M}$ be the preimage under this bijection of the point $m \in \mathrm{Sym}^7(\mathbb{C})$ given by

$$m(-2 - i) = 1, \qquad m(3 + 4i) = 2, \qquad m(\nu e^{i\nu}) = 4,$$

and $m(z) = 0$ for all other $z \in \mathbb{C}$. We choose a representative $w^\nu := (u^\nu, \Phi^\nu, \Psi^\nu) \in \widetilde{\mathcal{M}}^{1,p}$ of W^ν. Then

$$\deg_{w^\nu}(-2 - i) = 1, \qquad \deg_{w^\nu}(3 + 4i) = 2, \qquad \deg_{w^\nu}(\nu e^{i\nu}) = 4,$$

and $\deg_{w^\nu}(z) = 0$ otherwise. We choose a vortex $w_1 \in \widetilde{\mathcal{M}}^{1,p}$ such that

$$\deg_{w_1}(-2 - i) = 1, \qquad \deg_{w_1}(3 + 4i) = 2$$

and $\deg_{w_1}(z) = 0$ otherwise, and we choose a vortex $w_2 \in \widetilde{\mathcal{M}}^{1,p}$ such that $\deg_{w_2}(0) = 4$ and $\deg_{w_2}(z) = 0$ for $z \neq 0$. Then w^ν converges to the stable map consisting of the sets $V := \{1,2\}$, $\bar{T} := \{0\}$, the tree relation $E := \{(1,0),(0,1),(2,0),(0,2)\}$, the vortices w_1 and w_2, the $\bar{J}$-holomorphic sphere $\bar{u}_0 \equiv \mathrm{pt}$, the nodal points $z_{10} := z_{20} := \infty$, $z_{01} := 1$, $z_{02} := 2 \in S^2 \cong \mathbb{C} \cup \{\infty\}$, and the marked point $(\alpha_0, z_0) := (0, \infty) \in T \times S^2$. This follows from Proposition 3.13 below.

Remark 3.9 The purpose of the additional marked point (α_0, z_0) is to be able to formulate the second part of condition (iii). For $\alpha \in \bar{T}$ and $\nu \in \mathbb{N}$ the map $\mathrm{G}u^\nu \circ \varphi_\alpha^\nu$ is only defined on the subsets $(\varphi_\alpha^\nu)^{-1}(\mathbb{C}) \subseteq S^2$. Since by condition (i) we have $\varphi_\alpha^\nu(z_{\alpha,0}) = \infty$, the composition $\mathrm{G}u^\nu \circ \varphi_\alpha^\nu : Q \to M/\mathrm{G}$ is well-defined for each compact subset $Q \subseteq S^2 \setminus (Z_\alpha \cup \{z_{\alpha,0}\})$. Hence the the second part of condition (iii) makes sense.

Remark 3.10 One conceptual difficulty in defining the notion of convergence is the following. The symplectic vortex equations are invariant under rotation. This means the following. Let $(u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ be a map and $\theta \in \mathbb{R}$ be a number (corresponding to the angle of rotation). We define $(\widetilde{u}, \widetilde{\Phi}, \widetilde{\Psi}) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ by

$$\widetilde{w} := (\widetilde{u}, \widetilde{\Phi}, \widetilde{\Psi})(z) := \big(u(e^{i\theta}z), \cos(\theta)\Phi(e^{i\theta}z) + \sin(\theta)\Psi(e^{i\theta}z), -\sin(\theta)\Phi(e^{i\theta}z) + \cos(\theta)\Psi(e^{i\theta}z)\big) \quad (3.10)$$

Then $\widetilde{w}$ also solves the vortex equations. Therefore, the following question arises about the definition of convergence. Namely, denoting by $\mathrm{Isom}^+(\mathbb{C})$ the group of orientation preserving isometries of $\mathbb{C}$, in the case $\alpha \in V$ should we allow φ_α^ν to be any element of $\mathrm{Isom}^+(\mathbb{C})$ rather than just a translation? The answer is no. The reason is the following. There is a notion of equivalence of stable maps, see Definition 3.12. If in the definition of convergence we allowed $\varphi_\alpha^\nu \in \mathrm{Isom}^+(\mathbb{C})$ for $\alpha \in V$, then we would have to adjust the definition of equivalence, allowing $\varphi_\alpha \in \mathrm{Isom}^+(\mathbb{C})$ for $\alpha \in V$. Let EG be contractible topological space on which G acts continuously and freely. Then in general, there is no evaluation map $\mathrm{ev} : \widetilde{\mathcal{M}}^{1,p} \times S^2 \to (M \times \mathrm{EG})/\mathrm{G}$ such that $\mathrm{ev}(\widetilde{w}, e^{-i\theta}z) = \mathrm{ev}(w, z)$ for every $w \in \widetilde{\mathcal{M}}^{1,p}$, $z \in S^2$ and $\theta \in \mathbb{R}$, where $\widetilde{w}$ is given by formula (3.10). This implies that there is no evaluation map on the set of equivalence classes of vortices, if in the definition of equivalence we allow $\varphi_\alpha \in \mathrm{Isom}^+(\mathbb{C})$ for $\alpha \in V$. However, we need such an evaluation map.

Remark 3.11 We could replace the part of condition (iii) concerning the vortices by the following condition. Namely, for every $\alpha \in V$, $(g_\alpha^\nu)^* w^\nu$ converges to w_α for $\nu \to \infty$, uniformly with all derivatives on every compact subset of $\mathbb{C} \setminus Z_\alpha$. Combining the argument of the proof of Theorem 3.2. in the paper by K. Cieliebak et al. [CGMS] and Theorem E.8, it is possible to show compactness w.r.t. this alternative definition. However, we do not need this sharpened compactness theorem.

Note also that the condition $\mathrm{G}u^\nu \circ \varphi_\alpha^\nu \to \bar{u}_\alpha$ in $C^1(K, M^*/\mathrm{G})$ is strong enough to ensure that the evaluation map (see section 6) is continuous and that in the limit the equivariant homology class is preserved, see Proposition 5.4. Alternatively, for $\alpha \in \bar{T}$ one could impose a condition involving a lift u_α of $\bar{u}_\alpha$ and maps Φ_α, Ψ_α, defined on S^2 minus a finite number of points, and with target $\mathfrak{g}$. However, the above definition seems simpler to me.

Let (T, E), (T', E') be trees and $f : T \to T'$ be a map. Recall from subsection 1.5 that f is called a tree isomorphism iff it is bijective and $\alpha E \beta$ implies $f(\alpha)E'f(\beta)$. If f is a tree isomorphism and $\alpha' E' \beta'$ then $f^{-1}(\alpha')Ef^{-1}(\beta')$.

Definition 3.12 *Let*

$$(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}) := \big(V, \bar{T}, E, (w_\alpha)_{\alpha\in V}, (\bar{u}_\alpha)_{\alpha\in\bar{T}}, (z_{\alpha\beta})_{\alpha E\beta}, (\alpha_i, z_i)_{i=0,\dots,k-1}\big)$$

and

$$(\mathbf{w}', \bar{\mathbf{u}}', \mathbf{z}') := \big(V', \bar{T}', E', (w'_{\alpha'})_{\alpha'\in V'}, (\bar{u}'_{\alpha'})_{\alpha'\in\bar{T}'}, (z'_{\alpha'\beta'})_{\alpha' E'\beta'}, (\alpha'_i, z'_i)_{i=0,\dots,k-1}\big)$$

be two stable maps with k marked points. Then $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ and $(\mathbf{w}', \bar{\mathbf{u}}', \mathbf{z}')$ are called equivalent *iff there is a tree isomorphism $f : T \to T'$, there are Möbius transformations φ_α, for $\alpha \in T$ and there are gauge transformations $g_\alpha \in \mathrm{G}^{2,p}$ for $\alpha \in V$ such that the following conditions hold.*

(i)
- *We have $f(V) = V'$.*
- *For every $i = 0, \dots, k-1$ we have $f(\alpha_i) = \alpha'_i$.*
- *For $\alpha \in V$ the map φ_α is a translation.*

In the following, for $\alpha \in T$ we abbreviate $\alpha' := f(\alpha)$.

(ii) If $\alpha E \beta$ then $z'_{\alpha'\beta'} = \varphi_\alpha^{-1}(z_{\alpha\beta})$. Furthermore, $z'_{\alpha'_i} = \varphi_{\alpha_i}^{-1}(z_{\alpha_i})$ for every $i = 0, \dots, k-1$.

(iii) For $\alpha \in V$ we have $g_\alpha^*(w_\alpha \circ \varphi_\alpha) = w'_{\alpha'}$. Furthermore, for $\alpha \in \bar{T}$ we have $\bar{u}'_{\alpha'} = \bar{u}_\alpha \circ \varphi_\alpha$.

We denote by $\overline{\mathcal{M}}_k$ the set of all equivalence classes of stable maps with k marked points.

3.2 An example

Consider the action of $\mathrm{G} := S^1 \subseteq \mathbb{C}$ on $M := \mathbb{C}$ by multiplication. This action is Hamiltonian, a moment map $\mu : \mathbb{C} \to \mathrm{Lie}(S^1) = i\mathbb{R}$ is given by $\mu(z) := i/2(1-|z|^2)$. Let $J := i$ be the standard complex structure on $\mathbb{C}$. The symplectic quotient $\mu^{-1}(0)/S^1 = S^1/S^1$ consists only of the orbit S^1. Hence the only $\bar{J}$-holomorphic map is the constant map $\bar{u} \equiv S^1$. Furthermore, we can explicitly describe what it means for a sequence of vortices on $\mathbb{C}$ to converge against a stable map. Every finite energy vortex $w = (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$ determines a local degree map $\deg_w : \mathbb{C} \to \mathbb{N} \cup \{0\}$ and a degree $\deg(w)$, as in example 3.8. We fix a nonnegative integer d. Then as in that example, the map

$$\mathcal{M}_d := \{w \in \widetilde{\mathcal{M}}^{1,p} \mid \deg(w) = d\}/\mathcal{G}^{2,p} \to \mathrm{Sym}^d(\mathbb{C}) = \mathbb{C}^d/S_d, \qquad w \mapsto \deg_w$$

is a bijection. For $0 \le d' \le d$ there is an inclusion

$$\iota : \mathrm{Sym}^{d'}(\mathbb{C}) \to \mathrm{Sym}^d(S^2), \qquad \iota([z_1, \ldots, z_{d'}]) := [z_1, \ldots, z_{d'}, \infty, \ldots, \infty], \tag{3.11}$$

where we identify $S^2 \cong \mathbb{C} \cup \{\infty\}$. The next proposition gives an equivalent condition for convergence against a stable map. Here we drop the constant maps to the symplectic quotient from the notation for a stable map, since no information gets lost.

Proposition 3.13 *Let* $\mathrm{G} := S^1 \subseteq \mathbb{C}$ *act on* $M := \mathbb{C}$ *by multiplication with moment map* $\mu : \mathbb{C} \to i\mathbb{R}$*, and let* $J := i$*, as above. Let* $w^\nu \in \widetilde{\mathcal{M}}^{1,p}$ *be a sequence of finite energy vortices,* k *be a nonnegative integer, let* $z_1^\nu, \ldots, z_k^\nu \in \mathbb{C}$ *be sequences of points and let*

$$(\mathbf{w}, \mathbf{z}) := \big(\bar{T}, V, E, (w_\alpha)_{\alpha \in V}, (z_{\alpha\beta})_{\alpha E \beta}, (\alpha_i, z_i)_{i=0,\ldots,k}\big)$$

be a stable map. Then the following conditions are equivalent.

(i) The sequence $(w^\nu, z_0^\nu := \infty, z_1^\nu, \ldots, z_k^\nu)$ *converges to* $(\mathbf{w}, \mathbf{z})$*.*

(ii) For large enough ν *we have*

$$\deg(w^\nu) = \sum_{\alpha \in V} \deg(w_\alpha) =: d. \tag{3.12}$$

Furthermore, there are Möbius transformations $\varphi_\alpha^\nu : S^2 \to S^2$ *for* $\alpha \in T$ *and* $\nu \in \mathbb{N}$ *such that conditions (i), (ii) and (iv) of Definition 3.6 are satisfied and for every* $\alpha \in V$ *the point in the symmetric product* $\deg_{w_\nu} \circ \varphi_\alpha^\nu \in \mathrm{Sym}^d(\mathbb{C}) \subseteq \mathrm{Sym}^d(S^2)$ *converges to* $\iota(\deg_{w_\alpha}) \in \mathrm{Sym}^d(S^2)$ *for* $\nu \to \infty$*.*

For the proof of Proposition 3.13 we need the following lemma.

Lemma 3.14 *Let* k *be a positive integer,* $\varphi_0^\nu, \ldots, \varphi_k^\nu$ *be sequences of Möbius transformations, and let* $z_0, \ldots, z_{k-1}$ *and* $w_1, \ldots, w_k \in S^2$ *be points. Assume that* $z_1 \neq w_1, \ldots, z_{k-1} \neq w_{k-1}$*, and that*

$$(\varphi_i^\nu)^{-1} \circ \varphi_{i+1}^\nu \to z_i,$$

uniformly on compact subsets of $S^2 \setminus \{w_{i+1}\}$*, for* $i = 0, \ldots, k-1$*. Let* $Q \subseteq S^2 \setminus \{z_0\}$*,* $Q' \subseteq S^2 \setminus \{w_k\}$ *be compact subsets. Then for* ν *large enough we have*

$$\varphi_0^\nu(Q) \cap \varphi_k^\nu(Q') = \emptyset. \tag{3.13}$$

Proof of Lemma 3.14: For $i = 1, \ldots, k-1$ we choose a compact neighbourhood $Q_i \subseteq S^2$ of z_i that does not contain w_i. Furthermore, we choose a compact neighbourhood $Q_0 \subseteq S^2$ of z_0 such that $Q_0 \cap Q = \emptyset$, and we set $Q_k := Q'$. We denote $\varphi^\nu_{i,j} := (\varphi^\nu_i)^{-1} \circ \varphi^\nu_j$ for $i, j = 0, \ldots, k$. By assumption, there exists an integer ν_0 such that for $i = 0, \ldots, k-1$ and $\nu \geq \nu_0$ we have

$$\varphi^\nu_{i,i+1}(Q_{i+1}) \subseteq Q_i.$$

It follows that

$$\begin{aligned}
\varphi^\nu_{0,k}(Q') &= \varphi^\nu_{0,1} \circ \cdots \circ \varphi^\nu_{k-1,k}(Q') \\
&\subseteq \varphi^\nu_{0,1} \circ \cdots \circ \varphi^\nu_{k-2,k-1}(Q_{k-1}) \\
&\subseteq \ldots \\
&\subseteq \varphi^\nu_{0,1}(Q_1) \\
&\subseteq Q_0 \subseteq S^2 \setminus Q.
\end{aligned}$$

Hence

$$\varphi^\nu_k(Q') \subseteq \varphi^\nu_0(S^2 \setminus Q) = S^2 \setminus \varphi^\nu_0(Q),$$

and therefore condition (3.13) holds. This proves Lemma 3.14. □

Proof of Proposition 3.13: Assume that (i) holds. By Proposition D.20 we have $E(w_\nu) = \deg(w_\nu)\pi$ and $E(w_\alpha) = \deg(w_\alpha)\pi$ for $\alpha \in V$, and therefore by (3.9)

$$\begin{aligned}
\lim_{\nu\to\infty} \deg(w_\nu)\pi &= \lim_{\nu\to\infty} E(w_\nu) \\
&= \sum_{\alpha \in V} \deg(w_\alpha)\pi + 0.
\end{aligned}$$

It follows that $\deg(w_\nu) = d := \sum_{\alpha\in V} \deg(w_\alpha)$ for ν large enough, as claimed. Furthermore, by assumption, there exist Möbius transformations φ^ν_α for $\alpha \in T$ and $\nu \in \mathbb{N}$ such that conditions 3.6(i)-(iv) are satisfied. Let $\alpha \in V$. By condition 3.6(iii) the condition (i) of Proposition D.23 with w_ν, w replaced by $w^\nu_\alpha := w_\nu \circ \varphi^\nu_\alpha, w_\alpha$ is satisfied. Thus Proposition D.23 implies that

$$\deg_{w_\nu} \circ \varphi^\nu_\alpha = \deg_{w_\nu \circ \varphi^\nu_\alpha} \to \iota(\deg_{w_\alpha}) \in \mathrm{Sym}^d(S^2).$$

Therefore, condition (ii) holds.

Suppose now on the contrary, that condition (ii) holds. Similarly as above, equation (3.12) implies (3.9). Let φ^ν_α for $\alpha \in T$ be as in condition (ii). We have to show condition 3.6(iii). By Proposition D.23 with w_ν, w replaced by w^ν_α, w_α, for $\alpha \in V$ there exist gauge transformations $g^\nu_\alpha \in \mathcal{G}^{2,p}$ such that for every compact subset $Q \subseteq \mathbb{C}$ the maps $(g^\nu_\alpha)^{-1}(u_\nu \circ \varphi^\nu_\alpha)$ converge to u_α in $C^1(Q)$ and the maps $(g^\nu_\alpha)^*\big((\Phi_\nu, \Psi_\nu) \circ \varphi^\nu_\alpha\big)$ converge to $(\Phi_\alpha, \Psi_\alpha)$ in $C^0(Q)$.

Let now $\alpha \in \bar{T}$, and let $Q \subseteq S^2 \setminus (Z_\alpha \cup \{z_{\alpha,0}\})$ be a compact subset. We have to show that for ν large enough we have $u^\nu \circ \varphi_\alpha^\nu(Q) \subseteq M^*$ and that $\mathrm{G}u^\nu \circ \varphi_\alpha^\nu$ converges to the constant $\mathrm{pt} \in S^1/S^1$ in $C^1(Q)$. It suffices to prove that for every subsequence $(\nu_i)_{i\in\mathbb{N}}$ there exists a further subsequence $(i_j)_{j\in\mathbb{N}}$ such that $u^{\nu_{i_j}} \circ \varphi_\alpha^{\nu_{i_j}}(Q) \subseteq M^*$ for every $j \in \mathbb{N}$ and that $\mathrm{G}u^{\nu_{i_j}} \circ \varphi_\alpha^{\nu_{i_j}} \to \mathrm{pt}$, for $j \to \infty$. We fix a subsequence $(\nu_i)_{i\in\mathbb{N}}$. We choose Möbius transformation ψ such that $\psi(\infty) = z_{\alpha,0}$. Then by condition 3.6(i) we have

$$\varphi_\alpha^\nu \circ \psi(\infty) = \varphi_\alpha^\nu(z_{\alpha,0}) = \infty,$$

and hence there are numbers $\lambda_\alpha^\nu \in \mathbb{C} \setminus \{0\}$ and $z_\alpha^\nu \in \mathbb{C}$ such that $\varphi_\alpha^\nu \circ \psi(z) = \lambda_\alpha^\nu z + z_\alpha^\nu$. By the same condition, λ_α^ν converges to ∞. We identify $\mathfrak{g} \times \mathfrak{g}$ with the complexified Lie algebra $\mathfrak{g}^\mathbb{C}$. Then the hypotheses of Proposition 4.3 with $R_i := |\lambda_\alpha^{\nu_i}|, R_0 := \infty, r_i := i$ and w_i replaced by

$$\big(u_{\nu_i}, \bar{\lambda}_\alpha^{\nu_i} \cdot (\Phi_\alpha^{\nu_i} + i\Psi_\alpha^{\nu_i})\big)\varphi_\alpha^{\nu_i} \circ \psi$$

are satisfied. Therefore there exist a finite subset $Z \subseteq \mathbb{C}$, an ∞-vortex $w_0 := (u_0, \Phi_0, \Psi_0) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, M \times \mathfrak{g} \times \mathfrak{g})$ and a subsequence (i_j) such that conditions 4.3(i)-(iii) hold. By condition 4.3(ii) there exist gauge transformations $g_j \in W^{2,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, \mathrm{G})$ such that $g_j^{-1}(u_{\nu_{i_j}} \circ \varphi_\alpha^{\nu_{i_j}} \circ \psi)$ converges to u_0 in C^1 on every compact subset of $\mathbb{C} \setminus Z$. Let $Q' \subseteq S^2 \setminus \psi(Z \cup \{\infty\})$ be a compact subset. Then $u_\alpha^j := g_j^{-1} \circ \psi^{-1}(u_{\nu_{i_j}} \circ \varphi_\alpha^{\nu_{i_j}})$ converges to $u_0 \circ \psi^{-1}$ in $C^1(Q')$. Let $\delta > 0$ be so small that $M_\delta := \{x \in M \,|\, |\mu(x)| \le \delta\} \subseteq M^*$. For j large enough we have $u_\alpha^j(Q') \subseteq M_\delta \subseteq M^*$. Furthermore, $\mathrm{G}u_\alpha^j$ converges to $\mathrm{G}u_0 \circ \psi^{-1} \equiv \mathrm{pt} \equiv \bar{u}_\alpha$ in C^1 on every compact subset of $S^2 \setminus \psi(Z \cup \{\infty\})$, where $\mathrm{pt} \in \bar{M}$ equals the orbit $S^1 \subseteq \mathbb{C}$. So it suffices to prove the following claim.

Claim 1 *The set $\psi(Z \cup \{\infty\})$ is contained in $Z_\alpha \cup \{z_{\alpha,0}\}$.*

Proof of Claim 1: We choose a number $R > 0$ so large that

$$\sum_{\beta \in V} E(w_\beta, \mathbb{C} \setminus B_R) < \frac{\pi}{2}. \tag{3.14}$$

For every vertex $\beta \in V$ Lemma 4.9 implies that

$$E(w^\nu, \varphi_\beta^\nu(\bar{B}_R)) = E((g_\beta^\nu)^*(w^\nu \circ \varphi_\beta^\nu), \bar{B}_R) \to E(w_\beta, \bar{B}_R),$$

for $\nu \to \infty$. We choose an index j_0 so large that for $j \ge j_0$ and every $\beta \in V$ we have

$$E(w_{\nu_{i_j}}, \varphi_\beta^{\nu_{i_j}}(B_R)) > E(w_\beta, B_R) - \frac{\pi}{2|V|}. \tag{3.15}$$

Let $\beta \neq \gamma \in T$. Let $(\beta, \beta_1, \dots, \beta_{k-1}, \gamma)$ be the chain of edges connecting β with γ. We write $\beta_0 := \beta$, $\beta_k := \gamma$. Because of conditions 3.2(i) and 3.6(ii) the hypothesis of Lemma 3.14 with

$$\varphi_i^\nu := \varphi_{\beta_i}^\nu, \quad z_i := z_{\beta_i \beta_{i+1}}, \quad w_i := z_{\beta_i \beta_{i-1}}$$

are satisfied. It follows that for every compact subset $Q \subseteq S^2 \setminus Z_\beta$ and $Q' \subseteq S^2 \setminus Z_\gamma$ we have for ν large enough

$$\varphi_\beta^\nu(Q) \cap \varphi_\gamma^\nu(Q') = \emptyset. \tag{3.16}$$

Applying this several times with $\beta, \gamma \in V$ and $Q := Q' := \bar{B}_R$, we have that for ν large enough the sets $\varphi_\beta^\nu(\bar{B}_R)$, $\beta \in V$, are disjoint. Increasing j_0 we may assume w.l.o.g. that this holds for $\nu \geq \nu_{i_{j_0}}$. Therefore, for $j \geq j_0$

$$\begin{aligned} E\Big(w_{\nu_{i_j}}, \bigcup_{\beta \in V} \varphi_\beta^{\nu_{i_j}}(B_R)\Big) &= \sum_{\beta \in V} E(w^{\nu_{i_j}}, \varphi_\beta^{\nu_{i_j}}(B_R)) \\ &> \left(\sum_{\beta \in V} E(w_\beta, B_R)\right) - \frac{|V|\pi}{2|V|} \\ &= \sum_{\beta \in V} E(w_\beta, \mathbb{C}) - \sum_{\beta \in V} E(w_\beta, \mathbb{C} \setminus B_R) - \frac{\pi}{2} \\ &> \pi d - \pi. \end{aligned} \tag{3.17}$$

Here in the second line we have used (3.15) and in the last line we have used Proposition D.20 and (3.14).

Let now $z \in S^2 \setminus (Z_\alpha \cup \{z_{\alpha,0}\})$. We show that z does not belong to $\psi(Z \cup \{\infty\})$. Since $\psi(\infty) = z_{\alpha,0}$, we have $\psi^{-1}(z) \neq \infty$. We choose a number $\varepsilon > 0$ so small that $\bar{B}_\varepsilon(\psi^{-1}(z)) \subseteq \mathbb{C} \setminus \psi^{-1}(Z_\alpha)$. We define $Q' := \psi(\bar{B}_\varepsilon(\psi^{-1}(z)))$. By (3.16), increasing j_0 we may assume w.l.o.g. that for $\nu \geq \nu_{i_{j_0}}$ and $\beta \in V$ we have

$$\varphi_\alpha^\nu(Q') \cap \varphi_\beta^\nu(\bar{B}_R) = \emptyset.$$

Therefore, increasing j_0 so that (3.12) holds for $\nu \geq \nu_{i_{j_0}}$, we have by (3.17)

$$\begin{aligned} E(w_\nu, \varphi_\alpha^{\nu_{i_j}}(Q') &\leq E\Big(w^{\nu_{i_j}}, \mathbb{C} \setminus \bigcup_{\beta \in V} \varphi_\beta^{\nu_{i_j}}(B_R)\Big) \\ &< E(w^{\nu_{i_j}}, \mathbb{C}) - \pi d + \pi \\ &= \pi \deg(w^{\nu_{i_j}}) - \pi d + \pi = \pi, \end{aligned}$$

for $j \geq j_0$. On the other hand, if $\psi^{-1}(z)$ belonged to Z, then by condition 4.3(iii) we would have

$$\begin{aligned} \liminf_{\nu \to \infty} E(w_\nu, \varphi_\alpha^\nu(Q')) &= \liminf_{\nu \to \infty} E^{|\lambda_\alpha^\nu|}\Big(\big(u^\nu, \bar{\lambda}_\alpha^\nu \cdot (\Phi^\nu + i\Psi^\nu)\big) \circ \varphi_\alpha^\nu \circ \psi, B_\varepsilon(\psi^{-1}(z))\Big) \\ &\geq \pi. \end{aligned}$$

This contradiction proves that $\psi^{-1}(z) \notin Z$. This proves Claim 1. $\square$

It follows that condition (iii) of Definition 3.6 is satisfied. This proves that $(w_\nu, z_0^\infty, \dots, z_k^\nu)$ converges to $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$, and terminates the proof of Proposition 3.13. $\square$

4 Compactification for vortices on $\mathbb{C}$

Let (M, ω) be a symplectic manifold (without boundary), let G a compact connected Lie group with Lie algebra $\mathfrak{g}$ and let $\langle \cdot, \cdot \rangle$ be a G-invariant inner product on $\mathfrak{g}$. Assume that G acts on M in a Hamiltonian way, with moment map $\mu : M \to \mathfrak{g}$. We fix an ω-compatible G-invariant almost complex structure J on M. Remember that we always impose the hypothesis (H1), i.e. that G acts freely on $\mu^{-1}(0)$ and that μ is proper. We fix a number $p > 2$.

4.1 Bubbling

Theorem 4.1 (Bubbling) *Assume that the hypotheses (H2) (Convexity at ∞) and (H3) (Symplectic asphericity) hold, as stated in section 0. Let $w^\nu = (u^\nu, \Phi^\nu, \Psi^\nu) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ be a sequence of positive energy solutions of the vortex equations (0.1), (0.2) such that $\overline{u_\nu(\mathbb{C})}$ is compact, let $k \geq 0$ be an integer, and let $z_1^\nu, \ldots, z_k^\nu \in \mathbb{C}$ be sequences of points. Assume that*

$$\sup_{\nu \in \mathbb{N}} E(w^\nu) < \infty$$

and that

$$\limsup_{\nu \to \infty} |z_i^\nu - z_j^\nu| > 0,$$

for $i \neq j$. Then there exists a subsequence of $(w^\nu, z_0^\nu := \infty, z_1^\nu, \ldots, z_k^\nu)$ that converges to some stable map of vortices on $\mathbb{C}$ and pseudo-holomorphic spheres in $\bar{M} = \mu^{-1}(0)/\mathrm{G}$ with $k+1$ marked points.

The proof of this theorem is postponed to subsection 4.4. It combines Gromov compactness for pseudo-holomorphic spheres with Uhlenbeck compactness. It involves versions for vortices on $\mathbb{C}$ of quantization of energy, an a priori Lemma, compactness modulo bubbling, compactness with bounded energy density, hard rescaling, soft rescaling, and of an annulus lemma. Furthermore, it uses an extension of Uhlenbeck compactness to principal G-bundles with noncompact base, which is proved in the book [Weh] by K. Wehrheim. A standard reference for Gromov compactness is the book by D. McDuff and D. A. Salamon [MS3].

4.2 Compactness modulo bubbling for rescaled vortices

We fix a value $R \in [0, \infty]$. In the case $R < \infty$ the R-vortex equations on $\mathbb{C}$ are the equations for a map $w := (u, \Phi, \Psi) : \mathbb{C} \to M \times \mathfrak{g} \times \mathfrak{g}$ given by

$$\partial_s u + X_\Phi(u) + J(u)(\partial_t u + X_\Psi(u)) = 0, \tag{4.1}$$

$$\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + R^2 \mu \circ u = 0, \tag{4.2}$$

and in the case $R = \infty$ the ∞-vortex equations on $\mathbb{C}$ are the equation (4.1) and the equation $\mu \circ u = 0$. We call a solution w of (4.1), (4.2) an R-vortex. In the case $R := 1$ these are the usual vortex equations (0.1), (0.2). Assume that $0 < R < \infty$. Then equations (4.1), (4.2) arise from the equations (0.1), (0.2) by rescaling, as follows. We fix a map $w := (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ and define

$$\widetilde{w} := (\widetilde{u}, \widetilde{\Phi}, \widetilde{\Psi}) := (u, R\Phi, R\Psi)(R\cdot). \tag{4.3}$$

Then w solves the 1-vortex equations (0.1), (0.2) if and only if $\widetilde{w}$ solves the R-vortex equations (4.1), (4.2). We abbreviate

$$\kappa := \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi].$$

If $0 < R < \infty$ then we define the *R-energy density of* w to be

$$e^R_w := \frac{1}{2}\big(|\partial_s u + X_\Phi \circ u|^2 + |\partial_t u + X_\Psi \circ u|^2 + R^{-2}|\kappa|^2 + R^2 |\mu \circ u|^2\big). \tag{4.4}$$

Furthermore, if $R = 0$ and $\kappa = 0$ or if $R = \infty$ and $\mu \circ u = 0$ we define

$$e^R_w := \frac{1}{2}\big(|\partial_s u + X_\Phi \circ u|^2 + |\partial_t u + X_\Psi \circ u|^2\big). \tag{4.5}$$

For every measurable subset $X \subseteq \mathbb{C}$ we define the *R-energy of* w *on* X to be

$$E^R(w, X) := \int_X e^R_w \, ds\, dt \in [0, \infty]. \tag{4.6}$$

In the case $R = 1$ we abbreviate $e_w := e^1_w$ and $E(w, X) := E^1(w, X)$ and call this the *energy density* of w and the *energy* of w on X for short. In the case $X = \mathbb{C}$ we write $E^R(w) := E^R(w, \mathbb{C})$. Moreover, we define the energy of a map $\bar{u} \in W^{1,p}(X, \bar{M})$ on a measurable subset $X \subseteq S^2$ to be

$$E(\bar{u}, X) := \frac{1}{2} \int_X |d\bar{u}|^2 \mathrm{dvol}_{S^2}, \tag{4.7}$$

where the norm of the differential $d\bar{u}$ is taken w.r.t. the standard metric on S^2 and w.r.t. the metric $\bar{g}_{\omega,J} = \bar{\omega}(\cdot, \bar{J}\cdot)$ on $\bar{M}$. We abbreviate $E(\bar{u}) := E(\bar{u}, S^2)$.

In this subsection we consider pairs of sequences $((R_\nu), (w_\nu))$, where for each ν R_ν is a positive number and w_ν is an R_ν-vortex, and the sequence (R_ν) converges to some value $R_0 \in (0, \infty]$. The main result of this subsection is Proposition 4.3. Assuming that (M, ω) is symplectically aspherical, it says that if the energies $E_{R_\nu}(w_\nu)$ are uniformly bounded then there is a subsequence such that the R_ν-vortices w_ν converge modulo gauge and bubbling to some R_0-vortex $w_0 := (u_0, \Phi_0, \Psi_0)$. In the case $R_0 = \infty$ it follows that $\mathrm{G}u_\nu : \mathbb{C} \to M/\mathrm{G}$ converges to the $\bar{J}$-holomorphic map $\mathrm{G}u_0 : S^2 \to \bar{M}$, away from the bubbling points. On the other hand, in the case $R_0 < \infty$ no bubbling takes place.

We define

$$\begin{aligned} E_V &:= \inf\big\{E(w)\,|\,w\in W^{1,p}_{\mathrm{loc}}(\mathbb{C},M\times\mathfrak{g}\times\mathfrak{g}):\\ &\qquad (0.1),(0.2),E(w)>0,\overline{u(\mathbb{C})}\text{ compact}\big\},\\ \bar{E} &:= \inf\big\{E(\bar{u})\,|\,\bar{u}\in C^\infty(S^2,\bar{M}):\bar\partial_{\bar{J}}(\bar{u})=0,\,E(\bar{u})>0\big\},\\ \mathrm{E}_{\min} &:= \min\{E_V,\bar{E}\}. \end{aligned} \tag{4.8}$$

Here we use the convention that the infimum over the empty set is ∞. Lemma D.1 implies that $E_V>0$ and Proposition 4.1.4 in the book by D. McDuff and D. A. Salamon [MS3] implies that $\bar{E}>0$. Therefore, the number $\mathrm{E}_{\min}$ is positive.

Remark 4.2 If $w:=(u,\Phi,\Psi)\in W^{1,p}_{\mathrm{loc}}(\mathbb{C},M\times\mathfrak{g}\times\mathfrak{g})$ is a solution of the first vortex equation (0.1) such that $\mu\circ u=0$, then by Proposition E.11 the map $\mathrm{G}u:\mathbb{C}\to\bar{M}$ is $\bar{J}$-holomorphic and

$$E^\infty(u,\Phi,\Psi)=E(\mathrm{G}u).$$

If this energy is finite, then by removal of singularities (see Theorem 4.1.2. in the book [MS3]), the map $\mathrm{G}u$ extends to a $\bar{J}$-holomorphic map $\bar{u}:S^2\to\bar{M}$. It follows that $E^\infty(w)\geq\mathrm{E}_{\min}$, provided that $E^\infty(w)>0$.

Proposition 4.3 (Compactness modulo bubbling) *Assume that the hypothesis (H3) (symplectic asphericity) holds. Let $R_\nu>0$ be a sequence that converges to some value $R_0\in(0,\infty]$, let $r_\nu>0$ be a sequence that converges to ∞ and for every $\nu\in\mathbb{N}$ let $w_\nu=(u_\nu,\Phi_\nu,\Psi_\nu)\in W^{1,p}_{\mathrm{loc}}(B_{r_\nu},M\times\mathfrak{g}\times\mathfrak{g})$ be a solution of (4.1), (4.2) with $R:=R_\nu$. Assume that there is a compact subset $K\subseteq M$ such that $u_\nu(B_{r_\nu})\subseteq K$ for every ν. Suppose also that $\sup_\nu E^{R_\nu}(w_\nu,B_{r_\nu})<\infty$. Then there exist a finite subset $Z\subseteq\mathbb{C}$ and an R_0-vortex $w_0=(u_0,\Phi_0,\Psi_0)\in W^{1,p}_{\mathrm{loc}}(\mathbb{C}\setminus Z,M\times\mathfrak{g}\times\mathfrak{g})$ such that passing to some subsequence the following holds.*

(i) If $R_0<\infty$ then $Z=\emptyset$.

(ii) There are gauge transformations $g_\nu\in W^{2,p}_{\mathrm{loc}}(\mathbb{C}\setminus Z,\mathrm{G})$ such that for every compact subset $Q\subseteq\mathbb{C}\setminus Z$ the gauged transformed maps $g_\nu^{-1}u_\nu$ converge to u_0 in $C^1(Q,M)$ and the maps $g_\nu^*(\Phi_\nu,\Psi_\nu)$ converge to (Φ_0,Ψ_0) in $C^0(Q,\mathfrak{g}\times\mathfrak{g})$.

(iii) Fix a point $z\in Z$ and a number $\varepsilon_0>0$ so small that $B_{\varepsilon_0}(z)\cap Z=\{z\}$. Then for every $0<\varepsilon<\varepsilon_0$ the limit $E_z(\varepsilon):=\lim_{\nu\to\infty}E^{R_\nu}(w_\nu,B_\varepsilon(z))$ exists and $\mathrm{E}_{\min}\leq E_z(\varepsilon)<\infty$. Furthermore, the function $(0,\varepsilon_0)\ni\varepsilon\mapsto E_z(\varepsilon)\in[\mathrm{E}_{\min},\infty)$ is continuous.

The proof of this proposition will be given on page 89. Its strategy is the following. Consider the energy densities $e_\nu := e^{R_\nu}_{w_\nu}$. If for every compact subset $Q \subseteq \mathbb{C}$ we have $\sup_\nu ||e_\nu||_{L^\infty(Q)} < \infty$, then the statement of Proposition 4.3 with $Z = \emptyset$ follows by combining Uhlenbeck compactness and elliptic bootstrapping for perturbed pseudo-holomorphic curves. Otherwise we rescale the maps w_ν by zooming in near the point z_0 in order to get a positive energy $\widetilde{R}_0$-vortex in the limit, for some number $\widetilde{R}_0 < R_0$. If $R_0 < \infty$ then $\widetilde{R}_0 = 0$, and we get a J-holomorphic sphere in M. This contradicts hypothesis (H3) (symplectic asphericity) and thus the bubbling point z_0 cannot exist. If $R_0 = \infty$ then either a vortex or a pseudo-holomorphic sphere in $\bar{M}$ bubbles off, and we lose at least the energy $\mathrm{E}_{\min}$ at z_0. Our assumption that the energy of w_ν is uniformly bounded implies that there can only be finitely many bubbling points.

For the proof of the next lemma see also step 5, case 3 in the proof of Theorem A in the paper [GS] by R. Gaio and D. A. Salamon and Theorem 3.2 in the paper [CGMS] by K. Cieliebak et al. Note that in the latter theorem the domain of the vortices is a compact principal G-bundle P, while our situation corresponds to the trivial bundle $P := (\mathbb{C} \setminus Z) \times \mathrm{G}$, where $Z \subseteq \mathbb{C}$ is a finite set.

Lemma 4.4 (Compactness with bounded energy density) *Let $p > 2$, $Z \subseteq \mathbb{C}$ be a finite subset, $R^\nu \geq 0$ be a sequence of numbers that converges to some value $R_0 \in [0, \infty]$, let $\Omega_1 \subseteq \Omega_2 \subseteq \ldots \subseteq \mathbb{C} \setminus Z$ be open subsets such that $\bigcup_\nu \Omega_\nu = \mathbb{C} \setminus Z$ and for $\nu \in \mathbb{N}$ let $w_\nu = (u_\nu, \Phi_\nu, \Psi_\nu) \in W^{1,p}_{\mathrm{loc}}(\Omega_\nu, M \times \mathfrak{g} \times \mathfrak{g})$ be an R^ν-vortex. Suppose that there is a compact subset $K \subseteq M$ such that for ν large enough*

$$u_\nu(\Omega_\nu) \subseteq K, \tag{4.9}$$

and for every compact subset $Q \subseteq \mathbb{C} \setminus Z$

$$\sup_{\nu \geq \nu_Q} ||e^{R_\nu}_{w_\nu}||_{L^\infty(Q)} < \infty, \tag{4.10}$$

for every $\nu \geq \nu_Q$, where $\nu_Q \in \mathbb{N}$ is such that $Q \subseteq \Omega_\nu$ if $\nu \geq \nu_Q$. Then passing to some subsequence there are gauge transformations $g_\nu \in W^{2,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, \mathrm{G})$ and there is an R_0-vortex $w_0 = (u_0, \Phi_0, \Psi_0) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, M \times \mathfrak{g} \times \mathfrak{g})$ such that the following holds. Let $Q \subseteq \mathbb{C} \setminus Z$ be a compact subset. Then $g_\nu^{-1} u_\nu$ converges to u_0 in $C^1(Q)$ and $g_\nu^*(\Phi_\nu, \Psi_\nu)$ converges to (Φ_0, Ψ_0) in $C^0(Q)$.

Proof of Lemma 4.4: We write

$$\kappa_\nu := \partial_s \Psi_\nu - \partial_t \Phi_\nu + [\Phi_\nu, \Psi_\nu].$$

Let $Q \subseteq \mathbb{C} \setminus Z$ be a compact subset and let $\nu_Q \in \mathbb{N}$ be so large that $Q \subseteq \Omega_{\nu_Q}$.

Claim 1

$$\sup_{\nu \geq \nu_Q} ||\kappa_\nu||_{L^p(Q)} < \infty. \tag{4.11}$$

Proof of the Claim: Consider the **case** $0 \leq R_0 < \infty$. Then by the second vortex equation (4.2) we have

$$\begin{aligned} ||\kappa_\nu||_{L^p(Q)} &= ||R_\nu^2 \mu(u_\nu)||_{L^p(Q)} \\ &\leq C||\mu(u_\nu)||_{L^\infty(Q)} \leq C', \end{aligned}$$

where the constants $C, C' < \infty$ do not depend on ν. The last inequality follows from the assumption (4.9).

Consider now the **case** $R_0 = \infty$. Let $\Omega \subseteq \mathbb{C}$ be an open subset containing Q such that $\bar{\Omega}$ is compact and contained in $\mathbb{C} \setminus Z$. We check the conditions of Lemma C.3. Since

$$e_{w_\nu}^{R_\nu} = |\partial_s u_\nu + X_{\Phi_\nu}(u_\nu)|^2 + R_\nu^2 |\mu \circ u_\nu|^2,$$

assumption (4.10) implies that

$$c_0 := \sup_\nu ||\partial_s u_\nu + X_{\Phi_\nu}(u_\nu)||_{L^\infty(\Omega)} < \infty,$$

and

$$\sup_\nu R_\nu ||\mu \circ u_\nu||_{L^\infty(\Omega)} \leq \sup_\nu ||\sqrt{e_{w_\nu}}||_{L^\infty(\Omega)} < \infty. \tag{4.12}$$

Let $0 < \delta$ be so small that G acts freely on

$$M_\delta := \{x \in M \,|\, |\mu(x)| \leq \delta\}.$$

Since μ is proper the set M_δ is compact. Therefore there is a constant $c_1 > 0$ such that for every $x \in M_\delta$, $\xi \in \mathfrak{g}$ we have

$$|\xi| \leq c_1 |L_x \xi|.$$

Let c be the maximum of c_0 and c_1. Since by assumption $R_\nu \to \infty$, it follows from (4.12) that there is $\nu_0 \in \mathbb{N}$ such that for $\nu \geq \nu_0$ we have $||\mu \circ u_\nu||_{L^\infty(\Omega)} \leq \delta$. Therefore the conditions of Lemma C.3 with $w = w_\nu$ are satisfied and therefore

$$\sup_\nu R_\nu^2 ||\mu \circ u_\nu||_{L^p(Q)} \;<\; \infty. \tag{4.13}$$

Estimate (4.11) now follows from (4.13) and the second vortex equation (4.2). This proves Claim 1. □

Let $\nu \in \mathbb{N}$ be so large that $\bar{B}_1 \setminus \bigcup_{z\in Z} B_1(z) \subseteq \Omega_\nu$. Let $\ell_\nu \in \mathbb{N}$ be the largest number ℓ such that

$$Y_\ell := \bar{B}_\ell \setminus \bigcup_{z\in Z} B_{\frac{1}{\ell}}(z) \subseteq \Omega_\nu.$$

We check the conditions of the Uhlenbeck compactness Theorem E.8 with $X := \mathbb{C} \setminus Z$, $P := (\mathbb{C} \setminus Z) \times \mathrm{G}$, $X_\nu := Y_{\ell_\nu}$ and the connection one forms $A^\nu \in \mathcal{A}^{1,p}(P|_{X_\nu})$ given by

$$A^\nu_{(z,g)}(\zeta, g\xi) := \xi + g^{-1}(\zeta_1\Phi(z) + \zeta_2\Psi(z))g,$$

for $(z,g) \in P$ and $(\zeta, g\xi) \in T_{(z,g)}P = \mathbb{C} \times g\mathfrak{g}$. To see that condition (E.26) is satisfied, observe that

$$(F_{A_\nu})_{(z,g)}((\zeta, g\xi),(\zeta', g\xi')) = (\zeta_1\zeta_2' - \zeta_2\zeta_1')g^{-1}\kappa_\nu(z)g, \qquad (4.14)$$

for $(\zeta, g\xi),(\zeta', g\xi') \in T_{(z,g)}P = \mathbb{C} \times g\mathfrak{g}$ and $(z,g) \in P$, where

$$\kappa_\nu := \partial_s\Psi_\nu - \partial_t\Phi_\nu + [\Phi_\nu, \Psi_\nu],$$

see Theorem 7.2, (4) in the book by Dupont [Du]. By (4.14), the second vortex equation (0.2) and by the Claim we have for every compact subset $Q \subseteq X$

$$\sup_{\nu\geq\nu_Q} ||F_{A_\nu}||_{L^p(Q)} = \sup_{\nu\geq\nu_Q} ||\kappa_\nu||_{L^p(Q)} < \infty.$$

Therefore, condition (E.26) of Theorem E.8 is satisfied. Consider the smooth connection one form $\hat{A}$ on $\mathbb{C} \times \mathrm{G}$ given by

$$\hat{A}_{(z,g)}(\zeta, g\xi) := \xi,$$

for $(z,g) \in P$ and $(\zeta, g\xi) \in T_{(z,g)}P$. By Theorem E.8, passing to some subsequence, there is a connection one form $A^0 \in \hat{A} + W^{1,p}_{\mathrm{loc}}(\mathbb{C},(P \times \mathfrak{g})/\mathrm{G})$ and there are gauge transformations $\widetilde{g}_\nu \in W^{2,p}_{\mathrm{loc}}(\mathbb{C},(P \times \mathrm{G})/\mathrm{G})$ such that $\widetilde{g}_\nu^* A^\nu - \hat{A}$ converges to $A^0 - \hat{A}$, weakly in $W^{1,p}(X_{\nu_0},(P \times \mathfrak{g})/\mathrm{G})$, for every $\nu_0 \in \mathbb{N}$. We define $(\Phi_0, \Psi_0) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, \mathfrak{g} \times \mathfrak{g})$ by

$$\zeta_1\Phi_0(z) + \zeta_2\Psi_0(z) := A^0_{(z,g)}(\zeta, 0),$$

$g_\nu \in W^{2,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, \mathrm{G})$ by $g_\nu(z) := \widetilde{g}_\nu(z,e)$ and $w'_\nu := (u'_\nu, \Phi'_\nu, \Psi'_\nu) := g_\nu^* w_\nu \in W^{1,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, M \times \mathfrak{g} \times \mathfrak{g})$. Then (Φ'_ν, Ψ'_ν) converges to (Φ_0, Ψ_0), weakly in $W^{1,p}(X_{\nu_0}, \mathfrak{g} \times \mathfrak{g})$, for every $\nu_0 \in \mathbb{N}$. By the Kondrachov compactness theorem there is a subsequence $(\nu^1_j)_{j\in\mathbb{N}}$ of ν, a subsequence $(\nu^2_j)_{j\in\mathbb{N}}$ of (ν^1_j), and so on, such that $(\Phi'_{\nu^\ell_j}, \Psi'_{\nu^\ell_j})$ converges to (Φ_0, Ψ_0) in $C^0(X_{\nu_0}, \mathfrak{g} \times \mathfrak{g})$ for every $\ell \in \mathbb{N}$. Passing to the diagonal subsequence $(\nu^j_j)_{j\in\mathbb{N}}$ we may assume w.l.o.g. that (Φ'_ν, Ψ'_ν) converges in $C^0(X_{\nu_0})$ for every $\nu_0 \in \mathbb{N}$. It follows from Lemma B.1

that w'_ν solves the R-vortex equations (4.1), (4.2) with $R = R_\nu$, for every $\nu \in \mathbb{N}$.

We now show that the hypotheses of Proposition E.9 with $k := 1$ are satisfied. Let $Q \subseteq \mathbb{C} \setminus Z$ be a compact subset. Since the sequence (Φ'_ν, Ψ'_ν) converges to (Φ_0, Ψ_0), weakly in $W^{1,p}(Q)$, we have

$$\sup_{\nu \geq \nu_Q} ||(\Phi'_\nu, \Psi'_\nu)||_{W^{1,p}(Q)} < \infty, \tag{4.15}$$

where $\nu_Q \in \mathbb{N}$ is so large that $Q \subseteq \Omega_{\nu_Q}$. Condition (E.27) is satisfied by the assumption (4.9). We check condition (E.28). The assumption (4.9) implies that there is a constant C independent of ν such that

$$|X_\xi(u'_\nu(z))| \leq C|\xi|, \tag{4.16}$$

for every $z \in Q$, $\xi \in \mathfrak{g}$ and $\nu \in \mathbb{N}$. Furthermore,

$$\begin{aligned} \sup_{\nu \geq \nu_Q} ||\partial_s u'_\nu||_{L^p(Q)} &\leq \sup_{\nu \geq \nu_Q} (||\partial_s u'_\nu + X_{\Phi'_\nu}(u'_\nu)||_{L^p(Q)} + ||X_{\Phi'_\nu}(u'_\nu)||_{L^p(Q)}) \\ &\leq C' \sup_{\nu \geq \nu_Q} \big(||\partial_s u'_\nu + X_{\Phi'_\nu}(u'_\nu)||_{L^\infty(Q)} + ||\Phi'_\nu||_{L^p(Q)}\big) \\ &< \infty, \end{aligned} \tag{4.17}$$

for some constant C'. Here the second inequality uses (4.16) and the last inequality uses (4.10) and (4.15). Analogously, it follows that

$$\sup_{\nu \geq \nu_Q} ||\partial_t u'_\nu||_{L^p(Q)} < \infty. \tag{4.18}$$

Therefore condition (E.28) holds. Furthermore, by the Whitney embedding theorem we may assume w.l.o.g. that M is a submanifold of $\mathbb{R}^{4n+1}$. (See Theorem II.1. in the article by H. Whitney [Wh] or Theorem 3.2.14 p. 55 in the book by M. W. Hirsch [Hi].) Therefore, we can define the map

$$f : M \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}^{4n+1}, \qquad f(x, \xi, \eta) := X_\xi(x) + J(x)X_\eta(x).$$

We fix an index $\nu \geq \nu_Q$. Then by the first vortex equation (4.1) we have

$$\partial_s u'_\nu + J(u'_\nu)\partial_t u'_\nu = -X_{\Phi'_\nu}(u'_\nu) - J(u'_\nu)X_{\Psi'_\nu}(u'_\nu). \tag{4.19}$$

Since $u'_\nu(Q)$ is contained in the compact set $\mathrm{G} \cdot K \subseteq M$, this implies that

$$\begin{aligned} ||d(\partial_s u'_\nu + J(u'_\nu)\partial_t u'_\nu)||_{L^p(Q)} &= ||d(f \circ w'_\nu)||_{L^p(Q)} \\ &= ||df(w'_\nu)dw'_\nu||_{L^p(Q)} \leq C'', \end{aligned} \tag{4.20}$$

where $C'' > 0$ is a constant independent of ν. The last inequality uses (4.15), (4.17) and (4.18), and that $u'_\nu(Q) \subseteq \mathrm{G} \cdot K$. It follows that condition (E.29) of Proposition E.9 with $k = 1$ is satisfied. Thus this Proposition implies that passing to some subsequence u'_ν converges to some map $u_0 \in W^{2,p}(\mathbb{C} \setminus Z)$, weakly in $W^{2,p}$ and strongly in C^1 on every compact subset of $\mathbb{C} \setminus Z$. The limit (u_0, Φ_0, Ψ_0) solves the first vortex equation, (4.1). Assume now that $0 \leq R_0 < \infty$. Since w'_ν converges weakly in $W^{1,p}(X_\ell)$ for every $\ell \in \mathbb{N}$, w_0 solves the second vortex equation (4.2) with $R = R_0$. In the case $R_0 = \infty$ we have for ν so large that $R_\nu \neq 0$

$$R_\nu^{-2}(\partial_s \Psi'_\nu - \partial_t \Phi'_\nu + [\Phi'_\nu, \Psi'_\nu]) + \mu \circ u'_\nu = 0.$$

It follows that w_0 solves the equation $\mu \circ u_0 = 0$. This means that w_0 is an ∞-vortex. This proves Lemma 4.4. □

Remark 4.5 Note that in the proof of Lemma 4.4 we use Theorem E.8, which is a version of Uhlenbeck compactness with noncompact base manifold X. In our case $X := \mathbb{C} \setminus Z$. It is necessary to use this extension of ordinary Uhlenbeck compactness (Theorem E.7), because of the following. Theorem E.7 only provides gauge transformations on a fixed compact subset $K \subseteq \mathbb{C}$. The only chance to get the gauge transformations g_ν, using only Theorem E.7, is the following. Let $K_0 := K \subseteq K_1 \subseteq \ldots \subseteq \mathbb{C} \setminus Z$ be an exhausting sequence of compact subsets. Applying Theorem E.7 to $X := K_i$, we get gauge transformations g_i^ν, defined on K_i. Now one could try to extend g_i^ν to all of $\mathbb{C}$ and then use a diagonal subsequence argument to get the g_ν's. However, this does not work, since for $j > i$ g_i^ν need not be the restriction of g_j^ν to K_i^ν.

Recall the definition of $\mathrm{E}_{\min} > 0$, (4.8).

Proposition 4.6 (Hard rescaling) *Assume that hypothesis (H3) (symplectic asphericity) holds. Let $\Omega \subseteq \mathbb{C}$ be an open subset, $0 < R_\nu < \infty$ be a sequence such that $\inf_\nu R_\nu > 0$ and let $w_\nu \in W^{1,p}_{\mathrm{loc}}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ be an R_ν-vortex for $\nu \in \mathbb{N}$. Assume that there is a compact subset $K \subseteq M$ such that $u_\nu(\Omega) \subseteq K$ for every ν and that $\sup_\nu E^{R_\nu}(w_\nu, \Omega) < \infty$. Then the following holds.*

(i) For every compact subset $Q \subseteq \Omega$ we have

$$\sup_\nu R_\nu^{-2} ||e_{w_\nu}^{R_\nu}||_{C^0(Q)} < \infty.$$

(ii) If there is a compact subset $Q \subseteq \Omega$ such that $\sup_\nu ||e_{w_\nu}^{R_\nu}||_{C^0(Q)} = \infty$ then there is $z_0 \in Q$ with the following property. For every $\varepsilon > 0$ so small that $B_\varepsilon(z_0) \subseteq \Omega$ we have

$$\limsup_{\nu \to \infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)) \geq \mathrm{E}_{\min}\,. \tag{4.21}$$

Remark 4.7 Under the assumptions of Proposition 4.6, if $\sup_\nu R_\nu < \infty$ then for every compact subset $Q \subseteq \Omega$ we have $\sup_\nu ||e^{R_\nu}_{w_\nu}||_{C^0(Q)} < \infty$. This follows from (i).

The proof of Proposition 4.6 is built on a bubbling argument, as in step 5, in the proof of Theorem A in the paper by R. Gaio and D. A. Salamon [GS]. We need the following two lemmata. The first one is due to Hofer, see Lemma 4.6.4 in the book by McDuff and Salamon [MS3].

Lemma 4.8 (Hofer) *Let (X, d) be a metric space, $f : X \to [0, \infty)$ be a continuous function, $x \in X$ and $\delta > 0$. Assume that the closed ball $\bar{B}_{2\delta}(x)$ is complete. Then there is a $\xi \in X$ and a number $0 < \varepsilon \leq \delta$ such that*

$$d(x, \xi) < 2\delta, \qquad \sup_{B_\varepsilon(\xi)} f \leq 2f(\xi), \qquad \varepsilon f(\xi) \geq \delta f(x).$$

Proof: This is Lemma 4.6.4 in the book [MS3]. □

The next lemma ensures that for a suitably convergent sequence of rescaled vortices on an open subset $\Omega \subseteq \mathbb{C}$, in the limit $\nu \to \infty$ no energy gets lost on any compact subset of Ω.

Lemma 4.9 *Let $\Omega \subseteq \mathbb{C}$ be an open subset, $Q \subseteq \Omega$ a compact subset, $K \subseteq M$ be a compact subset, $R_\nu \geq 0$ be a sequence of numbers that converges to some value $R_0 \in [0, \infty]$, let $w_0 = (u_0, \Phi_0, \Psi_0) \in W^{1,p}_{\mathrm{loc}}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ be a R_0-vortex and for $\nu \in \mathbb{N}$ let $w_\nu = (u_\nu, \Phi_\nu, \Psi_\nu) \in W^{1,p}_{\mathrm{loc}}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ be an R_ν-vortex. Assume that $u_\nu(\Omega) \subseteq K$, that u_ν converges to u_0 in $C^1(\bar{\Omega})$ and that (Φ_ν, Ψ_ν) converges to (Φ_0, Ψ_0) in $C^0(\bar{\Omega})$. Then the energy densities $e^{R_\nu}_{w_\nu}$ converge to $e^{R_0}_{w_0}$ in $C^0(Q)$.*

Proof of Lemma 4.9: Recall that if $R < \infty$ and w is an R-vortex then

$$e^R_w = |\partial_s u_\nu + X_\Phi(u)|^2 + R^2|\mu(u)|^2. \tag{4.22}$$

In the **case** $R_0 < \infty$ the statement of the lemma follows directly from this formula.

Consider the **case** $R_0 = \infty$. Let $\delta > 0$ be so small that G acts freely on $M_\delta := \{x \in M \,|\, |\mu(x)| \leq \delta\}$. Let $c > 0$ be so big that

$$\sup_\nu ||\partial_s u_\nu + X_{\Phi_\nu}(u_\nu)||_{L^\infty(\Omega)} \leq c,$$

$|\mu|_{C^0(K)} \leq c$ and $|\xi| \leq c|L_x\xi|$ for every $x \in M_\delta$ and $\xi \in \mathfrak{g}$. Let $C_p > 0$, $\widetilde{R}_0 > 0$ be constants as in Lemma C.3, with R_0 replaced by $\widetilde{R}_0$. By assumption, condition (C.7) of Lemma C.3 is satisfied. Since $\mu \circ u_\nu \to \mu \circ u_0 = 0$ in $C^0(\Omega)$ there is $\nu_0 \in \mathbb{N}$ such that for $\nu \geq \nu_0$ we have $u_\nu(z) \in M_\delta$, for every $z \in \Omega$.

Increasing ν_0 we may assume that for every $\nu \geq \nu_0$ we have $R_\nu \geq \widetilde{R}_0$. We fix an index $\nu \geq \nu_0$. Then the conditions (C.8) and (C.9) of Lemma C.3 are satisfied with $w := w_\nu$. Applying that Lemma we have

$$\sup_Q |\mu \circ u_\nu| \leq C_p R_\nu^{2/p-2}.$$

This implies that

$$||R_\nu^2 |\mu \circ u_\nu|^2||_{C^0(Q)} \leq C_p R_\nu^{4/p-2} \to 0. \tag{4.23}$$

If follows from (4.22) and (4.23) that $e_{w_\nu}^{R_\nu}$ converges to $e_{w_0}^\infty = |\partial_s u_\nu + X_{\Phi_\nu}(u_\nu)|^2$ in $C^0(Q)$. This completes the proof of Lemma 4.9. □

Proof of Proposition 4.6 (Hard rescaling): Consider the function

$$f_\nu := |\partial_s u_\nu + X_{\Phi_\nu}(u_\nu)| + R_\nu |\mu \circ u_\nu| \in C^0(\Omega, [0, \infty)).$$

Claim 1 *Suppose that the hypotheses of Proposition 4.6 are satisfied and that there is a sequence $z_\nu \in \Omega$ that converges to some $z_0 \in \Omega$ such that*

$$f_\nu(z_\nu) \to \infty.$$

Then there is a value

$$0 < r_0 \leq \limsup_{\nu \to \infty} \frac{R_\nu}{f_\nu(z_\nu)} \, (\leq \infty) \tag{4.24}$$

and a positive energy r_0-vortex $w_0 \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ such that

$$E^{r_0}(w_0) \leq \limsup_{\nu \to \infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)), \tag{4.25}$$

for every $\varepsilon > 0$ so small that $B_\varepsilon(z_0) \subseteq \Omega$.

Proof of Claim 1: We define $\delta_\nu := f_\nu(z_\nu)^{-\frac{1}{2}}$ and denote $e_\nu := e_{w_\nu}^{R_\nu}$. Then for ν large enough we have $\bar{B}_{2\delta_\nu}(z_\nu) \subseteq \Omega$. We pass to some subsequence such that this holds for every ν. By Lemma 4.8 applied to

$$f := f_\nu, \qquad \delta := \delta_\nu, \qquad x := z_\nu$$

there are numbers $\zeta_\nu \in B_{2\delta_\nu}(z_0)$ and $\varepsilon_\nu \leq \delta_\nu$ such that

$$|\zeta_\nu - z_\nu| < 2 f_\nu(z_\nu)^{-\frac{1}{2}}, \tag{4.26}$$

$$\sup_{B_{\varepsilon_\nu}(\zeta_\nu)} f_\nu \leq 2 f_\nu(\zeta_\nu), \tag{4.27}$$

$$\varepsilon_\nu f_\nu(\zeta_\nu) \geq f_\nu(z_\nu)^{\frac{1}{2}}. \tag{4.28}$$

Since $f_\nu(z_\nu) \to \infty$, it follows from (4.26) that the sequence ζ_ν converges to z_0. We define $c_\nu := f_\nu(\zeta_\nu)$ and

$$\widetilde{w}_\nu := (\widetilde{u}_\nu, \widetilde{\Phi}_\nu, \widetilde{\Psi}_\nu) := (u_\nu, c_\nu^{-1}\Phi_\nu, c_\nu^{-1}\Psi_\nu)(c_\nu^{-1}\cdot + \zeta_\nu).$$

Passing to some subsequence we may assume that $\widetilde{R}_\nu := c_\nu^{-1}R_\nu$ converges to some $r_0 \in [0,\infty]$. Since $\varepsilon_\nu \leq \delta_\nu = f_\nu(z_\nu)^{-\frac{1}{2}}$ it follows from (4.28) that $f_\nu(z_\nu) \leq f_\nu(\zeta_\nu)$. It follows that (4.24) holds for the original sequence. We check the conditions of Lemma 4.4 with $Z := \emptyset$ and $\Omega_\nu := c_\nu \cdot (\Omega - \zeta_\nu)$ and R_ν, w_ν replaced by $\widetilde{R}_\nu, \widetilde{w}_\nu$. By hypothesis there is a compact subset $K \subseteq M$ such that for ν large enough we have $u_\nu(\Omega_\nu) \subseteq K$. So it suffices to check condition (4.10) with $Q := \bar{B}_R$, for every $R > 0$. We have

$$(\partial_s \widetilde{u}_\nu + X_{\widetilde{\Phi}_\nu}(\widetilde{u}_\nu))(\widetilde{z}) = c_\nu^{-1}(\partial_s u_\nu + X_{\Phi_\nu}(u_\nu))(c_\nu^{-1}\widetilde{z} + \zeta_\nu) = c_\nu^{-1} f_\nu(c_\nu^{-1}\widetilde{z} + \zeta_\nu).$$

Therefore, (4.27) implies that for every $\widetilde{z} \in B_{\varepsilon_\nu c_\nu}(0)$ we have

$$\begin{aligned} (|\partial_s \widetilde{u}_\nu + X_{\widetilde{\Phi}_\nu}(\widetilde{u}_\nu)| + \widetilde{R}_\nu|\mu(\widetilde{u}_\nu)|)(\widetilde{z}) &= c_\nu^{-1} f_\nu(c_\nu^{-1}\widetilde{z} + \zeta_\nu) \\ &\leq 2c_\nu^{-1} f_\nu(\zeta_\nu) = 2. \end{aligned} \tag{4.29}$$

Let $R > 0$. We choose an integer ν_R such that $f_\nu(z_\nu)^{\frac{1}{2}} > R$ for every $\nu \geq \nu_R$. By (4.28) we have

$$\varepsilon_\nu c_\nu = \varepsilon_\nu f_\nu(\zeta_\nu) > R$$

for every ν. Therefore, (4.29) implies that $||e^{\widetilde{R}_\nu}_{\widetilde{w}_\nu}||_{L^\infty(B_R)} \leq 8$ and so (4.10) is satisfied with $Q = \bar{B}_R$. By Lemma 4.4 there is an r_0-vortex $w_0 = (u_0, \Psi_0, \Psi_0) \in W^{1,p}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ with the following property. Passing to some subsequence there are $g_\nu \in W^{2,p}(\mathbb{C}, \mathrm{G})$ such that the map $g_\nu^{-1}\widetilde{u}_\nu$ converges to u_0 in $C^1(Q)$ and $g_\nu^*(\widetilde{\Phi}_\nu, \widetilde{\Psi}_\nu)$ converges to (Φ_0, Ψ_0) in $C^0(Q)$, for every compact subset $Q \subseteq \mathbb{C}$. By Lemma 4.9 with $\Omega \subseteq \mathbb{C}$ any bounded open subset such that $Q \subseteq \Omega$ we have

$$e^{\widetilde{R}_\nu}_{\widetilde{w}_\nu} = e^{\widetilde{R}_\nu}_{g_\nu^*\widetilde{w}_\nu} \to e^{r_0}_{w_0}, \tag{4.30}$$

in $C^0(Q)$ for every compact subset $Q \subseteq \mathbb{C}$. It follows that

$$\begin{aligned} e^{r_0}_{w_0}(0) &= \lim_{\nu\to\infty} e^{\widetilde{R}_\nu}_{\widetilde{w}_\nu}(0) \\ &= c_\nu^{-2} e^{R_\nu}_{w_\nu}(z_\nu) \geq \frac{1}{2}, \end{aligned} \tag{4.31}$$

and therefore, $E^{r_0}(w_0) > 0$.

Let $\varepsilon > 0$ be so small that $B_\varepsilon(z_0) \subseteq \Omega$. In order to show (4.25) we choose any $\delta > 0$. Let $R > 0$ be so large that $E^{r_0}(w_0, \mathbb{C} \setminus B_R) < \delta$. By (4.30) it follows that

$$\begin{aligned} \lim_{\nu\to\infty} E^{R_\nu}(w_\nu, B_{c_\nu^{-1}R}(\zeta_\nu)) &= \lim_{\nu\to\infty} E^{\widetilde{R}_\nu}(\widetilde{w}_\nu, B_R) \\ &= E^{r_0}(w_0, B_R) \\ &> E^{r_0}(w_0) - \delta. \end{aligned} \tag{4.32}$$

On the other hand, since $c_\nu \to \infty$ and $\zeta_\nu \to z_0$, for ν large enough the ball $B_{c_\nu^{-1}R}(\zeta_\nu)$ is contained in $B_\varepsilon(z_0)$ and therefore

$$E^{R_\nu}(w_\nu, B_{c_\nu^{-1}R}(\zeta_\nu) \leq E^{R_\nu}(w_\nu, B_\varepsilon(z_0)).$$

Together with (4.32) this implies that

$$\limsup_{\nu\to\infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)) \geq E^{r_0}(w_0) - \delta.$$

Since this holds for every $\delta > 0$ this proves (4.25) for the original sequence.

It remains to prove that $r_0 > 0$. Assume by contradiction that $r_0 = 0$. By Proposition D.2 there is a gauge transformation $g \in \mathcal{G}^{2,p}$ such that $w_0' := (u_0', \Phi_0', \Psi_0') := g^* w_0$ is smooth. By the second R-vortex equation (4.2) with $R := 0$ we have $\partial_s \Psi_0' - \partial_t \Phi_0' + [\Phi_0', \Psi_0'] = 0$. Therefore, by Proposition D.4 there is $h \in \mathcal{G}$ such that $h^*(\Phi_0', \Psi_0') = 0$. By the first vortex equation the map $u_0'' := h^{-1} u_0' : \mathbb{C} \to M$ is J-holomorphic. We have for every $\varepsilon > 0$ such that $B_\varepsilon(z_0) \subseteq \Omega$

$$\begin{aligned} E(u_0'') &= \int_{\mathbb{C}} |\partial_s u_0''|^2 \, ds\, dt \\ &= E^0(w_0'') \\ &= E^0(w_0) \\ &\leq \limsup_{\nu\to\infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)) \\ &\leq \sup_\nu E^{R_\nu}(w_\nu, \Omega) < \infty. \end{aligned} \tag{4.33}$$

Here in the forth line we have used inequality (4.25) and the last inequality holds by hypothesis. By (4.33) and removal of singularities, Theorem 4.1.2 in [MS3], it follows that u_0'' extends to a smooth J-holomorphic map from S^2 to M. Since $E(u_0'') = E^0(w_0) > 0$ such a map can not exist by the hypothesis (H3). This contradiction proves that $r_0 > 0$ and so concludes the proof of Claim 1. □

Proof of (i): Assume by contradiction that there was a compact subset $Q \subseteq \Omega$ such that

$$\sup_\nu R_\nu^{-2} ||e_{w_\nu}^{R_\nu}||_{C^0(Q)} = \infty.$$

Passing to some subsequence we may assume that

$$\lim_{\nu\to\infty} R_\nu^{-1} ||f_\nu||_{C^0(Q)} = \infty. \tag{4.34}$$

Let $z_\nu \in Q$ be such that $f_\nu(z_\nu) = ||f_\nu||_{C^0(Q)}$. Since by assumption $\inf_\nu R_\nu > 0$, it follows that $f_\nu(z_\nu) \to \infty$. Passing to some subsequence, we may assume that z_ν converges to some point $z_0 \in Q$. Therefore, applying Claim 1 we arrive at a contradiction, since by (4.34) we have

$$\limsup_{\nu\to\infty} \frac{R_\nu}{f_\nu(z_\nu)} = 0,$$

and thus the value r_0 of Claim 1 cannot exist. This proves (i).

Proof of (ii): Assume that there is a compact subset $Q \subseteq \Omega$ such that $\sup_\nu ||e^{R_\nu}_{w_\nu}||_{C^0(Q)} = \infty$. Let $z_\nu \in Q$ be such that $f_\nu(z_\nu) \to \infty$. By Claim 1 there is a value

$$0 < r_0 \leq \limsup_{\nu\to\infty} \frac{R_\nu}{f_\nu(z_\nu)}$$

and an r_0-vortex $w_0 \in W^{1,p}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ such that

$$0 < E^{r_0}(w_0) \leq \limsup_{\nu\to\infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0))$$

for every $\varepsilon > 0$ with $B_\varepsilon(z_0) \subseteq \Omega$. By the definition of $\mathrm{E}_{\min}$ and by Remark 4.2 it follows that $E^{r_0}(w_0) \geq \mathrm{E}_{\min}$, and therefore

$$\limsup_{\nu\to\infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)) \geq E^{r_0}(w_0) \geq \mathrm{E}_{\min},$$

for every $\varepsilon > 0$ such that $B_\varepsilon(z_0) \subseteq \Omega$. This proves (ii) and concludes the proof of Proposition 4.6. □

Proof of Proposition 4.3: We abbreviate $e_\nu := e^{R_\nu}_{w_\nu}$.

Claim 1 *For every $\ell \in \mathbb{N} \cup \{0\}$ there is a finite subset $Z_\ell \subseteq \mathbb{C}$ such that the following holds. If $R_0 < \infty$ then $Z_\ell = \emptyset$, and if $|Z_\ell| < \ell$ then*

$$\sup_{\nu \geq \nu_Q} ||e_\nu||_{C^0(Q)} < \infty, \tag{4.35}$$

for every compact subset $Q \subseteq \mathbb{C} \setminus Z_\ell$, where $\nu_Q \in \mathbb{N}$ is so large that $Q \subseteq B_{r_\nu}$ for $\nu \geq \nu_Q$. Furthermore, for every $z \in Z_\ell$ and every $\varepsilon > 0$ we have

$$\limsup_{\nu\to\infty} E^{R_\nu}(w_\nu, B_\varepsilon(z)) \geq \mathrm{E}_{\min}\,. \tag{4.36}$$

Proof of Claim 1: For $\ell = 0$ the assertion holds with $Z_0 := \emptyset$. We prove by induction that it holds for every $\ell \geq 1$. Fix $\ell \geq 1$. By induction hypothesis there is a finite subset $Z_{\ell-1} \subseteq \mathbb{C}$ such that the assertion with ℓ replaced by $\ell - 1$ holds. If (4.35) holds for any compact subset $Q \subseteq \mathbb{C} \setminus Z_{\ell-1}$ then the statement for ℓ holds with $Z_\ell := Z_{\ell-1}$. So assume that there is a compact subset $Q \subseteq \mathbb{C} \setminus Z_{\ell-1}$ such that

$$\sup_{\nu \geq \nu_Q} ||e_\nu||_{C^0(Q)} = \infty. \tag{4.37}$$

It follows from the induction hypothesis that

$$|Z_{\ell-1}| \geq \ell - 1. \tag{4.38}$$

Let $\Omega \subseteq \mathbb{C}$ be any bounded open subset such that $Q \subseteq \Omega$. Applying Proposition 4.6 to the sequence $(w_\nu|_\Omega)_{\nu \geq \nu_Q}$, by condition (ii) of that proposition there is a point $z_0 \in Q$ such that for every $\varepsilon > 0$ so small that $B_\varepsilon(z_0) \subseteq \Omega$ inequality (4.36) holds. We set $Z_\ell := Z_{\ell-1} \cup \{z_0\}$. By induction hypothesis (4.36) holds for every $z \in Z_{\ell-1}$ and thus for every $z \in Z_\ell$. We claim that $R_0 = \infty$. By condition (i) of Proposition 4.6 we have

$$||e_\nu||_{C^0(Q)} \leq C R_\nu^2,$$

for every ν so large that $\Omega \subseteq B_{r_\nu}$, where $C := \sup_\nu R_\nu^{-2} ||e_\nu||_{C^0(Q)}$. Thus (4.37) implies that $\sup_\nu R_\nu = \infty$ and therefore $R_0 = \infty$. Finally, since $z_0 \in Q \subseteq \mathbb{C} \setminus Z_{\ell-1}$, it follows from (4.38) that $|Z_\ell| \geq \ell$. So all the assertions of Claim 1 for ℓ are satisfied. This proves Claim 1. □

We fix an integer

$$\ell > \frac{\sup_\nu E^{R_\nu}(w_\nu, B_{r_\nu})}{\mathrm{E}_{\min}} \tag{4.39}$$

and a finite subset $Z := Z_\ell \subseteq \mathbb{C}$ that satisfies the conditions of Claim 1. Then condition (i) of Proposition 4.3 is satisfied. Furthermore, by (4.36) there are consecutive subsequences $(\nu_i^{(1)})_{i \in \mathbb{N}}, (\nu_i^{(2)})_{i \in \mathbb{N}}, \ldots$ (i.e. $(\nu_i^{(2)})_{i \in \mathbb{N}}$ is a subsequence of $(\nu_i^{(1)})_{i \in \mathbb{N}}, \ldots$) such that the limit

$$E_j(z) := \lim_{i \to \infty} E^{R_{\nu_i^{(j)}}}(w_{\nu_i^{(j)}}, B_{1/j}(z)) \tag{4.40}$$

exists and $E_j(z) \geq \mathrm{E}_{\min}$ for every $j \in \mathbb{N}$ and $z \in Z$. Passing to the diagonal subsequence $(\nu_i^{(i)})$ we may assume that $\lim_{\nu \to \infty} E^{R_\nu}(w_\nu, B_{1/j}(z))$ exists for every $j \in \mathbb{N}$ and $z \in Z$. Let j_0 be so large that $1/j_0 < 1/2|z - z'|$ for each two points $z \neq z' \in Z$. Then

$$\begin{aligned}
|Z|\,\mathrm{E}_{\min} &\leq \sum_{z \in Z} \lim_{\nu \to \infty} E^{R_\nu}(w_\nu, B_{1/j_0}(z)) \\
&= \lim_{\nu \to \infty} E^{R_\nu}\big(w_\nu, \bigcup_{z \in Z} B_{1/j_0}(z)\big) \\
&\leq \sup_\nu E^{R_\nu}(w_\nu, B_{r_\nu}).
\end{aligned}$$

This together with (4.39) implies that $\ell > |Z|$ and therefore inequality (4.35) is satisfied for every compact subset $Q \subseteq \mathbb{C} \setminus Z$. Thus the assumptions of Lemma 4.4 (Compactness with bounded energy density) with $\Omega_\nu := B_{r_\nu} \setminus Z$ are satisfied. So passing to some subsequence there are gauge transformations $g_\nu \in W^{2,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, \mathrm{G})$ and there is an R_0-vortex $w_0 = (u_0, \Phi_0, \Psi_0) \in$

$W^{1,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, M \times \mathfrak{g} \times \mathfrak{g})$ such that for every compact subset $Q \subseteq \mathbb{C} \setminus Z$ the maps $g_\nu^{-1} u_\nu$ converge to u_0 in $C^1(Q, M)$ and the maps $g_\nu^*(\Phi_\nu, \Psi_\nu)$ converge to (Φ_0, Ψ_0) in $C^0(Q, \mathfrak{g} \times \mathfrak{g})$. It follows that condition (ii) of Proposition 4.3 is satisfied.

To see that condition (iii) holds, let $z \in Z$ and fix $\varepsilon_0 > 0$ so small that $B_{\varepsilon_0}(z) \cap Z = \{z\}$. Let $0 < \varepsilon < \varepsilon_0$ and choose numbers $\varepsilon < \varepsilon' < \varepsilon_0$ and $j \in \mathbb{N}$ such that $1/j < \varepsilon$. Then Lemma 4.9 with $\Omega := B_{\varepsilon'}(z) \setminus \bar{B}_{1/2j}(z)$ implies that the limit $\lim_{\nu\to\infty} E^{R_\nu}(w_\nu, A(z, 1/j, \varepsilon))$ exists and equals $E^{R_0}(w_0, A(z, 1/j, \varepsilon))$. Since by our choice of a subsequence the limit $E_j(z) := \lim_{\nu\to\infty} E^{R_\nu}(w_\nu, B_{1/j}(z))$ exists and $E_j(z) \geq \mathrm{E}_{\min}$, it follows that the same holds for the limit $E_z(\varepsilon) := \lim_{\nu\to\infty} E^{R_\nu}(w_\nu, B_\varepsilon(z))$. Since $E^{R_0}(w_0, A(z, 1/j, \varepsilon))$ depends continuously on ε, the same holds for $E_z(\varepsilon)$. This implies condition (iii) and completes the proof of Proposition 4.3. □

4.3 Soft rescaling

The next proposition is an adaption of Proposition 4.7.1. in [MS3] to vortices instead of J-holomorphic curves.

Proposition 4.10 (Soft rescaling) *Assume that the hypothesis (H2) (convexity at ∞) and (H3) (symplectic asphericity) hold. Let $r > 0$, $z_0 \in \mathbb{C}$, $R_\nu > 0$ be a sequence that converges to ∞ and for every $\nu \in \mathbb{N}$ let $w_\nu \in W^{1,p}(B_r(z_0), M \times \mathfrak{g} \times \mathfrak{g})$ be an R_ν-vortex such that the following conditions are satisfied.*

(a) There exists a compact subset $K \subseteq M$ such that $u_\nu(B_r(z_0)) \subseteq K$ for every ν.

(b) For every $0 < \varepsilon \leq r$ the limit $E(\varepsilon) := \lim_{\nu\to\infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0))$ exists and $\mathrm{E}_{\min} \leq E(\varepsilon) < \infty$. Furthermore, the function $(0, r] \ni \varepsilon \mapsto E(\varepsilon) \in [\mathrm{E}_{\min}, \infty)$ is continuous.

Then passing to some subsequence there exist a value $R_0 \in \{1, \infty\}$, a finite subset $Z \subseteq \mathbb{C}$, an R_0-vortex $w_0 \in W^{1,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, M \times \mathfrak{g} \times \mathfrak{g})$ and sequences $\varepsilon_\nu > 0$ and $z_\nu \in \mathbb{C}$ such that the following conditions hold.

(i) The sequence z_ν converges to z_0. Furthermore, if $R_0 = 1$ then $\varepsilon_\nu = R_\nu^{-1}$ for every ν, and if $R_0 = \infty$ then ε_ν converges to 0 and $\varepsilon_\nu R_\nu$ converges to ∞.

(ii) There exists a sequence of gauge transformations $g_\nu \in W^{2,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, \mathrm{G})$ such that for every compact subset $Q \subseteq \mathbb{C} \setminus Z$ the sequence $g_\nu^{-1}(u_\nu(\varepsilon_\nu \cdot + z_\nu))$ converges to u_0 in $C^1(Q)$ and $g_\nu^*\big((\varepsilon_\nu \Phi_\nu, \varepsilon_\nu \Psi_\nu)(\varepsilon_\nu \cdot + z_\nu)\big)$ converges to (Φ_0, Ψ_0) in $C^0(Q)$.

(iii) *Fix $z \in Z$ and a number $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(z) \cap Z = \{z\}$. Then for every $0 < \varepsilon < \varepsilon_0$ the limit*

$$E_z(\varepsilon) := \lim_{\nu \to \infty} E^{\varepsilon_\nu R_\nu}\big((u_\nu, \varepsilon_\nu \Phi_\nu, \varepsilon_\nu \Psi_\nu)(\varepsilon_\nu \cdot + z_\nu), B_\varepsilon(z)\big)$$

exists and $\mathrm{E}_{\min} \leq E_z(\varepsilon) < \infty$. Furthermore, the function $(0, \varepsilon_0) \ni \varepsilon \mapsto E_z(\varepsilon) \in [\mathrm{E}_{\min}, \infty)$ is continuous.

(iv)

$$\lim_{R \to \infty} \limsup_{\nu \to \infty} E^{R_\nu}\big(w_\nu, B_{R^{-1}}(z_0) \setminus B_{R\varepsilon_\nu}(z_\nu)\big) = 0. \tag{4.41}$$

(v) *If $R_0 = 1$ then $Z = \emptyset$ and $E(w_0) > 0$. If $R_0 = \infty$ and $E^\infty(w_0) = 0$ then $|Z| \geq 2$.*

The proof of Proposition 4.10 is given on page 98. We need the following Lemma. For $0 \leq r, R \leq \infty$ we denote the closed and the open annulus around 0 with radii r, R by

$$A(r,R) := \bar{B}_R \setminus B_r, \qquad \mathring{A}(r,R) := B_R \setminus \bar{B}_r.$$

Note that in the case $r > R$ $A(r,R) = \emptyset$. Furthermore, $A(0,R) = \bar{B}_R$ and $A(r,\infty) = \mathbb{C} \setminus B_r$. Let

$$d : \bigcup_{M'} M' \times M' \to [0, \infty)$$

be the distance function induced by the Riemannian metric $g_{\omega,J}$, where M' runs over all connected components of M. Since by assumption G is connected, the action of G on M leaves each connected component of M invariant. We can therefore define

$$\bar{d} : \bigcup_{M'} M'/\mathrm{G} \times M'/\mathrm{G} \to [0,\infty), \qquad \bar{d}(\mathrm{G}x, \mathrm{G}y) := \min_{x' \in \mathrm{G}x,\, y' \in \mathrm{G}y} d(x', y'). \tag{4.42}$$

By Lemma A.10, for each connected component $M' \subseteq M$ the restriction of $\bar{d}$ to $M' \times M'$ is a distance function that induces the quotient topology.

Lemma 4.11 (Annulus Lemma) *For every compact subset $K \subseteq M$ and every number $r_0 > 0$ there are constants $E_1 > 0$, $a > 0$ and $C_1 > 0$ such that the following holds. Assume that $r_0 \leq r < R \leq \infty$ and $w \in W^{1,p}_{\mathrm{loc}}(\mathring{A}(r,R), M \times \mathfrak{g} \times \mathfrak{g})$ is a vortex such that $u(z) \in K$ for every $z \in A(r,R)$, and suppose that $E := E(w, A(r,R)) \leq E_1$. Then for every $\lambda \geq 2$ we have*

$$E(w, A(\lambda r, \lambda^{-1} R)) \leq \frac{C_1 E}{\lambda^a}, \tag{4.43}$$

$$\sup_{z,z' \in A(\lambda r, \lambda^{-1} R)} \bar{d}(\mathrm{G}u(z), \mathrm{G}u(z')) \leq \frac{C_1 \sqrt{E}}{\lambda^a}. \tag{4.44}$$

Remark 4.12 If $\lambda > \sqrt{R/r}$ then $A(\lambda r, \lambda^{-1}R) = \emptyset$ and hence the statement of the Lemma is void.

Proof: Let $K \subseteq M$ be a compact subset and let $r_0 > 0$ be a number. We choose constants $\delta_0, c > 0$ as in Lemma A.15 (Isoperimetric inequality) and $E_0 > 0$ as in the a-priori-Lemma C.1, with K replaced by $\mathrm{G} \cdot K$. We define

$$E_1 := \min\left\{E_0, \frac{\delta_0^2}{128\pi}, \frac{\pi\delta_0^2 r_0^2}{8}\right\}, \qquad a := \frac{\min\{1, r_0^2\}}{2c}.$$

Let $r_0 \leq r < R \leq \infty$ and $w = (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathring{A}(r,R), M \times \mathfrak{g} \times \mathfrak{g})$ be a vortex such that $u(z) \in K$ for every $z \in A(r,R)$ and $E \leq E_1$.

We consider the case $R < \infty$ and $w \in W^{1,p}(\mathring{A}(r,R), M \times \mathfrak{g} \times \mathfrak{g})$. By Theorem C.6 there exists a gauge transformation $g \in W^{2,p}(\mathring{A}(r,R), \mathrm{G})$ such that g^*w is smooth. Replacing K by $\mathrm{G} \cdot K$ and w by g^*w we may therefore assume w.l.o.g. that w is smooth. We identify $\mathbb{R} \times S^1 \cong \mathbb{C}/\sim$, where the equivalence relation $\sim$ is defined by $z \sim z + 2\pi$ for every $z \in \mathbb{C}$. The exponential map $\mathbb{C} \ni \tau + i\varphi \mapsto e^{\tau + i\varphi}$ descends to a map $\exp_{\mathbb{C}} : \mathbb{R} \times S^1 \cong \mathbb{C}/\sim \to \mathbb{C}$. We denote by $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the complexified Lie algebra and define the map $\widetilde{w} : \mathbb{R} \times S^1 \to M \times \mathfrak{g}^{\mathbb{C}}$ by

$$\widetilde{w} := (\widetilde{u}, \widetilde{\Phi} + i\widetilde{\Psi}) := \big(u \circ \exp_{\mathbb{C}}, \overline{\exp_{\mathbb{C}}} \cdot ((\Phi + i\Psi) \circ \exp_{\mathbb{C}})\big).$$

Claim 1 . *For every $1 \leq \lambda \leq \sqrt{R/r}$ we have*

$$E(w, A(\lambda r, \lambda^{-1}R)) \leq \left(\frac{2}{\lambda}\right)^a E. \tag{4.45}$$

Proof of Claim 1: It suffices to prove (4.45) for $\lambda \geq 2$.

Claim 2 *Let $\tau_0 \in [\log(2r), \log(R/2)]$. Then*

$$|\mu(\widetilde{u}(\tau_0, \varphi))| \leq \delta_0, \forall \varphi \in S^1, \tag{4.46}$$

$$\ell(\widetilde{u}(\tau_0, .), \widetilde{\Psi}(\tau_0, .)) \leq \delta_0. \tag{4.47}$$

Proof of Claim 2: Let $\varphi_0 \in S^1 \cong \mathbb{R}/(2\pi\mathbb{Z})$ and set $z_0 := \exp_{\mathbb{C}}(\tau_0 + i\varphi_0)$. Than the ball $B_{e^{\tau_0}-r}(z_0)$ is contained in the annulus $A(r,R)$. Therefore, condition (C.2) of Lemma C.1 is satisfied with r replaced by $e^{\tau_0} - r$. It follows that

$$\begin{aligned} e_w(z_0) &\leq \frac{8}{\pi(e^{\tau_0} - r)^2} E(w, B_{e^{\tau_0}-r}(z_0)) \\ &\leq \frac{32}{\pi e^{2\tau_0}} E. \end{aligned} \tag{4.48}$$

The last inequality uses the assumption $\tau_0 \geq \log(2r)$. Inequality (4.48) implies that

$$\begin{aligned} |\mu(\widetilde{u}(\tau_0,\varphi_0))| &= |\mu(u(e^{\tau_0+i\varphi_0}))| \\ &\leq \sqrt{e_w(z_0)} \\ &< \sqrt{\frac{8E_1}{\pi r_0^2}} < \delta_0. \end{aligned}$$

This proves (4.46). The $\exp_{\mathbb{C}}$-energy density of $\widetilde{w}$ is the map $\widetilde{e} := e_{\widetilde{w}}^{\exp_{\mathbb{C}}} : \mathbb{R}\times S^1 \to [0,\infty)$ given by

$$\widetilde{e}(\tau+i\varphi) = |\partial_\tau \widetilde{u} + X_{\widetilde{\Phi}}(\widetilde{u})|^2(\tau+i\varphi) + e^{2\tau}|\mu\circ\widetilde{u}(\tau+i\varphi)|^2,$$

see definition (B.10) and Proposition B.3. Furthermore, the energy of $\widetilde{w}$ on a measurable subset $X\subseteq \mathbb{R}\times S^1$ is given by

$$E^{\exp_{\mathbb{C}}}(\widetilde{w},X) = \int_X \widetilde{e}\,d\tau\,d\varphi.$$

By Proposition B.3 we have

$$\widetilde{e} = e^{2\tau} e_w \circ \exp_{\mathbb{C}},$$

and therefore

$$E^{\exp_{\mathbb{C}}}(\widetilde{w},X) = E(w,\exp_{\mathbb{C}}(X)),$$

for every measurable subset $X\subseteq\mathbb{R}\times S^1$. Furthermore, the map $\widetilde{w}$ solves the $\exp_{\mathbb{C}}$-vortex equations

$$\begin{aligned} \partial_\tau\widetilde{u} + X_{\widetilde{\Phi}}(\widetilde{u}) + J(\partial_\varphi\widetilde{u} + X_{\widetilde{\Psi}}(\widetilde{u})) &= 0, \\ \partial_\tau\widetilde{\Psi} - \partial_\varphi\widetilde{\Phi} + [\widetilde{\Phi},\widetilde{\Psi}] + e^{2\tau}\mu\circ\widetilde{u} &= 0. \end{aligned}$$

It follows that

$$\begin{aligned} |\partial_\tau\widetilde{u} + X_{\widetilde{\Phi}}(\widetilde{u})|^2 &= \frac{1}{2}\big(|\partial_\tau\widetilde{u} + X_{\widetilde{\Phi}}(\widetilde{u})|^2 + |\partial_\varphi\widetilde{u} + X_{\widetilde{\Psi}}(\widetilde{u})|^2\big) \\ &\leq \widetilde{e} \\ &= e^{2\tau} e_w \circ \varphi \end{aligned} \tag{4.49}$$

The last equality uses Proposition B.3. By (4.49) and (4.48)

$$\begin{aligned} |\partial_\tau\widetilde{u} + X_{\widetilde{\Phi}}(\widetilde{u})|(z_0) &\leq e^{\tau_0}\sqrt{e_w(z_0)} \\ &\leq 2\sqrt{\frac{8E}{\pi}} \\ &< 2\sqrt{\frac{8E_1}{\pi}} \\ &\leq \frac{\delta_0}{2\pi}. \end{aligned}$$

Therefore,

$$\ell(\widetilde{u}(\tau_0,.),\widetilde{\Psi}(\tau_0,.)) = \int_0^{2\pi} |\partial_\tau \widetilde{u} + X_{\widetilde{\Phi}}(\widetilde{u})|(\tau_0,\varphi)\, d\varphi \leq \delta_0.$$

This proves (4.47) and completes the proof of Claim 2. □

Let now $\log(2) \leq T \leq \log(R/r)/2$ be a number. We define $E(T) := E(w, A(re^T, Re^{-T}))$. The Claim guarantees that the local equivariant symplectic actions of $(\widetilde{u},\widetilde{\Psi})(\log(R)-T,.)$ and $(\widetilde{u},\widetilde{\Psi})(\log(r)+T,.)$ are well-defined, as in Definition A.14, and that the hypotheses of Lemma A.15 are satisfied. Therefore,

$$\begin{aligned}
&|\mathcal{A}((\widetilde{u},\widetilde{\Psi})(\log(R)-T,.))| \\
&\leq c\int_0^{2\pi} \big(|\partial_\varphi \widetilde{u} + X_{\widetilde{\Psi}}(\widetilde{u})|^2 + |\mu(\widetilde{u})|^2\big)(\log(R)-T,\varphi)\, d\varphi \\
&\leq c \max\{1, r_0^{-2}\} \int_0^{2\pi} \widetilde{e}(\log(R)-T,\varphi)\, d\varphi \\
&= -a^{-1}\frac{d}{dT}\int_{\log(Rr)/2}^{\log(R)-T}\int_0^{2\pi} \widetilde{e}(\tau,\varphi)\, d\varphi\, d\tau.
\end{aligned}$$

In the second inequality we have used the fact that $e^{2(T-\log R)} \leq r_0^{-2}$. Similarly,

$$|\mathcal{A}((\widetilde{u},\widetilde{\Psi})(\log(r)+T.))| \leq -a^{-1}\frac{d}{dT}\int_{\log(r)+T}^{\log(Rr)/2}\int_0^{2\pi} \widetilde{e}(\tau,\varphi)\, d\varphi\, d\tau.$$

Using Proposition B.3 and Lemma C.8 we have

$$\begin{aligned}
E(T) &= E^{\exp_{\mathbb{C}}}(\widetilde{w}, [\log(r)+T, \log(R)-T]\times S^1) \\
&= -\mathcal{A}((\widetilde{u},\widetilde{\Psi})(\log(R)-T,.)) + \mathcal{A}((\widetilde{u},\widetilde{\Psi})(\log(r)+T,.)) \\
&\leq |\mathcal{A}((\widetilde{u},\widetilde{\Psi}))(\log(R)-T,.))| + |\mathcal{A}((\widetilde{u},\widetilde{\Psi})(\log(r)+T,.))| \\
&\leq -a^{-1}\frac{d}{dT}\int_{\log(r)+T}^{\log(R)-T}\int_0^{2\pi} \widetilde{e}(\tau,\varphi)\, d\varphi\, d\tau \\
&= -a^{-1}\frac{d}{dT}E(\widetilde{w}, [\log(r)+T, \log(R)-T]\times S^1) = -a^{-1}E'(T).
\end{aligned}$$

Therefore,

$$(Ee^{aT})' = (E' + aE)e^{\frac{T}{c}} \leq (E' - E')e^{aT} = 0.$$

So

$$E(T)e^{aT} \leq E(\log(2))e^{a\log(2)} \leq 2^a E.$$

It follows that for $2 \leq \lambda \leq \sqrt{R/r}$

$$\begin{aligned}
E(w, A(\lambda r, \lambda^{-1}R)) &= E(\log(\lambda)) \\
&\leq 2^a e^{-\frac{\log(\lambda)}{c}} E \\
&= \left(\frac{2}{\lambda}\right)^a E.
\end{aligned}$$

This proves Claim 1. □

We define $\bar{\tau} := (\log(R) + \log(r))/2$.

Claim 3 *For every $\tau \in [\bar{\tau}, \log(R/2)]$, $\varphi \in S^1$ we have*

$$\widetilde{e}(\tau,\varphi) \leq CR^{-\frac{1}{c}} e^{\frac{\tau}{c}} E,$$

and for every $\tau \in [\log(2r), \bar{\tau}]$, $\varphi \in S^1$ we have

$$\widetilde{e}(\tau,\varphi) \leq Cr^{\frac{1}{c}} e^{-\frac{\tau}{c}} E,$$

where $C := 2^{2a+5}/\pi$.

Proof of Claim 3: Let $\tau \in [\log(2r), \log(R/2)]$ and $\varphi \in S^1$. Assume first that $\tau \geq \bar{\tau}$. We set $\lambda := \frac{R}{2e^\tau} \geq 1$, Then

$$\begin{aligned}
\widetilde{e}(\tau,\varphi) &= e^{2\tau} e_w(e^{\tau+i\varphi}) \\
&\leq \frac{32}{\pi} E(w, B_{\frac{e^\tau}{2}}(e^{\tau+i\varphi})) \\
&\leq \frac{32}{\pi} E(w, A(\lambda r, \lambda^{-1}R)) \\
&\leq \frac{32}{\pi} \left(\frac{2}{\lambda}\right)^{\frac{1}{c}} E \\
&= CR^{-\frac{1}{c}} e^{\frac{\tau}{c}} E.
\end{aligned}$$

The first line follows from (B.12), the second from Lemma C.1 and the third from the fact $B_{\frac{e^\tau}{2}}(e^{\tau+i\varphi}) \subseteq A(\lambda r, \lambda^{-1}R)$. The forth line follows from Claim 1. The case $\tau \in [\log(2r), \bar{\tau}]$ can be treated similarly with $\lambda := \frac{e^\tau}{2r} \geq 1$. This proves Claim 3. □

Claim 4 *For every $2 \leq \lambda \leq \sqrt{R/r}$ and $z_0 \in A(\lambda r, \lambda^{-1}R)$ we have*

$$\bar{d}(\mathrm{G}u(\sqrt{Rr}), \mathrm{G}u(z_0)) \leq \frac{C'\sqrt{E}}{\lambda^a}, \tag{4.50}$$

where $C' := \sqrt{C}(a^{-1} + 2\pi)$ and the constant C is as in Claim 3.

Proof of Claim 4: Let $\tau_0 \in \mathbb{R}$, $\varphi_0 \in [0, 2\pi)$ be such that $z_0 = e^{\tau_0 + i\varphi_0}$. Consider the **case** $\tau_0 \geq \bar{\tau}$. We define paths γ_1, γ_2 in M as follows. Let $g_1 : [\bar{\tau}, \tau_0] \to \mathrm{G}$ be the unique solution of the ordinary differential equation

$$\dot{g}_1(t) = g_1(t)\widetilde{\Phi}(t, 0), \qquad g_1(\bar{\tau}) = \mathrm{id}. \tag{4.51}$$

We define $\gamma_1 := g_1\widetilde{u}(\cdot,0) : [\bar{\tau},\tau_0] \to M$. Furthermore, let $g_2 : [0,\varphi_0] \to \mathrm{G}$ be the unique solution of

$$\dot{g}_2(t) = g_2(t)\widetilde{\Psi}(\tau_0,t), \qquad g_2(0) = g_1(\tau_0). \tag{4.52}$$

We define $\gamma_2 := g_2\widetilde{u}(\tau_0,\cdot) : [0,\varphi_0] \to M$. Then $\gamma_1\#\gamma_2$ is a path from the point $u(\sqrt{Rr})$ to $g_2(\varphi_0)u(z_0)$. Furthermore,

$$\begin{aligned} \dot{\gamma}_1 &= \left(g_1\widetilde{u}(\cdot,0)\right)^{\cdot} \\ &= g_1\big(X_{g_1^{-1}\dot{g}_1}(\widetilde{u}(\cdot,0)) + \partial_\tau\widetilde{u}(\cdot,0)\big) \\ &= g_1\big(X_{\widetilde{\Phi}(\cdot,0)}(\widetilde{u}(\cdot,0)) + \partial_\tau\widetilde{u}(\cdot,0)\big). \end{aligned}$$

Here in the last equality we have used (4.51). Claim 3 implies that

$$\begin{aligned} |\dot{\gamma}_1(t)| &= |\partial_\tau\widetilde{u}(t,0) + X_{\widetilde{\Phi}(t,0)}(\widetilde{u}(t,0))| \\ &\le \sqrt{\widetilde{e}(t,0)} \\ &\le \sqrt{CE}R^{-a}e^{at}. \end{aligned}$$

Analogously, we have

$$|\dot{\gamma}_2(t)| \le \sqrt{CE}R^{-a}e^{at}.$$

Therefore,

$$\begin{aligned} \bar{d}(\mathrm{G}u(\sqrt{Rr}),\mathrm{G}u(z_0)) &\le \ell(\gamma_1) + \ell(\gamma_2) \\ &\le \sqrt{CE}R^{-a}\left(\int_{\bar{\tau}}^{\tau_0} e^{a\tau}\,d\tau + \int_0^{\varphi_0} e^{a\tau_0}\,d\varphi\right) \\ &= \sqrt{CE}R^{-a}\left(a^{-1}e^{a\tau}\Big|_{\bar{\tau}}^{\tau_0} + \varphi_0 e^{a\tau_0}\right) \\ &\le C'\sqrt{E}\left(\frac{e^{\tau_0}}{R}\right)^a \\ &\le \frac{C'\sqrt{E}}{\lambda^a}. \end{aligned}$$

Here the last inequality follows from the fact $|z_0| = e^{\tau_0} \le R/\lambda$.

In the **case** $\tau_0 < \bar{\tau}$ inequality (4.50) follows analogously. This proves Claim 4. □

Recalling that we have set $a = 1/(2c)$ and setting $C_1 := \max\{2^a, 2C'\}$, inequality (4.43) follows from Claim 1. Furthermore, by Claim 4 we have for every $z,z' \in A(\lambda r, \lambda^{-1}R)$

$$\begin{aligned} \bar{d}(\mathrm{G}u(z),\mathrm{G}u(z')) &\le \bar{d}(\mathrm{G}u(z),\mathrm{G}u(\sqrt{Rr})) + \bar{d}(\mathrm{G}u(\sqrt{Rr}),\mathrm{G}u(z')) \\ &\le \frac{2C'\sqrt{E}}{\lambda^a}, \end{aligned}$$

and hence inequality (4.44) holds.

Consider now the general case $r_0 \leq r < R \leq \infty$ **and** $w \in W^{1,p}_{\mathrm{loc}}(\mathring{A}(r,R), M \times \mathfrak{g} \times \mathfrak{g})$**.** We choose sequences $r_\nu > r$ converging to r and $R_\nu < R$ converging to R. We fix a number $\lambda \leq 2$. What we have already proved, implies that

$$\begin{aligned} E(w, A(\lambda r_\nu, \lambda^{-1} R_\nu)) &\leq \frac{C_1 E}{\lambda^a}, \\ \sup_{z,z' \in A(\lambda r_\nu, \lambda^{-1} R_\nu)} \bar{d}(\mathrm{G}u(z), \mathrm{G}u(z')) &\leq \frac{C_1 \sqrt{E}}{\lambda^a}. \end{aligned}$$

Inequalities (4.43) and (4.44) follow from this. This proves Lemma 4.11. □

Proof of Proposition 4.10: By hypothesis (b) the function

$$(0, r] \ni \varepsilon \mapsto \lim_{\nu \to \infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)) \in [\mathrm{E}_{\min}, \infty)$$

is well-defined. Since it is increasing, the limit

$$m_0 := \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)) \in [\mathrm{E}_{\min}, \infty) \tag{4.53}$$

exists. Fix a compact subset $K \subseteq M$ as in hypothesis (a). We choose a constant $E_0 > 0$ as in Lemma C.1, depending on K.

Claim 1 *We may assume w.l.o.g. that $z_0 = 0$ and that*

$$||e^{R_\nu}_{w_\nu}||_{C^0(\bar{B}_r)} = e^{R_\nu}_{w_\nu}(0). \tag{4.54}$$

Proof of Claim 1: Suppose that we have already proved the proposition under these additional assumptions, and let r, z_0, R_ν, w_ν be as in the hypotheses of the proposition. We choose $0 < \widehat{r} \leq r/4$ so small that

$$\lim_{\nu \to \infty} E^{R_\nu}(w_\nu, B_{4\widehat{r}}(z_0)) < m_0 + \frac{E_0}{2}. \tag{4.55}$$

For $\nu \in \mathbb{N}$ we choose $\widetilde{z}_\nu \in \bar{B}_{2\widehat{r}}(z_0)$ such that

$$e^{R_\nu}_{w_\nu}(\widetilde{z}_\nu) = ||e^{R_\nu}_{w_\nu}||_{C^0(\bar{B}_{2\widehat{r}}(z_0))}. \tag{4.56}$$

Claim 2 *The sequence $\widetilde{z}_\nu$ converges to z_0.*

Proof of Claim 2: Recall that

$$A(z, r_1, r_2) := \bar{B}_{r_1}(z) \setminus B_{r_2},$$

denotes the closed annulus of radii $0 \le r_1 \le r_2 \le \infty$ with center $z \in \mathbb{C}$. Fix $0 < \varepsilon \le 2\widehat{r}$. By hypothesis (b) there exists a number $\nu(\varepsilon) \in \mathbb{N}$ such that for every $\nu \ge \nu(\varepsilon)$

$$E^{R_\nu}(w_\nu, B_{\varepsilon/2}) > m_0 - \frac{E_0}{2}. \tag{4.57}$$

By (4.55), by increasing $\nu(\varepsilon)$ we may also assume that for $\nu \ge \nu(\varepsilon)$

$$E^{R_\nu}(w_\nu, B_{4\widehat{r}}) < m_0 + \frac{E_0}{2}. \tag{4.58}$$

Let $z \in A(z_0, \varepsilon, 2\widehat{r})$ and $\nu \ge \nu(\varepsilon)$. Consider the 1-vortex

$$\widehat{w}_\nu := \big(u_\nu(R_\nu^{-1}\cdot), R_\nu^{-1}\Phi_\nu(R_\nu^{-1}\cdot), R_\nu^{-1}\Psi_\nu(R_\nu^{-1}\cdot)\big) \in W^{1,p}(B_{rR_\nu}, M \times \mathfrak{g} \times \mathfrak{g}).$$

We check the condition (C.2) of the a priori Lemma C.1 with $w := \widehat{w}_\nu$ and z_0, r replaced by $R_\nu z$ and $R_\nu\varepsilon/2$. We have

$$\begin{aligned} E(\widehat{w}_\nu, B_{R_\nu\varepsilon/2}(R_\nu z)) &= E^{R_\nu}(w_\nu, B_{\varepsilon/2}(z)) \\ &\le E^{R_\nu}(w_\nu, A(\varepsilon/2, 4\widehat{r})) < E_0. \end{aligned}$$

Here the last inequality follows from (4.57) and (4.58). So condition (C.2) is satisfied and thus Lemma C.1 implies that

$$\begin{aligned} e^{R_\nu}_{w_\nu}(z) &= R_\nu^2 e_{\widehat{w}_\nu}(R_\nu z) \\ &\le \frac{32}{\pi\varepsilon^2} E(\widehat{w}_\nu, B_{R_\nu\varepsilon/2}(R_\nu z)) \\ &< \frac{32E_0}{\pi\varepsilon^2}. \end{aligned} \tag{4.59}$$

We choose a number

$$0 < \delta \le \min\left\{2\widehat{r}, \varepsilon \cdot \sqrt{\frac{m_0}{64E_0}}\right\}.$$

By assumption (b), increasing $\nu(\varepsilon)$ we may assume that $E^{R_\nu}(w_\nu, B_\delta(z_0)) > m_0/2$ for $\nu \ge \nu(\varepsilon)$. Let $\nu \ge \nu(\varepsilon)$. Then

$$\begin{aligned} ||e^{R_\nu}_{w_\nu}||^2_{C^0(\bar{B}_\delta(z_0))} &\ge \frac{1}{\pi\delta^2} E^{R_\nu}(w_\nu, B_\delta(z_0)) \\ &> \frac{m_0}{2\pi\delta^2} \ge \frac{32E_0}{\pi\varepsilon^2}. \end{aligned} \tag{4.60}$$

Since inequality (4.59) holds for $z \in A(z_0, \varepsilon, 2\widehat{r})$, it follows from (4.56) and (4.60) that $\widetilde{z}_\nu \in B_\varepsilon(z_0)$. This proves Claim 2. □

By Claim 2 we may pass to some subsequence such that $|\widetilde{z}_\nu| < \widehat{r}$ for every ν. We define now the map $\widehat{w}_\nu : B_{\widehat{r}} \to M \times \mathfrak{g} \times \mathfrak{g}$ by $\widehat{w}_\nu(z) := w_\nu(z + \widetilde{z}_\nu)$.

Then (4.54) with w_ν, r replaced by $\widehat{w}_\nu, \widehat{r}$ is satisfied. We check the hypotheses of Proposition 4.10 with (w_ν, r, z_0) replaced by $(\widehat{w}_\nu, \widehat{r}, 0)$. Hypothesis (a) for the sequence $\widehat{w}_\nu$ holds by the assumption for the sequence w_ν. In order to see that condition (b) holds for the sequence $\widehat{w}_\nu$, we show that for every $0 < \varepsilon \leq \widehat{r}$ we have

$$\liminf_{\nu\to\infty} E^{R_\nu}(\widehat{w}_\nu, B_\varepsilon) \geq \lim_{\nu\to\infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)) \geq \limsup_{\nu\to\infty} E^{R_\nu}(\widehat{w}_\nu, B_\varepsilon). \tag{4.61}$$

Let $0 < \varepsilon \leq \widehat{r}$. We fix a number $0 < \delta < \varepsilon$. Then for ν so large that $|\widetilde{z}_\nu - z_0| \leq \delta$ we have

$$B_{\varepsilon-\delta}(z_0) \subseteq B_\varepsilon(z_\nu).$$

It follows that

$$\lim_{\nu\to\infty} E^{R_\nu}(w_\nu, B_{\varepsilon-\delta}(z_0)) \leq \liminf_{\nu\to\infty} E^{R_\nu}(\widehat{w}_\nu, B_\varepsilon).$$

Since this holds for every $0 < \delta < \varepsilon$ and by hypothesis (b) for the sequence w_ν the map

$$(0, r] \ni \varepsilon \mapsto \lim_{\nu\to\infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)) \in [\mathrm{E}_{\min}, \infty)$$

is continuous, it follows that

$$\lim_{\nu\to\infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)) \leq \liminf_{\nu\to\infty} E^{R_\nu}(\widehat{w}_\nu, B_\varepsilon).$$

Analogously

$$\lim_{\nu\to\infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)) \geq \limsup_{\nu\to\infty} E^{R_\nu}(\widehat{w}_\nu, B_\varepsilon),$$

and thus (4.61) holds. This implies that for every $0 < \varepsilon \leq \widehat{r}$ the limit $\lim_{\nu\to\infty} E^{R_\nu}(\widehat{w}_\nu, B_\varepsilon)$ exists and equals $\lim_{\nu\to\infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0))$ and thus implies condition (b) for the sequence $\widehat{w}_\nu$.

Assuming that we have already proved the Proposition for $\widehat{w}_\nu$, passing to some subsequence, there exist a value $R_0 \in \{1, \infty\}$, a finite subset $Z \subseteq \mathbb{C}$, an R_0-vortex $w_0 \in W^{1,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, M \times \mathfrak{g} \times \mathfrak{g})$ and sequences $\varepsilon_\nu > 0$ and $\widehat{z}_\nu \in \mathbb{C}$ such that conditions (i)-(v) of Proposition 4.10 hold with (w_ν, z_ν, z_0) replaced by $(\widehat{w}_\nu, \widehat{z}_\nu, 0)$. Setting $z_\nu := \widehat{z}_\nu + \widetilde{z}_\nu$, it follows that conditions (i)-(iii) and (v) hold for the tuple $(w_\nu, R_0, Z, w_0, (\varepsilon_\nu, z_\nu))$. To see that also (iv) is satisfied for the sequence w_ν, fix a number $R \geq \widehat{r}^{-1}$. Since by Claim 2 the sequence $\widetilde{z}_\nu$ converges to z_0, we have for ν large enough

$$B_{R^{-1}}(z_0) \subseteq B_{2/R}(\widetilde{z}_\nu),$$

and therefore

$$\begin{aligned} E^{R_\nu}\big(w_\nu, B_{R^{-1}}(z_0) \setminus B_{R\varepsilon_\nu}(z_\nu)\big) &\leq E^{R_\nu}\big(w_\nu, B_{2/R}(\widetilde{z}_\nu) \setminus B_{(R/2)\varepsilon_\nu}(z_\nu)\big) \\ &= E^{R_\nu}\big(\widehat{w}_\nu, B_{2/R} \setminus B_{(R/2)\varepsilon_\nu}(\widehat{z}_\nu)\big). \end{aligned}$$

Taking $\limsup_{\nu\to\infty}$ and then $R \to \infty$ and using condition (iv) for the sequence $\widehat{w}_\nu$, we get

$$\begin{aligned}&\lim_{R\to\infty}\limsup_{\nu\to\infty} E^{R_\nu}\big(w_\nu, B_{R^{-1}}(z_0)\setminus B_{R\varepsilon_\nu}(z_\nu)\big)\\ &\le \lim_{R\to\infty}\limsup_{\nu\to\infty} E^{R_\nu}\big(\widehat{w}_\nu, B_{2/R}\setminus B_{(R/2)\varepsilon_\nu}(\widehat{z}_\nu)\big) = 0. \qquad (4.62)\end{aligned}$$

It follows that the assertion of Proposition 4.10 for w_ν is satisfied. This proves Claim 1. □

So we assume w.l.o.g. that $z_0 = 0$ and that (4.54) holds. Recalling that we have chosen $E_0 > 0$ as in the a-priori Lemma C.1, we fix any constant $\delta > 0$ smaller than $\mathrm{E}_{\min}$ and $E_0/2$. By assumption (b) we have for ν large enough

$$E^{R_\nu}(w_\nu, B_r(z_0)) > m_0 - \delta. \qquad (4.63)$$

We pass to some subsequence such that (4.63) holds for every $\nu \in \mathbb{N}$ and choose $0 < \widehat{\varepsilon}_\nu < r$ such that

$$E^{R_\nu}(w_\nu, B_{\widehat{\varepsilon}_\nu}) = m_0 - \delta. \qquad (4.64)$$

We claim that

$$\widehat{\varepsilon}_\nu \to 0. \qquad (4.65)$$

To see this let $\varepsilon > 0$. By assumption (b) there is a number $\nu_0 \in \mathbb{N}$ such that $E^{R_\nu}(w_\nu, B_\varepsilon) > m_0 - \delta$ for $\nu \ge \nu_0$. By (4.64) it follows that $\widehat{\varepsilon}_\nu < \varepsilon$ for $\nu \ge \nu_0$. This proves the claim.

Abbreviating

$$C_\nu := e^{R_\nu}_{w_\nu}(0),$$

we have by (4.54)

$$E^{R_\nu}(w_\nu, B_{\widehat{\varepsilon}_\nu}) \le \pi\widehat{\varepsilon}_\nu^2 ||e^{R_\nu}_{w_\nu}||_{C^0(\bar{B}_{\widehat{\varepsilon}_\nu})} \le \pi\widehat{\varepsilon}_\nu^2 C_\nu. \qquad (4.66)$$

It follows that

$$\inf_\nu \widehat{\varepsilon}_\nu^2 R_\nu^2 \ge \inf_\nu \frac{E^{R_\nu}(w_\nu, B_{\widehat{\varepsilon}_\nu})R_\nu^2}{\pi C_\nu} = \frac{m_0-\delta}{\pi}\inf_\nu \frac{R_\nu^2}{C_\nu}. \qquad (4.67)$$

Here the first inequality follows from (4.66) and the second step uses (4.64). We apply now Proposition 4.6 (Hard rescaling) with $\Omega := B_r$ and the sequences R_ν and w_ν. Note that the hypotheses of this Proposition are satisfied, since by assumption $R_\nu \to \infty$ and because of hypotheses (a) and (b). Thus by assertion (i) of that proposition with $Q := \{0\}$ we have

$$\inf_\nu \frac{R_\nu^2}{C_\nu} > 0. \qquad (4.68)$$

Since $m_0 \geq \mathrm{E}_{\min}$ and $\delta < \mathrm{E}_{\min}$, it follows from (4.67) and (4.68) that

$$\inf_{\nu} \widehat{\varepsilon}_\nu^2 R_\nu^2 > 0. \tag{4.69}$$

Therefore, passing to some subsequence, we may assume that the limit

$$\widehat{R}_0 := \lim_{\nu\to\infty} \widehat{\varepsilon}_\nu R_\nu \in (0,\infty] \tag{4.70}$$

exists. We define

$$\begin{aligned} (R_0, \varepsilon_\nu) &:= \begin{cases} (\infty, \widehat{\varepsilon}_\nu), & \text{if } \widehat{R}_0 = \infty, \\ (1, R_\nu^{-1}), & \text{otherwise,} \end{cases} \\ \widetilde{R}_\nu &:= \varepsilon_\nu R_\nu. \end{aligned} \tag{4.71}$$

and $\widetilde{R}_\nu := \varepsilon_\nu R_\nu$. Consider the sequence of $\widetilde{R}_\nu$-vortices $\widetilde{w}_\nu \in W^{1,p}_{\mathrm{loc}}(B_{r/\varepsilon_\nu}, M \times \mathfrak{g} \times \mathfrak{g})$ defined by

$$\widetilde{w}_\nu(z) := (\widetilde{u}_\nu, \widetilde{\Phi}_\nu, \widetilde{\Psi}_\nu)(z) := \big(u_\nu(\varepsilon_\nu z), \varepsilon_\nu \Phi_\nu(\varepsilon_\nu z), \varepsilon_\nu \Psi_\nu(\varepsilon_\nu z)\big). \tag{4.72}$$

By Proposition 4.3 with R_ν, w_ν replaced by $\widetilde{R}_\nu$, $\widetilde{w}_\nu$ and $r_\nu := r/\varepsilon_\nu$ there exist a finite subset $Z \subseteq \mathbb{C}$ and an R_0-vortex $w_0 = (u_0, \Phi_0, \Psi_0) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, M \times \mathfrak{g} \times \mathfrak{g})$ such that passing to some subsequence conditions (i)-(iii) of Proposition 4.3 hold.

We check the conditions (i)-(v) of Proposition 4.10 with $z_\nu := z_0 := 0$.

Condition 4.10(i) holds by the definition (4.71) of ε_ν, by (4.65) and by (4.70). Condition 4.10(ii) follows from 4.3(ii) and condition 4.10(iii) follows from 4.3(iii).

We prove condition 4.10(iv). We choose constants $E_1 > 0$, $a > 0$ and $C_1 > 0$ as in Lemma 4.11, corresponding to K and

$$r_0 := \inf_\nu \varepsilon_\nu R_\nu.$$

By inequality (4.69) the number r_0 is positive. We choose numbers $0 < \varepsilon < E_1/2$ and

$$\lambda > \max\left\{2, \frac{1}{r}, \left(\frac{C_1 E_1}{\varepsilon}\right)^{\frac{1}{a}}\right\}, \tag{4.73}$$

such that

$$\lim_{\nu\to\infty} E^{R_\nu}(w_\nu, B_{\lambda^{-1}}) < m_0 + \varepsilon.$$

We choose an index $\nu_0 \in \mathbb{N}$ so large that for $\nu \geq \nu_0$ we have

$$\varepsilon_\nu < \lambda^{-3} \tag{4.74}$$

$$E^{R_\nu}(w_\nu, B_{\lambda^{-1}}) < m_0 + \varepsilon. \tag{4.75}$$

Fix $\nu \geq \nu_0$. We check the requirements of Lemma 4.11 with r, R, w_ν replaced by $\varepsilon_\nu R_\nu$, $\lambda^{-1} R_\nu$ and

$$\widehat{w}_\nu := (u_\nu, R_\nu^{-1}\Phi_\nu, R_\nu^{-1}\Psi_\nu)(R_\nu^{-1}\cdot).$$

Since $\lambda > 2$ we have by (4.74) that $\varepsilon_\nu R_\nu < \lambda^{-1} R_\nu$, and therefore

$$\begin{aligned} E\big(\widehat{w}_\nu, A(\varepsilon_\nu R_\nu, \lambda^{-1} R_\nu)\big) &= E^{R_\nu}(w_\nu, A(\varepsilon_\nu, \lambda^{-1})) \\ &= E^{R_\nu}(w_\nu, B_{\lambda^{-1}}) - E^{R_\nu}(w_\nu, B_{\varepsilon_\nu}) \\ &< m_0 + E_1/2 - m_0 + \delta < E_1. \end{aligned} \tag{4.76}$$

Here the third step uses (4.75) and (4.64). It follows that the requirements of Lemma 4.11 are satisfied, and thus by that Lemma

$$\begin{aligned} E^{R_\nu}(w_\nu, A(\lambda\varepsilon_\nu, \lambda^{-2})) &= E(\widehat{w}_\nu, A(\lambda R_\nu \varepsilon_\nu, \lambda^{-2} R_\nu)) \\ &\leq \frac{C_1 E(\widehat{w}_\nu, A(R_\nu \varepsilon_\nu, \lambda^{-1} R_\nu))}{\lambda^a} \\ &= \frac{C_1 E^{R_\nu}(w_\nu, A(\varepsilon_\nu, \lambda^{-1}))}{\lambda^a} \\ &< \frac{C_1 E_1}{\lambda^a} < \varepsilon. \end{aligned} \tag{4.77}$$

Here in the third step we use (4.76) and in the last step we use (4.73). It follows that for every $\varepsilon > 0$ there exists a $\lambda > 0$ such that

$$\limsup_{\nu\to\infty} E^{R_\nu}(w_\nu, A(\lambda\varepsilon_\nu, \lambda^{-2})) < \varepsilon.$$

Since $z_\nu = z_0 = 0$, it follows that **condition (iv) is satisfied.**

To see that **condition 4.10(v)** holds, assume first that $R_0 = 1$. Then $Z = \emptyset$ by 4.3(i). Since $\widehat{\varepsilon}_\nu R_\nu \to \widehat{R}_0$, for ν large enough we have $B_{2\widehat{R}_0/R_\nu} \supseteq B_{\widehat{\varepsilon}_\nu}$ and therefore

$$\begin{aligned} E(\widetilde{w}_\nu, B_{2\widehat{R}_0}) &= E^{R_\nu}(w_\nu, B_{2\widehat{R}_0/R_\nu}) \\ &\geq E^{R_\nu}(w_\nu, B_{\widehat{\varepsilon}_\nu}) \\ &= m_0 - \delta > 0, \end{aligned} \tag{4.78}$$

where the third step follows from (4.64). By condition 4.3(ii) we have $Z = \emptyset$. Thus condition 4.3(ii) and Lemma 4.9 imply that

$$E(w_0, B_{2\widehat{R}_0}) = \lim_{\nu\to\infty} E(\widetilde{w}_\nu, B_{2\widehat{R}_0}) > 0.$$

Assume now that $R_0 = \infty$ and $E^\infty(w_0) = 0$.

Claim 3 *The set Z is not contained in the open ball B_1.*

Proof of Claim 3: By 4.10(iv) there is $\lambda > 0$ so that

$$\limsup_{\nu\to\infty} E^{R_\nu}(w_\nu, A(z_\nu, \lambda\varepsilon_\nu, \lambda^{-1})) < \delta. \tag{4.79}$$

Since $R_0 = \infty$, we have $\widehat{\varepsilon}_\nu = \varepsilon_\nu$. Therefore by (4.64)

$$E^{R_\nu}(w_\nu, A(\varepsilon_\nu, \lambda^{-1})) = E^{R_\nu}(w_\nu, B_{\lambda^{-1}}) - m_0 + \delta,$$

for every ν, and hence by the definition (4.53) of m_0

$$\lim_{\nu\to\infty} E^{R_\nu}(w_\nu, A(\varepsilon_\nu, \lambda^{-1})) \geq m_0 - m_0 + \delta = \delta.$$

Together with (4.79) this implies that

$$\begin{aligned}\liminf_{\nu\to\infty} E^{\widetilde{R}_\nu}(\widetilde{w}_\nu, A(1,\lambda)) &= \liminf_{\nu\to\infty} E^{R_\nu}(w_\nu, A(\varepsilon_\nu, \lambda\varepsilon_\nu)) \\ &= \lim_{\nu\to\infty} E^{R_\nu}(w_\nu, A(\varepsilon_\nu, \lambda^{-1})) \\ &\quad - \limsup_{\nu\to\infty} E^{R_\nu}(w_\nu, A(\lambda\varepsilon_\nu, \lambda^{-1})) > 0. \end{aligned} \tag{4.80}$$

Suppose by contradiction that $Z \subseteq B_1$. Then by 4.3(ii) there exists a sequence of gauge transformations $g_\nu \in W^{2,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, \mathrm{G})$ such that the maps $g_\nu^{-1}\widetilde{u}_\nu$ converge to u_0 in $C^1(A(1,\lambda))$ and the maps $g_\nu^*(\widetilde{\Phi}_\nu, \widetilde{\Psi}_\nu)$ converge to (Φ_0, Ψ_0) in $C^0(A(1,\lambda))$. By Lemma 4.9 it follows that

$$\begin{aligned} 0 &= E^{R_0}(w_0) \\ &\geq E^{R_0}(w_0, A(1,\lambda)) \\ &= \lim_{\nu\to\infty} E^{\widetilde{R}_\nu}(\widetilde{w}_\nu, A(1,\lambda)) > 0. \end{aligned}$$

Here in the last step we have used (4.80). This contradiction proves Claim 3. □

We claim that $0 \in Z$. To see this, observe that by Claim 3 $Z \setminus B_1$ is nonempty. We choose a point $z \in Z \setminus B_1$ and a number $\varepsilon_0 > 0$ so small that $B_{\varepsilon_0}(z) \cap Z = \{z\}$ and fix $0 < \varepsilon < \varepsilon_0$. Condition (iii) implies that there is a ν_0 such that for $\nu \geq \nu_0$

$$E^{\widetilde{R}_\nu}(\widetilde{w}_\nu, B_\varepsilon(z)) \geq \frac{\mathrm{E}_{\min}}{2}.$$

Increasing ν_0 we may assume that $\varepsilon\varepsilon_\nu < r$ for $\nu \geq \nu_0$. We fix an index $\nu \geq \nu_0$. By the definition (4.72) of $\widetilde{w}_\nu$ and by (4.54) we have

$$e^{\widetilde{R}_\nu}_{\widetilde{w}_\nu}(0) = ||e^{\widetilde{R}_\nu}_{\widetilde{w}_\nu}||_{C^0(\bar{B}_\varepsilon(z))}.$$

It follows that

$$e_{\widetilde{w}_\nu}^{\widetilde{R}_\nu}(0)^2 \geq \frac{E^{\widetilde{R}_\nu}(\widetilde{w}_\nu, B_\varepsilon(z))}{\pi\varepsilon^2} \geq \frac{\mathrm{E}_{\min}}{2\pi\varepsilon^2}.$$

Therefore, given any number $C > 0$, choosing

$$0 < \varepsilon \leq \frac{\sqrt{\mathrm{E}_{\min}}}{\sqrt{2\pi C}},$$

there is an integer ν_0 such that for every $\nu \geq \nu_0$

$$e_{\widetilde{w}_\nu}^{\widetilde{R}_\nu}(0) \geq C.$$

This means that

$$e_{\widetilde{w}_\nu}^{\widetilde{R}_\nu}(0) \to \infty. \tag{4.81}$$

If 0 did not belong to Z, then choosing $\varepsilon > 0$ sufficiently small, by 4.3(ii) with $Q := \{0\}$ and Lemma 4.9 with $\Omega := B_\varepsilon$ and $Q := \{0\}$ we would have $e_{\widetilde{w}_\nu}^{\widetilde{R}_\nu}(0) \to e_{w_0}^{\infty}(0)$, a contradiction to (4.81). So indeed $0 \in Z$. Since by Claim 3 the set Z is not contained in B_1, it follows that Z contains at least 2 points. Thus condition (v) of Proposition 4.10 is satisfied.

This concludes the proof of Proposition 4.10. □

Remark 4.13 In the proof of condition (v) of Proposition 4.10, assuming that $R_0 = \infty$ and $E^\infty(w_0) = 0$, we showed that $Z \not\subseteq B_1$. We can say more than that. Namely, since for every $R > 1$

$$\lim_{\nu\to\infty} E^{\widetilde{R}_\nu}(\widetilde{w}_\nu, A(1,R)) = \lim_{\nu\to\infty} E^{\widetilde{R}_\nu}(\widetilde{w}_\nu, B_R) - m_0 + \delta \leq m_0 - m_0 + \delta < \mathrm{E}_{\min},$$

it follows that no point of Z lies outside the *closed* ball $\bar{B}_1$. So at least one point of Z lies on the circle S^1.

The next lemma is Exercise 5.1.2. in the book [MS3].

Lemma 4.14 *Let $k \in \mathbb{N} \cup \{0\}$ be a number, (T, E) be a finite tree, $\alpha_1, \ldots, \alpha_k \in T$ be vertices, $f : T \to [0, \infty)$ be a function and $E_0 > 0$ be a number. Assume that for every vertex $\alpha \in T$ we have*

$$f(\alpha) \geq E_0 \qquad \text{or} \qquad \#\{\beta \in T \,|\, \alpha E \beta\} + \#\{i \in \{1, \ldots, k\} \,|\, \alpha_i = \alpha\} \geq 3. \tag{4.82}$$

Then

$$\#T \leq \frac{2\sum_{\alpha\in T} f(\alpha)}{E_0} + k. \tag{4.83}$$

Proof of Lemma 4.14: If T is empty or $\#T = 1$ then (4.83) is true. So assume that $\#T \geq 2$. For $\alpha \in T$ we denote

$$n_\alpha := \#\{\beta \in T \mid \alpha E \beta\} + \#\{i \in \{1, \dots, k\} \mid \alpha_i = \alpha\}.$$

Since $\#T \geq 2$, we have $n_\alpha \geq 1$ for every $\alpha \in T$. Since the number of unoriented edges of T is $\#T - 1$, we have

$$\sum_{\alpha \in T} n_\alpha = \sum_{\alpha \in T} \#\{\beta \in T \mid \alpha E \beta\} + k = 2\#T - 2 + k.$$

Therefore

$$\begin{aligned}
\#T &= 3\#T - \sum_{\alpha \in T} n_\alpha + k - 2 \\
&= \sum_{\alpha \in T} (3 - n_\alpha) + k - 2 \\
&= \sum_{\alpha : f(\alpha) = 0} (3 - n_\alpha) + \sum_{\alpha : f(\alpha) \geq E_0} (3 - n_\alpha) + k - 2 \\
&\leq 0 + \sum_{\alpha : f(\alpha) \geq E_0} 2 + k - 2 \\
&\leq \frac{2 \sum_{\alpha \in T} f(\alpha)}{E_0} + k - 2.
\end{aligned}$$

Here in the forth line we have used the hypothesis $n_\alpha \geq 3$ if $f(\alpha) = 0$, and that $n_\alpha \geq 1$ for every $\alpha \in T$. This proves Lemma 4.14. □

Recall the definition (4.42) of $\bar{d}$.

Lemma 4.15 *Let $r_0 > 0$ be a number, $\bar{u} : \mathbb{C} \cup \{\infty\} \setminus B_{r_0} \to M/\mathrm{G}$ be a continuous map, $R_\nu > 0$ be a sequence of numbers, and let $w_\nu = (u_\nu, \Phi_\nu, \Psi_\nu) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C} \setminus B_{r_0}, M \times \mathfrak{g} \times \mathfrak{g})$ be an R_ν-vortex for $\nu \in \mathbb{N}$. Assume that $\inf_\nu R_\nu > 0$, and that there is a compact subset $K \subseteq M$ such that $u_\nu(\mathbb{C}) \subseteq K$ for every ν and that $\mathrm{G}u_\nu(z) \to \bar{u}(z)$ for every $z \in \mathbb{C} \setminus B_{r_0}$. Then the following conditions hold.*

(A) Suppose also that

$$\lim_{R \to \infty} \limsup_{\nu \to \infty} E^{R_\nu}(w_\nu, \mathbb{C} \setminus B_R) = 0. \tag{4.84}$$

Then for every $\varepsilon > 0$ there are numbers $R \geq r_0$ and $\nu_0 \in \mathbb{N}$ such that for every $\nu \geq \nu_0$ and every $z \in \mathbb{C} \setminus B_R$ we have

$$\bar{d}(\bar{u}(\infty), \mathrm{G}u_\nu(z)) < \varepsilon. \tag{4.85}$$

(B) Let now in addition $z_0 \in \mathbb{C}$ *be a point,* $\bar{u}' : B_{r_0^{-1}}(z_0) \to M/\mathrm{G}$ *be another continuous map and* λ_ν, $z_\nu \in \mathbb{C}$ *be sequences of numbers. Assume that* $\mathrm{G}u_\nu(\lambda_\nu(z - z_\nu)) \to \bar{u}'(z)$ *for every* $z \in B_{r_0^{-1}}(z_0) \setminus \{z_0\}$*, and that*

$$\lambda_\nu \to \infty, \tag{4.86}$$

$$z_\nu \to z_0, \tag{4.87}$$

$$\lim_{R\to\infty} \limsup_{\nu\to\infty} E^{R_\nu}\big(w_\nu, B_{|\lambda_\nu|/R}(\lambda_\nu(z_0 - z_\nu)) \setminus B_R\big) = 0. \tag{4.88}$$

Then

(i) For every $\varepsilon > 0$ *there are numbers* $R > r_0$ *and* $\nu_0 \in \mathbb{N}$ *so that for every* $\nu \geq \nu_0$ *and every* $z \in \bar{B}_{|\lambda_\nu|/R}(\lambda_\nu(z_0 - z_\nu)) \setminus B_R$ *we have*

$$\bar{d}(\bar{u}(\infty), \mathrm{G}u_\nu(z)) + \bar{d}(\bar{u}'(z_0), \mathrm{G}u_\nu(z)) < \varepsilon. \tag{4.89}$$

(ii)

$$\bar{u}(\infty) = \bar{u}'(z_0).$$

Proof of Lemma 4.15: Let r_0, $\bar{u}$, R_ν, w_ν and K be as in the hypothesis. Let E_1, C_1 and $a > 0$ be constants as in Lemma 4.11, corresponding to K and r_0 replaced by $R/2\,\inf_\nu R_\nu$.

We prove (A). Assume that (4.84) holds. Let $\varepsilon > 0$ be a number. We choose $R > r_0$ so large that

$$\limsup_{\nu\to\infty} E^{R_\nu}(w_\nu, \mathbb{C} \setminus B_{R/2}) < \min\left\{E_1, \frac{4^a\varepsilon^2}{9C_1^2}\right\}, \tag{4.90}$$

$$\bar{d}(\bar{u}(\infty), \bar{u}(R)) < \frac{\varepsilon}{3}. \tag{4.91}$$

We choose $\nu_0 \in \mathbb{N}$ so large that for $\nu \geq \nu_0$

$$E^{R_\nu}(w_\nu, \mathbb{C} \setminus B_{R/2}) < \min\left\{E_1, \frac{4^a\varepsilon^2}{9C_1^2}\right\}. \tag{4.92}$$

By assumption $\mathrm{G}u_\nu(R)$ converges to $\bar{u}(R)$. Thus increasing ν_0 we may assume that for $\nu \geq \nu_0$

$$\bar{d}(\bar{u}(R), \mathrm{G}u_\nu(R)) < \frac{\varepsilon}{3}. \tag{4.93}$$

We fix $\nu \geq \nu_0$. Then by (4.92)

$$E(w_\nu(\cdot/R_\nu), \mathbb{C} \setminus B_{RR_\nu/2}) = E^{R_\nu}(w_\nu, \mathbb{C} \setminus B_{R/2}) < E_1,$$

and hence the hypotheses of the annulus Lemma 4.11 with $w := w_\nu(\cdot/R_\nu)$ and r, R replaced by $RR_\nu/2, \infty$ are satisfied. Setting $\lambda := 2$ it follows that for each two points $z, z' \in \mathbb{C} \setminus B_R$ we have

$$\bar{d}\big(\mathrm{G}u_\nu(z), \mathrm{G}u_\nu(z')\big) \leq \frac{C_1\sqrt{E(w, \mathbb{C} \setminus B_{RR_\nu/2})}}{2^a} < \frac{\varepsilon}{3}. \tag{4.94}$$

Here in the last step we have used (4.92). Setting $z' := R$ we get, combining (4.91), (4.93) and (4.94),

$$\begin{aligned}\bar{d}(\bar{u}(\infty), \mathrm{G}u_\nu(z)) &\leq \bar{d}(\bar{u}(\infty), \bar{u}(R)) + \bar{d}(\bar{u}(R), \mathrm{G}u_\nu(R)) \\ &\quad + \bar{d}(\mathrm{G}u_\nu(R), \mathrm{G}u_\nu(z)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

and hence (4.85) holds. **This proves (A)**.

We prove (B). Let z_0, $\bar{u}'$, λ_ν and z_ν be as required, satisfying (4.86), (4.87) and (4.88). We show that **(i)** holds. Let $\varepsilon > 0$. We choose $R > 2r_0$ so large that

$$\limsup_{\nu\to\infty} E^{R_\nu}\Big(w_\nu, B_{7|\lambda_\nu|/R}(\lambda_\nu(z_0 - z_\nu)) \setminus B_{R/2}\Big) < \min\left\{E_1, \frac{4^a\varepsilon^2}{36C_1^2}\right\}, \quad (4.95)$$

$$\bar{d}(\bar{u}(\infty), \bar{u}(R)) < \frac{\varepsilon}{6}, \quad (4.96)$$

$$\bar{d}\Big(\bar{u}'(z_0), \bar{u}'\big(z_0 + \frac{2}{R}\big)\Big) < \frac{\varepsilon}{6}. \quad (4.97)$$

We choose $\nu_0 \in \mathbb{N}$ so large that for $\nu \geq \nu_0$ we have

$$E^{R\nu}\Big(w_\nu, B_{7|\lambda_\nu|/R}\big(\lambda_\nu(z_0 - z_\nu)\big)\big) \setminus B_{R/2}\Big) < \min\left\{E_1, \frac{4^a\varepsilon^2}{36C_1^2}\right\}. \quad (4.98)$$

By (4.87) and (4.86), increasing ν_0 we may assume that for $\nu \geq \nu_0$

$$\begin{aligned}|z_0 - z_\nu| &< \frac{1}{R}, &(4.99)\\ |\lambda_\nu| &\geq R^2. &(4.100)\end{aligned}$$

Let $\nu \geq \nu_0$. Then (4.99) implies that $B_{6|\lambda_\nu|/R} \subseteq B_{7|\lambda_\nu|/R}(\lambda_\nu(z_0 - z_\nu))$. Therefore, setting

$$\widetilde{w}_\nu(z) := (\widetilde{u}_\nu, \widetilde{\Phi}_\nu, \widetilde{\Psi}_\nu)(z) := \big(u_\nu, \Phi_\nu/R_\nu, \Psi_\nu/R_\nu\big)(z/R_\nu),$$

we have

$$\begin{aligned}E\big(w, A(RR_\nu/2, 6R_\nu|\lambda_\nu|/R)\big) &= E^{R_\nu}(w_\nu, A(R/2, 6|\lambda_\nu|/R)) \\ &\leq E^{R\nu}\Big(w_\nu, B_{7|\lambda_\nu|/R}\big(\lambda_\nu(z_0 - z_\nu)\big)\big) \setminus B_{R/2}\Big) \\ &< \min\left\{E_1, \frac{4^a\varepsilon^2}{36C_1^2}\right\}. \quad (4.101)\end{aligned}$$

Here in the last step we have used (4.98). It follows that the hypotheses of the annulus Lemma 4.11 with r, R replaced by $RR_\nu/2$, $6R_\nu|\lambda_\nu|/R$ are satisfied. Hence by that Lemma, setting $\lambda := 2$ we have for each two points $z, z' \in A(RR_\nu, 3R_\nu|\lambda_\nu|/R)$

$$\begin{aligned}\bar{d}\big(\mathrm{G}u_\nu(z/R_\nu), \mathrm{G}u_\nu(z'/R_\nu)\big) &= \bar{d}(\mathrm{G}\widetilde{u}_\nu(z), \mathrm{G}\widetilde{u}_\nu(z')) \\ &\leq \frac{C_1\sqrt{E\big(\widetilde{w}_\nu, A(RR_\nu/2, 6R_\nu|\lambda_\nu|/R)\big)}}{2^a} \\ &< \frac{\varepsilon}{6}. \qquad (4.102)\end{aligned}$$

Here in the last step we have used (4.101).

By assumption $\mathrm{G}u_\nu(R)$ converges to $\bar{u}(R)$ and $\mathrm{G}u_\nu\big(\lambda_\nu(z_0 + \frac{2}{R} - z_\nu)\big)$ converges to $\bar{u}'\big(z_0 + \frac{2}{R}\big)$. Hence increasing ν_0 further we may assume that for $\nu \geq \nu_0$

$$\bar{d}(\bar{u}(R), \mathrm{G}u_\nu(R)) < \frac{\varepsilon}{6}, \qquad (4.103)$$

$$\bar{d}\Big(\bar{u}'\big(z_0 + \frac{2}{R}\big), \mathrm{G}u_\nu\big(\lambda_\nu(z_0 + \frac{2}{R} - z_\nu)\big)\Big) < \frac{\varepsilon}{6}. \qquad (4.104)$$

Let $\nu \geq \nu_0$. We set $z' := R$. Combining (4.96), (4.102) and (4.103) we get for every $z \in A(R, 3|\lambda_\nu|/R)$

$$\begin{aligned}\bar{d}(\bar{u}(\infty), \mathrm{G}u_\nu(z)) &\leq \bar{d}\big(\bar{u}(\infty), \bar{u}(R)\big) + \bar{d}\big(\bar{u}(R), \mathrm{G}u_\nu(R)\big) \\ &\quad + \bar{d}\big(\mathrm{G}u_\nu(R), \mathrm{G}u_\nu(z)\big) \qquad (4.105) \\ &< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}. \qquad (4.106)\end{aligned}$$

Similarly, we set $z' := \lambda_\nu(z_0 + \frac{2}{R} - z_\nu)$. By (4.99) we have

$$\frac{1}{R} \leq \left| z_0 + \frac{2}{R} - z_\nu \right| \leq \frac{3}{R}.$$

Hence by (4.100) we have $z' \in A(R, 3|\lambda_\nu|/R)$. Therefore, the inequalities (4.97), (4.102) and (4.104) imply that

$$\begin{aligned}\bar{d}(\bar{u}'(z_0), \mathrm{G}u_\nu(z)) &\leq \bar{d}\big(\bar{u}'(z_0), \bar{u}'(z_0 + \frac{2}{R})\big) + \bar{d}\big(\bar{u}'(z_0 + \frac{2}{R}), \mathrm{G}u_\nu(z')\big) \\ &\quad + \bar{d}(\mathrm{G}u_\nu(z'), \mathrm{G}u_\nu(z)) \\ &< \frac{\varepsilon}{2}, \qquad (4.107)\end{aligned}$$

for every $z \in A(R, 3|\lambda_\nu|/R)$. By (4.99) we have

$$\bar{B}_{|\lambda_\nu|/R}(\lambda_\nu(z_0 - z_\nu)) \setminus B_R \subseteq A(R, 3|\lambda_\nu|/R),$$

hence statement **(i)** follows from (4.107) and (4.106).

To see that statement **(ii)** holds, let $\varepsilon > 0$ be arbitrary. By **(i)** there are numbers $R > r_0$ and $\nu_0 \in \mathbb{N}$ such that (4.89) holds for every $z \in \bar{B}_{|\lambda_\nu|/R}(\lambda_\nu(z_0 - z_\nu)) \setminus B_R$ and every $\nu \geq \nu_0$. Choosing an integer $\nu \geq \nu_0$ so large that $z := R \in \bar{B}_{|\lambda_\nu|/R}(\lambda_\nu(z_0 - z_\nu))$, it follows that

$$\bar{d}(\bar{u}(\infty), \bar{u}'(z_0)) \leq \bar{d}(\bar{u}(\infty), \mathrm{G}u_\nu(R)) + \bar{d}(\mathrm{G}u_\nu(R), \bar{u}'(z_0)) < 2\varepsilon.$$

Since this holds for arbitrary $\varepsilon > 0$, it follows that $\bar{d}(\bar{u}(\infty), \bar{u}'(z_0)) = 0$, and therefore $\bar{u}(\infty) = \bar{u}'(z_0)$. This proves **(ii)** and therefore **(B)**.

This completes the proof of Lemma 4.15. □

We will use the following facts about sequences of Möbius transformations in the proof of the Bubbling Theorem in the case that there are at least two marked points.

Remark 4.16 Let $x, y \in S^2$ be points and φ_ν be a sequence of Möbius transformations that converges to y, uniformly on every compact subset of $S^2 \setminus \{x\}$. Then φ_ν^{-1} converges to x, uniformly on every compact subset of $S^2 \setminus \{y\}$. To see this, let $Q \subseteq S^2 \setminus \{y\}$ be a compact subset and $U \subseteq S^2$ be an open neighbourhood of x. Since φ_ν converges to y, uniformly on $S^2 \setminus U$, there exists an index ν_0 such that $\varphi_\nu(S^2 \setminus U) \subseteq S^2 \setminus Q$ for every $\nu \geq \nu_0$. This means that $\varphi_\nu^{-1}(Q) \subseteq U$. It follows that φ_ν^{-1} converges to x, uniformly on Q.

Lemma 4.17 *Let φ_ν be a sequence of Möbius transformations and $x, y \in S^2$ be points. Suppose there exist convergent sequences $x_1^\nu, x_2^\nu, y^\nu \in S^2$ such that*

$$\begin{aligned} x \neq \lim_{\nu\to\infty} x_1^\nu \neq \lim_{\nu\to\infty} x_2^\nu \neq x, &\qquad y \neq \lim_{\nu\to\infty} y^\nu, \\ \lim_{\nu\to\infty} \varphi_\nu(x_1^\nu) = \lim_{\nu\to\infty} \varphi_\nu(x_2^\nu) = y, &\qquad \lim_{\nu\to\infty} \varphi_\nu^{-1}(y^\nu) = x. \end{aligned}$$

Then φ_ν converges to y, uniformly with all derivatives on every compact subset of $S^2 \setminus \{x\}$.

Proof: This follows from Lemma D.1.4 and from Lemma 4.6.6 in the book [MS3]. □

Lemma 4.18 (Middle rescaling) *Let $x, x_\nu, y \in S^2$ be points and φ_ν be a sequence of Möbius transformations that converges to y, uniformly on compact subsets of $S^2 \setminus \{x\}$, such that x_ν converges to x and $\varphi_\nu(x_\nu)$ converges to y. Then there exists a sequence of Möbius transformations ψ_ν such that $\psi_\nu(1) = x_\nu$, ψ_ν converges to x, uniformly with all derivatives on compact subsets of $S^2 \setminus \{\infty\}$, and $\varphi_\nu \circ \psi_\nu$ converges to y, uniformly with all derivatives on compact subsets of $S^2 \setminus \{0\}$. Moreover, if $x' \neq x$ is any point in S^2 then we may choose ψ_ν such that $\psi_\nu(\infty) = x'$.*

Proof of Lemma 4.18: Let $x' \neq x$ and $y'' \neq y$ be any two points in S^2. We claim that for ν large enough the three points

$$x''_\nu := \varphi_\nu^{-1}(y''), \quad x_\nu, \quad x' \tag{4.108}$$

are all distinct. To see this, note that by hypothesis the point $\varphi_\nu(x_\nu)$ converges to y. Hence for ν large enough $\varphi_\nu(x_\nu) \neq y''$, i.e. $x_\nu \neq x''_\nu$. Since by hypothesis, x_ν converges to x, we have $x_\nu \neq x'$ for ν large enough. Furthermore, by Remark 4.16 the sequence φ_ν^{-1} converges to x, uniformly on compact subsets of $S^2 \setminus \{y\}$. It follows that x''_ν converges to x, and therefore $x''_\nu \neq x'$ for ν large enough. This proves the claim.

Let ν be any positive integer. If at least two of the three points x''_ν, x_ν and x' are equal then we choose ψ_ν to be any Möbius transformation such that $\psi_\nu(1) = x_\nu$. Otherwise, we define ψ_ν to be the unique Möbius transformation such that

$$\psi_\nu(0) = x''_\nu, \quad \psi_\nu(1) = x_\nu, \quad \psi_\nu(\infty) = x'.$$

Then the hypotheses of Lemma 4.17 with (φ_ν, x, y) replaced by (ψ_ν, ∞, x) and $x_1^\nu := 0$, $x_2^\nu := 1$ and $y_\nu := x'$ are satisfied. Hence by that Lemma the maps ψ_ν converge to x, uniformly with all derivatives on compact subsets of $S^2 \setminus \{\infty\}$. Moreover, the hypotheses of the same Lemma with (φ_ν, x) replaced by $(\varphi_\nu \circ \psi_\nu, 0)$ and $x_1^\nu := 1$, $x_2^\nu := \infty$ and $y_\nu := y''$ are satisfied. It follows that $\varphi_\nu \circ \psi_\nu$ converges to y, uniformly with all derivatives on compact subsets of $S^2 \setminus \{0\}$. This proves Lemma 4.18. □

4.4 Proof of Theorem 4.1 (Bubbling)

The proof of Theorem 4.1 combines Gromov-compactness with Uhlenbeck compactness. Uhlenbeck compactness enters the proof of Lemma 4.4 (Compactness with bounded energy density). The starting point in the proof of Gromov compactness for J-holomorphic curves is the following. Let (M, ω) be a compact symplectic manifold, J be an ω-compatible almost complex structure, and $u_\nu : S^2 \to M$ be a sequence of J-holomorphic curves such that the energies $E(u_\nu)$ are uniformly bounded above. In order to find the first bubble, we search for a point $z_0 \in S^2$ where the energy densities e_{u_ν} go to infinity. Then we rescale so much that the energy densities of the rescaled sequence stay bounded on each compact subset of S^2 minus one point. Using Hofer's Lemma, we can do this in such a way that the resulting bubble has positive energy. For details see the book by D. McDuff and D. A. Salamon [MS3].

Let now w_ν be a sequence of vortices as in Theorem 2. In order to find the first bubble in $\bar{M}$ or vortex on $\mathbb{C}$ we can try to do the same thing as for

J-holomorphic curves. However, this does not work. The problem is that bubbling does not occur at places in $\mathbb{C}$, but at ∞! This means that it can happen that

$$\lim_{R\to\infty}\limsup_{\nu\to\infty} E(w_\nu, \mathbb{C}\setminus B_R) > 0.$$

(Bubbling at places in $\mathbb{C}$ is excluded by hypothesis (H3). It would lead to J-holomorphic curves in M.) While for pseudo-holomorphic spheres we can apply a Möbius transformation in order to transform this to bubbling at some point in $\mathbb{C}$, this does not work for vortices. The problem is that a vortex on $\mathbb{C}$ does not extend to a map on S^2. (There are topological and analytical obstructions.) So the energy density of a vortex at infinity does not make sense! Note also that unlike the Cauchy-Riemann equations, the symplectic vortex equations are not invariant under biholomorphic maps of an open subset of $\mathbb{C}$ to another open subset. However, they transform according to formulae (B.8), (B.9) of section B.

The way out of this problem is the following. First, we rescale with factors that are so large that the energies of the rescaled vortices on $\mathbb{C}\setminus B_1(0)$ tend to zero. This ensures that no energy gets lost at ∞. Then we rescale back in order to catch the bubbles in $\bar{M}$ and the vortices on $\mathbb{C}$.

Proof of Theorem 4.1: We consider first the case $k=0$. Let $w_\nu \in \widetilde{\mathcal{M}}^{1,p}$ be a sequence of vortices with positive energy such that

$$\sup_\nu E(w_\nu) < \infty.$$

We prove that there exists a subsequence such that $(w_\nu, z_0^\nu := \infty)$ converges to some stable map with one marked point.

Passing to some subsequence we may assume that $E(w_\nu)$ converges to some constant $E \geq \mathrm{E}_{\min}$. We choose a sequence $R^\nu \geq 1$ such that

$$E(w^\nu, B_{R^\nu}) \to E. \tag{4.109}$$

and define $R_0^\nu := \nu R^\nu$ and

$$w_0^\nu := (u_0^\nu, \Phi_0^\nu, \Psi_0^\nu) := \big(u^\nu(R_0^\nu\cdot), R_0^\nu\Phi^\nu(R_0^\nu\cdot), R_0^\nu\Psi^\nu(R_0^\nu\cdot)\big).$$

We set $j_1 := 0$, $z_1 := 0$, $Z_0 := \{0\}$ and $z_0^\nu := 0$. Recall that an ∞-vortex on an open subset $\Omega \subseteq \mathbb{C}$ is a solution $(u,\Phi,\Psi) \in W^{1,p}_{\mathrm{loc}}(\Omega, M\times\mathfrak{g}\times\mathfrak{g})$ of (0.1) such that $\mu\circ u = 0$. If φ is a Möbius transformation and $w := (u, \Phi + i\Psi)$: $\mathbb{C} \to M\times\mathfrak{g}^{\mathbb{C}}$ is a map, then we define

$$\varphi^* w := \big(u\circ\varphi, \bar{\varphi}'\cdot((\Phi+i\Psi)\circ\varphi)\big) : S^2\setminus\varphi^{-1}(\infty) \to M\times\mathfrak{g}^{\mathbb{C}}.$$

Claim 1 *For every number $\ell \in \mathbb{N}$, passing to some subsequence, there exist a number $N := N(\ell) \in \mathbb{N}$, values $R_i \in \{1, \infty\}$, R_i-vortices $w_i = (u_i, \Phi_i, \Psi_i) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$, finite subsets $Z_i \subseteq \mathbb{C}$ and numbers $R_i^\nu > 0$, $z_i^\nu \in \mathbb{C}$ for $i = 1, \dots, N$, and numbers $j_i \in \{1, \dots, i-1\}$ and $z_i \in \mathbb{C}$ for $i = 2, \dots, N$, such that the following conditions hold.*

(i) For every $i = 2, \dots, N$ we have $z_i \in Z_{j_i}$. Moreover, if $i, i' \in \{2, \dots, N\}$ are such that $i \neq i'$ and $j_i = j_{i'}$ then $z_i \neq z_{i'}$.

(ii) Let $i = 1, \dots, N$. If $R_i = 1$ then $Z_i = \emptyset$ and $E(w_i) > 0$. If $R_i = \infty$ and $E^\infty(w_i) = 0$ then $|Z_i| \geq 2$.

(iii) Fix $i = 1, \dots, N$. If $R_i = 1$ then $R_i^\nu = 1$ for every ν, and if $R_i = \infty$ then $R_i^\nu \to \infty$. Furthermore,

$$\frac{R_i^\nu}{R_{j_i}^\nu} \to 0, \qquad \frac{z_i^\nu - z_{j_i}^\nu}{R_{j_i}^\nu} \to z_i. \tag{4.110}$$

In the following we set $\varphi_i^\nu(z) := R_i^\nu z + z_i^\nu$, for $i = 0, \dots, N$ and $\nu \in \mathbb{N}$.

(iv) For every $i = 1, \dots, N$ there exist gauge transformations $g_i^\nu \in \mathcal{G}^{2,p}$ such that for every compact subset $Q \subseteq \mathbb{C} \setminus Z_i$ the sequence $(g_i^\nu)^{-1} u^\nu \circ \varphi_i^\nu$ converges to u_i in $C^1(Q)$ and $(g_i^\nu)^*(\varphi_i^\nu)^*(\Phi_\nu, \Psi_\nu)$ converges to (Φ_i, Ψ_i) in $C^0(Q)$.

(v) Fix $i = 1, \dots, N$, $z \in Z_i$ and $\varepsilon_0 > 0$ so small that $B_{\varepsilon_0}(z) \cap Z_i = \{z\}$. Then for every $0 < \varepsilon < \varepsilon_0$ the limit

$$E_z(\varepsilon) := \lim_{\nu \to \infty} E^{R_i^\nu}\big((\varphi_i^\nu)^* w_\nu, B_\varepsilon(z)\big)$$

exists and $\mathrm{E}_{\min} \leq E_z(\varepsilon) < \infty$. Furthermore, the function $(0, \varepsilon_0) \ni \varepsilon \mapsto E_z(\varepsilon) \in [\mathrm{E}_{\min}, \infty)$ is continuous.

(vi) For $i = 1, \dots, N$

$$\lim_{R \to \infty} \limsup_{\nu \to \infty} E\big(w_\nu, B_{R_{j_i}^\nu / R}(z_{j_i}^\nu + R_{j_i}^\nu z_i) \setminus B_{R R_i^\nu}(z_i^\nu)\big) = 0.$$

(vii) If $\ell > N$ then for every $j = 1, \dots, N$ we have

$$Z_j = \{z_i \,|\, j < i \leq N, j_i = j\}.$$

Proof of Claim 1: We show that the statement holds for $\ell := 1$. We check the conditions of Proposition 4.10 (Soft rescaling) with $z_0 := 0$, $r := 1$ and R_ν, w_ν replaced by R_0^ν, w_0^ν. Condition 4.10(a) follows from assertion (B) of

Proposition D.6 (asymptotic behaviour). We check condition 4.10(b). Let $\varepsilon > 0$. Then for every $\nu \in \mathbb{N}$ such that $\nu \geq \varepsilon^{-1}$

$$E^{R_0^\nu}(w_0^\nu, B_\varepsilon) \geq E^{R_0^\nu}(w_0^\nu, B_{\frac{1}{\nu}}) = E(w_\nu, B_{R_\nu}). \tag{4.111}$$

On the other hand

$$E^{R_0^\nu}(w_0^\nu, B_\varepsilon) \leq E^{R_0^\nu}(w_0^\nu, \mathbb{C}) = E(w_\nu).$$

Therefore, since $E(w_\nu) \to E$, $E(w^\nu, B_{R^\nu}) \to E$ and $R_0^\nu = \nu R_\nu$, it follows that

$$\lim_{\nu\to\infty} E^{R_0^\nu}(w_0^\nu, B_\varepsilon) = E.$$

By Lemma D.1 (quantization of energy) we have $E(w_\nu) \geq \mathrm{E}_{\min}$ for every ν. Therefore, condition 4.10(b) is satisfied. Thus by Proposition 4.10, passing to some subsequence, there exist a value $R_0 \in \{1, \infty\}$, a finite subset $Z \subseteq \mathbb{C}$, a map $w_0 \in W^{1,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z_1, M \times \mathfrak{g} \times \mathfrak{g})$ and sequences $\varepsilon_\nu > 0$, z_ν such that the conclusions of Proposition 4.10 with w_ν replaced by w_0^ν hold. We define $N := N(1) := 1$, $R_1 := R_0$, $Z_1 := Z$, $w_1 := w_0$, $R_1^\nu := \varepsilon^\nu R_0^\nu$ and $z_1^\nu := R_0^\nu z_\nu$. We check conditions (i)-(vii) of Claim 1. Condition (i) is void. Condition (ii) follows from condition (v) of Proposition 4.10. Condition (iii) follows from 4.10(i), since

$$\frac{R_1^\nu}{R_0^\nu} = \varepsilon_1^\nu, \qquad \frac{z_1^\nu - z_0^\nu}{R_0^\nu} = z^\nu.$$

Condition (iv) follows from 4.10(ii), condition (v) follows from 4.10(iii), and (vi) follows from 4.10(iv). Condition (vii) is void. This proves the statement of the Claim for $\ell = 1$.

Let $\ell \in \mathbb{N}$ and assume, by induction, that we have already proved the statement of Claim 1 for ℓ. This means that there is a number $N := N(\ell)$ and there are R_i, w_i, Z_i, R_i^ν, z_i^ν, j_i and z_i for $i = 1, \dots, N$ such that conditions (i)-(vii) hold. If for every $j = 1, \dots, N$ we have $Z_j = \{z_i \,|\, j < i \leq N,\, j_i = j\}$ then conditions (i)-(vii) hold with $N(\ell + 1) := N$ and the Claim is proved. So assume that there is a $j_0 \in \{1, \dots, N\}$ such that

$$Z_{j_0} \neq \{z_i \,|\, j_0 < i \leq N,\, j_i = j_0\}. \tag{4.112}$$

Then we set $N(\ell + 1) := N + 1$ and choose an element

$$z_{N+1} \in Z_{j_0} \setminus \{z_i \,|\, j < i \leq N,\, j_i = j_0\}.$$

We fix any number $r > 0$ so small that $B_r(z_{N+1}) \cap Z_{j_0} = \{z_{N+1}\}$. We apply Proposition 4.10 with $z_0 := z_{N+1}$ and R_ν, w_ν replaced by $R_{j_0}^\nu$, $(\varphi_{j_0}^\nu)^* w^\nu$. Condition 4.10(a) holds by hypothesis. Furthermore, by condition (v), condition 4.10(b) is satisfied. So passing to some subsequence, we get a value

$R_0 \in \{1, \infty\}$, a finite subset $Z \subseteq \mathbb{C}$, an R_0-vortex $w_0 \in W^{1,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, M \times \mathfrak{g} \times \mathfrak{g})$ and sequences $\varepsilon^\nu > 0$, z^ν such that the conclusions of Proposition 4.10 are satisfied. We define $R_{N+1} := R_0$, $Z_{N+1} := Z$, $w_{N+1} := w_0$, $R^\nu_{N+1} := \varepsilon_\nu R^\nu_{j_0}$, $z^\nu_{N+1} := R^\nu_{j_0} z_\nu + z^\nu_{j_0}$ and $j_{N+1} := j_0$.

We check conditions (i)-(vi) of Claim 1 with N replaced by $N+1$. To see that (i) holds, observe that by the induction hypothesis, the statements hold for $i, i' \in \{2, \ldots, N\}$. Furthermore, by our choice of z_{N+1} we have $z_{N+1} \in Z_{j_{N+1}} = Z_{j_0}$ and $z_{N+1} \neq z_i$ for every $i \in \{1, \ldots, N\}$ such that $j_i = j_0$. So (i) is satisfied. As in the case $N = 1$, condition (ii) follows from 4.10(v), condition (iii) follows from 4.10(i), condition (iv) follows from 4.10(ii), condition (v) follows from 4.10(iii) and (vi) follows from 4.10(iv). We check (vii) with N replaced by $N+1$. By the induction hypothesis, it holds for N. Together with our assumption (4.112) this implies that $N \geq \ell$, i.e. $N + 1 \geq \ell + 1$. So there is nothing to check. This completes the induction and the proof of Claim 1. □

Let $\ell \in \mathbb{N}$ be any natural number and let $N := N(\ell)$, R_i, $w_i = (u_i, \Phi_i, \Psi_i)$, Z_i, R^ν_i, z^ν_i, j_i and z_i be as in Claim 1. Recall that $Z_0 = \{0\}$ and $z^\nu_0 := 0$. Fix $i = 0, \ldots, N$. We define $\varphi^\nu_i(z) := R^\nu_i z + z^\nu_i$, for every measurable subset $X \subseteq \mathbb{C}$ we abbreviate

$$E_i(X) := E^{R_i}(w_i, X), \qquad E^\nu_i(X) := E^{R^\nu_i}\big((u_\nu, R^\nu_i \Phi^\nu_i, R^\nu_i \Psi^\nu_i) \circ \varphi^\nu_i, X\big),$$

and for $z \in Z_i$ we define

$$m_i(z) := \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E^\nu_i(B_\varepsilon(z)). \tag{4.113}$$

For $i = 0$ it follows from (4.109) and $R^\nu_0 = \nu R_\nu$ that

$$m_0(0) = \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E(w_\nu, B_{\varepsilon \nu R_\nu}) = \lim_{\varepsilon \to 0} E(\mathbb{C}) = E. \tag{4.114}$$

For $i = 1, \ldots, N$ it follows from condition (v) that the limit (4.113) exists and that $m_i(z) \geq \mathrm{E}_{\min}$. For $j, k = 0, \ldots, N = N(\ell)$ we define

$$Z_{j,k} := Z_j \setminus \{z_i \,|\, j < i \leq k,\ j_i = j\}$$

We define the function $f : \{1, \ldots, N\} \to [0, \infty)$ by

$$f(i) := E_i(\mathbb{C} \setminus Z_i) + \sum_{z \in Z_{i,N}} m_i(z). \tag{4.115}$$

Claim 2

$$\sum_{i=1}^N f(i) = E. \tag{4.116}$$

Proof of Claim 2: We show by induction that for every $k = 1, \ldots, N$ we have

$$\sum_{i=1}^{k} \big(E_i(\mathbb{C} \setminus Z_i) + \sum_{z \in Z_{i,k}} m_i(z) \big) = E. \tag{4.117}$$

Claim 2 follows from this with $k = N$.

Claim 3 *For every $i = 1, \ldots, N$ we have*

$$m_{j_i}(z_i) = E_i(\mathbb{C} \setminus Z_i) + \sum_{z \in Z_i} m_i(z). \tag{4.118}$$

Proof of Claim 3: Fix any $i = 1, \ldots, N$. We choose a number $\varepsilon > 0$ so small that

$$\bar{B}_\varepsilon(z_i) \cap Z_{j_i} = \{z_i\}, \qquad Z_i \subseteq B_{\varepsilon^{-1} - \varepsilon},$$

and if $z \neq z'$ are points in Z_i then $2\varepsilon < |z - z'|$. By condition (v) of Claim 1, the limits

$$\lim_{\nu \to \infty} E^\nu_{j_i}(B_\varepsilon(z_i)), \qquad \lim_{\nu \to \infty} E^\nu_i(B_\varepsilon(z)), \quad z \in Z_i$$

exist. Applying Lemma 4.9 with

$$\Omega := B_{2\varepsilon^{-1}} \setminus \bigcup_{z \in Z_i} \bar{B}_{\varepsilon/2}(z), \qquad Q := \bar{B}_{\varepsilon^{-1}} \setminus \bigcup_{z \in Z_i} B_\varepsilon(z),$$

we have

$$\lim_{\nu \to \infty} E^\nu_i(B_{\varepsilon^{-1}}) = E_i\big(B_{\varepsilon^{-1}} \setminus \bigcup_{z \in Z_i} B_\varepsilon(z)\big) + \sum_{z \in Z_i} \lim_{\nu \to \infty} E^\nu_i(B_\varepsilon(z)). \tag{4.119}$$

It follows from the definitions of $E^\nu_{j_i}$ and E^ν_i that

$$\begin{aligned} E^\nu_{j_i}(B_\varepsilon(z_i)) &= E(w_\nu, B_{\varepsilon R^\nu_{j_i}}(z^\nu_{j_i} + R^\nu_{j_i} z_i)) \\ &= E\big(w_\nu, B_{\varepsilon R^\nu_{j_i}}(z^\nu_{j_i} + R^\nu_{j_i} z_i) \setminus B_{\varepsilon^{-1} R^\nu_i}(z^\nu_i)\big) + E^\nu_i(B_{\varepsilon^{-1}}). \end{aligned}$$

Taking the limits $\nu \to \infty$ and then $\varepsilon \to 0$, we get

$$\begin{aligned} m_{j_i}(z_i) &= \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E^\nu_{j_i}(B_\varepsilon(z_i)) \\ &= \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E\big(w_\nu, B_{\varepsilon R^\nu_{j_i}}(z^\nu_{j_i} + R^\nu_{j_i} z_i) \setminus B_{\varepsilon^{-1} R^\nu_i}(z^\nu_i)\big) + \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E^\nu_i(B_{\varepsilon^{-1}}) \\ &= 0 + \lim_{\varepsilon \to 0} E_i\big(B_{\varepsilon^{-1}} \setminus \bigcup_{z \in Z_i} B_\varepsilon(z)\big) + \sum_{z \in Z_i} \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E^\nu_i(B_\varepsilon(z)) \\ &= E_i(\mathbb{C} \setminus Z_i) + \sum_{z \in Z_i} m_i(z). \end{aligned}$$

Here in the third equality we have used condition (vi) of Claim 1 and equality (4.119). This proves Claim 3. □

Since $Z_{1,1} = Z_1$, equality (4.117) for $k = 1$ follows from Claim 3 and (4.114). Let now $k = 1, \dots, N-1$ and assume that we have proved (4.117) for k. For $i = 1, \dots, k+1$ we have

$$Z_{i,k+1} = \begin{cases} Z_{i,k}, & \text{if } i \le k \text{ and } i \ne j_{k+1}, \\ Z_{i,k} \setminus \{z_{k+1}\}, & \text{if } i \le k, \text{ and } i = j_{k+1}, \\ Z_{k+1}, & \text{if } i = k+1, \end{cases}$$

Therefore,

$$\sum_{i=1}^{k+1} \sum_{z \in Z_{i,k+1}} m_i(z) \quad = \quad \sum_{i=1}^{k} \sum_{z \in Z_{i,k}} m_i(z) - m_{j_{k+1}}(z_{k+1}) + \sum_{z \in Z_{k+1}} m_{k+1}(z).$$

It follows that

$$\begin{aligned} &\sum_{i=1}^{k+1} \Big(E_i(\mathbb{C} \setminus Z_i) + \sum_{z \in Z_{i,k+1}} m_i(z) \Big) = \\ &E + E_{k+1}(\mathbb{C} \setminus Z_{k+1}) - m_{j_{k+1}}(z_{k+1}) + \sum_{z \in Z_{k+1}} m_{k+1}(z) = E \quad . \end{aligned}$$

Here in the first equality we have used the induction hypothesis and in the second equality Claim 3 with $i := k+1$. It follows that (4.117) holds for every $k = 1, \dots, N$. This proves Claim 2. □

Claim 4 *The estimate*

$$N \le \frac{2E}{\mathrm{E}_{\min}} + 1 \tag{4.120}$$

holds.

Proof: Consider the relation E on $T := \{1, \dots, N\}$ defined by iEi' iff $i = j_{i'}$ or $i' = j_i$. This is a tree relation. We apply Lemma 4.14 with f as in (4.115), $k := 1$, $\alpha_1 := 1 \in T$ and $E_0 := \mathrm{E}_{\min}$. To see that the hypothesis (4.82) holds, suppose that $i_0 \in \{1, \dots, N\}$ is a vertex such that $f(i_0) < E_0 = \mathrm{E}_{\min}$. It follows that $E_{i_0}(\mathbb{C} \setminus Z_{i_0}) < \mathrm{E}_{\min}$ and therefore $E_{i_0}(\mathbb{C} \setminus Z_{i_0}) = 0$. Condition (ii) of Claim 1 implies that $\#Z_{i_0} \ge 2$. Furthermore, $Z_{i_0,N}$ is empty, because by condition (v) of Claim 1 we have for every element $z \in Z_{i_0}$

$$m_{i_0}(z) = \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E_{i_0}^{\nu}(B_\varepsilon(z)) \ge \mathrm{E}_{\min} \,.$$

This means that for every $z \in Z_{i_0}$ there is an index $i_0 < i \le N$ such that $z_i = z$. If $i_0 = 1$ then

$$n_{i_0} := \#\{i \in T \,|\, iEi_0\} + \#\{i \in \{1, \dots, k\} \,|\, \alpha_i = i_0\} = \#Z_{i_0} + 1 \ge 3.$$

Assume now that $i_0 \neq 1$**.** Then the vertices adjoint to i_0 are the elements of Z_{i_0} and the vertex j_{i_0}. Therefore

$$n_{i_0} = \#Z_{i_0} + 1 \geq 3.$$

So hypothesis (4.82) of Lemma 4.14 is satisfied, and therefore by that Lemma

$$N = \#T \leq \frac{2\sum_{i=1}^{N} f(i)}{\mathrm{E}_{\min}} + 1 = \frac{2E}{\mathrm{E}_{\min}} + 1.$$

Here in the last equality we have used Claim 2. This proves Claim 4. □

We choose now an integer $\ell > 2E/\mathrm{E}_{\min} + 1$ and assume that

$$N = N(\ell), \quad R_i, \quad w_i = (u_i, \Phi_i, \Psi_i), \quad Z_i, \quad R_i^\nu, \quad z_i^\nu, \quad j_i, \quad z_i$$

are as in Claim 1. Then by (4.120) we have

$$\ell > \frac{2E}{\mathrm{E}_{\min}} + 1 \geq N,$$

and therefore condition (vii) of Claim 1 implies that for every $j = 1, \ldots, N$ we have

$$Z_j = \{z_i \,|\, j < i \leq N,\, j_i = j\}. \tag{4.121}$$

This means that all the bubble points have been resolved. We define

$$T := \{1, \ldots, N\}, \quad V := \{i \in T \,|\, R_i = 1\}, \quad \bar{T} := T \setminus V,$$

and the tree relation E on T by

$$iEi' \iff i = j_{i'} \text{ or } i' = j_i.$$

Furthermore, for $i, i' \in T$ such that iEi' we define the nodal points

$$z_{ii'} := \begin{cases} \infty, & \text{if } i' = j_i, \\ z_{i'}, & \text{if } i = j_{i'}. \end{cases}$$

Moreover, we define the marked point

$$(\alpha_0, z_0) := (1, \infty) \in T \times S^2.$$

Claim 5 *Let $i \in T$. If $i \in V$ then $w_i \in \widetilde{\mathcal{M}}^{1,p}$, and if $i \in \bar{T}$ then the map*

$$\mathrm{G}u_i : \mathbb{C} \setminus Z_i \to \bar{M} = \mu^{-1}(0)/\mathrm{G}$$

extends to a smooth $\bar{J}$-holomorphic map

$$\bar{u}_i : S^2 \cong \mathbb{C} \cup \{\infty\} \to \bar{M}.$$

Proof: By condition (iv) of Claim 1 there exist gauge transformations $g_i^\nu \in \mathcal{G}^{2,p}$ such that for every compact subset $Q \subseteq \mathbb{C}$ the maps

$$u_i^\nu := (g_i^\nu)^{-1}(u^\nu \circ \varphi_i^\nu)$$

converge to u_i in $C^1(Q)$, and the maps

$$(\Phi_i^\nu, \Psi_i^\nu) := (g_i^\nu)^*\big((R_i^\nu \Phi_\nu, R_i^\nu \Psi_\nu) \circ \varphi_i^\nu\big)$$

converge to (Φ_i, Ψ_i) in $C^0(Q)$.

Assume that $i \in V$. By Fatou's lemma, we have

$$\begin{aligned}
E(w_i) &= \int_{\mathbb{C}} \big(|\partial_s u_i + X_{\Phi_i}(u_i)|^2 + |\mu \circ u_i|^2\big)\, ds\, dt \\
&\leq \liminf_{\nu\to\infty} \int_{\mathbb{C}} \big(|\partial_s u_i^\nu + X_{\Phi_i^\nu}(u_i^\nu)|^2 + |\mu \circ u_i^\nu|^2\big)\, ds\, dt \\
&= \liminf_{\nu\to\infty} E(w_i^\nu) \\
&= \lim_{\nu\to\infty} E(w_\nu) \\
&= E < \infty.
\end{aligned}$$

Furthermore, by Proposition D.6(B) there exists a G-invariant compact subset $K_0 \subseteq M$ such that $u_i^\nu(\mathbb{C}) \subseteq K_0$. Since u_i^ν converges to u_i pointwise, it follows that $u_i(\mathbb{C}) \subseteq K_0$. Hence $w_i \in \widetilde{\mathcal{M}}^{1,p}$.

Assume now that $i \in \bar{T}$. By Proposition E.11 the map

$$\mathrm{G}u_i : \mathbb{C} \setminus Z_i \to \bar{M} = \mu^{-1}(0)/\mathrm{G}.$$

is $\bar{J}$-holomorphic, and $e_{\mathrm{G}u_i} = e^\infty_{w_i}$. Using again Fatou's lemma, it follows that

$$\begin{aligned}
E(\mathrm{G}u_i, \mathbb{C} \setminus Z_i) &= E^\infty(w_i, \mathbb{C} \setminus Z_i) \\
&= \int_{\mathbb{C}\setminus Z_i} |\partial_s u_i + X_{\Phi_i}(u_i)|^2\, ds\, dt \\
&\leq \liminf_{\nu\to\infty} \int_{\mathbb{C}} \big(|\partial_s u_i^\nu + X_{\Phi_i^\nu}|^2 + (R_i^\nu)^2 |\mu \circ u_i^\nu|^2\big)\, ds\, dt \\
&= \liminf_{\nu\to\infty} E^{R_i^\nu}(w_i^\nu) \\
&= \lim_{\nu\to\infty} E(w_\nu) \\
&= E < \infty.
\end{aligned}$$

Therefore, by removal of singularities, Theorem 4.1.2 in the book [MS3], it follows that $\mathrm{G}u_i$ extends to a smooth $\bar{J}$-holomorphic map $\bar{u}_i : S^2 \to \bar{M}$. This proves Claim 5. $\square$

Claim 6 *The tuple*

$$(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}) := \big(V, \bar{T}, E, (w_i)_{i\in V}, (\bar{u}_i)_{i\in\bar{T}}, (z_{ii'})_{iEi'}, (\alpha_0, z_0)\big)$$

is a stable map in the sense of Definition 3.2 and the sequence $(w^\nu, z_0^\nu) = (w^\nu, \infty)$ *converges to* $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ *in the sense of Definition 3.6.*

Proof of Claim 6: We check the conditions of Definition 3.2. To see that condition (i) holds, observe that the first part of this condition is satisfied, since $z_0 = \infty$. Fix $i_0 \in T$.

Consider the case $i_0 = 1$**.** We have $1Ei$ if and only if $1 = j_i$. Therefore, the special points at the vertex 1 are the nodal points $z_{1i} = z_i$ with $i \in T$ such that $j_i = 1$, together with the marked point $z_0 = \infty$. It follows from condition (i) of Claim 1 that these points are all distinct.

Consider the case $i_0 \neq 1$**.** Then the special points at i_0 are the nodal points $z_{i_0 i} = z_i$ with i such that $j_i = i_0$ together with the nodal point $z_{i_0 j_{i_0}} = \infty$. By condition 1(i) they are all distinct. To check the last part of condition (i) of Definition 3.2, let $i \in V$ and $i' \in T$ be such that iEi'. Then by condition (ii) of Claim 1 the set Z_i is empty, so $i' = j_i$, and therefore $z_{ii'} = \infty$. So we have proved condition (i) of Definition 3.2.

In order to check **condition (ii) of Definition 3.2**, we fix an index $i = 2, \dots, N$. We define $\bar{u}_i : S^2 \to M/\mathrm{G}$ to be the unique continuous extension of the map

$$\bar{u}_i := \mathrm{G}u_i : \mathbb{C} \setminus Z_i \to M/\mathrm{G}.$$

If $i \in V$ then it follows from Proposition D.6(A) (asymptotic behaviour) that $\bar{u}_i$ is well-defined, and if $i \in \bar{T}$ then this follows from Claim 5. Let $r_0 > 0$ be so big that $B_{r_0^{-1}}(z_i) \cap Z_{j_i} = \{z_i\}$ and $Z_i \subseteq B_{r_0}$. We apply part (B) of Lemma 4.15 with $z_0 := z_i$, $\bar{u} := \bar{u}_i$, $\bar{u}' := \bar{u}_{j_i}$, $R_\nu := R_i^\nu$, $\lambda_\nu := R_{j_i}^\nu / R_i^\nu$, $z_\nu := (z_i^\nu - z_{j_i}^\nu)/R_{j_i}^\nu$ and $w_\nu = (u_\nu, \Phi_\nu, \Psi_\nu)$ replaced by $(\varphi_i^\nu)^* w_\nu$. Observe that the conditions (4.86) and (4.87) follow from Claim 1(iii), and condition (4.88) follows from Claim 1(vi). Furthermore, it follows from Claim 1(iv) that $\mathrm{G}u_i^\nu(z)$ converges to $\bar{u}_i(z) \in M/\mathrm{G}$ for every $z \in \mathbb{C} \setminus B_{r_0^{-1}}$ and $\mathrm{G}u_{j_i}^\nu(z)$ converges to $\bar{u}_{j_i}(z)$ for every $z \in B_{r_0}(z_i) \setminus \{z_i\}$. So all hypotheses of Lemma 4.15(B) are satisfied and thus the Lemma implies that

$$\mathrm{ev}_\infty(w_i) = \bar{u}_i(\infty) = \bar{u}_{j_i}(z_i) = \mathrm{ev}_{z_0}(w_{j_i}).$$

This proves **condition 3.2(ii)**.

We check the stability **condition (iii)** of that Definition. Let $i \in T$. By Claim 1(ii) we have $E(w_i) > 0$ if $i \in V$, and $|Z_i| \geq 2$ if $i \in \bar{T}$ and

$E(\bar{u}_i) = E^\infty(w_i) = 0$. **Assume that** $i \in \bar{T}$ **and** $E(\bar{u}_i) = 0$**.** Then the number of special points at i equals $|Z_i| + 1 \geq 3$. Here we have used that the vertex $i = 1$ carries the marked point $(\alpha_0, z_0) = (1, \infty)$. This proves **condition 3.2(iii)**. So all conditions of Definition 3.2 are satisfied.

We check now the **conditions of Definition 3.6**. To see that **condition (3.9)** holds, observe that by (4.121) the set $Z_{i,N}$ is empty for every $i = 1, \ldots, N$, and therefore by Claim 2

$$\begin{aligned} \lim_{\nu\to\infty} E(w_\nu) = E &= \sum_{i=1}^N f(i) \\ &= \sum_{i\in V} E(w_i) + \sum_{i\in\bar{T}} E^\infty(w_i, \mathbb{C}\setminus Z_i) \\ &= \sum_{i\in V} E(w_i) + \sum_{i\in\bar{T}} E(\bar{u}_i). \end{aligned}$$

This proves condition (3.9).

To see **condition 3.6(i)**, observe first that if $i \in V$, then by Claim 1(iii) we have $R_i^\nu = 1$, and thus $\varphi_i^\nu(z) = z + z_i^\nu$. Furthermore, for every $i = 1, \ldots, N$ we have $z_{i,0} = \infty$ and therefore $\varphi_i^\nu(z_{i,0}) = \infty$, where $z_{i,0}$ is defined as in (3.7) and (3.8). Finally, if $i \in \bar{T}$, choosing $\psi_i := \mathrm{id}$ the derivatives $(\varphi_i^\nu \circ \psi_i)'(z) = R_i^\nu$ converge to ∞ if $z \in \mathbb{C}$. This proves condition (i).

By Lemma B.9, **condition 3.6(ii)** is equivalent to

$$\frac{R_i^\nu}{R_j^\nu} \to 0, \qquad \frac{z_i^\nu - z_j^\nu}{R_j^\nu} \to z_{ji},$$

for each pair $(j, i) \in \{1, \ldots, N\}^2$ of adjacent vertices such that $z_{ji} \neq \infty$. If (j, i) is such a pair then $j = j_i$. Thus 3.6(ii) follows from Claim 1(iii).

Condition 3.6(iii) follows from Claim 1(iv). Finally, **condition 3.6(iv)** is void, since $k = 0$. This proves Claim 6. □

So we have proved Theorem 4.1 in the case $k = 0$.

We prove now by induction that the Theorem holds for every $k \geq 1$.

Let $k \geq \mathbb{N}$ be an integer and assume that we have already proved the assertion of the Theorem with k replaced by $k - 1$. Let $w_\nu \in \widetilde{\mathcal{M}}^{1,p}$ be a sequence of vortices and $z_1^\nu, \ldots, z_k^\nu \in \mathbb{C}$ be sequences of marked points such that the hypotheses of Theorem 4.1 are satisfied. We prove that there exists a subsequence of $(w_\nu, z_0^\nu = \infty, z_1^\nu, \ldots, z_k^\nu)$ that converges to some stable map with $k + 1$ marked points.

By our assumption, there exists a subsequence such that $(w_\nu, z_0^\nu, \dots, z_{k-1}^\nu)$ converges to some stable map

$$(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}) := \big(V, \bar{T}, E, (w_\alpha)_{\alpha \in V}, (\bar{u}_\alpha)_{\alpha \in \bar{T}}, (z_{\alpha\beta})_{\alpha E \beta}, (\alpha_i, z_i)_{i=0,\dots,k-1}\big),$$

via sequences of Möbius transformations φ_α^ν, $\alpha \in T$. By hypothesis, we have

$$\limsup_{\nu \to \infty} |z_i^\nu - z_k^\nu| > 0,$$

for every $i = 1, \dots, k-1$. Passing to some subsequence, we may assume that the limits

$$z_{ki} := \lim_{\nu \to \infty} (z_k^\nu - z_i^\nu) \in \mathbb{C} \cup \{\infty\}$$

exist for $i = 1, \dots, k-1$, and

$$z_{ki} \neq 0, \tag{4.122}$$

and that the limit

$$z_{\alpha k} := \lim_{\nu \to \infty} (\varphi_\alpha^\nu)^{-1}(z_k^\nu) \in S^2$$

exists for every $\alpha \in T$. We set $z_{k0} := \infty$. There are three cases.

Case (I) There exists a vertex $\alpha \in T$ such that

$$z_{\alpha k} \notin Y_\alpha := Z_\alpha \cup \{z_i \,|\, \alpha_i = \alpha,\, i = 0, \dots, k-1\}.$$

Case (II) There exists an index $i \in \{0, \dots, k-1\}$ such that $z_{\alpha_i k} = z_i$.

Case (III) There exists an edge $\alpha E \beta$ such that $z_{\alpha k} = z_{\alpha\beta}$ and $z_{\beta k} = z_{\beta\alpha}$.

These three cases exclude each other. For the combination of the cases (II) and (III) this follows from condition (i) (distinctness of the special points) of Definition 3.2.

Claim 7 *One of the three cases always applies.*

Proof of Claim 7: Assume that neither Case (I) nor Case (II) applies. This means that for every $\alpha \in T$ we have $z_{\alpha k} \in Y_\alpha$ and there exists no index $i \in \{0, \dots, k-1\}$ such that $z_{\alpha_i k} = z_i$. It follows that for every $\alpha \in T$ the point $z_{\alpha k}$ is a nodal point. We choose any vertex $\beta_0 \in T$. Consider the chain of edges $\beta_1, \beta_2, \dots$, where for each $j \geq 1$ β_j is the unique vertex such that $z_{\beta_{j-1}\beta_j} = z_{\beta_{j-1}k}$. Since T is finite, there exist two indices $j \neq j' \geq 0$ such that $\beta_j = \beta_{j'}$. Furthermore, since T is a tree, there are no cycles, hence $j = j' + 2$ or $j' = j + 2$. Since $z_{\beta_j \beta_{j'}} = z_{\beta_j k}$ and $z_{\beta_{j'} \beta_j} = z_{\beta_{j'} k}$, the condition of Case (III) with $\alpha := \beta_j$ and $\beta := \beta_{j'}$ is satisfied. This proves Claim 7. □

Assume that Case (I) holds. We fix a vertex $\alpha \in T$ such that $z_{\alpha k} \notin Y_\alpha$. (This vertex is actually unique, but we do not need this.) We define $\alpha_k := \alpha$ and introduce a new marked point

$$z_k^{\text{new}} := z_{\alpha_k k}$$

on the α_k-sphere. Then $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ augmented by z_k^{new} is again a stable map and the sequence $(w_\nu, z_0^\nu, \ldots, z_k^\nu)$ converges to this new stable map along $\{\varphi_\alpha\}_{\alpha \in T}$.

Assume that Case (II) holds. We fix an index $0 \le i \le k-1$ such that $z_{\alpha_i k} = z_i$. (It is unique.) We extend the tree T be introducing an additional vertex γ which is adjacent to α_i. If $z_{ki} = \infty$ then the new vertex corresponds to a bubble in $\bar{M}$, otherwise it corresponds to a vortex. Furthermore, we move the i-th marked point from the vertex α_i to the vertex γ and introduce an additional marked point on γ. More precisely, we define

$$\bar{T}^{\text{new}} := \begin{cases} \bar{T} \sqcup \{\gamma\}, & \text{if } z_{ki} = \infty, \\ \bar{T}, & \text{otherwise,} \end{cases}$$

$$T^{\text{new}} := T \sqcup \{\gamma\}, \quad V^{\text{new}} := T^{\text{new}} \setminus \bar{T}^{\text{new}},$$
$$\alpha_i^{\text{new}} := \alpha_k := \gamma, \quad z_{\gamma\alpha_i}^{\text{new}} := \infty, \quad z_{\alpha_i\gamma}^{\text{new}} := z_i$$
$$z_i^{\text{new}} := 0, \quad z_k^{\text{new}} := \begin{cases} z_{ki}, & \text{if } \alpha_i \in V, \\ 1, & \text{if } \alpha_i \in \bar{T}. \end{cases}$$

Assume that $\gamma \in V^{\text{new}}$. This means that $z_{ki} \neq \infty$. Our assumption $z_{\alpha_i k} = z_i$ implies that $\alpha_i \in \bar{T}$. We define $w_\gamma : \mathbb{C} \to M \times \mathfrak{g} \times \mathfrak{g}$ to be the constant map $(x_0, 0, 0)$, where x_0 is any point in the orbit $\bar{u}_{\alpha_i}(z_i)$. If $\gamma \in \bar{T}^{\text{new}}$ then we define $\bar{u}_\gamma : S^2 \to \bar{M}$ to be the constant map

$$\bar{u}_\gamma \equiv \bar{u}_{\alpha_i}(z_i).$$

Note that the new component γ is a "ghost", i.e. the map w_γ (or $\bar{u}_\gamma$) has 0 energy. The tuple $(\mathbf{w}^{\text{new}}, \bar{\mathbf{u}}^{\text{new}}, \mathbf{z}^{\text{new}})$ obtained from $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ in this way is again a stable map.

We define the sequence of Möbius transformations $\varphi_\gamma^\nu : S^2 \to S^2$ by

$$\varphi_\gamma^\nu(z) := \begin{cases} z + z_i^\nu, & \text{if } \gamma \in V^{\text{new}}, \\ (z_k^\nu - z_i^\nu)z + z_i^\nu, & \text{if } \gamma \in \bar{T}^{\text{new}},\ i \ge 1, \\ \frac{z_k^\nu - \varphi_{\alpha_0}^\nu(w)}{z} + \varphi_{\alpha_0}^\nu(w), & \text{if } \gamma \in \bar{T}^{\text{new}},\ i = 0, \end{cases} \tag{4.123}$$

where $w \in S^2 \setminus \{z_0\}$ is chosen such that $\varphi_{\alpha_0}^\nu(w) \neq z_k^\nu$ for all ν. Note that the last line makes sense, since $\varphi_{\alpha_0}^\nu(w) \neq \varphi_{\alpha_0}^\nu(z_0) = \infty$. Here we use the convention that $\infty^{-1} := 0$.

Claim 8 *There exists a subsequence of* $(w_\nu, z_0^\nu, \ldots, z_k^\nu)$ *that converges to* $(\mathbf{w}^{\text{new}}, \bar{\mathbf{u}}^{\text{new}}, \mathbf{z}^{\text{new}})$ *along the Möbius transformations* $\{\varphi_\alpha^\nu\}_{\alpha\in T^{\text{new}},\nu\in\mathbb{N}}$.

Proof of Claim 8: Condition (3.9) (energy conservation) still holds, since the new component γ carries no energy.

We check condition (i) of Definition 3.6. The first part of this condition is satisfied by the definition of φ_γ^ν. We show that the second part holds, i.e. that

$$\varphi_\alpha^\nu(z_{\alpha,0}^{\text{new}}) = \infty, \tag{4.124}$$

for every $\alpha \in \bar{T}^{\text{new}}$, where $z_{\alpha,0}^{\text{new}} \in S^2$ is defined as in (3.7) and (3.8) by using the new marked and nodal points. We have

$$z_{\alpha,0}^{\text{new}} = \begin{cases} z_{\alpha,0}, & \text{if } \alpha \neq \gamma, \\ z_0^{\text{new}} = 0, & \text{if } \alpha = \gamma,\ i = 0, \\ z_{\gamma\alpha_i}^{\text{new}} = \infty, & \text{if } \alpha = \gamma, i \geq 1. \end{cases}$$

Hence for $\alpha \neq \gamma$ condition (4.124) is satisfied since $\varphi_\alpha^\nu(z_{\alpha,0}) = \infty$, and for $\alpha = \gamma$ it follows from the definition (4.123) of φ_γ^ν. We check the third part of condition (i). We fix a vertex $\alpha \in \bar{T}$. It suffices to show that there exists a Möbius transformation ψ_α such that $\psi_\alpha(\infty) = z_{\alpha,0}$ and

$$(\varphi_\alpha^\nu \circ \psi_\alpha)'(z) \to \infty, \qquad \forall z \in \mathbb{C}. \tag{4.125}$$

If $\alpha \neq \gamma$ then this holds by the choice of the Möbius transformations φ_α^ν. Assume that $\alpha = \gamma$. In the case $i \geq 1$ we set $\psi_\gamma :=$ id. Then (4.125) follows from (4.123), since $\gamma \in \bar{T}^{\text{new}}$ means that $\lim_{\nu\to\infty}(z_k^\nu - z_i^\nu) = z_{ki} = \infty$.

Assume now that $i = 0$**.** We define

$$\psi_\gamma(z) := \frac{1}{z}.$$

By the definition (4.123) we have to show that

$$z_k^\nu - \varphi_{\alpha_0}^\nu(w) \to \infty. \tag{4.126}$$

Assume that $\alpha_0 \in V$**.** Then there exist points $z_0^\nu \in \mathbb{C}$ such that $\varphi_{\alpha_0}^\nu(z) = z + z_0^\nu$. By our assumption that we are in Case (II) and that $i = 0$, we have

$$\begin{aligned} \infty = z_0 &= z_{\alpha_0 k} \\ &= \lim_{\nu\to\infty} (\varphi_{\alpha_0}^\nu)^{-1}(z_k^\nu) \\ &= \lim_{\nu\to\infty} (z_k^\nu - z_0^\nu). \end{aligned}$$

It follows that

$$z_k^\nu - \varphi_{\alpha_0}^\nu(w) = z_k^\nu - z_0^\nu - w \to \infty,$$

so (4.126) holds in this case.

Assume now that $\alpha_0 \in \bar{T}$. Then we choose a Möbius transformation ψ_{α_0} such that $\psi_{\alpha_0}(\infty) = z_0$ and define

$$\varphi_\nu := \varphi^\nu_{\alpha_0} \circ \psi_{\alpha_0}.$$

Since $\varphi_\nu(\infty) = \infty$ there exist numbers $\lambda_\nu \in \mathbb{C} \setminus \{0\}$ and points $z_\nu \in \mathbb{C}$ such that $\varphi_\nu(z) = \lambda_\nu z + z_\nu$. By condition 3.6(i) the numbers λ_ν converge to ∞. Furthermore,

$$\begin{aligned} z^\nu_k - \varphi^\nu_{\alpha_0}(w) &= \lambda_\nu \left(\frac{z^\nu_k - z_\nu}{\lambda_\nu} - \psi^{-1}_{\alpha_0}(w) \right) \\ &= \lambda_\nu \big(\varphi^{-1}_\nu (z^\nu_k) - \psi^{-1}_{\alpha_0}(w) \big) \\ &= \lambda_\nu \big(\psi^{-1}_{\alpha_0} \circ (\varphi^\nu_{\alpha_0})^{-1}(z^\nu_k) - \psi^{-1}_{\alpha_0}(w) \big). \end{aligned} \tag{4.127}$$

By assumption we have

$$(\varphi^\nu_{\alpha_0})^{-1}(z^\nu_k) \to z_0.$$

Since $w \neq z_0$, it follows that

$$\psi^{-1}_{\alpha_0} \circ (\varphi^\nu_{\alpha_0})^{-1}(z^\nu_k) - \psi^{-1}_{\alpha_0}(w) \to \psi^{-1}_{\alpha_0}(z_0) - \psi^{-1}_{\alpha_0}(w) = \infty.$$

Since $\lambda_\nu \to \infty$, (4.127) implies (4.126). So the last part of condition (i) of Definition 3.6 holds also in the case $\alpha_0 \in \bar{T}$.

We check condition 3.6(ii). It is enough to consider the cases $(\alpha, \beta) := (\alpha_i, \gamma)$ and $(\alpha, \beta) := (\gamma, \alpha_i)$. Consider the case $(\alpha, \beta) := (\alpha_i, \gamma)$. We define

$$x := z^{\mathrm{new}}_{\gamma\alpha_i} = \infty, \quad y := z^{\mathrm{new}}_{\alpha_i\gamma} = z_i, \quad x^\nu_1 := z^{\mathrm{new}}_i = 0,$$

$$x^\nu_2 := \begin{cases} z^\nu_k - z^\nu_i, & \text{if } \gamma \in V^{\mathrm{new}}, \\ z^{\mathrm{new}}_k = 1, & \text{if } \gamma \in \bar{T}^{\mathrm{new}}, \end{cases}$$

$$y_\nu := \begin{cases} z_{\alpha_i,0}, & \text{if } \gamma \in V^{\mathrm{new}} \text{ or } (\gamma \in \bar{T}^{\mathrm{new}} \text{ and } i \geq 1), \\ w, & \text{if } \gamma \in \bar{T}^{\mathrm{new}},\ i = 0. \end{cases}$$

Then the hypotheses of Lemma 4.17 with $\varphi_\nu := \varphi^\nu_{\alpha_i\gamma} := (\varphi^\nu_{\alpha_i})^{-1} \circ \varphi^\nu_\gamma$ are satisfied, and therefore by that Lemma $\varphi^\nu_{\alpha_i\gamma}$ converges to $y = z^{\mathrm{new}}_{\alpha_i\gamma}$, uniformly with all derivatives on every compact subset of $S^2 \setminus \{x\} = S^2 \setminus \{z^{\mathrm{new}}_{\gamma\alpha_i}\}$. By Remark 4.16 it follows that $\varphi^\nu_{\gamma\alpha_i} = (\varphi^\nu_{\alpha_i\gamma})^{-1}$ converges to $z^{\mathrm{new}}_{\gamma\alpha_i}$, uniformly on every compact subset of $S^2 \setminus \{z^{\mathrm{new}}_{\alpha_i\gamma}\}$. This proves condition 3.6(ii).

We check condition 3.6(iii). Given an open subset $\Omega \subseteq S^2$, a compact subset $Q \subseteq \Omega$, a sequence of maps $\bar{u}_\nu : \Omega \to M/\mathrm{G}$ and a map $\bar{u} : \Omega \to M^*/\mathrm{G}$, we say that $\bar{u}_\nu$ converges to $\bar{u}$ in $C^1(Q)$, iff the following holds. There exists an index $\nu_0 \in \mathbb{N}$ such that for $\nu \geq \nu_0$ we have $\bar{u}_\nu(Q) \subseteq M^*/\mathrm{G}$ and

$\bar{u}_\nu \in C^1(Q, M^*/\mathrm{G})$. Furthermore, the maps $u_\nu|_Q$ for $\nu \geq \nu_0$ converge to $\bar{u}|_Q$ in $C^1(Q, M^*/\mathrm{G})$.

Assume that $\gamma \in V^{\mathrm{new}}$**.** This means that $z_{ki} \neq \infty$. Since $z_{k0} = \infty$, it follows that $i \neq 0$. Furthermore, since we are in Case (II), it follows that $\alpha_i \in \bar{T}$. For $\alpha \in T$ we write

$$w^\nu_\alpha := (u^\nu_\alpha, \Phi^\nu_\alpha, \Psi^\nu_\alpha) := w_\nu \circ \varphi^\nu_\alpha.$$

By Proposition 4.3 (Compactness modulo bubbling) with $R_\nu := 1$, $r_\nu := \nu$ and w_ν replaced by w^ν_γ, passing to some subsequence, there exist gauge transformations $\widetilde{g}^\nu_\gamma \in \mathcal{G}^{2,p}$ and a vortex $\widetilde{w}_\gamma := (\widetilde{u}_\gamma, \widetilde{\Phi}_\gamma, \widetilde{\Psi}_\gamma) \in \widetilde{\mathcal{M}}^{1,p}$ such that for every compact subset $Q \subseteq \mathbb{C}$ the maps $(\widetilde{g}^\nu_\gamma)^{-1} u^\nu_\gamma$ converge to $\widetilde{u}_\gamma$ in $C^1(Q)$ and the maps $(\widetilde{g}^\nu_\gamma)^*(\Phi^\nu_\gamma, \Psi^\nu_\gamma)$ converge to $(\Phi_\gamma, \Psi_\gamma)$ in $C^0(Q)$. It follows that $\mathrm{G}u^\nu_\gamma : \mathbb{C} \to M/\mathrm{G}$ converges to $\mathrm{G}\widetilde{u}_\gamma : \mathbb{C} \to M/\mathrm{G}$, uniformly on every compact subset of $\mathbb{C}$. On the other hand, since $\alpha_i \in \bar{T}^{\mathrm{new}}$ and $i \neq 0$, the maps $\mathrm{G}u^\nu_{\alpha_i}$ converge to $\bar{u}_{\alpha_i}$, in C^1 on every compact subset of $S \setminus Z_{\alpha_i}$, and hence on every small enough neighbourhood of z_i. Since $\varphi^\nu_{\alpha_i\gamma}$ converges to $z_{\alpha_i\gamma} = z_i$, uniformly on every compact subset of $S^2 \setminus \{z^{\mathrm{new}}_{\gamma\alpha_i}\} = \mathbb{C}$, it follows that

$$\mathrm{G}u^\nu_\gamma = \mathrm{G}u^\nu_{\alpha_i} \circ \varphi^\nu_{\alpha_i\gamma} \to \bar{u}_{\alpha_i}(z_i),$$

uniformly on every compact subset of $\mathbb{C}$. It follows that $\mathrm{G}\widetilde{u}_\gamma \equiv \bar{u}_{\alpha_i}(z_i)$. Recall that $w_\gamma = (u_\gamma, \Phi_\gamma, \Psi_\gamma) := (x_0, 0, 0)$, where x_0 is any point in the orbit $\bar{u}_{\alpha_i}(z_i)$. For every $z \in \mathbb{C}$ we define $g(z) \in \mathrm{G}$ to be the unique element such that

$$\widetilde{u}_\gamma(z) = g(z)x_0.$$

Since $\widetilde{u}_\gamma \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$, it follows that $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, \mathrm{G})$, and Lemma B.2 implies that $g \in W^{2,p}_{\mathrm{loc}}(\mathbb{C}, \mathrm{G})$. We define $g^\nu_\gamma := \widetilde{g}^\nu_\gamma g \in \mathcal{G}^{2,p}$. Then for every compact subset $Q \subseteq \mathbb{C}$

$$(g^\nu_\gamma)^{-1} u^\nu_\gamma = g^{-1}(\widetilde{g}^\nu_\gamma)^{-1} u^\nu_\gamma \to g^{-1}\widetilde{u}_\gamma = u_\gamma \equiv x_0,$$

in $C^1(Q)$, and similarly

$$(g^\nu_\gamma)^*(\Phi^\nu_\gamma, \Psi^\nu_\gamma) \to (\Phi_\gamma, \Psi_\gamma) \equiv (0, 0),$$

uniformly on Q. This proves condition 3.6(iii) in the case $\gamma \in V^{\mathrm{new}}$.

Assume now that $\gamma \in \bar{T}^{\mathrm{new}}$**. Suppose also that** $i \geq 1$**.** Then $\mathrm{G}u^\nu_{\alpha_i}$ converges to $\bar{u}_\gamma \equiv \bar{u}_{\alpha_i}(z_i)$, in C^1 on every compact subset of $S^2 \setminus Z_{\alpha_i}$ and hence on every small enough neighbourhood of z_i. Fix a compact subset $Q \subseteq \mathbb{C} = S^2 \setminus Z_\gamma$. Since $\varphi^\nu_{\alpha_i\gamma}$ converges to z_i, in $C^1(Q)$, it follows that $\mathrm{G}u^\nu_\gamma = \mathrm{G}u^\nu_{\alpha_i} \circ \varphi^\nu_{\alpha_i\gamma}$ converges to $\bar{u}_\gamma$ in $C^1(Q)$, as required.

Suppose now that $i = 0$**.** For every point $z \in S^2$ and every number $r > 0$ we denote by $B_r^{S^2}(z)$ the open ball of radius r around z in S^2, w.r.t. the standard metric on S^2. Recall the definition (4.42) of $\bar{d}$. Fix a number $\varepsilon > 0$. As in the proof of Proposition 5.4 (see inequality (5.31)) there exist numbers $\delta > 0$ and $\nu_0 \in \mathbb{N}$ such that for every $\nu \geq \nu_0$ and every $z \in B_\delta^{S^2}(z_0) \setminus \{z_0\}$ we have

$$\bar{d}(\bar{u}_{\alpha_0}(z_0), \mathrm{G}u^\nu_{\alpha_0}(z)) < \varepsilon.$$

Let $Q \subseteq S^2 \setminus (Z^{\mathrm{new}}_{\alpha_0} \sqcup \{z_0^{\mathrm{new}}\}) = \mathbb{C} \setminus \{0\}$ be a compact subset. Since $\varphi^\nu_{\alpha_0\gamma}$ converges to $z_{\alpha_0\gamma} = z_0$, uniformly on every compact subset of $\mathbb{C} = S^2 \setminus \{z_{\gamma\alpha_0}\}$, it follows that there exists an index $\nu_1 \geq \nu_0$ such that for $\nu \geq \nu_1$

$$\varphi^\nu_{\alpha_0\gamma}(Q) \subseteq B_\delta^{S^2}(z_0) \setminus \{z_0\}.$$

Hence for every $\nu \geq \nu_1$ and every $z \in Q$ we have

$$\bar{d}(\bar{u}_{\alpha_0}(z_0), \mathrm{G}u^\nu_\gamma(z)) = \bar{d}\big(\bar{u}_{\alpha_0}(z_0), \mathrm{G}u^\nu_{\alpha_0} \circ \varphi^\nu_{\alpha_0\gamma}(z)\big) < \varepsilon.$$

It follows that $\mathrm{G}u^\nu_\gamma$ converges to the constant map $\bar{u}_\gamma \equiv \bar{u}_{\alpha_0}(z_0)$, uniformly on every compact subset of $\mathbb{C} \setminus \{0\}$. We show that passing to some subsequence the convergence is in C^1 on every compact subset of $\mathbb{C} \setminus \{0\}$. To see this, we define $R_\nu > 0$, $\varphi_\nu \in [0, 2\pi)$ and $\widetilde{\varphi}^\nu_\gamma$ by

$$R_\nu e^{i\varphi_\nu} := z_k^\nu - \varphi^\nu_{\alpha_0}(w), \qquad \widetilde{\varphi}^\nu_\gamma(z) := \varphi^\nu_\gamma(e^{i\varphi_\nu}/z) = R_\nu z + \varphi^\nu_{\alpha_0}(w),$$

$$\widetilde{w}^\nu_\gamma := (\widetilde{u}^\nu_\gamma, \widetilde{\Phi}^\nu_\gamma, \widetilde{\Psi}^\nu_\gamma) := \big(u_\nu, R_\nu\Phi_\nu, R_\nu\Psi_\nu\big) \circ \widetilde{\varphi}^\nu_\gamma.$$

By (4.126) the sequence R_ν converges to $R_0 := \infty$. Hence by Proposition 4.3 with $r_\nu := \nu$ and w_ν replaced by $\widetilde{w}^\nu_\gamma$ there exist a finite subset $Z \subseteq \mathbb{C}$ and an ∞-vortex $\widetilde{w}_\gamma := (\widetilde{u}_\gamma, \widetilde{\Phi}_\gamma, \widetilde{\Psi}_\gamma)$ such that passing to some subsequence the assertions 4.3(i)-(iii) hold. By 4.3(ii) there exist gauge transformations $g^\nu_\gamma \in W^{2,p}_{\mathrm{loc}}(\mathbb{C} \setminus Z, \mathrm{G})$ such that for every compact subset $Q \subseteq \mathbb{C} \setminus Z$ the maps $(g^\nu_\gamma)^{-1}\widetilde{u}^\nu_\gamma$ converge to $\widetilde{u}_\gamma$ in $C^1(Q)$ and the maps $(\widetilde{g}^\nu_\gamma)^*(\widetilde{\Phi}^\nu_\gamma, \widetilde{\Psi}^\nu_\gamma)$ converge to $(\widetilde{\Phi}_\gamma, \widetilde{\Psi}_\gamma)$ in $C^0(Q)$.

Claim 9 *We have* $Z \subseteq \{0\}$.

Proof of Claim 9: For $z \in \mathbb{C}$ and $\varepsilon > 0$ we define

$$E_z(\varepsilon) := \liminf_{\nu\to\infty} E^{R_\nu}(\widetilde{w}^\nu_\gamma, B_\varepsilon(z)). \tag{4.128}$$

Condition 4.3(iii) implies that $E_z(\varepsilon) \geq \mathrm{E}_{\min}$ for every $z \in Z$. Let $z \in \mathbb{C} \setminus \{0\}$. We set $\varepsilon := |z|/2$. As in Claim 1 of the proof of Proposition 5.4 (Conservation of the homology class), we have

$$\lim_{\delta\to 0} \limsup_{\nu\to\infty} E\big(w_\nu, \varphi^\nu_{\alpha_0}(B_\delta^{S^2}(z_0) \setminus \{z_0\})\big) = 0. \tag{4.129}$$

We fix a number $\delta > 0$ so small that

$$\limsup_{\nu\to\infty} E\big(w_\nu, \varphi^\nu_{\alpha_0}(B^{S^2}_\delta(z_0)\setminus\{z_0\})\big) < \mathrm{E}_{\min}\,. \tag{4.130}$$

Since the maps $\varphi^\nu_{\alpha_0\gamma}$ converge to $z^{\mathrm{new}}_{\alpha_0\gamma} = z_0$, uniformly on every compact subset of $\mathbb{C} = S^2\setminus\{z^{\mathrm{new}}_{\gamma\alpha_0}\}$, the maps $\widetilde{\varphi}^\nu_{\alpha_0\gamma} := (\varphi^\nu_{\alpha_0})^{-1}\circ\widetilde{\varphi}^\nu_\gamma$ converge to z_0, uniformly on every compact subset of $S^2\setminus\{0\}$. Hence for ν large enough

$$\widetilde{\varphi}^\nu_\gamma(B_\varepsilon(z)) = \varphi^\nu_{\alpha_0}\circ\widetilde{\varphi}^\nu_{\alpha_0\gamma}(B_\varepsilon(z)) \subseteq \varphi^\nu_{\alpha_0}(B^{S^2}_\delta(z_0)),$$

and therefore by (4.130)

$$\begin{aligned} E^{R_\nu}(\widetilde{w}^\nu_\gamma, B_\varepsilon(z)) &= E\big(w_\nu, \widetilde{\varphi}^\nu_\gamma(B_\varepsilon(z))\big) \\ &\le E\big(w_\nu, \varphi^\nu_{\alpha_0}(B^{S^2}_\delta(z_0)\setminus\{z_0\})\big) < \mathrm{E}_{\min}\,. \end{aligned}$$

Hence $E_z(\varepsilon) < \mathrm{E}_{\min}$, where $E_z(\varepsilon)$ is as in (4.128), and therefore $z \notin Z$. This proves Claim 9. □

Let $Q \subseteq \mathbb{C}\setminus\{0\} \subseteq \mathbb{C}\setminus Z$ be a compact subset. Then by Claim 9 the maps $(\widetilde{g}^\nu_\gamma)^{-1}\widetilde{u}^\nu_\gamma$ converge to $\widetilde{u}_\gamma$ in $C^1(Q)$, and hence the maps $\mathrm{G}\widetilde{u}^\nu_\gamma = \mathrm{G}(\widetilde{g}^\nu_\gamma)^{-1}\widetilde{u}^\nu_\gamma$ converge to $\mathrm{G}\widetilde{u}_\gamma$ in $C^1(Q)$. Passing to some subsequence, we may assume that the numbers φ_ν converge to some number $\varphi_0 \in [0, 2\pi]$. It follows that $\mathrm{G}u^\nu_\gamma$ converges to the map $\mathbb{C}\setminus\{0\} \ni z \mapsto \mathrm{G}\widetilde{u}_\gamma(e^{i\varphi_0}/z) \in \bar{M}$, in $C^1(Q)$. On the other hand, the maps $\mathrm{G}u^\nu_\gamma$ converge to $\bar{u}_\gamma \equiv \bar{u}_{\alpha_0}(z_0)$, uniformly on every compact subset of $\mathbb{C}\setminus\{0\}$, and therefore $\mathrm{G}\widetilde{u}_\gamma(e^{i\varphi_0}/\cdot) \equiv \bar{u}_{\alpha_0}(z_0)$. This proves condition 3.6(iii) in the case $\gamma \in \bar{T}^{\mathrm{new}}$, $i = 0$.

Condition 3.6(iv) follows from the definition (4.123) of φ^ν_γ. Note that if $\gamma \in \bar{T}^{\mathrm{new}}$ and $i = 0$ then there is nothing to check.

This proves Claim 8. □

Assume now that Case (III) holds. This means that there exists an edge $\alpha E \beta$, such that

$$z_{\alpha k} = \lim_{\nu\to\infty}(\varphi^\nu_\alpha)^{-1}(z^\nu_k) = z_{\alpha\beta}, \qquad z_{\beta k} = \lim_{\nu\to\infty}(\varphi^\nu_\beta)^{-1}(z^\nu_k) = z_{\beta\alpha}.$$

In this case we introduce a new vertex γ between α and β, corresponding to a bubble in $\bar{M}$. Hence α and β are no longer adjacent, but are separated by γ. We define

$$\bar{T}^{\mathrm{new}} := \bar{T} \sqcup \{\gamma\}, \quad V^{\mathrm{new}} := V, \quad T^{\mathrm{new}} := T \sqcup \{\gamma\},$$

$$z^{\mathrm{new}}_{\alpha\gamma} := z_{\alpha\beta}, \quad z^{\mathrm{new}}_{\beta\gamma} := z_{\beta\alpha}, \quad z^{\mathrm{new}}_{\gamma\alpha} := 0, \quad z^{\mathrm{new}}_{\gamma\beta} := \infty, \quad \alpha^{\mathrm{new}}_k := \gamma, \quad z^{\mathrm{new}}_k := 1.$$

If $\alpha \in V$ then we define $\bar{u}_\gamma : S^2 \to \bar{M}$ to be the constant map equal to $\overline{\mathrm{ev}}_\infty(w_\alpha)$, where $\overline{\mathrm{ev}}$ is defined as in (3.5). If $\alpha \in \bar{T}$ then we define $\bar{u}_\gamma :\equiv \bar{u}_\alpha(z_{\alpha\beta})$. So in both cases the new component is a "ghost", i.e. carries no energy. The tuple $(\mathbf{w}^{\mathrm{new}}, \bar{\mathbf{u}}^{\mathrm{new}}, \mathbf{z}^{\mathrm{new}})$ obtained from $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ in this way is again a stable map. For every $\alpha \in T^{\mathrm{new}}$ we define $z^{\mathrm{new}}_{\alpha,0}$ as in (3.7) and (3.8), with $i := 0$ and w.r.t. to the new tree T^{new}. By interchanging α and β if necessary, we may assume w.l.o.g. that β is contained in the chain of edges from α to α_0. It follows that for $\alpha \neq \gamma$, $z^{\mathrm{new}}_{\alpha,0} = z_{\alpha,0}$, where $z_{\alpha,0}$ is defined as in (3.7) and (3.8), with $i := 0$ and w.r.t. to the old tree T, and $z^{\mathrm{new}}_{\gamma,0} = z^{\mathrm{new}}_{\gamma,\beta} = \infty$. Furthermore, the hypotheses of Lemma 4.18 (Middle rescaling) with

$$x := z^{\mathrm{new}}_{\beta\gamma}, \quad x' := z_{\beta,0}, \quad x_\nu := z^\nu_{\beta k} := (\varphi^\nu_\beta)^{-1}(z^\nu_k), \quad y := z^{\mathrm{new}}_{\alpha\gamma},$$
$$\varphi_\nu := \varphi^\nu_{\alpha\beta} = (\varphi^\nu_\alpha)^{-1} \circ \varphi^\nu_\beta$$

are satisfied. It follows that there exists a sequence of Möbius transformations ψ_ν such that $\psi_\nu(1) = z^\nu_{\beta k}$, ψ_ν converges to $z^{\mathrm{new}}_{\beta\gamma}$, uniformly with all derivatives on compact subsets (u.c.s.) of $S^2 \setminus \{\infty\}$, $\varphi^\nu_{\alpha\beta} \circ \psi_\nu$ converges to $z^{\mathrm{new}}_{\alpha\gamma}$, u.c.s. on $S^2 \setminus \{0\}$, and $\psi_\nu(\infty) = z_{\beta,0}$. We define

$$\varphi^\nu_\gamma := \varphi^\nu_\beta \circ \psi_\nu.$$

Claim 10 *With these choices the sequence* $(w_\nu, z^\nu_0, \ldots, z^\nu_k)$ *converges to* $(\mathbf{w}^{\mathrm{new}}, \bar{\mathbf{u}}^{\mathrm{new}}, \mathbf{z}^{\mathrm{new}})$ *along the sequence of collections of Möbius transformations* $\{\varphi^\nu_\alpha\}_{\alpha \in T^{\mathrm{new}}, \nu \in \mathbb{N}}$.

Proof of Claim 10: Condition (3.9) (energy conservation) still holds, since the new vertex carries no energy. We verify condition 3.6(i). For the first part, there is nothing to check, since $\gamma \in \bar{T}$. Furthermore,

$$\begin{aligned} \varphi^\nu_\gamma(z^{\mathrm{new}}_{\gamma,0}) &= \varphi^\nu_\gamma(\infty) \\ &= \varphi^\nu_\beta \circ \psi_\nu(\infty) \\ &= \varphi^\nu_\beta(z_{\beta,0}) = \infty. \end{aligned} \tag{4.131}$$

Hence the second part holds. To see that the third part holds, observe that (4.131) implies that there exist numbers $\lambda^\nu_\gamma \in \mathbb{C} \setminus \{0\}$ and $z^\nu_\gamma \in \mathbb{C}$ such that $\varphi^\nu_\gamma(z) = \lambda^\nu_\gamma z + z^\nu_\gamma$. We choose a Möbius transformation ψ_α such that $\psi_\alpha(\infty) = z_{\alpha,0}$. We define $\widetilde{\varphi}^\nu_\alpha := \varphi^\nu_\alpha \circ \psi_\alpha$ and $\varphi_\nu := \psi_\alpha^{-1} \circ \varphi^\nu_{\alpha\beta} \circ \psi_\nu$. Then $\widetilde{\varphi}^\nu_\alpha(\infty) = \infty$ and $\varphi_\nu(\infty) = \infty$, and hence there exist numbers $\lambda^\nu_\alpha, \lambda_\nu \in \mathbb{C} \setminus \{0\}$ and $z^\nu_\alpha, z_\nu \in \mathbb{C}$ such that

$$\widetilde{\varphi}_\alpha(z) = \lambda^\nu_\alpha z + z^\nu_\alpha, \qquad \varphi_\nu(z) = \lambda_\nu z + z_\nu.$$

By the condition 3.6(i) for α, we have $\lambda^\nu_\alpha \to \infty$. Furthermore, since $\varphi^\nu_{\alpha\beta} \circ \psi_\nu$ converges to $z_{\alpha\beta}$, uniformly on every compact subset of $S^2 \setminus \{z_{\beta\alpha}\}$, it follows that $\lambda_\nu \to \infty$. Since

$$\lambda^\nu_\gamma \cdot + z^\nu_\gamma = \varphi^\nu_\gamma = \widetilde{\varphi}^\nu_\alpha \circ \varphi_\nu = \lambda^\nu_\alpha(\lambda_\nu \cdot + z_\nu) + z^\nu_\alpha,$$

it follows that λ_γ^ν converges to ∞, for $\nu \to \infty$. This proves part three of 3.6(i) for the vertex γ.

We check **condition 3.6(ii)**. Let $\delta E \delta'$ be an edge. It suffices to consider the cases $(\delta,\delta') = (\alpha,\gamma)$ and $(\delta,\delta') = (\beta,\gamma)$. The condition holds, since $\varphi^\nu_{\beta\gamma} = \psi_\nu$ converges to $z^{\mathrm{new}}_{\beta\gamma}$, u.c.s. on $S^2 \setminus \{\infty\} = S^2 \setminus \{z^{\mathrm{new}}_{\gamma\beta}\}$, and $\varphi^\nu_{\alpha\gamma} = \varphi^\nu_{\alpha\beta} \circ \psi_\nu$ converges to $z^{\mathrm{new}}_{\alpha\gamma}$, u.c.s. on $S^2 \setminus \{0\} = S^2 \setminus \{z^{\mathrm{new}}_{\gamma\alpha}\}$.

To see that **condition 3.6(iii)** holds, fix a number $\varepsilon > 0$. As in the proof of Proposition 5.4 (see inequality (5.32)) there exist numbers $R > 0$, $\nu_0 \in \mathbb{N}$ such that for every $\nu \geq \nu_0$ and every

$$z \in B^{S^2}_{R^{-1}}(z^{\mathrm{new}}_{\beta\gamma}) \setminus \varphi^\nu_{\beta\gamma}(B_R)$$

we have

$$\bar{d}(\bar{u}_\beta(z^{\mathrm{new}}_{\beta\gamma}), \mathrm{G}u^\nu_\beta(z)) < \varepsilon.$$

As in Case (II) with $\gamma \in \bar{T}^{\mathrm{new}}$ and $i = 0$ it follows that $\mathrm{G}u^\nu_\gamma$ converges to the constant map $\bar{u}_\gamma \equiv \bar{u}_\beta(z^{\mathrm{new}}_{\beta\gamma})$, uniformly on compact subsets of $\mathbb{C} \setminus \{0\} = S^2 \setminus \{z^{\mathrm{new}}_{\gamma\alpha}, z^{\mathrm{new}}_{\gamma\beta}\}$. To see that the convergence is in C^1 on every compact subset of $\mathbb{C} \setminus \{0\}$, writing $\lambda^\nu_\gamma =: R^\nu_\gamma e^{i\varphi^\nu_\gamma}$ we define

$$w^\nu_\gamma := (u^\nu_\gamma, \Phi^\nu_\gamma, \Psi^\nu_\gamma) := \big(u_\nu, R^\nu_\gamma \Phi_\nu, R^\nu_\gamma \Psi_\nu\big)(R^\nu_\gamma \cdot + z^\nu_\gamma).$$

Since $R^\nu_\gamma \to \infty$, the hypotheses of Proposition 4.3 with $r_\nu := \nu$ and R_ν, w_ν replaced by $R^\nu_\gamma, w^\nu_\gamma$ are satisfied. It follows that there exist a finite subset $Z \subseteq \mathbb{C}$ and an ∞-vortex $w_\gamma := (u_\gamma, \Phi_\gamma, \Psi_\gamma)$ such that passing to some subsequence the assertions 4.3(i)-(iii) hold. Condition 3.6(iii) follows now as in Case (II) with $\gamma \in \bar{T}^{\mathrm{new}}$ and $i = 0$. Finally, by construction

$$(\varphi^\nu_{\alpha^{\mathrm{new}}_k})^{-1}(z^\nu_k) = (\varphi^\nu_\gamma)^{-1}(z^\nu_k) = 1 = z^{\mathrm{new}}_k,$$

and hence the last **condition (iv)** of Definition 3.6 is also satisfied. This proves Claim 10. □

This proves the induction step and hence terminates the proof of Theorem 4.1 in the general case. □

5 Conservation of the equivariant homology class

Fix a contractible topological space EG on which G acts continuously and freely. For a proof that such a space exists see the book [Hu] by Husemoller. We abbreviate $M_{\mathrm{G}} := (M \times \mathrm{EG})/\mathrm{G}$. We fix a number $p > 2$. Every finite energy solution $w \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ of the vortex equations (0.1), (0.2) on the plane $\mathbb{C}$ such that $\overline{u(\mathbb{C})} \subseteq M$ is compact carries an equivariant homology class $[w]_{\mathrm{G}} \in H_2(M_{\mathrm{G}}, \mathbb{Z})$. More generally, we will define such a class for every stable $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ of vortices on $\mathbb{C}$ and pseudo-holomorphic spheres in the symplectic quotient $\bar{M}$. The main result of this section states that if $(w_\nu, z_0^\nu, \ldots, z_k^\nu)$ is a sequence of vortices with marked points that converges to a stable map $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ then for ν large enough the equivariant homology class of w_ν equals the equivariant homology class of $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$.

Recall that M^* denotes the set of all points $x \in M$ for which $gx = x$ implies that $g = \mathrm{id}$. By the Corollary A.6 of the local slice Theorem, M^* is an open subset of M and the canonical projection $\pi : M^* \to M^*/\mathrm{G}$ defines a principal G-bundle. The map

$$\pi_{\mathrm{G}} : M^*_{\mathrm{G}} := (M^* \times \mathrm{EG})/\mathrm{G} \to M^*/\mathrm{G}, \qquad \pi_{\mathrm{G}}([x, e]) := \pi(x)$$

defines a continuous fibre bundle with fibre EG. We denote by

$$\iota_{\mathrm{G}} : M^*_{\mathrm{G}} \to M_{\mathrm{G}} := (M \times \mathrm{EG})/\mathrm{G} \tag{5.1}$$

the map induced by the inclusion $\iota : M^* \to M$. Since EG is contractible, there exists a continuous section $s : M^*/\mathrm{G} \to M^*_{\mathrm{G}}$. For every continuous equivariant map $\theta : M^* \to \mathrm{EG}$ we define the section $s_\theta : M^*/\mathrm{G} \to M^*_{\mathrm{G}}$ by

$$s_\theta(\mathrm{G}x) := [x, \theta(x)]. \tag{5.2}$$

By Theorem I.4.8.1 (p.46) in the book [Hu] by Husemoller, the map $\theta \mapsto s_\theta$ from the set of continuous equivariant maps $M^* \to \mathrm{EG}$ to the set of continuous sections of M^*_{G} is a bijection. We denote the fundamental class of a closed oriented manifold X of dimension $m \in \mathbb{N} \cup \{0\}$ by $[X] \in H_m(X, \mathbb{Z})$.

Definition 5.1 (Equivariant homology class) *Let $w := (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$ be a finite energy vortex such that $\overline{u(\mathbb{C})}$ is compact. Let $f : S^2 \to (M \times \mathrm{EG})/\mathrm{G}$ be a continuous map for which there exists a continuous map $\mathsf{e} : \mathbb{C} \to \mathrm{EG}$ such that*

$$f(z) = [u(z), \mathsf{e}(z)], \qquad \forall z \in \mathbb{C}.$$

We define the equivariant homology class $[w]_{\mathrm{G}}$ *to be the pushforward of the fundamental class $[S^2]$ under f,*

$$[w]_{\mathrm{G}} := f_*[S^2] \in H_2(M_{\mathrm{G}}, \mathbb{Z}).$$

Moreover, we define the equivariant homology class *of a stable map of vortices on the plane and pseudo-holomorphic spheres in* $\bar{M}$

$$(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}) := \big(V, \bar{T}, E, (w_\alpha)_{\alpha\in V}, (\bar{u}_\alpha)_{\alpha\in\bar{T}}, (z_{\alpha\beta})_{\alpha E\beta}, (\alpha_i, z_i)_{i=0,\dots,k}\big)$$

to be the class

$$[\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}]_{\mathrm{G}} := \sum_{\alpha\in V}[w_\alpha]_{\mathrm{G}} + \sum_{\alpha\in\bar{T}}(\iota_{\mathrm{G}} \circ s \circ \bar{u}_\alpha)_*[S^2],$$

where ι_{G} *is as in (5.1) and* $s : M^*/\mathrm{G} \to M^*_{\mathrm{G}}$ *is a section.*

Lemma 5.2 *The equivariant homology class* $[w]_{\mathrm{G}}$ *is welldefined, i.e. the required map* f *exists and* $[w]_{\mathrm{G}}$ *does not depend on the choice of* f*. Furthermore, if* $g \in \mathcal{G}^{2,p}$ *is a gauge transformation then* $[g^*w]_{\mathrm{G}} = [w]_{\mathrm{G}}$*. Moreover,* $[\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}]_{\mathrm{G}}$ *does not depend on the choice of* s*.*

Proof of Lemma 5.2: Fix a vortex $w := (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$. By part (A) of Proposition D.6 the map $\mathrm{G}u : \mathbb{C} \to M/\mathrm{G}$ extends continuously to a map $\bar{u} : S^2 = \mathbb{C} \cup \{\infty\} \to M/\mathrm{G}$, the extension is unique, and $\bar{u}(\infty)$ lies in the symplectic quotient $\bar{M} \subseteq M/\mathrm{G}$.

Claim 1 *There exists a number* $R > 0$ *such that for* $z \in S^2 \setminus B_R$ *we have* $\bar{u}(z) \in M^*/\mathrm{G}$*.*

Proof of Claim 1: By hypothesis (H1) the restriction of the action of G to $\mu^{-1}(0)$ is free. Therefore we have $\mu^{-1}(0) \subseteq M^*$, and hence

$$\bar{M} \subseteq M^*/\mathrm{G}. \tag{5.3}$$

Since the subset $M^* \subseteq M$ is open, Lemma A.2 implies that subset $M^*/\mathrm{G} \subseteq M/\mathrm{G}$ is open. Hence by the continuity of $\bar{u}$ the subset $\bar{u}^{-1}(M^*/\mathrm{G}) \subseteq S^2$ is open. Furthermore, it contains the point $\infty \in S^2$, because of $\bar{u}(\infty) \in \bar{M}$ and (5.3). Claim 1 follows from this. □

We define the map $f : S^2 \to M_{\mathrm{G}} = (M \times \mathrm{EG})_{\mathrm{G}}$ as follows. Fix a continuous equivariant map $\theta : M^* \to \mathrm{EG}$ and let $R > 0$ be as in Claim 1. Consider the map

$$\theta \circ u|_{\mathbb{C}\setminus B_R} : \mathbb{C} \setminus B_R \to \mathrm{EG}\,.$$

Since EG is contractible, we can extend this map continuously to a map $\mathsf{e} : \mathbb{C} \to \mathrm{EG}$. Let $s_\theta : M^*/\mathrm{G} \to M^*_{\mathrm{G}}$ be as in (5.2). We define $f : S^2 = \mathbb{C} \cup \{\infty\} \to M_{\mathrm{G}}$ by

$$f(z) := \begin{cases} [u(z), \mathsf{e}(z)], \text{if } z \in \mathbb{C}, \\ s_\theta \circ \bar{u}(\infty), \text{if } z = \infty. \end{cases}$$

Then the map f satisfies

$$f|_{S^2\setminus B_R} = s_\theta \circ \bar{u}|_{S^2\setminus B_R}.$$

Since $\bar{u} : S^2 \to M/\mathrm{G}$ is continuous, the same holds for the restriction $f|_{S^2\setminus B_R}$. It follows that the map $f : S^2 \to M_{\mathrm{G}}$ satisfies all the requirements of Definition 5.1.

Assume now that $f, f' : S^2 \to M_{\mathrm{G}}$ are two maps that satisfy the requirements of Definition 5.1. We choose any two maps $\mathsf{e}, \mathsf{e}' : \mathbb{C} \to \mathrm{EG}$ as in Definition 5.1, corresponding to f and f'. We claim that we may assume w.l.o.g. that there exists a continuous map $g_0 : S^1 \to \mathrm{G}$ and a point $x_0 \in \mu^{-1}(0)$ such that

$$\lim_{r\to\infty} \sup_{z\in S^1} d(u(rz), g_0(z)x_0) = 0, \tag{5.4}$$

$$g_0(1) = \mathrm{id}. \tag{5.5}$$

To see this, assume for the moment that we have already proved the Lemma in this case. By Proposition D.3 there is gauge transformation $g \in \mathcal{G}^{2,p}$ such that $(\widehat{u}, \widehat{\Phi}, \widehat{\Psi}) := g^*w$ is smooth and in radial gauge outside some ball containing the origin. Thus by Proposition D.7(B) with $p := 2$ there exist a point $x_0 \in \mu^{-1}(0)$ and a map $g_0 \in W^{1,2}(S^1, \mathrm{G})$ such that (5.4) holds with u replaced by $\widehat{u}$. Defining $\widehat{\mathsf{e}} := g^{-1}\mathsf{e}, \widehat{\mathsf{e}}' := g^{-1}\mathsf{e}' : \mathbb{C} \to \mathrm{EG}$ we have

$$[\widehat{u}, \widehat{\mathsf{e}}] = [g^{-1}u, g^{-1}\mathsf{e}] = [u, \mathsf{e}] = f$$

on the subset $\mathbb{C} \subseteq S^2$, and analogously $[\widehat{u}, \widehat{\mathsf{e}}'] = f'$ on $\mathbb{C}$. Thus our assumption implies that $f_*[S^2] = f'_*[S^2]$. So we assume that there are a function g_0 and a point x_0 as above.

We define the map $\psi_0 : \mathbb{D} \to S^2 \cong \mathbb{C} \cup \{\infty\}$ by

$$\psi_0(z) := \begin{cases} \tan\left(\frac{\pi|z|}{2}\right)\frac{z}{|z|}, & \text{if } z \neq 0,\ |z| < 1 \\ 0, & \text{if } z = 0, \\ \infty, & \text{if } |z| = 1. \end{cases}$$

Then $\psi_0|_{B_1} : B_1 \to \mathbb{C}$ is an orientation preserving homeomorphism. By (5.4), we can continuously extend the map $u \circ \psi_0 : B_1 \to M$ to a map $v : \mathbb{D} \to M$, such that

$$v|_{S^1} = g_0 x_0. \tag{5.6}$$

Claim 2 *The maps* $\mathsf{e} \circ \psi_0, \mathsf{e}' \circ \psi_0 : B_1 \to \mathrm{EG}$ *can be continuously extended to maps* $\widetilde{\mathsf{e}}, \widetilde{\mathsf{e}}' : \mathbb{D} \to \mathrm{EG}$ *such that*

$$\widetilde{\mathsf{e}}|_{S^1} = g_0\widetilde{\mathsf{e}}(1), \tag{5.7}$$

$$\widetilde{\mathsf{e}}'|_{S^1} = g_0\widetilde{\mathsf{e}}'(1). \tag{5.8}$$

Proof of Claim 2: We choose an open subset $U \subseteq M^*$ containing the orbit $\bar{u}(\infty)$, on which there exists a local equivariant trivialization. This means that there is a function $\mathsf{g} : U \to \mathrm{G}$, equivariant w.r.t. to left multiplication on G, such that the map

$$(\pi|_U, \mathsf{g}) : U \to U/\mathrm{G} \times \mathrm{G}$$

is a diffeomorphism. The local trivialization $(\pi|_U, \mathsf{g})$ induces the local trivialization

$$(\pi_\mathrm{G}|_{(U\times \mathrm{EG})/\mathrm{G}}, e_\mathrm{G}) : U_\mathrm{G} := (U \times \mathrm{EG})/\mathrm{G} \to U/\mathrm{G} \times \mathrm{EG}$$

of the fibre bundle $M^*_\mathrm{G} = (M^* \times \mathrm{EG})/\mathrm{G}$. Here the map $e_\mathrm{G} : U_\mathrm{G} \to \mathrm{EG}$ is defined by

$$e_\mathrm{G}([x, e]) := \mathsf{g}(x)^{-1}e. \tag{5.9}$$

Since the map $\bar{u}$ is continuous and by Lemma A.2 the subset $U/\mathrm{G} \subseteq M/\mathrm{G}$ is open, it follows that the subset $\bar{u}^{-1}(U/\mathrm{G}) \subseteq S^2$ is open. It contains the point ∞, since $\bar{u}(\infty) \subseteq U$. Hence there is a number $R > 0$ such that for every $z \in \mathbb{C} \setminus B_R$ we have $u(z) \in U$. It follows that for every $z \in \mathbb{C} \setminus B_R$

$$\mathsf{e}(z) = \mathsf{g}(u(z)) \cdot e_\mathrm{G}([u(z), \mathsf{e}(z)]) = (\mathsf{g} \circ u(z)) \cdot e_\mathrm{G} \circ f(z).$$

Defining $r := \psi_0^{-1}(R)$, this means that on $B_1 \setminus B_r$ we have

$$\mathsf{e} \circ \psi_0 = (\mathsf{g} \circ u \circ \psi_0) \cdot e_\mathrm{G} \circ f \circ \psi_0 = (\mathsf{g} \circ v) \cdot e_\mathrm{G} \circ f \circ \psi_0 : B_1 \setminus B_r \to \mathrm{EG}\,.$$

Since the maps $\mathsf{g} : U \to \mathrm{G}$, $v : \mathbb{D} \to M$, $e_\mathrm{G} : U_\mathrm{G} \to \mathrm{EG}$ and $f : S^2 \to M_\mathrm{G}$ are continuous, it follows that we can uniquely extend the map $\mathsf{e} \circ \psi_0 : B_1 \to \mathrm{EG}$ to a continuous map $\widetilde{\mathsf{e}} : \mathbb{D} \to \mathrm{EG}$ such that

$$\widetilde{\mathsf{e}}(z) = \mathsf{g}(v(z)) \cdot e_\mathrm{G}(f(\infty)), \qquad \forall z \in S^1.$$

Observe that

$$\begin{aligned} f(\infty) &= \lim_{r\to\infty} f(r) \\ &= \lim_{r\to\infty} [u(r), \mathsf{e}(r)] \\ &= [g_0(1)x_0, \widetilde{e}(1)] \\ &= [x_0, \widetilde{e}(1)]. \end{aligned}$$

Here in the third step we have used (5.4), and in the last step we have used (5.5). It follows that

$$e_\mathrm{G}(f(\infty)) = \mathsf{g}(x_0)^{-1}\widetilde{e}(1),$$

and therefore

$$\begin{aligned} \widetilde{e}(z) &= \mathsf{g}(g_0(z)x_0)e_\mathrm{G}(f(\infty)) \\ &= g_0(z)\mathsf{g}(x_0)e_\mathrm{G}(f(\infty)) \\ &= g_0(z)\widetilde{e}(1). \end{aligned}$$

Hence condition (5.7) is satisfied. Similarly, it follows that the map $\mathsf{e}' \circ \psi_0 : B_1 \to \mathrm{EG}$ can be extended to a continuous map $\widetilde{e}' : \mathbb{D} \to \mathrm{EG}$ such that condition (5.8) holds. This proves Claim 2. □

Let $\mathbb{D}'$ be a copy of the unit disk $\mathbb{D}$ and $\widetilde{\mathsf{e}} : \mathbb{D} \to \mathrm{EG}$ and $\widetilde{\mathsf{e}}' : \mathbb{D}' \to \mathrm{EG}$ be as in Claim 2. Consider the connected sum

$$X := \mathbb{D}\#([0,1] \times S^1)\#\mathbb{D}'$$

obtained from the cylinder $[0,1] \times S^1$ and two disks $\mathbb{D}, \mathbb{D}'$ by identifying the boundary of $\mathbb{D}$ with the left boundary of $[0,1] \times S^1$ and the boundary of $\mathbb{D}'$ with the right boundary of $[0,1] \times S^1$. The space X is homeomorphic to S^2. We choose a continuous path $\gamma : [0,1] \to \mathrm{EG}$ joining $\widetilde{\mathsf{e}}(1)$ with $\widetilde{\mathsf{e}}'(1)$. By (5.7) and (5.8) the maps $\widetilde{\mathsf{e}}$, $\widetilde{\mathsf{e}}'$ and and the map

$$[0,1] \times S^1 \ni (\lambda, z) \mapsto g_0(z)\gamma(\lambda) \in \mathrm{EG}$$

can be connected to give a continuous map

$$E : X \to \mathrm{EG}\,.$$

Since EG is contractible, there exists a continuous map

$$h : [0,1] \times \mathbb{D} \to \mathrm{EG}$$

such that $h|_{\partial X} = E$, i.e.

$$\begin{aligned} h(0,z) = \widetilde{\mathsf{e}}(z), \qquad h(1,z) &= \widetilde{\mathsf{e}}'(z), \qquad \forall z \in \mathbb{D}, \\ h(\lambda, z) &= g_0(z)\gamma(\lambda), \quad \forall \lambda \in [0,1],\ z \in S^1. \end{aligned}$$

We define $F : [0,1] \times S^2 \to M_{\mathrm{G}} = (M \times \mathrm{EG})/\mathrm{G}$ by

$$F(\lambda, z) := \begin{cases} [u(z), h(\lambda, \psi_0^{-1}(z))], & \text{if } z \neq \infty, \\ [v(1), \gamma(\lambda)], & \text{if } z = \infty. \end{cases}$$

We denote by $\mathrm{pr}_1 : \mathbb{D} \times [0,1] \to [0,1]$ the projection to the first factor and by $\mathrm{id} \times \psi_0$ the map

$$[0,1] \times \mathbb{D} \ni (\lambda, z) \mapsto (\lambda, \psi_0(z)) \in [0,1] \times S^2.$$

It follows that

$$F \circ (\mathrm{id} \times \psi_0) = [v \circ \mathrm{pr}_1, h] : [0,1] \times \mathbb{D} \to M_{\mathrm{G}}.$$

Since the map $[v \circ \mathrm{pr}_1, h]$ is continuous and the map $\mathrm{id} \times \psi_0$ is open, it follows that the map $F : [0,1] \times S^2 \to M_{\mathrm{G}}$ is continuous. Furthermore, $F(0,z) = f(z)$ and $F(1,z) = f'(z)$ for every $z \in S^2$, so F is a homotopy from f to f'. Therefore, the pushforwards $f_*[S^2]$ and $f'_*[S^2]$ agree. This shows that $[w]_{\mathrm{G}}$ does not depend on the choice of f.

To prove the second statement of the lemma, let $g \in \mathcal{G}^{2,p}$ be a gauge transformation. We fix a continuous map $f : S^2 \to M_{\mathrm{G}}$ such that there exists a continuous map $e : \mathbb{C} \to \mathrm{EG}$ with $f|_{\mathbb{C}} = [u, e]$. Defining $w' := g^* w$ and $e' := g^{-1} e$, we have

$$f|_{\mathbb{C}} = [u, e] = [g^{-1}u, g^{-1}e] = [u', e'],$$

and hence $[w']_{\mathrm{G}} = f_*[S^2] = [w]_{\mathrm{G}}$.

To prove the last statement of the lemma, observe that each two sections $s, s' : M^*/\mathrm{G} \to M^*_{\mathrm{G}}$ are homotopic to each other, since the fibre EG is contractible. It follows that for every $\alpha \in \bar{T}$ the equivariant homology classes $(\iota_{\mathrm{G}} \circ s \circ \bar{u}_\alpha)_*[S^2]$ and $(\iota_{\mathrm{G}} \circ s' \circ \bar{u}_\alpha)_*[S^2]$ agree.

This proves Lemma 5.2. □

Remark 5.3 For a vortex, in [GS], D. Salamon and R. Gaio define the equivariant homology class in a different but equivalent way, constructing a certain principal G-bundle over S^2. The above definition is more adapted for the proof of the main result, which is the following proposition.

Proposition 5.4 (Conservation of homology class) *Assume that (H2) (convexity at ∞) holds. Let $(w^\nu, z_0^\nu := \infty, z_1^\nu, \dots, z_k^\nu) \in \widetilde{\mathcal{M}}^{1,p} \times \{\infty\} \times \mathbb{C}^k$ be a sequence of vortices with marked points that converges to some stable map $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$. Then for ν large enough $[w^\nu]_{\mathrm{G}} = [\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}]_{\mathrm{G}}$.*

Remark 5.5 Intuitively, one might expect that even the equivariant *homotopy* class of w^ν is conserved in the limit. However, it is not obvious how to define the equivariant *homotopy* class of a vortex or a stable map, even in the case $\mathrm{G} = \{\mathbf{1}\}$. One attempt to define such a homotopy class is the following. Let $\mathrm{G} := \{\mathbf{1}\}$, let (M, J) be an almost complex manifold and $x_0 \in M$ be a fixed point. We denote the set of (free) homotopy classes of maps from $S^2 \to M$ by $[S^2, M]$, and we denote the homotopy class of a continuous map $u : S^2 \to M$ by $[u] \in [S^2, M]$. For $k \in \mathbb{N}$ we denote the based homotopy class of a continuous map $u : S^k \to M$ by $[u]_0$. The fundamental group $\pi_1(M, x_0)$ acts on the group $\pi_2(M, x_0)$ as follows. Let $[\gamma]_0 \in \pi_1(M, x_0)$ and $[u]_0 \in \pi_2(M, x_0)$ and let $\gamma \# u : S^2 \to M$ be the map obtained by running through γ on the lower hemisphere and through u on the upper hemisphere. We define

$$([\gamma]_0)_*[u]_0 := [\gamma \# u]_0 \in \pi_2(M, x_0)$$

The map

$$\pi_2(M,x_0)/\pi_1(M,x_0) \ni \pi_1(M,x_0)_*[u]_0 \mapsto [u] \in [S^2, M]$$

is welldefined and a bijection. Let

$$(\mathbf{u},\mathbf{z}) = (T, (z_{\alpha\beta})_{\alpha E \beta}, (z_i)_{i=1,\dots,k}, (u_\alpha)_{\alpha\in T})$$

be a stable of J-holomorphic spheres in M. One attempt to define the homotopy class of $(\mathbf{u},\mathbf{z})$ would be

$$[\mathbf{u}] := \sum_{\alpha\in T} [\gamma_\alpha \# u_\alpha] \in \pi_2(M,x_0)/\pi_1(M,x_0) = [S^2, M]. \tag{5.10}$$

Here $\gamma_\alpha : [0,1] \to M$ is a path connecting x_0 with $u_\alpha(0) \in M$. However, this does not always make sense, since the addition on $\pi_2(M,x_0)$ need not descend to a group structure on $\pi_2(M,x_0)/\pi_1(M,x_0)$, as the following example shows. Consider the manifold

$$M := \mathbb{R}^4 \setminus \big(\mathbb{R}\times\{(0,0,0)\} \cup \mathbb{R}^2 \times \{(1,0)\}\big) = \mathbb{R}\times\big(\mathbb{R}^3\setminus(\{(0,0,0)\}\cup\mathbb{R}\times\{(1,0)\})\big)$$

and let $x_0 \in M$ be a point. There is a deformation retraction $r : M \to X := S^1 \vee S^2$. For $k \in \mathbb{N}$ and $y_0 := r(x_0)$ this induces isomorphisms $r_* : \pi_k(M,x_0) \cong \pi_k(X,y_0)$. Furthermore, for $a \in \pi_1(M,x_0)$, $b \in \pi_2(M,x_0)$ we have $r_*(a_* b) = (r_* a)_*(r_* b)$. Therefore, it is enough to show that the addition on $\pi_2(X,y_0)$ does not descend to a group structure on $\pi_2(X,y_0)/\pi_1(X,y_0)$. Let $\gamma : S^1 \to X = S^1 \vee S^2$ and $\iota : S^2 \to S^1 \vee S^2$ be the inclusions. Then $\pi_2(X,y_0)$ is the free abelian group generated by $e_i := ([\gamma]_0)^i_*[\iota]_0$, for $i \in \mathbb{Z}$ (see for example 16.5.9 Beispiel in the book [SZ] by R. Stöcker and H. Zieschang). Consider the classes $B_1 := B_2 := \pi_1(X,y_0)_* e_0 \in \pi_2(X,y_0)/\pi_1(X,y_0)$. Then $B_1 + B_2$ is not welldefined. Namely, let $b_1 := b_2 := b'_1 := e_0$ and $b'_2 := e_1$. Then $B_i = \pi_1(X,y_0)_* b_i = \pi_1(X,y_0)_* b'_i$ for $i = 1,2$. Let $a \in \pi_1(X,y_0) \cong \pi_1(S^1,1) \cong \mathbb{Z}$. Let $i \in \mathbb{Z}$ be such that $[\gamma]^i_0 = a$. Then

$$a_*(b_1 + b_2) = 2e_i \neq b'_1 + b'_2 = e_i + e_{i+1}.$$

Therefore,

$$\pi_1(X,y_0)_*(b_1+b_2) \neq \pi_1(X,y_0)_*(b'_1+b'_2).$$

This shows that $B_1 + B_2$ is not welldefined and therefore (5.10) does not make sense.

Proof of Proposition 5.4: Let w_ν, $z_0^\nu, \dots, z_k^\nu$ and $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ be as in the hypothesis and let φ_α^ν be a sequence of Möbius transformations for $\alpha \in T := V \sqcup \bar{T}$ so that conditions (i)-(iv) of Definition 3.6 are satisfied. We may assume W.l.o.g. that $z_0 = \infty$ and for every $\alpha \in \bar{T}$ there are sequences of numbers $\lambda_\alpha^\nu \in \mathbb{C} \setminus \{0\}$ and $z_\alpha^\nu \in \mathbb{C}$ such that

$$\varphi_\alpha^\nu(z) = \lambda_\alpha^\nu z + z_\alpha^\nu. \tag{5.11}$$

To see this we choose Möbius transformations ψ_α for $\alpha \in \bar{T}$ so that $\psi_\alpha(\infty) = z_{\alpha,0}$, where $z_{\alpha,0}$ is given by (3.7), (3.8), set $\psi_\alpha :=$ id for $\alpha \in V$, and define

$$\bar{v}_\alpha := \bar{u}_\alpha \circ \psi_\alpha, \qquad \widetilde{\varphi}_\alpha^\nu := \varphi_\alpha^\nu \circ \psi_\alpha, \qquad \widetilde{z}_{\alpha\beta} := \psi_\alpha^{-1}(z_{\alpha\beta}), \qquad \widetilde{z}_i := \psi_{\alpha_i}^{-1}(z_i),$$

$$(\widetilde{\mathbf{w}}, \bar{\mathbf{v}}, \widetilde{\mathbf{z}}) := \big(V, \bar{T}, E, (w_\alpha)_{\alpha\in V}, (\bar{v}_\alpha)_{\alpha\in\bar{T}}, (\widetilde{z}_{\alpha\beta})_{\alpha E\beta}, (\widetilde{z}_i)_{i=0,\dots,k}\big)$$

Then conditions 3.2(i)-(iii) and 3.6(i)-(iv) are satisfied with $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ and φ_α^ν replaced by $(\widetilde{\mathbf{w}}, \bar{\mathbf{v}}, \widetilde{\mathbf{z}})$ and $\widetilde{\varphi}_\alpha^\nu$. Furthermore,

$$[\widetilde{\mathbf{w}}, \bar{\mathbf{v}}, \widetilde{\mathbf{z}}]_{\mathrm{G}} = [\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}]_{\mathrm{G}}.$$

So we assume that $z_0 = \infty$ and $\varphi_\alpha^\nu(z) = \lambda_\alpha^\nu z + z_\alpha^\nu$ for some numbers $\lambda_\alpha^\nu \in \mathbb{C} \setminus \{0\}$ and $z_\alpha^\nu \in \mathbb{C}$, for $\alpha \in \bar{T}$. For $\alpha \in V$ let $z_\alpha^\nu \in \mathbb{C}$ be such that $\varphi_\alpha^\nu(z) = z + z_\alpha^\nu$.

We introduce the following ordering $>$ on T. For $\alpha \neq \beta \in T$ we write $\alpha > \beta$ iff the chain of edges $[\alpha_0, \beta]$ contains α. It follows that $\alpha_0 > \alpha$ for every $\alpha_0 \neq \alpha \in T$. We denote by $\widetilde{E} \subseteq T \times T$ the subrelation of E consisting of all pairs $(\alpha, \beta) \in E$ such that $\alpha > \beta$. We write $\alpha\widetilde{E}\beta$ iff $(\alpha, \beta) \in \widetilde{E}$.

Claim 1 *For every pair $\alpha\widetilde{E}\beta$ we have*

$$\lim_{R\to\infty} \limsup_{\nu\to\infty} E\Big(w_\nu, B_{R^{-1}|\lambda_\alpha^\nu|}(\lambda_\alpha^\nu z_{\alpha\beta} + z_\alpha^\nu) \setminus B_{R|\lambda_\beta^\nu|}(z_\beta^\nu)\Big) = 0. \tag{5.12}$$

Furthermore,

$$\lim_{R\to\infty} \limsup_{\nu\to\infty} E\big(w_\nu, \mathbb{C} \setminus \varphi_{\alpha_0}^\nu(B_R)\big) = 0. \tag{5.13}$$

Proof of Claim 1: We prove (5.12). Let $\delta > 0$ be a constant. For every $\alpha \in T$ and $R > 0$ we denote

$$K_{\alpha,R} := \bar{B}_R \setminus \bigcup_{\alpha\widetilde{E}\beta} B_{R^{-1}}(z_{\alpha\beta}).$$

We choose $R > 0$ so large that

$$\bar{B}_{R^{-1}}(z_{\alpha\beta}) \subseteq B_R, \qquad \text{if } \alpha\widetilde{E}\beta, \tag{5.14}$$

$$\bar{B}_{R^{-1}}(z_{\alpha\beta}) \cap \bar{B}_{R^{-1}}(z_{\alpha\gamma}) = \emptyset, \qquad \text{if } \alpha\widetilde{E}\beta,\ \alpha\widetilde{E}\gamma, \tag{5.15}$$

$$\textstyle\sum_{\alpha\in T} E(w_\alpha, \mathbb{C} \setminus K_{\alpha,R}) < \delta, \tag{5.16}$$

where by abuse of notation we set $w_\alpha := \bar{u}_\alpha$ if $\alpha \in \bar{T}$. Fix a pair $\alpha\widetilde{E}\beta$. Denoting

$$A_\nu := B_{(2R)^{-1}|\lambda_\alpha^\nu|}(\lambda_\alpha^\nu z_{\alpha\beta} + z_\alpha^\nu) \setminus B_{2R|\lambda_\beta^\nu|}(z_\beta^\nu),$$

we claim that

$$\limsup_{\nu\to\infty} E(w_\nu, A_\nu) < \delta. \tag{5.17}$$

To see this, observe that by conditions 3.6(i), 3.6(ii) and Lemma B.9 we have

$$\frac{\lambda_\beta^\nu}{\lambda_\alpha^\nu} \to 0, \qquad \frac{z_\beta^\nu - z_\alpha^\nu}{\lambda_\alpha^\nu} \to z_{\alpha\beta}.$$

Therefore there is a number ν_0 such that for every $\nu \geq \nu_0$ we have

$$B_{2R|\lambda_\beta^\nu|/|\lambda_\alpha^\nu|}\left(\tfrac{z_\beta^\nu - z_\alpha^\nu}{\lambda_\alpha^\nu}\right) \subseteq B_{(2R)^{-1}}(z_{\alpha\beta}), \tag{5.18}$$

$$(\varphi_\alpha^\nu)^{-1}(A_\nu) = B_{(2R)^{-1}}(z_{\alpha\beta}) \setminus B_{2R|\lambda_\beta^\nu|/|\lambda_\alpha^\nu|}\left(\tfrac{z_\beta^\nu - z_\alpha^\nu}{\lambda_\alpha^\nu}\right) \subseteq B_{R^{-1}}(z_{\alpha\beta}). \tag{5.19}$$

It follows from condition (ii) of Definition 3.6 that increasing ν_0 if necessary, we have for every $\nu \geq \nu_0$ and every pair $\gamma\widetilde{E}\gamma'$

$$(\varphi_\gamma^\nu)^{-1} \circ \varphi_{\gamma'}^\nu(\bar{B}_R) \subseteq B_{R^{-1}}(z_{\gamma\gamma'}). \tag{5.20}$$

Fix $\nu \geq \nu_0$. Fix a pair $\gamma > \gamma'$. Setting $(\gamma_0, \ldots, \gamma_\ell) := [\gamma, \gamma']$ and using (5.20), it follows that

$$\begin{aligned} \varphi_{\gamma'}^\nu(\bar{B}_R) &\subseteq \varphi_{\gamma_{\ell-1}}^\nu(B_{R^{-1}}(z_{\gamma_{\ell-1}\gamma'})) \\ &\subseteq \varphi_{\gamma_{\ell-1}}^\nu(B_R) \\ &\subseteq \ldots \\ &\subseteq \varphi_\gamma^\nu(B_{R^{-1}}(z_{\gamma\gamma_1})). \end{aligned} \tag{5.21}$$

For $\gamma \in T$ we define

$$K_\gamma^\nu := \varphi_\gamma^\nu(K_{\gamma,R}),$$

It follows from (5.21) that for every pair $\gamma > \gamma'$ we have

$$K_{\gamma'}^\nu \subseteq \varphi_{\gamma'}^\nu(\bar{B}_R) \subseteq \varphi_\gamma^\nu(B_{R^{-1}}(z_{\gamma\gamma_1})) \subseteq \mathbb{C} \setminus K_\gamma^\nu,$$

$$\text{ie} \quad K_\gamma^\nu \cap K_{\gamma'}^\nu = \emptyset. \tag{5.22}$$

By (5.19) and (5.21) we have for every $\gamma > \alpha$

$$\begin{aligned} A_\nu &\subseteq \varphi_\alpha^\nu(B_{R^{-1}}(z_{\alpha\beta})) \\ &\subseteq \varphi_\alpha^\nu(B_R) \\ &\subseteq \varphi_\gamma^\nu(B_{R^{-1}}(z_{\gamma\gamma_1})) \\ &\subseteq \mathbb{C} \setminus K_\gamma^\nu, \end{aligned}$$

where $[\gamma,\alpha]=(\gamma,\gamma_1,\ldots,\alpha)$. Similarly, for every $\gamma\leq\alpha$ we have

$$K_\gamma^\nu\subseteq\mathbb{C}\setminus A_\nu.$$

It follows that

$$A_\nu\cap K_\gamma^\nu=\emptyset, \tag{5.23}$$

for every $\gamma\in T$. It follows from (5.22) and (5.23) that

$$E(w_\nu,A_\nu)\leq E(w_\nu,\mathbb{C})-\sum_{\gamma\in T}E(w_\nu,K_\gamma^\nu).$$

Taking the limit $\nu\to\infty$ we have

$$\limsup_{\nu\to\infty}E(w_\nu,A_\nu)\leq\lim_{\nu\to\infty}E(w_\nu,\mathbb{C})-\sum_{\gamma\in T}\liminf_{\nu\to\infty}E(w_\nu,K_\gamma^\nu). \tag{5.24}$$

Fix a $\gamma\in T$. If $\gamma\in V$ then it follows from condition 3.6(iii) with $Q:=\bar{B}_R$ that

$$\begin{aligned}\lim_{\nu\to\infty}E(w_\nu,K_\gamma^\nu) &= \lim_{\nu\to\infty}E\big((g_\gamma^\nu)^*(w_\nu(\cdot+z_\gamma^\nu)),\bar{B}_R\big)\\ &= E(w_\gamma,\bar{B}_R).\end{aligned} \tag{5.25}$$

Assume now that $\gamma\in\bar{T}$. Abbreviating $w_\gamma^\nu:=(u_\gamma^\nu,\Phi_\gamma^\nu,\Psi_\gamma^\nu):=(\varphi_\gamma^\nu)^*w_\nu$, since w_γ^ν is a $|\lambda_\gamma^\nu|$-vortex, Proposition B.3 implies that

$$\begin{aligned}E(w_\nu,K_\gamma^\nu) &= E^{|\lambda_\gamma^\nu|}\big(w_\gamma^\nu,K_{\gamma,R}\big)\\ &= \int_{K_{\gamma,R}}\big(|\partial_s u_\gamma^\nu+X_{\Phi_\gamma^\nu}(u_\gamma^\nu)|^2+|\lambda_\gamma^\nu|^2|\mu\circ u_\gamma^\nu|^2\big)\,ds\,dt.\end{aligned} \tag{5.26}$$

By condition 3.6(iii) we have for ν large enough

$$u_\nu(K_\gamma^\nu)=u_\gamma^\nu(K_{\gamma,R})\subseteq M^*,$$

and therefore by the definition of the Riemannian metric $\bar{g}$ on M^*/G

$$|\partial_s u_\gamma^\nu+X_{\Phi_\gamma^\nu}(u_\gamma^\nu)|\geq|\partial_s\mathrm{G}u_\gamma^\nu|.$$

Thus it follows from (5.26) and condition 3.6(iii) that

$$\begin{aligned}E(w_\nu,K_\gamma^\nu) &\geq \int_{K_{\gamma,R}}|\partial_s\mathrm{G}u_\gamma^\nu|^2\,ds\,dt\\ &\to \int_{K_{\gamma,R}}|\partial_s\bar{u}_\gamma|^2\,ds\,dt\\ &= E(\bar{u}_\gamma,K_{\gamma,R}).\end{aligned} \tag{5.27}$$

Combining (5.24), (5.25), (5.27) and (3.9), it follows that

$$\begin{aligned}\limsup_{\nu\to\infty} E(w_\nu, A_\nu) &\leq \sum_{\gamma\in V}\big(E(w_\gamma) - E(w_\gamma, \bar B_R)\big) + \sum_{\gamma\in\bar T}\big(E(\bar u_\gamma) - E(\bar u_\gamma, K_{\gamma,R})\big)\\ &= \sum_{\gamma\in T} E(w_\gamma, \mathbb{C}\setminus K_{\gamma,R}) < \delta.\end{aligned}$$

Here the last inequality follows from (5.16). This proves (5.17). It follows that (5.12) is satisfied. The proof of (5.13) is analogous. This proves Claim 1. □

Recall that $M^* \subseteq M$ denotes the set of points x such that if $ax = x$ then $a = e$, for every $a \in \mathrm{G}$. We denote by exp the exponential map on M w.r.t. the Riemannanian metric $g_{\omega,J} := \omega(\cdot, J\cdot)$, by $\overline{\exp}$ the exponential map on M^*/G w.r.t. the Riemannian metric $\bar g_{\omega,J}$ given as in (1.10), and by $d^{\bar g}$ the distance function on the components of M^*/G induced by $\bar g$. Recall also the definition (4.42) of the distance function $\bar d$ on the components of M/G. Since the set $\bar M \subseteq M^*/\mathrm{G}$ is compact, by Lemma A.11 there is a number $\varepsilon > 0$ such that for any two points $\bar x, \bar y \in \bar M$ satisfying $\bar d(\bar x, \bar y) < 2\varepsilon$ we have

$$\bar d(\bar x, \bar y) = d^{\bar g}(\bar x, \bar y).$$

We fix a G-invariant compact subset $K_0 \subseteq M$ such that $u(\mathbb{C}) \subseteq K_0$ for every vortex $(u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$, as in part (B) of Proposition D.6. Furthermore, we choose an open set $K_0 \subseteq U_0 \subseteq M$ such that $\bar U_0$ is compact. Decreasing ε if necessary, we may also assume that the maps

$$\begin{aligned}(\mathrm{id}, \exp) &: \{(x, v) \in TM \,|\, x \in U_0, |v| < \varepsilon\}\\ &\to \{(x, y) \in U_0 \times M \,|\, d(x, y) < \varepsilon\}\\ (x, v) &\mapsto (x, \exp_x v),\end{aligned}$$

$$\begin{aligned}(\mathrm{id}, \overline{\exp}) &: \{(\bar x, \bar v) \in T(M^*/\mathrm{G}) \,|\, \bar x \in \bar M,\ |\bar v| < 2\varepsilon\}\\ &\to \{(\bar x, \bar y) \in \bar M \times M^*/\mathrm{G} \,|\, d^{\bar g}(\bar x, \bar y) < 2\varepsilon\},\\ (\bar x, \bar v) &\mapsto (\bar x, \overline{\exp}_{\bar x}\bar v)\end{aligned}$$

are well-defined and diffeomorphisms. Recall the definition 3.1 of $\overline{\mathrm{ev}}$. We choose $r_0 > 0$ so large that conditions (5.14), (5.15) are satisfied with $R := r_0$ and such that for every $\alpha \in T$, $z \in \mathbb{C}\setminus B_{r_0}$

$$\bar d(\overline{\mathrm{ev}}_\infty(w_\alpha), \overline{\mathrm{ev}}_z(w_\alpha)) < \varepsilon, \tag{5.28}$$

and for every pair $\alpha\widetilde{E}\beta$ and every $z \in B_{r_0^{-1}}(z_{\alpha\beta})$

$$\bar d(\bar u_\alpha(z_{\alpha\beta}), \bar u_\alpha(z)) < \varepsilon. \tag{5.29}$$

Here by abuse of notation we write $w_\alpha := \bar{u}_\alpha$ if $\alpha \in \bar{T}$.

For $\alpha \in V$ we choose a sequence of gauge transformations $g_\alpha^\nu \in \mathcal{G}^{2,p}$ as in condition 3.6(iii), and we define

$$w_\alpha^\nu := (u_\alpha^\nu, \Phi_\alpha^\nu, \Psi_\alpha^\nu) := (g_\alpha^\nu)^*(w_\nu(\cdot + z_\alpha^\nu)),$$

and $\bar{u}_\alpha : S^2 = \mathbb{C} \cup \{\infty\} \to M/\mathrm{G}$ by

$$\bar{u}_\alpha(z) := \begin{cases} \mathrm{G}u_\alpha(z), & \text{if } z \in \mathbb{C}, \\ \lim_{r\to\infty} \mathrm{G}u_\alpha(r), & \text{if } z = \infty. \end{cases} \tag{5.30}$$

That the limit exists follows from part (A) of Proposition D.6 (Asymptotic Behaviour). Furthermore, for $\alpha \in \bar{T}$ we set

$$u_\alpha^\nu := u_\nu \circ \varphi_\alpha^\nu.$$

We check the hypotheses of part (A) of Lemma 4.15 with $\bar{u} := \bar{u}_{\alpha_0}$, and w_ν replaced by the $|\lambda_{\alpha_0}^\nu|$-vortex $w_{\alpha_0}^\nu$. The condition that there is a compact subset $K \subseteq M$ such that $u_{\alpha_0}^\nu(\mathbb{C}) \subseteq K$ for every ν follows from Proposition D.6(B). That $\mathrm{G}u_{\alpha_0}^\nu(z) \to \bar{u}_{\alpha_0}(z)$ for every $z \in \mathbb{C} \setminus B_{r_0}$ follows from condition (iii) of Definition 3.6. Condition (4.84) is satisfied by assertion (5.13) of Claim 1. So applying part (A) of Lemma 4.15, it follows that there exist numbers $R_0 \geq r_0$ and $\nu_0 \in \mathbb{N}$ such that for every $\nu \geq \nu_0$ and every $z \in \mathbb{C} \setminus B_{R_0}$ we have

$$\bar{d}(\bar{u}_{\alpha_0}(\infty), \mathrm{G}u_{\alpha_0}^\nu(z)) < \varepsilon. \tag{5.31}$$

Assume that $\alpha \in \bar{T}$, $\beta \in T$ are two vertices such that $\alpha \widetilde{E} \beta$. We check the hypothesis of part (B) of Lemma 4.15 with $z_0 := z_{\alpha\beta}$, $\bar{u} := \bar{u}_\beta$, $\bar{u}' := \bar{u}_\alpha$, $R_\nu := |\lambda_\beta^\nu|$, $\lambda_\nu := \lambda_\alpha^\nu/\lambda_\beta^\nu$, $z_\nu := (z_\beta^\nu - z_\alpha^\nu)/\lambda_\alpha^\nu$, and w_ν replaced by the $|\lambda_\beta^\nu|$-vortex w_β^ν. The condition that there is a compact subset $K \subseteq M$ such that $u_\beta^\nu(\mathbb{C}) \subseteq K$ for every ν is satisfied by Proposition D.6(B), and conditions (4.86) and (4.87) follow from condition 3.6(ii) and Lemma B.9. Furthermore, condition 3.6(iii) implies that $\mathrm{G}u_\beta^\nu(z) \to \bar{u}_\beta(z)$ for every $z \in \mathbb{C} \setminus B_{r_0}$, and that

$$\mathrm{G}u_\beta^\nu(\lambda_\nu(z - z_\nu)) = \mathrm{G}u_\alpha^\nu(z) \to \bar{u}_\alpha,$$

for every $z \in B_{r_0^{-1}}(z_{\alpha\beta}) \setminus \{z_{\alpha\beta}\}$. Finally, condition (4.88) follows from assertion (5.12) of Claim 1. So applying part (B) of Lemma 4.15 there are numbers $R_{\alpha\beta} > r_0$ and $\nu_{\alpha\beta} \in \mathbb{N}$ such that for every $\nu \geq \nu_{\alpha\beta}$ and every

$$z \in \bar{B}_{R_{\alpha\beta}^{-1}|\lambda_\nu|}(\lambda_\nu(z_{\alpha\beta} - z_\nu)) \setminus B_{R_{\alpha\beta}}$$

we have

$$\bar{d}(\bar{u}_\beta(\infty), \mathrm{G}u_\beta^\nu(z)) < \varepsilon. \tag{5.32}$$

We set

$$R := \max\{R_0\} \cup \{R_{\alpha\beta} \,|\, \alpha\widetilde{E}\beta\}.$$

Fix an $\alpha \in T$. By condition 3.6(iii) with $Q := \bar{B}_R \setminus \bigcup_{\alpha\widetilde{E}\beta} B_{R^{-1}}(z_{\alpha\beta})$ there is a number $\nu_\alpha \in \mathbb{N}$ such that for every $\nu \geq \nu_\alpha$ the following holds. If $\alpha \in V$ then for every $z \in \bar{B}_R$ we have

$$d(u^\nu_\alpha(z), u_\alpha(z)) < \varepsilon, \tag{5.33}$$

and if $\alpha \in \bar{T}$ then for every $z \in \bar{B}_R \setminus \bigcup_{\alpha\widetilde{E}\beta} B_{R^{-1}}(z_{\alpha\beta})$ we have

$$\bar{d}(\mathrm{G}u^\nu_\alpha(z), \bar{u}_\alpha(z)) < \varepsilon. \tag{5.34}$$

Condition (ii) of Definition 3.6 implies that there is an integer $\nu_1 \in \mathbb{N}$ such that for every $\nu \geq \nu_1$ and every pair $\alpha\widetilde{E}\beta$ we have

$$(\varphi^\nu_\alpha)^{-1} \circ \varphi^\nu_\beta(\bar{B}_R) \subseteq B_{R^{-1}}(z_{\alpha\beta}), \tag{5.35}$$

We set

$$\nu_2 := \max\Big(\{\nu_0, \nu_1\} \cup \{\nu_\alpha \,|\, \alpha \in T\} \cup \{\nu_{\alpha\beta} \,|\, \alpha\widetilde{E}\beta\}\Big).$$

Claim 2 *For every $\nu \geq \nu_2$ the equivariant homology class of w_ν equals the equivariant homology class of $(\mathbf{w}, \bar{\mathbf{u}})$.*

Proof of Claim 2: We fix a continuous equivariant map $\theta : M^* \to \mathrm{EG}$ and define

$$s_\theta : M^*/\mathrm{G} \to M_\mathrm{G} := (M \times \mathrm{EG})/\mathrm{G}, \qquad s_\theta(\mathrm{G}x) := [x, \theta(x)].$$

Fix $\nu \geq \nu_2$. For $\alpha \in T$ we define maps $f^\nu_\alpha : S^2 = \mathbb{C} \sqcup \{\infty\} \to M_\mathrm{G}$ as follows. For every $\alpha \neq \alpha_0$ we choose a continuous map

$$\bar{v}^\nu_\alpha : S^2 \setminus B_R \to B^{\bar{d}}_\varepsilon(\bar{u}_\alpha(\infty)),$$

such that

$$\bar{v}^\nu_\alpha(Re^{i\varphi}) = \mathrm{G}u^\nu_\alpha(Re^{i\varphi}),$$

for every $\varphi \in \mathbb{R}$. To see that this map exists, observe that by (5.32) we have

$$\mathrm{G}u^\nu_\alpha(Re^{i\varphi}) \in B^{\bar{d}}_\varepsilon(\bar{u}_\alpha(\infty)), \tag{5.36}$$

for every $\varphi \in \mathbb{R}$, and that by the choice of $\varepsilon > 0$ the ball $B^{\bar{d}}_\varepsilon(\bar{u}_\alpha(\infty))$ is contractible. We set $\bar{v}^\nu_{\alpha_0} := \bar{u}^\nu_{\alpha_0}$.

We fix a vertex $\alpha \in V$. Since EG is contractible, we can extend the map

$$S^1_R \ni z \mapsto \theta \circ u^\nu_\alpha(z) \in \mathrm{EG}$$

to a continuous map $\mathsf{e}_\alpha^\nu : \bar{B}_R \to \mathrm{EG}$. We define $f_\alpha^\nu : S^2 \cong \mathbb{C} \cup \{\infty\} \to M_{\mathrm{G}}$ by

$$f_\alpha^\nu(z) := \begin{cases} [u_\alpha^\nu(z), \mathsf{e}_\alpha^\nu(z)], & \text{if } z \in \bar{B}_R, \\ s_\theta \circ \bar{v}_\alpha^\nu(z), & \text{if } z \in S^2 \setminus B_R. \end{cases}$$

Note that the composition in the second line makes sense, since $\bar{v}_\alpha^\nu$ takes values in $B_\varepsilon^{\bar{d}}(\bar{u}_\alpha(\infty)) \subseteq M^*/\mathrm{G}$. For $\alpha = \alpha_0$ this follows from (5.31), since $\nu \geq \nu_2 \geq \nu_0$.

We fix now a vertex $\alpha \in \bar{T}$. For every vertex $\beta \in T$ such that $\alpha \widetilde{E} \beta$ we choose a continuous map

$$\bar{v}_{\alpha\beta}^\nu : B_{R^{-1}}(z_{\alpha\beta}) \to B_\varepsilon^{\bar{d}}(\bar{u}_\alpha(z_{\alpha\beta})) \subseteq M^*/\mathrm{G},$$

such that for every $z \in S^1_{R^{-1}}(z_{\alpha\beta})$ we have

$$\bar{v}_{\alpha\beta}^\nu(z) = \mathrm{G}u_\alpha^\nu(z).$$

Since $B_\varepsilon^{\bar{d}}(\bar{u}_\alpha(z_{\alpha\beta}))$ is contractible, such a map exists by (5.34). We define $\widehat{u}_\alpha^\nu : S^2 \to M/\mathrm{G}$ by

$$\widehat{u}_\alpha^\nu(z) := \begin{cases} \mathrm{G}u_\alpha^\nu(z), & \text{if } z \in \bar{B}_R \setminus \bigcup_{\alpha\widetilde{E}\beta} B_{R^{-1}}(z_{\alpha\beta}), \\ \bar{v}_\alpha^\nu(z), & \text{if } z \in S^2 \setminus B_R, \\ \bar{v}_{\alpha\beta}^\nu(z), & \text{if } z \in B_{R^{-1}}(z_{\alpha\beta}),\ \alpha\widetilde{E}\beta. \end{cases}$$

We claim that

$$\bar{d}(\widehat{u}_\alpha^\nu(z), \bar{u}_\alpha(z)) < 2\varepsilon, \tag{5.37}$$

for every $z \in S^2$. To see this observe that by (5.34), the estimate holds for every $z \in \bar{B}_R \setminus \bigcup_{\alpha\widetilde{E}\beta} B_{R^{-1}}(z_{\alpha\beta})$. Furthermore, for $z \in S^2 \setminus B_R$ we have by the choice of $\bar{v}_\alpha^\nu$ that

$$\widehat{u}_\alpha^\nu(z) = \bar{v}_\alpha^\nu(z) \in B_\varepsilon^{\bar{d}}(\bar{u}_\alpha(\infty)),$$

and hence by (5.28)

$$\bar{d}(\widehat{u}_\alpha^\nu(z), \bar{u}_\alpha(z)) \leq \bar{d}(\bar{v}_\alpha^\nu(z), \bar{u}_\alpha(\infty)) + \bar{d}(\bar{u}_\alpha(\infty), \bar{u}_\alpha(z)) < \varepsilon + \varepsilon.$$

Similarly, if $\beta \in T$ is such that $\alpha\widetilde{E}\beta$ and $z \in B_{R^{-1}}(z_{\alpha\beta})$, then by the choice of $\bar{v}_{\alpha\beta}^\nu$ we have

$$\widehat{u}_\alpha^\nu(z) = \bar{v}_{\alpha\beta}^\nu(z) \in B_\varepsilon^{\bar{d}}(\bar{u}_\alpha(z_{\alpha\beta})),$$

and hence by (5.29)

$$\bar{d}(\widehat{u}_\alpha^\nu(z), \bar{u}_\alpha(z)) \leq \bar{d}(\bar{v}_{\alpha\beta}^\nu(z), \bar{u}_\alpha(z_{\alpha\beta})) + \bar{d}(\bar{u}_\alpha(z_{\alpha\beta}), \bar{u}_\alpha(z)) < \varepsilon + \varepsilon.$$

It follows that (5.37) holds for every $z \in S^2$.

We define $f_\alpha^\nu := s_\theta \circ \widehat{u}_\alpha^\nu : S^2 \to (M^* \times \mathrm{EG})/\mathrm{G}$, for $\alpha \in \bar{T}$. It follows from (5.37) that this composition is welldefined.

Claim 3 *For every $\alpha \in V$ we have*

$$(f_\alpha^\nu)_*[S^2] = [w_\alpha]_{\mathrm{G}}. \tag{5.38}$$

Furthermore, for every $\alpha \in \bar{T}$ the maps $s_\theta \circ \bar{u}_\alpha$ and f_α^ν are homotopic. Moreover,

$$\sum_{\alpha \in T} (f_\alpha^\nu)_*[S^2] = [w^\nu]_{\mathrm{G}}. \tag{5.39}$$

Proof of Claim 3: Let $\alpha \in V$**.** Since $R \geq r_0$ we have by (5.28) for every $z \in \mathbb{C} \setminus B_R$

$$\bar{d}(\overline{\mathrm{ev}}_\infty(w_\alpha), \mathrm{G}u_\alpha(z)) < \varepsilon,$$

and hence $\mathrm{G}u_\alpha(z) \in M^*/\mathrm{G}$, i.e. $u_\alpha(z) \in M^*$. Therefore, the map

$$S_R^1 \ni z \mapsto \theta(u_\alpha(z)) \in \mathrm{EG}$$

is welldefined. Since EG is contractible, we can extend this to a continuous map $\mathsf{e}_\alpha : \bar{B}_R \to \mathrm{EG}$. We define $f_\alpha : S^2 \to M_{\mathrm{G}}$ by

$$f_\alpha(z) := \begin{cases} [u_\alpha(z), \mathsf{e}_\alpha(z)], & \text{if } z \in B_R, \\ s_\theta \circ \overline{\mathrm{ev}}_z(w_\alpha), & \text{if } z \in S^2 \setminus B_R. \end{cases}$$

It follows that f_α is continuous and

$$[w_\alpha]_{\mathrm{G}} = (f_\alpha)_*[S^2]. \tag{5.40}$$

We show that f_α is homotopic to f_α^ν. To see this, observe that by (5.33) for every $z \in \bar{B}_R$ we have $d(u_\alpha^\nu(z), u_\alpha(z)) < \varepsilon$ and hence by the choice of ε there exists a unique vector $\xi_\alpha^\nu(z) \in T_{u_\alpha(z)}M$ such that

$$\begin{aligned} |\xi_\alpha^\nu(z)| &< \varepsilon, \\ \exp_{u_\alpha(z)} \xi_\alpha^\nu(z) &= u_\alpha^\nu(z). \end{aligned} \tag{5.41}$$

Furthermore, since u_α^ν is continuous, the map $\xi_\alpha^\nu : \bar{B}_R \to TM$ is continuous. We define

$$h_\alpha^\nu : [0,1] \times \bar{B}_R \to M, \qquad h_\alpha^\nu(\lambda, z) := \exp_{u_\alpha(z)} \lambda \xi_\alpha^\nu(z).$$

Then for every $\lambda \in [0,1]$, $z \in S_R^1$

$$\begin{aligned} \bar{d}(\mathrm{G}h_\alpha^\nu(\lambda,z), \overline{\mathrm{ev}}_\infty(w_\alpha)) &\leq \bar{d}(\mathrm{G}h_\alpha^\nu(\lambda,z), \mathrm{G}u_\alpha(z)) + \bar{d}(\mathrm{G}u_\alpha(z), \overline{\mathrm{ev}}_\infty(w_\alpha)) \\ &\leq d(h_\alpha^\nu(\lambda,z), u_\alpha(z)) + \bar{d}(\mathrm{G}u_\alpha(z), \overline{\mathrm{ev}}_\infty(w_\alpha)) < \varepsilon + \varepsilon. \end{aligned}$$

Here we have used (5.41) and (5.28). Furthermore, the restriction of the map $\bar{u}_\alpha$ to the boundary $S_R^1 = \partial(S^2 \setminus B_R)$ agrees with the restriction

$$\mathrm{G}h_\alpha^\nu(0,\cdot) : \{0\} \times S_R^1 \to B_\varepsilon^{\bar{d}}(\overline{\mathrm{ev}}_\infty(w_\alpha)),$$

and the restriction of the map $\bar{u}_\alpha^\nu$ to S_R^1 agrees with the restriction

$$\mathrm{G}h_\alpha^\nu(1,\cdot):\{1\}\times S_R^1\to B_{2\varepsilon}^{\bar{d}}(\overline{\mathrm{ev}}_\infty(w_\alpha)).$$

Thus we can connect the maps

$$\bar{u}_\alpha|_{S^2\setminus B_R},\qquad \mathrm{G}h_\alpha^\nu|_{[0,1]\times S_R^1},\qquad \bar{u}_\alpha^\nu|_{S^2\setminus B_R}$$

to obtain a map

$$\widetilde{u}_\alpha^\nu : S^2\setminus B_R\#([0,1]\times S_R^1)\# S^2\setminus B_R\to B_{2\varepsilon}^{\bar{d}}(\overline{\mathrm{ev}}_\infty(w_\alpha)).$$

Note that the domain of this map is homeomorphic to S^2. Since the ball $B_{2\varepsilon}^{\bar{d}}(\overline{\mathrm{ev}}_\infty(w_\alpha))$ is contractible, the map $\widetilde{u}_\alpha^\nu$ is homotopic to a constant. This means that there exists a continuous map

$$\widetilde{h}_\alpha^\nu:[0,1]\times S^2\setminus B_R\to B_{2\varepsilon}^{\bar{d}}(\overline{\mathrm{ev}}_\infty(w_\alpha)),$$

whose restriction to the boundary

$$\partial([0,1]\times S^2\setminus B_R)=S^2\setminus B_R\#([0,1]\times S_R^1)\# S^2\setminus B_R$$

equals the map $\widetilde{u}_\alpha^\nu$. Since for every $z\in S_R^1$ we have $\mathsf{e}_\alpha(z)=\theta\circ u_\alpha(z)$ and $\mathsf{e}_\alpha^\nu(z)=\theta\circ u_\alpha^\nu(z)$, we can connect the map $\mathsf{e}_\alpha:\bar{B}_R\to\mathrm{EG}$, the map

$$[0,1]\times S_R^1\ni(\lambda,z)\mapsto\theta\circ h_\alpha^\nu(\lambda,z)\in\mathrm{EG}$$

and the map $\mathsf{e}_\alpha^\nu:\bar{B}_R\to\mathrm{EG}$ to obtain a continuous map

$$\widetilde{e}_\alpha^\nu:\bar{B}_R\#([0,1]\times S_R^1)\#\bar{B}_R\to\mathrm{EG}\,.$$

Since EG is contractible, we can extend this map to a continuous map

$$E:[0,1]\times\bar{B}_R\to\mathrm{EG}\,.$$

We define now the map $F:[0,1]\times S^2\to M_{\mathrm{G}}$ by

$$F(\lambda,z):=\begin{cases}[h_\alpha^\nu(\lambda,z),E(\lambda,z)], & \text{if } \lambda\in[0,1],\ z\in B_R,\\ s_\theta\circ\widetilde{h}_\alpha^\nu(\lambda,z), & \text{if } \lambda\in[0,1],\ z\in S^2\setminus B_R.\end{cases}$$

The map F is a homotopy from f_α to f_α^ν. Hence using (5.40), it follows that

$$(f_\alpha^\nu)_*[S^2]=(f_\alpha)_*[S^2]=[w_\alpha]_{\mathrm{G}}.$$

This proves (5.38).

To see that the second statement of the claim holds, fix $\alpha \in \bar{T}$. By (5.37) and the choice of ε, for every $z \in S^2$ there exists a unique vector $\bar{\xi}_\alpha^\nu(z) \in T_{\bar{u}_\alpha(z)}(M^*/\mathrm{G})$ such that

$$\overline{\exp}_{\bar{u}_\alpha(z)}\bar{\xi}_\alpha^\nu(z) = \widehat{u}_\alpha^\nu(z).$$

The map $\xi_\alpha^\nu : S^2 \to T(M^*/\mathrm{G})$ is continuous. The map

$$[0,1] \times S^2 \ni (\lambda, z) \mapsto s_\theta \circ \overline{\exp}_{\bar{u}_\alpha(z)}\lambda\bar{\xi}_\alpha^\nu(z) \in M^*/\mathrm{G}$$

is a homotopy from $s_\theta \circ \bar{u}_\alpha$ to $f_\alpha^\nu = s_\theta \circ \widehat{u}_\alpha^\nu$, as required.

In order to see that (5.39) holds, note that by (5.35), since $\nu \geq \nu_2 \geq \nu_1$ we have for $\alpha\widetilde{E}\beta$

$$\varphi_\beta^\nu(\bar{B}_R) \subseteq \varphi_\alpha^\nu(B_{R^{-1}}).$$

It follows as in the proof of Claim 1 that S^2 is the disjoint union of the sets

$$S^2 \setminus \varphi_{\alpha_0}^\nu(\bar{B}_R), \qquad X_\alpha^\nu := \varphi_\alpha^\nu\Big(\bar{B}_R \setminus \bigcup_{\alpha\widetilde{E}\beta} B_{R^{-1}}(z_{\alpha\beta})\Big), \text{ for } \alpha \in T,$$

$$X_{\alpha\beta}^\nu := \varphi_\alpha^\nu(B_{R^{-1}}(z_{\alpha\beta})) \setminus \varphi_\beta^\nu(\bar{B}_R), \text{ for } \alpha\widetilde{E}\beta.$$

We fix a pair $\alpha\widetilde{E}\beta$. It follows from (5.32) that

$$\mathrm{G}u^\nu(z) \in B_\varepsilon^{\bar{d}}(\bar{u}_\beta(\infty)),$$

for every $z \in X_{\alpha\beta}^\nu$. Consider the connected sum

$$\bar{v}_{\alpha\beta}^\nu \circ (\varphi_\alpha^\nu)^{-1} \# \mathrm{G}u^\nu|_{X_{\alpha\beta}^\nu} \# \bar{v}_\beta^\nu \circ (\varphi_\beta^\nu)^{-1} : \varphi_\alpha^\nu(\bar{B}_{R^{-1}}(z_{\alpha\beta})) \# X_{\alpha\beta}^\nu \# \varphi_\beta^\nu(S^2 \setminus B_R) \to B_\varepsilon^{\bar{d}}(z_{\alpha\beta}).$$

Note that the domain of this map is homeomorphic to S^2. Since $B_\varepsilon^{\bar{d}}(z_{\alpha\beta})$ is contractible, this map is contractible. We define $f_{\alpha\beta}^\nu$ to be the composition of this map with the map $s_\theta : M^*/\mathrm{G} \to M_\mathrm{G}^*$. It follows that $f_{\alpha\beta}^\nu$ is contractible.

We define $f^\nu : S^2 \to M_\mathrm{G}$ by

$$f^\nu(z) := \begin{cases} [u^\nu(z), \mathsf{e}_\alpha^\nu \circ (\varphi_\alpha^\nu)^{-1}(z)], & \text{if } z \in \varphi_\alpha^\nu(\bar{B}_R),\ \alpha \in V, \\ s_\theta \circ \mathrm{G}u^\nu(z), & \text{if } z \in S^2 \setminus \bigcup_{\alpha \in V} \varphi_\alpha^\nu(\bar{B}_R). \end{cases}$$

The map f^ν is continuous and represents the equivariant homology class $[w^\nu]_\mathrm{G}$. It follows that

$$\begin{aligned} [w^\nu]_\mathrm{G} &= f_*^\nu[S^2] \\ &= \sum_{\alpha \in T}(f_\alpha^\nu)_*[S^2] + \sum_{\alpha\widetilde{E}\beta}(f_{\alpha\beta}^\nu)_*[\big(-\varphi_\alpha^\nu(\bar{B}_{R^{-1}}(z_{\alpha\beta}))\big)\#X_{\alpha\beta}^\nu\# - \varphi_\beta^\nu(S^2 \setminus B_R)] \\ &= \sum_{\alpha \in T}(f_\alpha^\nu)_*[S^2]. \end{aligned}$$

Here the $-$ indicates that we reverse the orientation. In the second line we have used that $\bar{v}^\nu_{\alpha_0} = \bar{u}^\nu_{\alpha_0}$ and that on chain level, for every $\alpha\widetilde{E}\beta$ the restriction

$$f^\nu_\alpha|_{\bar{B}_{R^{-1}}(z_{\alpha\beta})} = s_\theta \circ \bar{v}^\nu_{\alpha\beta}$$

cancels the restriction

$$f^\nu_{\alpha\beta}|_{-\varphi^\nu_\alpha(\bar{B}_{R^{-1}}(z_{\alpha\beta}))},$$

and the restriction $f^\nu_\beta|_{S^2\setminus B_R}$ cancels the restriction $f^\nu_{\alpha\beta}|_{-\varphi^\nu_\beta(S^2\setminus B_R)}$. In the third line we have used that $f^\nu_{\alpha\beta}$ is contractible for every pair $\alpha\widetilde{E}\beta$. This proves (5.39) and completes the proof of Claim 3. □

It follows from Claim 3 that

$$[w^\nu]_{\mathrm{G}} = \sum_{\alpha\in T}(f^\nu_\alpha)_*[S^2] = \sum_{\alpha\in V}[w_\alpha]_{\mathrm{G}} + \sum_{\alpha\in\bar{T}}(s_\theta\circ\bar{u}_\alpha)_*[S^2] = [\mathbf{w},\bar{\mathbf{u}},\mathbf{z}]_{\mathrm{G}},$$

for every $\nu \geq \nu_2$. This proves Claim 2 and concludes the proof of Proposition 5.4. □

6 A continuous evaluation map

This section contains a proof that there exists an evaluation map from the set of equivalence classes of stable maps with k marked points $\bar{\mathcal{M}}_k$ to $((M \times \mathrm{EG})/\mathrm{G})^k$, that is continuous w.r.t. convergence of a sequence of vortices against a stable map. Here EG is a contractible topological space on which G acts continuously and freely.

As a corollary to the theorem of Peter and Weyl, every compact connected Lie group G has an embedding as a closed subgroup $\iota : \mathrm{G} \to U(\ell)$ for ℓ large enough, see the book by Th. Bröcker and T. tom Dieck [BtD]. We may therefore assume w.l.o.g. that G is a closed subgroup of $U(\ell)$ for some $\ell \in \mathbb{N}$. We define EG to be the set of all orthonormal tuples $(v^1, \ldots, v^\ell)$, where $v^i = (v^i_j)_{j \in \mathbb{N}}$ is a sequence of complex numbers whose members vanish for large enough j. For every $N \geq \ell$ we denote by $\mathrm{EG}^N \subseteq \mathrm{EG}$ the subset of all orthonormal tuples $(v^1, \ldots, v^i)$ such that $v^i_j = 0$ if $j \geq N+1$. We identify EG^N with the set of all unitary $\ell \times N$ matrices. We endow the set EG with the topology in which a subset $U \subseteq \mathrm{EG}$ is open iff the intersection $U \cap \mathrm{EG}^N$ is open in EG^N for every N. The Lie group G acts continuously on EG by multiplication from the left. The quotient

$$\mathrm{BG} := \mathrm{EG}/\mathrm{G}$$

is the classifying space of G. We fix a regular value $\delta_0 > 0$ of $|\mu|^2 : M \to \mathbb{R}$ so small that G acts freely on $M_{2\delta_0} := \{x \in M \,|\, |\mu(x)| \leq 2\delta_0\}$. Then $M_{2\delta_0}$ is a smooth compact principal G-bundle (with boundary) and hence there exists a smooth G-equivariant map $\theta : M_{2\delta_0} \to \mathrm{EG}^m$ for some large enough integer m. We define

$$s_\theta : M_{\delta_0}/\mathrm{G} \to (M_{\delta_0} \times \mathrm{EG}^m)/\mathrm{G}, \qquad s_\theta(\mathrm{G}x) := [x, \theta(x)]. \tag{6.1}$$

Recall the notation

$$M_\mathrm{G} := (M \times \mathrm{EG})/\mathrm{G},$$

and the definition 3.1 of the map

$$\overline{\mathrm{ev}} : (C^0(S^2, M/\mathrm{G}) \sqcup \widetilde{\mathcal{M}}^{1,p}) \times S^2 \to M/\mathrm{G}.$$

We denote by

$$\pi_\mathrm{G} : M_\mathrm{G} \to M/\mathrm{G}$$

the canonical projection.

Proposition 6.1 (Existence of a continuous evaluation map) *Assume that hypothesis (H2) (Convexity at ∞) holds. Then for every $k \in \mathbb{N}$ there exists a map* $\mathrm{ev} : \overline{\mathcal{M}}_k \to M_\mathrm{G}^k$ *with the following properties.*

(i) **(Evaluation)** *Fix an equivalence class of stable maps*

$$[\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}] = \\ [\bar{T}, V, E, (w_\alpha := (u_\alpha, \Phi_\alpha, \Psi_\alpha))_{\alpha \in V}, (\bar{u}_\alpha)_{\alpha \in \bar{T}}, (z_{\alpha\beta})_{\alpha E \beta}, (\alpha_i, z_i)_{i=0,\dots,k-1}] \in \overline{\mathcal{M}}_k$$

and an index $i \in \{0, \dots, k-1\}$*. If* $\alpha_i \in V$ *and* $z_i \neq \infty$ *then*

$$\pi \circ \mathrm{ev}^i([\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}]) = \mathrm{G}u_{\alpha_i}(z_i). \tag{6.2}$$

Furthermore, assume that $\alpha_i \in \bar{T}$ *or the following condition is satisfied. The vertex* α_i *lies in* V *and*

$$z_i = \infty \qquad or \qquad |\mu \circ u_{\alpha_i}(z_i)| < \delta_0.$$

Then

$$\mathrm{ev}^i([\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}]) = s_\theta \circ \overline{\mathrm{ev}}_{z_i}(w_{\alpha_i}), \tag{6.3}$$

where by abuse of notation we write $w_{\alpha_i} := \bar{u}_{\alpha_i}$ *if* $\alpha_i \in \bar{T}$*.*

(ii) **(Continuity)** *Let* $w^\nu \in \widetilde{\mathcal{M}}^{1,p}$ *and* $z_1^\nu, \dots, z_{k-1}^\nu \in \mathbb{C}$ *be sequences of vortices and marked points such that* $(w_\nu, z_0^\nu := \infty, z_1^\nu, \dots, z_{k-1}^\nu)$ *converges to some stable map* $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ *of vortices on* $\mathbb{C}$ *and pseudo-holomorphic spheres in* $\bar{M}$*. Then*

$$\mathrm{ev}([w^\nu, z_0^\nu, \dots, z_{k-1}^\nu]) \to \mathrm{ev}([\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}]).$$

Remark 6.2 If $\mathrm{G} = \{\mathbf{1}\}$ then we can identify $\bar{\mathcal{M}}_k$ with the set of stable maps of J-holomorphic spheres in M with k marked points modelled over some tree $T = V \sqcup \bar{T}$ such that $z_{\alpha\beta} = \infty$ if $\alpha \in V$ and $\alpha E \beta$. Furthermore, $M_{\{\mathbf{1}\}} = M$ and the map

$$\overline{\mathcal{M}}_k \to M^k, \\ \big(V, \bar{T}, (u_\alpha)_{\alpha \in T}, (z_{\alpha\beta})_{\alpha E \beta}, (\alpha_i, z_i)_{i=0,\dots,k-1}\big) \mapsto (u_0(z_0), \dots, u_{k-1}(z_{k-1}))$$

satisfies the conditions of Proposition 6.1. If G is not the trivial Lie group then we have to take $((M \times \mathrm{EG})/\mathrm{G})^k$ as a target of the evaluation map, since it does not make sense to define an evaluation map from $\bar{\mathcal{M}}_k$ to M^k. To understand this observe that if $\alpha_i \in \bar{T}$ then there is no canonical choice of a representative in $\mu^{-1}(0)$ of the orbit $\bar{u}_{\alpha_i}(z_i) \in \bar{M}$. Furthermore, if $\alpha_i \in V$ then the problem is that the point $u_{\alpha_i}(z_i) \in M$ depends on the choice of a representative $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ of the equivalence class of stable maps, since gauge transforming the vortex $(u_{\alpha_i}, \Phi_{\alpha_i}, \Psi_{\alpha_i})$ we may get a point in the orbit of $u_{\alpha_i}(z_i)$.

The proof of Proposition 6.1 is based on the following Proposition.

Proposition 6.3 *Assume that (H2) holds. Then there exists a map*

$$\Theta : \widetilde{\mathcal{M}}^{1,p} \times \mathbb{C} \to \mathrm{EG}$$

such that the following holds.

(i) *For every $g \in \mathcal{G}^{2,p}$ and $(w,z) \in \widetilde{\mathcal{M}}^{1,p} \times \mathbb{C}$ we have*

$$\Theta(g^* w, z) = g(z)^{-1}\Theta(w,z).$$

(ii) **(Invariance under translation)** *Let $(w,z_0) \in \widetilde{\mathcal{M}}^{1,p}\times\mathbb{C}$ and let $\zeta \in \mathbb{C}$. Then*

$$\Theta(w(\cdot - \zeta), z_0 + \zeta) = \Theta(w, z_0).$$

(iii) *For every $(u,\Phi,\Psi;z_0) \in \widetilde{\mathcal{M}}^{1,p} \times \mathbb{C}$ such that $|\mu(u(z_0))| < \delta_0$ we have $\Theta(u,\Phi,\Psi;z_0) = \theta(u(z_0))$.*

(iv) *Let $w_\nu \in \widetilde{\mathcal{M}}^{1,p} \times \mathbb{C}$ be a sequence of vortices that converges to a vortex $w \in \widetilde{\mathcal{M}}^{1,p}$, uniformly on every compact subset of $\mathbb{C}$, and let $z_\nu \in \mathbb{C}$ is a sequence of points converging to some point $z \in \mathbb{C}$. Assume that $\sup_{\nu\in\mathbb{N}} E(w_\nu) < \infty$. Then*

$$\Theta(w_\nu, z_\nu) \to \Theta(w,z).$$

Proof of Proposition 6.3: We fix a smooth function $\rho : [0,\infty) \to [0,1]$ such that $\rho(a) = 1$ if $a \le \delta_0$ and $\rho(a) = 0$ if $a \ge 2\delta_0$. For every vortex $w \in \widetilde{\mathcal{M}}^{1,p}$, every point $z_0 \in \mathbb{C}$ and every nonnegative integer j we define

$$\rho_j^{w,z_0} := \rho(|\mu(u(z_0 + 2j))|).$$

Claim 1 *For every constant $E \ge 0$ there exists an integer n such that the following holds. If $(w,z_0) \in \widetilde{\mathcal{M}}^{1,p} \times \mathbb{C}$ is a pair such that $E(w, B_{2n}(z_0)) \le E$ then there exists an integer $j_0 \in \{0,\ldots,n\}$ such that $\rho_{j_0}^{w,z_0} = 1$.*

Proof: By Proposition D.6(B) there exists a compact subset $K_0 \subseteq M$ such that $u(\mathbb{C}) \subseteq K_0$ for every $(u,\Phi,\Psi) \in \widetilde{\mathcal{M}}^{1,p}$. We choose $E_0 > 0$ as in the a priori Lemma C.1, depending on K_0. Furthermore, we fix an integer n greater than $\frac{E}{E_0}$ and greater than $8E/(\pi\delta_0^2)$. Let (w,z_0) be as in the hypothesis. Since the balls $B_1(z_0+2j)$, $j = 0,\ldots,n-1$, are all disjoint, we have

$$E \ge E(w, B_{2n}(z_0)) \ge \sum_{j=0}^{n-1} E(w, B_1(z_0 + 2j)).$$

Hence there exists an integer $j_0 \in \{0,\ldots,n-1\}$ such that

$$E(w, B_1(z_0 + 2j_0)) \le \frac{E}{n}. \tag{6.4}$$

By the choice of n it follows that $E(w, B_1(z_0+2j_0)) < E_0$, and therefore the conditions of Lemma C.1 with $r := 1$ and z_0 replaced by z_0+2j are satisfied. Hence by that Lemma

$$e_w(z_0+2j_0) \le \frac{8E(w, B_1(z_0+2j_0))}{\pi} \le \frac{8E}{n\pi} < \delta_0^2.$$

Here in the second inequality we have used the estimate (6.4). It follows that

$$|\mu(u(z_0+2j_0))| \le \sqrt{e_w(z_0+2j_0)} < \delta_0,$$

and hence

$$\rho_{j_0} := \rho(|\mu(u(z_{j_0}))|) = 1.$$

This proves Claim 1. □

We define now the map $\Theta : \widetilde{\mathcal{M}}^{1,p} \times \mathbb{C} \to \mathrm{EG}$ as follows. Fix a vortex $w := (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$ and a point $z_0 \in \mathbb{C}$. Since $p > 2$, by the Morrey embedding theorem the maps $\Phi, \Psi : \mathbb{C} \to \mathfrak{g}$ are continuous. Applying Proposition E.13 repeatedly with $T := 1, 2, \ldots$, $\xi := \Phi(\cdot + z_0)|_{(0,1)}$, $\xi := \Phi(\cdot + z_0)|_{(0,2)}$, ... it follows that there exists a unique continuous solution $g_{z_0} : [0, \infty) \to \mathrm{GL}(\ell, \mathbb{R})$ of the linear integral equation

$$g_{z_0}(t) = \mathbf{1} - \int_0^t \Phi(s+z_0) g_{z_0}(s) ds.$$

Since Φ is continuous, it follows that g_{z_0} is continuously differentiable and that it solves the linear differential equation

$$\dot{g}_{z_0}(s) = -\Phi(s+z_0) g_{z_0}(s), \qquad g_{z_0}(0) = \mathbf{1}. \tag{6.5}$$

Furthermore, since Φ takes values in the Lie algebra $\mathfrak{g}$ of G, Proposition E.13 implies that g_{z_0} takes values in G. We choose a positive integer n as in Claim 1, corresponding to $E := E(w)$. For $j = 1, \ldots, n$ we set $z_j := z_0 + 2j$, and for $j = 0, \ldots, n$ we define $\rho_j := \rho(|\mu \circ u(z_j)|)$ and

$$\lambda_j := \rho_j \sqrt{(1-\rho_{j-1}^2) \cdots (1-\rho_0^2)}, \tag{6.6}$$

and $\Theta_j(w, z_0) \in \mathbb{C}^{\ell \times m}$ by

$$\Theta_j(w, z_0) := \begin{cases} \lambda_j g_{z_0}(2j)^{-1} \theta(u(z_j)), & \text{if } |\mu(u(z_j))| \le 2\delta_0, \\ 0, & \text{otherwise.} \end{cases}$$

We define

$$\Theta(w, z_0) := \begin{pmatrix} \Theta_0(w, z_0) & \ldots & \Theta_n(w, z_0) \end{pmatrix} \in \mathbb{C}^{\ell \times (n+1)m}. \tag{6.7}$$

Claim 2 *We have $\Theta(w,z_0) \in \mathrm{EG}^{(n+1)m}$, i.e. the rows of $\Theta(w,z_0)$ are an orthonormal system.*

Proof: Since the rows of the matrix $\theta(u(z_j)) \in \mathbb{C}^{\ell\times m}$ are orthonormal and $g_{z_0}(2j) \in \mathrm{U}(\ell)$ it follows that the rows of $\Theta_j(w,z_0)$ are orthonormal. This implies that the rows of $\Theta(w,z_0)$ are orthogonal. Fix $j=1,\dots,\ell$. We show that the j-th row of $\Theta(w,z_0)$ has norm 1. Since for every $j=1,\dots,n$ the j-th row of $\Theta_j(w,z_0)$ has norm 1, it is enough to show that

$$\sum_{j=0}^{n} \lambda_j^2 = 1. \tag{6.8}$$

To see this observe that by the statement of Claim 1 there exists an integer $j_0 \in \{0,\dots,n\}$ such that $\rho_{j_0}=1$, and hence

$$(1-\rho_n^2)\cdots(1-\rho_{j_0}^2)\cdots(1-\rho_0^2) = (1-\rho_n^2)\cdots 0\cdots(1-\rho_0^2) = 0.$$

It follows that

$$\begin{aligned}
\sum_{j=0}^{n}\lambda_j^2 &= \rho_0^2+\ldots+\rho_n^2(1-\rho_{n-1}^2)\cdots(1-\rho_0^2)+(1-\rho_n^2)\cdots(1-\rho_0^2)\\
&= \rho_0^2+\ldots+\rho_{n-1}^2(1-\rho_{n-2}^2)\cdots(1-\rho_0^2)\\
&\quad +(\rho_n^2+1-\rho_n^2)(1-\rho_{n-1}^2)\cdots(1-\rho_0^2)\\
&= \rho_0^2+\ldots+\rho_{n-2}^2(1-\rho_{n-3}^2)\cdots(1-\rho_0^2)\\
&\quad +(\rho_{n-1}^2+1-\rho_{n-1}^2)(1-\rho_{n-2}^2)\cdots(1-\rho_0^2)\\
&= \ldots\\
&= \rho_0^2+(1-\rho_0^2)=1.
\end{aligned}$$

It follows that all the rows of $\Theta(w,z_0)$ have norm 1 and hence $\Theta(w,z_0) \in \mathrm{EG}^{(n+1)m}$. This proves Claim 2. □

We check the conditions (i)-(iv). To see that condition **(i)** holds, let $g \in \mathcal{G}^{2,p}$ and $(w,z_0) \in \widetilde{\mathcal{M}}^{1,p}\times\mathbb{C}$ and define $w' := (u',\Phi',\Psi') := g^*w \in \widetilde{\mathcal{M}}^{1,p}$. Let $g_{z_0} : [0,\infty) \to \mathrm{G}$ be the unique continuously differentiable solution of (6.5). We define

$$g'_{z_0} : [0,\infty) \to \mathrm{G}, \qquad g'_{z_0}(s) := g(s+z_0)^{-1}g_{z_0}(s)g(z_0).$$

Then $g'_{z_0}(0)=\mathbf{1}$, g'_{z_0} is continuously differentiable, and

$$\begin{aligned}
\dot g'_{z_0}(s) &= g(s+z_0)^{-1}\Big(-\partial_s g(s+z_0)g(s+z_0)^{-1}g_{z_0}(s)+\dot g_{z_0}(s)\Big)g(z_0)\\
&= -g(s+z_0)^{-1}\Big(\partial_s g(s+z_0)+\Phi(s+z_0)g(s+z_0)\Big)g(s+z_0)^{-1}g_{z_0}(s)g(z_0)\\
&= -\Phi'(s+z_0)g'_{z_0}(s).
\end{aligned}$$

This means that g'_{z_0} solves equation (6.5) with Φ replaced by Φ'. We define λ_j as in (6.6), corresponding to u. Since

$$|\mu(u'(z_j))| = |\mu(u(z_j))|,$$

setting $z_j := z_0 + 2j$, it follows that

$$\begin{aligned}
\Theta_j(w', z_0) &= \lambda_j g'_{z_0}(2j)^{-1}\theta(u'(z_j)) \\
&= \lambda_j g(z_0)^{-1} g_{z_0}(2j)^{-1} g(z_j)\theta(g(z_j)^{-1}u(z_j)) \\
&= \lambda_j g(z_0)^{-1} g_{z_0}(2j)^{-1}\theta(u(z_j)) \\
&= g(z_0)^{-1}\Theta_j(w, z_0).
\end{aligned}$$

Therefore, $\Theta(g^*w, z_0) = g(z_0)^{-1}\Theta(w, z_0)$. This proves **(i)**.

(ii) follows similarly to (i) with $w' := w(\cdot - \zeta)$.

To prove **(iii)** observe that in the case $|\mu(u(z_0))| < \delta_0$ we have $\rho_0 := |\rho(|\mu(u(z))|) = 1$ and therefore $\Theta_0(w, z_0) = \theta(u(z_0))$. Furthermore, for $j = 1, \ldots, n$ we have

$$\lambda_j = \rho_j^2(1 - \rho_{j-1}^2)\cdots(1 - \rho_1^2)\cdot 0 = 0,$$

and hence $\Theta_j(w, z_0) = 0$. This implies **(iii)**.

We prove **(iv)**. Let $w_\nu := (u_\nu, \Phi_\nu, \Psi_\nu) \in \widetilde{\mathcal{M}}^{1,p}$ be a sequence of vortices that converges to some vortex $w := (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$, uniformly on every compact subset of $\mathbb{C}$, and let $z_\nu \in \mathbb{C}$ be a sequence of points converging to some point $z \in \mathbb{C}$. Assume that $E := \sup_\nu E(w_\nu) < \infty$. We choose an integer n as in Claim 1, corresponding to E. Fix an integer $j = 0, \ldots, n$. Since $z_\nu \to z$, there exists a number $R > 0$ such that $z_\nu \in \bar{B}_R$ for every ν. By assumption, the maps u_ν converge to u, uniformly on $\bar{B}_R$, and hence

$$d(u_\nu(z_\nu), u(z)) \le d(u_\nu(z_\nu), u(z_\nu)) + d(u(z_\nu), u(z)) \to 0. \tag{6.9}$$

We define $z_j := z + 2j$, $z_j^\nu := z_\nu + 2j$, $\rho_j := \rho(|\mu \circ u(z_j)|)$ and λ_j as in (6.6), $\rho_j^\nu := \rho(|\mu \circ u_\nu(z_j^\nu)|)$, and λ_j^ν as in (6.6) with ρ_j replaced by ρ_j^ν. We define $\widetilde{\Theta}_j \in \mathbb{C}^{\ell\times m}$ by

$$\widetilde{\Theta}_j := \begin{cases} \lambda_j\theta\circ u(z), & \text{if } |\mu\circ u(z)| \le 2\delta_0, \\ 0, & \text{otherwise,} \end{cases}$$

and $\widetilde{\Theta}_j^\nu \in \mathbb{C}^{\ell\times m}$ by

$$\widetilde{\Theta}_j^\nu := \begin{cases} \lambda_j^\nu\theta\circ u_\nu(z_\nu), & \text{if } |\mu\circ u_\nu(z_\nu)| \le 2\delta_0, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from (6.9), that

$$\widetilde{\Theta}_j^\nu \to \widetilde{\Theta}_j.$$

Let $g : [0,\infty) \to \mathrm{G}$ and $g_\nu : [0,\infty) \to \mathrm{G}$ be the unique continuously differentiable solutions of (6.5), corresponding to Φ, $z_0 := z$ and Φ_ν, $z_0 := z_\nu$ respectively. Then, abbreviating

$$\Theta_j := g(z_j)^{-1}\widetilde{\Theta}_j, \qquad \Theta_j^\nu := g_\nu(z_j^\nu)^{-1}\widetilde{\Theta}_j^\nu,$$

we have

$$\Theta(w,z) = \big(\Theta_0 \ldots \Theta_n\, 0 \ldots\big), \qquad \Theta(w_\nu, z_\nu) = \big(\Theta_0^\nu \ldots \Theta_n^\nu\, 0 \ldots\big).$$

Let $\langle\cdot,\cdot\rangle$ be an invariant inner product on $\mathfrak{g}$. By the last statement in Proposition E.13, with

$$T := 2j, \qquad \xi := \Phi(\cdot + z),\ \eta := \Phi_\nu(\cdot + z_\nu) : [0,2j] \to \mathfrak{g}$$

we have

$$\begin{aligned} d^{\langle\cdot,\cdot\rangle}(g_\nu(z_j^\nu), g(z_j)) &\leq ||\Phi(\cdot+z) - \Phi_\nu(\cdot + z_\nu)||_{L^1([0,2j])} \\ &\leq 2j||\Phi(\cdot+z) - \Phi_\nu(\cdot + z_\nu)||_{C^0([0,2j])} \\ &\leq 2j||\Phi(\cdot+z) - \Phi(\cdot + z_\nu)||_{C^0([0,2j])} \\ &\quad +2j||\Phi(\cdot+z_\nu) - \Phi_\nu(\cdot + z_\nu)||_{C^0([0,2j])} \to 0. \end{aligned}$$

Here in the last step we have used that $z_\nu \to z$ and that Φ_ν converges to Φ, uniformly on $\bar{B}_{R+2j}$. Since $\widetilde{\Theta}_j^\nu \to \widetilde{\Theta}_j$, this implies that $\Theta(w_\nu, z_\nu) \to \Theta(w,z)$. This completes the proof of **(iv)** and therefore of Proposition 6.3. □

Proof of Proposition 6.1: Fix a nonnegative integer k. We define the map $\mathrm{ev} : \bar{\mathcal{M}}_k \to M_{\mathrm{G}}^k$ as follows. Fix a map $\Theta : \widetilde{\mathcal{M}}^{1,p} \times \mathbb{C} \to E\mathrm{G}$ as in Proposition 6.3. Let $\mathbf{W} \in \mathcal{M}_k$ be an equivalence class of stable maps. We choose a representative

$$(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}) := \big(\bar{T}, V, E, (u_\alpha, \Phi_\alpha, \Psi_\alpha)_{\alpha\in V}, (\bar{u}_\alpha)_{\alpha\in\bar{T}}, (z_{\alpha\beta})_{\alpha E\beta}, (\alpha_i, z_i)_{i=0,\ldots,k-1}\big)$$

of $\mathbf{W}$. For every $i = 0, \ldots, k-1$ we define the i-th component of $\mathrm{ev}(\mathbf{W})$ by

$$\mathrm{ev}^i(\mathbf{W}) := \begin{cases} [u_{\alpha_i}(z_i), \Theta(w_{\alpha_i}, z_i)], & \text{if } \alpha_i \in V,\ z_i \neq \infty, \\ s_\theta \circ \overline{\mathrm{ev}}_\infty(w_{\alpha_i}), & \text{if } \alpha_i \in V,\ z_i = \infty, \\ s_\theta \circ \overline{\mathrm{ev}}_{z_i}(\bar{u}_{\alpha_i}), & \text{if } \alpha_i \in \bar{T}. \end{cases}$$

We claim that $\mathrm{ev}^i(\mathbf{W})$ is welldefined, i.e. does not depend on the choice of the representative $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ of $\mathbf{W}$. To see this, assume that

$$(\mathbf{w}', \bar{\mathbf{u}}', \mathbf{z}') := \big(V', \bar{T}', E', (u'_\alpha, \Phi'_\alpha, \Psi'_\alpha)_{\alpha\in V}, (\bar{u}'_\alpha)_{\alpha\in\bar{T}}, (z'_{\alpha\beta})_{\alpha E\beta}, (\alpha'_i, z'_i)_{i=0,\ldots,k-1}\big)$$

is another representative of $\mathbf{W}$. This means that $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ and $(\mathbf{w}', \bar{\mathbf{u}}', \mathbf{z}')$ are equivalent stable maps. Hence there exist a tree isomorphism $f : T := V \sqcup \bar{T} \to T' := V' \sqcup \bar{T}'$, Möbius transformations φ_α, for $\alpha \in T$ and gauge transformations $g_\alpha \in \mathcal{G}^{2,p}$ such that the conditions (i)-(iii) of Definition 3.12 are satisfied. Fix $i = 0, \ldots, k-1$. We denote $\alpha_i' := f(\alpha_i) \in T'$.

Assume that $\alpha_i \in V$ **and** $z_i \neq \infty$**.** Denoting

$$w_i := (u_{\alpha_i}, \Phi_{\alpha_i}, \Psi_{\alpha_i}), \quad w_i' := (u'_{\alpha_i'}, \Phi'_{\alpha_i'}, \Psi'_{\alpha_i'}), \quad \varphi_i := \varphi_{\alpha_i}, \quad g_i := g_{\alpha_i},$$

we have by condition 3.12(iii) $g_i^*(w_i \circ \varphi_i) = w_i'$. It follows that

$$\begin{aligned} \Theta(w_i', z_i') &= \Theta\big(g_i^*(w_i \circ \varphi_i), z_i'\big) \\ &= g_i(z_i')^{-1}\Theta(w_i \circ \varphi_i, z_i') \\ &= g_i(z_i')^{-1}\Theta(w_i, \varphi_i(z_i')) \\ &= g_i(z_i')^{-1}\Theta(w_i, z_i). \end{aligned}$$

Here the second equality follows from assertion (i) of Proposition 6.3, the third equality from assertion 6.3(ii), and the last equality from condition (ii) of Definition 3.12. It follows that

$$\begin{aligned} [u_i'(z_i'), \Theta(w_i', z_i')] &= [g_i(z_i')^{-1}u_i \circ \varphi_i(z_i'), g_i(z_i')^{-1}\Theta(w_i, z_i)] \\ &= [u_i(z_i), \Theta(w_i, z_i)]. \end{aligned}$$

Hence $\mathrm{ev}^i(\mathbf{W})$ does not depend on the choice of $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$ in the case $\alpha_i \in V$ and $z_i \neq \infty$. That this is true also in the other cases follows directly from the definition of $\mathrm{ev}^i(\mathbf{W})$.

We claim that ev satisfies the conditions of Proposition 6.1. **We check (i).** If $\alpha_i \in V$ and $z_i \neq \infty$, then condition (6.2) is satisfied by the definition of $\mathrm{ev}^i(\mathbf{W})$. Assume that $\alpha_i \in \bar{T}$ or $\alpha_i \in V$ and ($z_i = \infty$ or $|\mu \circ u_{\alpha_i}(z_i)| < \delta_0$). To see that condition (6.3) holds, note that in the case $\alpha_i \in \bar{T}$ or $\alpha_i \in V$ and $z_i = \infty$, the statement follows from the definition of $\mathrm{ev}^i(\mathbf{W})$. If $\alpha_i \in V$, $z_i \neq \infty$ and $|\mu \circ u_{\alpha_i}(z_i)| < \delta_0$ then (6.3) follows from condition (iii) of Proposition 6.3 and the definition (6.1) of s_θ.

In order to see that condition 6.1(ii) holds, let $w_\nu := (u_\nu, \Phi_\nu, \Psi_\nu) \in \widetilde{\mathcal{M}}^{1,p}$ and $z_1^\nu, \ldots, z_{k-1}^\nu \in \mathbb{C}$ be sequences of vortices and marked points such that $(w_\nu, z_0^\nu := \infty, z_1^\nu, \ldots, z_{k-1}^\nu)$ converges to some stable map $(\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z})$. Hence there exist Möbius transformations φ_α for $\alpha \in T$ such that the conditions (i)-(iii) of Definition 3.6 are satisfied. Fix an integer $i = 0, \ldots, k-1$.

Assume that $i \geq 1$ **and** $\alpha_i \in V$**.** Then by condition (i) of Definition 3.6 the maps $\varphi_i^\nu := \varphi_{\alpha_i}^\nu$ are translations, and by condition 3.6(iii) there exist gauge transformations $g_i^\nu \in \mathcal{G}^{2,p}$ such that $(g_i^\nu)^*(w_\nu \circ \varphi_i^\nu)$ converges to $w_i := w_{\alpha_i}$, uniformly on every compact subset of $\mathbb{C}$. Denoting $z_i'^\nu := (\varphi_i^\nu)^{-1}(z_i^\nu)$

and $w_i^\nu := (g_i^\nu)^*(w_\nu \circ \varphi_i^\nu)$, we have

$$\begin{aligned}\Theta(w_\nu, z_i^\nu) &= \Theta(w_\nu \circ \varphi_i^\nu, (\varphi_i^\nu)^{-1}(z_i^\nu)) \\ &= g_i^\nu((\varphi_i^\nu)^{-1}(z_i^\nu))\Theta\big((g_i^\nu)^*(w_\nu \circ \varphi_i^\nu), (\varphi_i^\nu)^{-1}(z_i^\nu)\big) \\ &= g_i^\nu(z_i'^\nu)\Theta(w_i^\nu, z_i'^\nu).\end{aligned}$$

Here the first equality follows from condition (ii) of Proposition 6.3 and the second one from 6.3(i). It follows that

$$\begin{aligned}\mathrm{ev}^i([w_\nu, z_0^\nu, \dots, z_{k-1}^\nu]) &= [u_\nu(z_i^\nu), \Theta(w_\nu, z_i^\nu)] \\ &= [u_\nu \circ \varphi_i^\nu(z_i'^\nu), g_i^\nu(z_i'^\nu)\Theta\big(w_i^\nu, z_i'^\nu\big)] \\ &= [g_i^\nu(z_i'^\nu)^{-1}u_\nu \circ \varphi_i^\nu(z_i'^\nu), \Theta\big(w_i^\nu, z_i'^\nu\big)] \end{aligned} \tag{6.10}$$

By condition (iv) of Definition 3.6 we have

$$z_i'^\nu = (\varphi_i^\nu)^{-1}(z_i^\nu) \to z_i.$$

By Remark 3.5 we have $z_i \neq \infty$. So, abbreviating $x_i^\nu := g_i^\nu(z_i'^\nu)^{-1}u_\nu \circ \varphi_i^\nu(z_i'^\nu)$, we have

$$d(x_i^\nu, u_i(z_i)) \leq d(x_i^\nu, u_i(z_i'^\nu)) + d(u_i(z_i'^\nu), u_i(z_i)).$$

Since

$$w_i^\nu = \Big((g_i^\nu)^{-1}(u_\nu \circ \varphi_i^\nu), (g_i^\nu)^*\big((\Phi_i^\nu, \Psi_i^\nu) \circ \varphi_i^\nu\big)\Big) \to w_i =: (u_i, \Phi_i, \Psi_i),$$

uniformly on every compact subset of $\mathbb{C}$, it follows that x_i^ν converges to $u_i(z_i)$. Furthermore, by condition (iv) of Proposition 6.3 we have

$$\Theta\big(w_i^\nu, z_i'^\nu\big) \to \Theta(w_i, z_i).$$

Hence equality (6.10) implies that $\mathrm{ev}^i([w_\nu, z_0^\nu, \dots, z_{k-1}])$ converges to

$$[u_i(z_i), \Theta(w_i, z_i)] = \mathrm{ev}^i([\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}]).$$

This proves condition (ii) in the case $i \geq 1$ and $\alpha_i \in V$.

Assume that $i \geq 1$ and $\alpha_i \in \bar{T}$. Then by condition 3.6(iv) we have

$$z_i'^\nu := (\varphi_{\alpha_i}^\nu)^{-1}(z_i^\nu) \to z_i.$$

We choose $\varepsilon > 0$ so small that

$$\bar{B}_\varepsilon^{S^2}(z_i) \cap Z_{\alpha_i} = \emptyset,$$

where for every $r > 0$ and $z \in S^2$ $\bar{B}_r^{S^2}(z)$ denotes the closed ball on S^2 w.r.t. the standard metric, with radius r and center z. By condition 3.6(iii) with $Q := \bar{B}_\varepsilon^{S^2}(z_i)$ the maps

$$\bar{u}_{\alpha_i}^\nu := \mathrm{G}u_{\alpha_i} \circ \varphi_{\alpha_i}^\nu : S^2 \setminus \{z_{\alpha_i,0}\} \to M/\mathrm{G}$$

converge to $\bar{u}_{\alpha_i}$ on $\bar{B}_\varepsilon^{S^2}(z_i)$, uniformly w.r.t. $\bar{d}$. It follows that

$$\begin{aligned}\bar{d}(\mathrm{G}u_\nu(z_i^\nu),\bar{u}_{\alpha_i}(z_i)) &= \bar{d}(\bar{u}_{\alpha_i}^\nu(z_i'^\nu),\bar{u}_{\alpha_i}(z_i)) \\ &\leq \bar{d}(\bar{u}_{\alpha_i}^\nu(z_i'^\nu),\bar{u}_{\alpha_i}(z_i'^\nu)) + \bar{d}(\bar{u}_{\alpha_i}(z_i'^\nu),\bar{u}_{\alpha_i}(z_i)) \to 0.\end{aligned}$$

Therefore for ν large enough we have

$$\mathrm{G}u_\nu(z_i^\nu) \subseteq M_{\delta_0}/\mathrm{G},$$

i.e. $|\mu \circ u_\nu(z_i^\nu)| \leq \delta_0$ and hence

$$\mathrm{ev}^i([w_\nu; z_0^\nu, \ldots, z_{k-1}^\nu]) = s_\theta \circ \mathrm{G}u_\nu(z_i^\nu),$$

and this converges to

$$s_\theta \circ \bar{u}_{\alpha_i}(z_i) = \mathrm{ev}^i([\mathbf{w}, \bar{\mathbf{u}}, \mathbf{z}]).$$

This proves condition (ii) in the case $i \geq 1$ and $\alpha_i \in \bar{T}$.

Assume now that $i = 0$. We claim that

$$\overline{\mathrm{ev}}_\infty(w_\nu) \to \overline{\mathrm{ev}}_{z_0}(w_{\alpha_0}), \qquad \text{for } \nu \to \infty, \tag{6.11}$$

where by abuse of notation we write $w_{\alpha_0} := \bar{u}_{\alpha_0}$, if $\alpha_0 \in \bar{T}$. To see this, let $\varepsilon > 0$. It follows as in the proof of Proposition 5.4 (Conservation of the equivariant homology class, inequality (5.31)), that there exist numbers $\delta > 0$ and $\nu_0 \in \mathbb{N}$ such that for every $\nu \geq \nu_0$ and every $z \in B_\delta^{S^2}(z_0) \setminus \{z_0\}$ we have $\varphi_{\alpha_0}^\nu(z) \in \mathbb{C}$ and

$$\bar{d}(\overline{\mathrm{ev}}_{z_0}(w_{\alpha_0}), \mathrm{G}u^\nu \circ \varphi_{\alpha_0}^\nu(z)) < \varepsilon.$$

Fix an integer $\nu \geq \nu_0$. By condition (i) of Definition 3.6 we have $\varphi_{\alpha_0}^\nu(z_0) = \infty$, hence for large enough $r > 0$ we have

$$(\varphi_{\alpha_0}^\nu)^{-1}(r) \in B_\delta^{S^2}(z_0). \tag{6.12}$$

On the other hand, by definition

$$\overline{\mathrm{ev}}_\infty(w_\nu) = \lim_{r\to\infty} \mathrm{G}u_\nu(r).$$

We choose $r > 0$ so large that

$$\bar{d}(\mathrm{G}u_\nu(r), \overline{\mathrm{ev}}_\infty(w_\nu)) < \varepsilon$$

and (6.12) is satisfied. It follows that

$$\bar{d}(\overline{\mathrm{ev}}_\infty(w_\nu), \overline{\mathrm{ev}}_{z_0}(w_{\alpha_0}))$$
$$\bar{d}(\overline{\mathrm{ev}}_\infty(w_\nu), \mathrm{G}u_\nu(r)) + \bar{d}(\mathrm{G}u_\nu(r), \overline{\mathrm{ev}}_{z_0}(w_{\alpha_0})) < \varepsilon + \varepsilon.$$

This proves (6.11).

It follows that

$$\mathrm{ev}^0([w_\nu;\infty,z_1^\nu,\ldots,z_{k-1}^\nu]) = s_\theta\circ\overline{\mathrm{ev}}_\infty(w_\nu)\to s_\theta\circ\overline{\mathrm{ev}}_{z_0}(w_{\alpha_0}) = \mathrm{ev}^0([\mathbf{w},\bar{\mathbf{u}},\mathbf{z}]).$$

This proves condition 6.1(ii) in the case $i=0$. So 6.1(ii) is satisfied in every case.

This proves Proposition 6.1. □

Remark 6.4 The map $\Theta:\widetilde{\mathcal{M}}\times\mathbb{C}\to\mathrm{EG}$ only uses local data, and hence the same holds for the evaluation map $\mathrm{ev}:\bar{\mathcal{M}}_k\to M_{\mathrm{G}}^k$ as defined in the proof of Proposition 6.1. More precisely, for every constant $E>0$ the component $\mathrm{ev}^i([\mathbf{w},\bar{\mathbf{u}},\mathbf{z}])\in M_{\mathrm{G}}$ only depends on the restriction $w_{\alpha_i}|_{\bar{B}_{R(E)}(z_i)}$, where $R(E)>0$ is a number independent of $[\mathbf{w},\bar{\mathbf{u}},\mathbf{z}]$.

Naively, we could try to define the evaluation map in the following alternative way, without using Proposition 6.3. We fix a continuous equivariant map $\theta:M^*\to\mathrm{EG}$ and set $s_\theta(\mathrm{G}x):=[x,\theta(x)]$. Let $\mathbf{W}\in\bar{\mathcal{M}}_k$ be an equivalence class of stable maps and $i\in\{0,\ldots,k-1\}$. We choose a representative $(\mathbf{w},\bar{\mathbf{u}},\mathbf{z})$ of $\mathbf{W}$. If $\alpha_i\in\bar{T}$ then we define as before

$$\mathrm{ev}^i(\mathbf{W}) := s_\theta\circ\bar{u}_{\alpha_i}(z_i).$$

Similarly, if $\alpha_i\in V$ and $z_i=\infty$ we set $\mathrm{ev}^i(\mathbf{W}):=s_\theta(\lim_{r\to\infty}\mathrm{G}u_{\alpha_i}(r))$. Assume now that $\alpha_i\in V$ and $z_i\neq\infty$. We choose a gauge transformation $g_i\in\mathcal{G}^{2,p}$ such that

$$w_i' := (u_i',\Phi_i',\Psi_i') := g_i^*(w_{\alpha_i}(\cdot+z_i-1))$$

is smooth and in radial gauge outside B_1, as in Proposition D.3. Then by Proposition D.7 $u_i'(r)$ converges to some point $x_i\in\mu^{-1}(0)$ for $r\to\infty$. The idea would be to define $\mathrm{ev}^i(\mathbf{W}):=[u_i'(z_i),\theta(x_i)]$. This does not depend on the choice of the representative $(\mathbf{w},\bar{\mathbf{u}},\mathbf{z})$ of $\mathbf{W}$ nor on the choice of g_i. However, the problem is that the map ev so defined is *not continuous* w.r.t. to convergence of a sequence of vortices against a stable map. This comes from the fact that it uses global data instead of local data. More precisely, the point $\mathrm{ev}^i([\mathbf{w},\bar{\mathbf{u}},\mathbf{z}])$ depends on the restriction of w to the ray parallel to the real axis emanating from the point z_i. This global information gets lost in the limit, if $(w_\nu,z_0^\nu:=\infty,z_1^\nu,\ldots,z_{k-1}^\nu)$ is a sequence of vortices with marked points converging to some stable map $(\mathbf{w},\bar{\mathbf{u}},\mathbf{z})$.

7 An application

This section contains an application of the techniques developed in this dissertation. It states that there exists a solution of the symplectic vortex equations on the complex plane with positive energy if at least one of the following conditions is satisfied.

- The symplectic vortex invariants do not vanish for some nonzero second equivariant homology class.
- The 3-point Gromov-Witten invariants of the symplectic quotient do not vanish for some nonzero homology class in the quotient.

Since I still have to carry out some details, the application is still a conjecture. However, the ingredients of its proof are all contained in this dissertation. In particular, it does not involve transversality for vortices on $\mathbb{C}$. For details about the Gromov-Witten invariants the reader is referred to the book by D. McDuff and D. A. Salamon [MS3]. The symplectic vortex invariants are defined in the paper by K. Cieliebak, R. Gaio, I. Mundet and D. A. Salamon [CGMS]. See also the papers [CGS], [CGMS], [CS], [GS], R. Gaio's dissertation [Ga], and my overview article [Zi2].

The 3-point Gromov-Witten invariants of a symplectic manifold are defined by means of a regular almost complex structure. We fix a symplectic manifold (M, ω) and an ω-compatible almost complex structure J on M. Then J is called regular iff the vertical differential of the Cauchy-Riemann operator $\bar{\partial}_J$ is surjective at every simple J-holomorphic sphere and all the edge evaluation maps are transverse to the diagonal. A J-holomorphic map $u : S^2 \to M$ is called simple iff there does not exist a J-holomorphic map $u' : S^2 \to M$ and a holomorphic map $\varphi : S^2 \to S^2$ of degree > 1 such that $u = u' \circ \varphi$. If the vertical differential of $\bar{\partial}_J$ at u is surjective for every simple J-holomorphic sphere u then for every spherical homology class $B \in H_2(M, \mathbb{Z})$ the set $\widetilde{\mathcal{M}}^*(B, J)$ of all simple J-holomorphic spheres representing B carries a canonical structure of a smooth finite dimensional manifold. The dimension of $\widetilde{\mathcal{M}}^*(B, J)$ is given by the Riemann-Roch formula

$$\dim \widetilde{\mathcal{M}}^*(B, J) = \dim M + 2\langle c_1(TM, \omega), B\rangle.$$

The condition that the edge evaluation maps are transverse to the diagonal implies that the set of simple stable maps in M carries a canonical structure of a smooth finite dimensional manifold.

Assume now that (M, ω) is closed and semipositive. The latter condition means that for every homotopy class $B \in \pi_2(M)$ we have

$$\langle [\omega], B\rangle > 0, \quad c_1(B) := \langle c_1(TM, \omega), B\rangle \geq 3 - n \qquad \Longrightarrow \qquad c_1(B) \geq 0.$$

Here $c_1(TM,\omega)$ denotes the first Chern class of the symplectic vector bundle (TM,ω). We denote by $H^*(M)$ the quotient of $H^*(M,\mathbb{Z})$ over its torsion subgroup. We call a homology class $B\in H_2(M,\mathbb{Z})$ *good*, iff there is no pair (k,B'), where $k\geq 2$ is an integer and B' is a spherical second homology class such that $B=kB'$ and $c_1(B')=0$. For every positive integer k, every spherical homology class $0\neq B\in H_2(M,\mathbb{Z})$ and every ω-compatible almost complex structure J on M we denote by $\mathcal{M}^*_k(B,J)$ the set of equivalence classes of tuples $(u,z_1,\dots,z_k)$, where $u:S^2\to M$ is a simple J-holomorphic map that represents B and $z_1,\dots,z_k\in S^2$ are distinct points. Semipositivity of (M,ω) implies that for every regular ω-compatible almost complex structure J and every good spherical homology class B the evaluation map

$$\mathrm{ev}_{B,J}:\mathcal{M}^*_k(B,J)\to M^k,\quad (u,z_1,\dots,z_k)\mapsto(u(z_1),\dots,u(z_k))$$

is a pseudo-cycle. The genus 0 three-point Gromov-Witten invariants

$$\mathrm{GW}^{M,\omega}_{3,B}:H^*(M)^3\to\mathbb{Z}$$

are given as follows. For each triple $(\alpha_1,\alpha_2,\alpha_3)\in H^*(M)^3$ such that

$$\sum_{i=1}^{3}\deg\alpha_i=\dim M+2c_1(B) \tag{7.1}$$

the number $\mathrm{GW}^{M,\omega}_{3,B}(\alpha_1,\alpha_2,\alpha_3)$ is defined to be the intersection number of the pseudo-cycles $\mathrm{ev}_{B,J}$ and f, where J is a regular ω-compatible almost complex structure and $f:X\to M^3$ is a pseudo-cycle Poincaré dual to $\pi_1^*\alpha_1\smile\pi_2^*\alpha_2\smile\pi_3^*\alpha_3\in H^*(M^3)$. Here $\pi_i:M^3\to M$ denotes the projection to the i-th factor. If the dimension condition (7.1) is not satisfied then $\mathrm{GW}^{M,\omega}_{3,B}(\alpha_1,\alpha_2,\alpha_3):=0$.

Let now (M,ω) be a symplectic manifold with a Hamiltonian action of a compact connected Lie group G, and let J be a G-invariant ω-compatible almost complex structure. Assume that G acts freely on $\mu^{-1}(0)$, and let the symplectic structure $\bar\omega$ on $\bar M:=\mu^{-1}(0)/\mathrm{G}$ be defined as in (1.2). Recall that J induces an ω-compatible almost complex structure $\bar J$ on the symplectic quotient $\bar M:=\mu^{-1}(0)/\mathrm{G}$, defined as in (1.9). For every equivariant homology class $B\in H_2^{\mathrm{G}}(M,\mathbb{Z})$ the *genus 0 symplectic vortex invariants corresponding to B* are given by a linear map

$$\Phi^{M,\omega,\mu}_B:H^*_{\mathrm{G}}(M)\to\mathbb{Q}.$$

Fix a class $\alpha\in H^*_{\mathrm{G}}(M)$. The class B determines an isomorphism class $[P]$ of principal G-bundles over S^2. Intuitively, the number $\Phi^{M,\omega,\mu}_B(\alpha)$ equals the number of gauge equivalence classes of solutions (u,A) of the symplectic vortex equations over P, such that $[u]_{\mathrm{G}}=B$ and (u,A) evaluates at the point $0\in S^2$ to a point in the "Poincaré dual" of α. Here $[u]_{\mathrm{G}}\in H_2(M_{\mathrm{G}},\mathbb{Z})$

denotes the equivariant homology class of u. For $\alpha \in H^*_{\mathrm{G}}(M)$ the number $\Phi^{M,\omega,\mu}_B(\alpha)$ can only be nonzero if the dimension condition

$$\deg \alpha = \dim M - 2\dim \mathrm{G} + 2\langle c_1^{\mathrm{G}}(TM), B\rangle$$

is satisfied. Here $c_1^{\mathrm{G}}(TM)$ denotes the first Chern class of the complex vector bundle $TM \times_{\mathrm{G}} \mathrm{EG} \to M \times_{\mathrm{G}} \mathrm{EG}$, and EG is a contractible topological space on which G acts continuously and freely.

Conjecture 7.1 (Existence of vortices on $\mathbb{C}$ with positive energy) *Assume that the hypotheses (H1), (H2) and (H3) are satisfied, that the symplectic quotient $(\bar{M}, \bar{\omega})$ is semipositive, and that $\bar{J}$ is regular. If at least one of the following conditions is satisfied, then there exists a vortex on $\mathbb{C}$ with positive energy.*

(i) There exists an equivariant homology class $0 \neq B \in H_2^{\mathrm{G}}(M, \mathbb{Z})$ such that

$$\Phi^{M,\omega,\mu}_B \not\equiv 0 \tag{7.2}$$

and the following condition holds. If there exists a spherical homology class $\bar{B} \in H_2(\bar{M}, \mathbb{Z})$ with $\kappa_{.}\bar{B} = B$ then $\bar{B}$ is good and every $\bar{J}$-holomorphic sphere representing $\bar{B}$ is simple. Here $\kappa_{.} : H_*(\bar{M}, \mathbb{Z}) \to H_*(M_{\mathrm{G}}, \mathbb{Z})$ denotes the Kirwan homomorphism.

(ii) There exists a good homology class $0 \neq \bar{B} \in H_2(\bar{M}, \mathbb{Z})$, such that

$$\mathrm{GW}^{\bar{M},\bar{\omega}}_{3,\bar{B}} \not\equiv 0$$

and every $\bar{J}$-holomorphic map $\bar{u} : S^2 \to \bar{M}$ that represents $\bar{B}$ is simple.

Example 7.2 For $n \geq 1$ consider the diagonal action of $\mathrm{G} := S^1$ on $M := \mathbb{C}^{n+1}$ with the standard symplectic structure ω_0, with moment map $\mu(\mathbf{z}) := \frac{i}{2}(1 - |\mathbf{z}|^2)$. It satisfies the hypotheses (H1), (H2), (H3). The symplectic quotient is $\bar{M} = \mathbb{C}\mathrm{P}^n$ with the Fubini-Studi form ω_{FS}. It is monotone and hence semipositive. Furthermore, $\bar{J}$ equals the standard complex structure on $\mathbb{C}\mathrm{P}^n$. By Proposition 7.4.3. in the book [MS3], it is regular. We denote by

$$L \in H_2(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$$

the class of the projective line $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^n$, and by

$$c \in H^2_{S^1}(\mathbb{C}^{n+1}) \cong H^2(\mathbb{C}\mathrm{P}^\infty)$$

the positive generator for which

$$\langle c, \kappa_{.} L\rangle = 1,$$

where $\kappa_. : H_*(\mathbb{C}\mathrm{P}^n, \mathbb{Z}) \to H_*^{S^1}(\mathbb{C}^{n+1}, \mathbb{Z})$ denotes the Kirwan homomorphism. Let d be a positive integer. Since $\mathbb{C}\mathrm{P}^n$ is monotone, the spherical class $dL \in H_2(\mathbb{C}\mathrm{P}^n)$ is good. We define $m := d(n+1)+n$. By Example 3.3. in the paper [CS], the genus 0 symplectic vortex invariants of $(d\kappa_. L, c^m)$ are given by

$$\Phi_{d\kappa_. L}^{\mathbb{C}^{n+1},\omega_0,\mu}(c^m) = 1.$$

It follows that condition (i) of Conjecture 7.1 is satisfied. Hence if the conjecture is true then there exists a vortex on $\mathbb{C}$ with positive energy. Note that also condition ii) is satisfied, since

$$\mathrm{GW}_{3,L}^{\mathbb{C}\mathrm{P}^n,\omega_{FS}}\big(\mathrm{PD}([\mathrm{pt}]), \mathrm{PD}([\mathrm{pt}]), \mathrm{PD}([\mathbb{C}\mathrm{P}^{n-1}])\big) = 1, \tag{7.3}$$

where $\mathrm{PD} : H_k(\mathbb{C}\mathrm{P}^n) \to H^{2n-k}(\mathbb{C}\mathrm{P}^n)$ denotes Poincaré duality, $\mathrm{pt} \subseteq \mathbb{C}\mathrm{P}^n$ is an oriented submanifold consisting of one point, and we embed $\mathbb{C}\mathrm{P}^{n-1}$ in $\mathbb{C}\mathrm{P}^n$. This formula follows from the fact that there exists exactly one complex projective line through two distinct points in $\mathbb{C}\mathrm{P}^n$.

Sketch of the proof of Conjecture 7.1: Assume that (i) holds. Let $0 \neq B \in H_2^{\mathrm{G}}(M,\mathbb{Z})$ be such that $\Phi_B^{M,\omega,\mu} \not\equiv 0$. Since G is connected, the classifying space $B\mathrm{G}$ is simply connected, and hence by Hurewicz's theorem the canonical map $\pi_2(B\mathrm{G}) \to H_2(B\mathrm{G})$ is an isomorphism. Let $C \in \pi_2(B\mathrm{G})$ be the image of B under the composition

$$H_2^{\mathrm{G}}(M,\mathbb{Z}) \to H_2(B\mathrm{G},\mathbb{Z}) \to \pi_2(B\mathrm{G}),$$

where the first map is induced by the canonical projection $(M \times \mathrm{EG})/\mathrm{G} \to B\mathrm{G} = \mathrm{EG}/\mathrm{G}$. We fix a principal G-bundle P over S^2 whose isomorphism class is determined by C. We denote by $C_{\mathrm{G}}^\infty(P,M)$ the space of smooth G-equivariant maps from P to M, and by $\mathcal{A}(P)$ the space of smooth connection one forms on P. For $\varepsilon > 0$ consider the ε-vortex equations on P for a pair $(u,A) \in C_{\mathrm{G}}^\infty(P,M) \times \mathcal{A}(P)$:

$$\bar\partial_{J,A}(u) = 0, \tag{7.4}$$

$$\varepsilon^2 * F_A + \mu \circ u = 0. \tag{7.5}$$

For every continuous equivariant map $u : P \to M$ we denote by $[u]_{\mathrm{G}} \in H_2(M_{\mathrm{G}},\mathbb{Z})$ its equivariant homology class, and we define

$$\widetilde{\mathcal{M}}_B^\varepsilon := \big\{(u,A) \in C_{\mathrm{G}}^\infty(P,M) \times \mathcal{A}(P) \,\big|\, (7.4),\ (7.5),\ [u]_{\mathrm{G}} = B\big\}. \tag{7.6}$$

We choose a constant $c > 0$ and a function $f : M \to [0,\infty)$ as in hypothesis (H2). Let $\bar B_r \subseteq \mathfrak{g}$ be the closed ball of radius $r := \langle [\omega-\mu]^{\mathrm{G}}, B\rangle/(4\pi)$. Here $[\omega-\mu]^{\mathrm{G}} \in H_{\mathrm{G}}^2(M)$ denotes the equivariant cohomology class of $\omega - \mu$. Since μ is proper, the inverse image $\mu^{-1}(\bar B_r) \subseteq M$ is compact, and hence

$f(\mu^{-1}(\bar{B}_r)) \subseteq [0,\infty)$ is compact. Therefore, increasing c if necessary, we may assume that $f(\mu^{-1}(\bar{B}_r)) \subseteq [0,c)$. This means that

$$f(x) \geq c \qquad \Longrightarrow \qquad |\mu(x)| > \frac{\langle [\omega - \mu]^{\mathrm{G}}, B\rangle}{4\pi}. \tag{7.7}$$

Therefore, as in the proof of Lemma 2.7 in [CGMS] with $\tau := 0$, it follows that for every $0 < \varepsilon \leq 1$ and every ε-vortex $(u,A) \in \widetilde{\mathcal{M}}^{\varepsilon}_B$ we have

$$u(P) \subseteq M_{<c} = \{x \in M \mid |\mu \circ u(x)| < c\}. \tag{7.8}$$

Increasing c further we may assume that it is a regular value of the map $|\mu|^2 : M \to [0,\infty)$. It follows that $M_c \subseteq M$ is a compact submanifold with boundary.

For $\delta > 0$ and $r > 0$ we define

$$\mathcal{B}^{\delta,r} := \left\{ (u,A) \in C^\infty_{\mathrm{G}}(P,M) \times \mathcal{A}(P) \,\middle|\, \exists z \in S^2 \sup_{B_r(z)} |\mu \circ u| < \delta \right\}. \tag{7.9}$$

Furthermore, we denote by $\mathcal{G}(P)$ the space of smooth gauge transformations on P, i.e. of smooth maps $g : P \to \mathrm{G}$ such that $g(ph) = h^{-1}g(p)h$ for every $h \in \mathrm{G}$ and $p \in P$. As a corollary to the theorem of Peter and Weyl we may assume w.l.o.g. that G is a closed subgroup of $U(\ell)$ for some integer ℓ. For every integer $m \geq \ell$ we define

$$\mathrm{EG}^m := \{\Theta \in \mathbb{C}^{\ell \times m} \mid \Theta\Theta^* = \mathbf{1}\}.$$

This is a finite dimensional approximation of the classifying space EG, which consists of all orthonormal ℓ-tuples of finite sequences of complex numbers. By our hypothesis (H1) we may choose a number $\delta > 0$ so small that G acts freely on M_δ. We also choose a smooth G-equivariant map $\theta : M_\delta \to \mathrm{EG}^m$ for some integer $m \geq \ell$. By Proposition 12.1 in the paper by R. Gaio and D. A. Salamon [GS], for every $r > 0$ there exist an integer $N \geq m$ and a map $\Theta : \mathcal{B}^{\delta,r} \times P \to \mathrm{EG}^N$ with the following properties.

(i) For every $g \in \mathcal{G}(P)$, $h \in \mathrm{G}$ and $(u,A,p) \in \mathcal{B}^{\delta,r} \times P$ we have

$$\Theta(g^{-1}u, g^*A, pg(p)^{-1}) = \Theta(u,A,p) = h\Theta(u,A,ph). \tag{7.10}$$

(ii) The map Θ is continuous w.r.t. the C^0-topology on $\mathcal{B}^{\delta,r}$.

(iii) Denoting by $\iota^{m,N} : \mathrm{EG}^m \to \mathrm{EG}^N$ the inclusion we have

$$|\mu \circ u(p)| < \delta \qquad \Longrightarrow \qquad \Theta(u,A,p) = \iota^{m,N} \circ \theta \circ u(p).$$

By Lemma 12.2 in the paper [GS] there exist positive constants r and ε_0 such that

$$\widetilde{\mathcal{M}}_B^\varepsilon \subseteq \mathcal{B}^{\delta,r},$$

for $0 < \varepsilon \leq \varepsilon_0$. We fix such constants ε_0 and r and define the evaluation map at the point $0 \in S^2$ to be the map

$$\mathrm{ev} : \mathcal{B}^{\delta,r} \to M_{<c} \times_{\mathrm{G}} \mathrm{EG}^N, \quad \mathrm{ev}(u, A) := [u(p), \Theta(u, A, p)], \tag{7.11}$$

where $p \in P$ is an arbitrary point in the fibre over $0 \in S^2$.

We define

$$k := \dim M - 2 \dim \mathrm{G} + 2\langle c_1^{\mathrm{G}}(TM), B\rangle.$$

By our assumption that $\Phi_B^{M,\omega,\mu} \not\equiv 0$, there exists a cohomology class $\alpha \in H^*_{\mathrm{G}}(M)$ of degree k such that $\Phi_B^{M,\omega,\mu}(\alpha) \neq 0$. Combining the inclusion $\mathrm{EG}^N \to \mathrm{EG}$ with the inclusion $M_c \hookrightarrow M$, we get a continuous map

$$M_c \times \mathrm{EG}^N \to M \times \mathrm{EG}$$

which descends to a map

$$\iota : M_c \times_{\mathrm{G}} \mathrm{EG}^N := (M_c \times \mathrm{EG}^N)/\mathrm{G} \to M_{\mathrm{G}} := (M \times \mathrm{EG})/\mathrm{G}. \tag{7.12}$$

We define $\widetilde{\alpha} := \iota^*\alpha \in H^*(M_c \times_{\mathrm{G}} \mathrm{EG}^N)$. By Theorem E.16 (de Rham isomorphism) we may identify $\widetilde{\alpha}$ with a de Rham cohomology class on the manifold with boundary $M_c \times_{\mathrm{G}} \mathrm{EG}^N$. By Theorem E.18 applied with α replaced by $\widetilde{\alpha}$ there exists a rational number $\lambda \neq 0$ and a compact oriented submanifold $X \subseteq M_c \times_{\mathrm{G}} \mathrm{EG}^N$ of codimension k, possibly with boundary $\partial X \subseteq \partial(M_c \times_{\mathrm{G}} \mathrm{EG}^N)$, such that

$$\lambda \alpha_X = \widetilde{\alpha},$$

where α_X is the de Rham cohomology class Poincaré dual to X defined in (E.50). We define the map

$$s_\theta : \bar{M} = \mu^{-1}(0)/\mathrm{G} \to \mu^{-1}(0) \times_{\mathrm{G}} \mathrm{EG}^N, \quad s_\theta(\bar{x}) := [x, \iota^{m,N} \circ \theta(x)], \tag{7.13}$$

where $x \in \mu^{-1}(0)$ is an arbitrary point in the orbit $\bar{x}$. Perturbing X slightly we may assume w.l.o.g. that it is transverse to the submanifold $\mu^{-1}(0) \times_{\mathrm{G}} \mathrm{EG}^N \subseteq M_c \times_{\mathrm{G}} \mathrm{EG}^N$ and to s_θ. It follows that

$$\bar{X} := s_\theta^{-1}(X \cap \mu^{-1}(0) \times_{\mathrm{G}} \mathrm{EG}^N) \subseteq \bar{M}$$

is a submanifold of codimension k. Fix a finite tree $(\bar{T}, E)$, a positive integer ℓ, vertices $\alpha_1, \ldots, \alpha_\ell \in \bar{T}$ and a collection $\bar{B}_\alpha \in H_2(\bar{M}, \mathbb{Z})$, $\alpha \in \bar{T}$ of spherical homology classes. We denote by $\widetilde{\mathcal{M}}^*_{\bar{T},(\alpha_i)}((\bar{B}_\alpha)_{\alpha\in\bar{T}}, \bar{J})$ the set of *simple* stable maps of $\bar{J}$-holomorphic spheres in $\bar{M}$ modelled over $\bar{T}$ with ℓ marked points corresponding to $\alpha_1, \dots, \alpha_\ell$. It consists of all stable maps of $\bar{J}$-holomorphic spheres

$$\big(\bar{T}, E, (\bar{u}_\alpha)_{\alpha \in \bar{T}}, (z_{\alpha\beta})_{\alpha E \beta}, (\alpha_i, z_i)_{i=1,\dots,\ell}\big)$$

(see Definition 5.1.1 in the book [MS3]), such that each $\bar{u}_\alpha$ is simple and $\bar{u}_\alpha(S^2) \neq \bar{u}_\beta(S^2)$ for any two vertices $\alpha \neq \beta$ such that $\bar{u}_\alpha$ and $\bar{u}_\beta$ are non-constant. Moreover, we denote by $\mathcal{M}^*_{\bar{T},(\alpha_i)}((\bar{B}_\alpha)_{\alpha \in \bar{T}}, \bar{J})$ the set of equivalence classes of such simple stable maps. Since by assumption $\bar{J}$ is regular, by Theorem 6.2.6 in the book [MS3] this set carries a canonical structure of a smooth manifold. We define the evaluation map

$$\begin{aligned} \overline{\mathrm{ev}}_{\bar{T},(\alpha_i),(\bar{B}_\alpha)} : \mathcal{M}^*_{\bar{T},(\alpha_i)}((\bar{B}_\alpha)_{\alpha \in \bar{T}}, \bar{J}) \to \bar{M}^\ell, \\ \big(\bar{T}, E, (\bar{u}_\alpha)_{\alpha \in \bar{T}}, (z_{\alpha\beta})_{\alpha E \beta}, (\alpha_i, z_i)_{i=1,\dots,\ell}\big) \mapsto \big(\bar{u}_{\alpha_1}(z_1), \dots, \bar{u}_{\alpha_\ell}(z_\ell)\big). \end{aligned} \tag{7.14}$$

We assume that for every finite tree $\bar{T}$, every vertex $\alpha_1 \in \bar{T}$ and every collection of spherical homology classes $\bar{B}_\alpha \in H_2(\bar{M}, \mathbb{Z})$, $\alpha \in \bar{T}$, the submanifold $\bar{X} \subseteq \bar{M}$ is transverse to $\overline{\mathrm{ev}}_{\bar{T},\alpha_1,(\bar{B}_\alpha)}$. It should be possible to arrange this by perturbing X slightly.

Since $\Phi^{M,\omega,\mu}_B(\alpha) \neq 0$, it should follow that for every number $0 \leq \varepsilon \leq \varepsilon_0$ there exists a smooth solution (u, A) of the ε-vortex equations (7.4), (7.5) on P such that $[u]_{\mathrm{G}} = B$ and

$$\mathrm{ev}(u, A) \in X,$$

where ev is defined as in (7.11). The idea to prove this is to use the functoriality axiom in the definition of the Euler class of a regular G-moduli problem (see the paper [CGMS], Theorem 6.4) and Lemma (E.19). We choose a sequence of numbers $0 < \varepsilon_\nu < \varepsilon_0$ converging to 0 and a sequence (u_ν, A_ν) of solutions of the ε_ν vortex equations on P such that $[u]_{\mathrm{G}} = B$ and $\mathrm{ev}(u_\nu, A_\nu) \in X$. By a bubbling argument analogous to Theorem 4.1 there should exist a subsequence of (u_ν, A_ν) that converges in a suitable sense to some stable map of vortices on $\mathbb{C}$ (with positive energy) and pseudo-holomorphic spheres in $\bar{M}$ with one marked point. If this stable map contains vortices than we are done. Hence assume by contradiction that it consists only of pseudo-holomorphic spheres in $\bar{M}$. It follows that the class B lies in the image of the Kirwan homomorphism $\kappa_. : H_2(\bar{M}, \mathbb{Z}) \to H_2(M_{\mathrm{G}}, \mathbb{Z})$. Let $\bar{B} \in H_2(\bar{M}, \mathbb{Z})$ be such that $\kappa_. \bar{B} = B$. By assumption, the limit stable map is a tuple

$$(\bar{\mathbf{u}}, \mathbf{z}) := \big(\bar{T}, E, (\bar{u}_\alpha)_{\alpha \in \bar{T}}, (z_{\alpha\beta})_{\alpha E \beta}, \alpha_1, z_1\big),$$

where $(\bar{T}, E)$ is a tree, the $\bar{u}_\alpha : S^2 \to \bar{M}$ are $\bar{J}$-holomorphic maps, the $z_{\alpha\beta} \in S^2$ are nodal points and $z_1 \in S^2$ is a marked point thought of as lying on the vertex $\alpha_1 \in T$. Convergence of the sequence (u_ν, A_ν) against $(\bar{\mathbf{u}}, \mathbf{z})$ should imply that

$$\bar{u}_{\alpha_1}(z_1) \in \bar{X}. \tag{7.15}$$

By Proposition 6.1.2. in the book [MS3] there exists a simple stable map with one marked point

$$(\bar{\mathbf{u}}', \mathbf{z}') := \big(\bar{T}', E', (\bar{u}_\alpha)_{\alpha\in\bar{T}'}, (z'_{\alpha\beta})_{\alpha E'\beta}, \alpha'_0, z'_0\big),$$

such that

$$\begin{aligned} \overline{\mathrm{ev}}_{\bar{T}',\alpha_1,(\bar{B}_\alpha)}(\bar{\mathbf{u}}', \mathbf{z}') &= \bar{u}_{\alpha_1}(z_1), \\ \bar{B} &= \sum_{\alpha\in\bar{T}'} m_\alpha \bar{B}'_\alpha. \end{aligned} \tag{7.16}$$

Here m_α are positive integers and $\bar{B}'_\alpha := [\bar{u}'_\alpha] \in H_2(\bar{M}, \mathbb{Z})$. Furthermore, since by assumption $(\bar{M}, \bar{\omega})$ is semipositive and $\bar{B}$ is good, it follows as in the proof of Theorem 6.6.1. in the book [MS3] that

$$\dim \mathcal{M}^*_{\bar{T}',\alpha_1}((\bar{B}'_\alpha)_{\alpha\in\bar{T}'}, \bar{J}) \leq k - 4. \tag{7.17}$$

By our choice of X the evaluation map $\overline{\mathrm{ev}}_{\bar{T}',\alpha_1,(\bar{B}'_\alpha)}$ is transverse to the submanifold $\bar{X} \subseteq \bar{M}$. Furthermore, by (7.16) and (7.15) we have

$$\overline{\mathrm{ev}}_{\bar{T}',\alpha_1,(\bar{B}'_\alpha)}(\bar{\mathbf{u}}', \mathbf{z}') = \bar{u}_{\alpha_1}(z_1) \in \bar{X}.$$

Since the codimension of $\bar{X}$ in $\bar{M}$ is k and by (7.17) the domain of the evaluation map $\overline{\mathrm{ev}}_{\bar{T}',(\bar{B}'_\alpha)}$ has dimension at most $k-4$, this is a contradiction. Hence there exists a vortex on $\mathbb{C}$ with positive energy.

Assume now that (ii) holds. The idea of proof of the conjecture is then the following. By our assumption there exists a good homology class $0 \neq \bar{B} \in H_2(\bar{M}, \mathbb{Z})$ and cohomology classes $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 \in H^*(\bar{M})$ such that $\mathrm{GW}^{\bar{M},\bar{\omega}}_{3,\bar{B}}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) \neq 0$. We define $B := \kappa_. \bar{B}$. Since the Kirwan homomorphism $\kappa^{\cdot} : H^*(M_{\mathrm{G}}) \to H^*(\bar{M})$ is surjective, there exist cohomology classes $\alpha_i \in H^*(M_{\mathrm{G}})$ such that $\kappa^{\cdot}\alpha_i = \bar{\alpha}_i$. As above, we choose a principal G-bundle P whose isomorphism class is determined by B, and a constant $c > 0$ such that for every number $0 < \varepsilon \leq 1$ and every ε-vortex $(u, A) \in \widetilde{\mathcal{M}}^\varepsilon_B$ we have $u(P) \subseteq M_{<c}$ and such that c is a regular value of the map $|\mu|^2 : M \to [0, \infty)$. Furthermore, we assume that G is a closed subgroup of $U(\ell)$, and we choose a number $\delta > 0$ so small that G acts freely on M_δ and a smooth G-equivariant map $\theta : M_\delta \to \mathrm{EG}^m$ for some $m \geq \ell$. We also choose positive constants ε_0 and r such that $\widetilde{\mathcal{M}}^\varepsilon_B \subseteq \mathcal{B}^{\delta,r}$, for $0 < \varepsilon \leq \varepsilon_0$, as in Lemma 12.2 in the paper [GS]. Here $\widetilde{\mathcal{M}}^\varepsilon_B$ and $\mathcal{B}^{\delta,r}$ are defined as in (7.6) and (7.9). Furthermore, we choose an integer $N \geq m$ and a map $\Theta : \mathcal{B}^{\delta,r} \times P \to \mathrm{EG}^N$ such that conditions (i), (ii) and (iii) above are satisfied. For every $0 \leq \varepsilon \leq \varepsilon_0$ we define $\mathcal{M}^\varepsilon_B := \widetilde{\mathcal{M}}^\varepsilon_B / \mathcal{G}(P)$ to be the set of gauge equivalence classes of smooth ε-vortices that represent B. For $\varepsilon := 0$ the map

$$\mathcal{M}^0_B \ni [u, A] \mapsto \mathrm{G}u \in \widetilde{\mathcal{M}}(\bar{B}, \bar{J}) := \big\{\bar{u} : S^2 \to \bar{M} \,\big|\, u \text{ is } \bar{J}\text{-holomorphic}, [\bar{u}] = \bar{B}\big\} \tag{7.18}$$

is a bijection, see for example Proposition 4.20 in my overview article [Zi2] or the Ph.D.-thesis by R. Gaio [Ga]. Since by hypothesis every $\bar{J}$-holomorphic map $\bar{u} : S^2 \to \bar{M}$ is simple and $\bar{J}$ is regular, by Theorem 3.1.5. in the book [MS3] the set $\widetilde{\mathcal{M}}(\bar{B}, \bar{J})$ carries a canonical structure of a smooth manifold of dimension

$$\dim \widetilde{\mathcal{M}}(\bar{B}, \bar{J}) = \dim M - 2 \dim \mathrm{G} + 2\langle c_1(T\bar{M}, \bar{\omega}), \bar{B}\rangle.$$

Hence the bijection (7.18) endows $\mathcal{M}_B^0$ with a canonical structure of a smooth manifold.

For every point $z \in S^2$ we define

$$\mathrm{ev}_z^\varepsilon : \mathcal{M}_B^\varepsilon \to M_{<c} \times_{\mathrm{G}} \mathrm{EG}^N, \quad \mathrm{ev}_z^\varepsilon([u, A]) := [u(p), \Theta(u, A, p)],$$

where $p \in P$ is an arbitrary point over the fibre of z, and we define the evaluation map

$$\mathrm{ev}^\varepsilon : \mathcal{M}_B^\varepsilon \to (M_{<c} \times_{\mathrm{G}} \mathrm{EG}^N)^3,$$
$$\mathrm{ev}^\varepsilon([u, A]) := \big(\mathrm{ev}_1^\varepsilon([u, A]), \mathrm{ev}_2^\varepsilon([u, A]), \mathrm{ev}_3^\varepsilon([u, A])\big).$$

We fix an index $i = 1, 2, 3$. We define $\widetilde{\alpha}_i := \iota^*\alpha_i \in H^*(M_c \times_{\mathrm{G}} \mathrm{EG}^N)$, where ι is as in (7.12), and identify this with a de Rham cohomology class on $M_c \times_{\mathrm{G}} \mathrm{EG}^N$. By Theorem E.18 there exists a rational number $\lambda_i \neq 0$ and a compact oriented submanifold $X_i \subseteq M_c \times_{\mathrm{G}} \mathrm{EG}^N$ of codimension $\deg \alpha_i$, possibly with boundary $\partial X_i \subseteq \partial(M_c \times_{\mathrm{G}} \mathrm{EG}^N)$, such that

$$\lambda_i \alpha_{X_i} = \widetilde{\alpha}_i,$$

where α_{X_i} is the de Rham cohomology class Poincaré dual to X_i defined in (E.50). We define s_θ as in (7.13). Perturbing the X_i's slightly we may assume w.l.o.g. that X_i is transverse to the submanifold $\mu^{-1}(0) \times_{\mathrm{G}} \mathrm{EG}^N \subseteq M_c \times_{\mathrm{G}} \mathrm{EG}^N$ and to the map s_θ, and that $X := X_1 \times X_2 \times X_3$ is transverse to $\mathrm{ev}^0 : \mathcal{M}_B^0 \to M_{<c} \times_{\mathrm{G}} \mathrm{EG}^N$. We also assume that for every finite tree $\bar{T}$, each three vertices $\alpha_1, \alpha_2, \alpha_3 \in \bar{T}$ and every collection $\bar{B}_\alpha \in H_2(\bar{M}, \mathbb{Z})$ of spherical homology classes the submanifold $\bar{X} := \bar{X}_1 \times \bar{X}_2 \times \bar{X}_3 \subseteq \bar{M}^3$ is transverse to the evaluation map

$$\overline{\mathrm{ev}}_{\bar{T},(\alpha_i),(\bar{B}_\alpha)} : \mathcal{M}^*_{\bar{T},(\alpha_i)}((\bar{B}_\alpha)_{\alpha \in \bar{T}}, \bar{J}) \to \bar{M}^3.$$

Here $\overline{\mathrm{ev}}_{\bar{T},(\alpha_i),(\bar{B}_\alpha)}$ is defined as in (7.14).

For $\varepsilon \geq 0$ define

$$\mathcal{M}_{B,X}^\varepsilon := \big\{[u, A] \in \mathcal{M}_B^\varepsilon \,\big|\, \mathrm{ev}^\varepsilon(u, A) \in X\big\}.$$

In Step 4 in the proof of Theorem A of the paper [GS], R. Gaio and D. A. Salamon constructed for $0 < \varepsilon \leq \varepsilon_1$ an injective map

$$\mathcal{T}_{B,X}^\varepsilon : \mathcal{M}_{B,X}^0 \to \mathcal{M}_{B,X}^\varepsilon,$$

where $0 < \varepsilon_1 \le \varepsilon_0$ is a small enough number. There are two cases.

Case: The map $\mathcal{T}^{\varepsilon}_{B,X}$ is surjective for some $0 < \varepsilon \le \varepsilon_1$. Then

$$\mathrm{GW}^{\bar{M},\bar{\omega}}_{\bar{B}}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) = \Phi^{M,\omega,\mu}_{B}(\alpha_1 \smile \alpha_2 \smile \alpha_3),$$

see the paper [GS]. Since by assumption $\mathrm{GW}^{\bar{M},\bar{\omega}}_{\bar{B}}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) \neq 0$, the argument of the first part of the sketch of a proof shows that there exists a vortex on $\mathbb{C}$ of positive energy.

Case: The map $\mathcal{T}^{\varepsilon}_{B,X}$ is not surjective for any $0 < \varepsilon \le \varepsilon_1$. Then for every integer $\nu \ge \varepsilon_1^{-1}$ we choose a gauge equivalence class of $(1/\nu)$-vortices

$$W_\nu \in \mathcal{M}^{\frac{1}{\nu}}_{B,X} \setminus \mathrm{im}\mathcal{T}^{\frac{1}{\nu}}_{B,X},$$

and a smooth representative (u_ν, A_ν) of W_ν. By a bubbling argument analogous to Theorem 4.1 there should exist a subsequence of (u_ν, A_ν) that converges in a suitable sense to some stable map of vortices on $\mathbb{C}$ (with positive energy) and pseudo-holomorphic spheres in $\bar{M}$ with three marked points. If it contains vortices on $\mathbb{C}$ than we are done. So assume by contradiction that it consists solely of $\bar{J}$-holomorphic spheres. So it is a tuple

$$(\bar{\mathbf{u}}, \mathbf{z}) := \big(\bar{T}, E, (\bar{u}_\alpha)_{\alpha\in\bar{T}}, (z_{\alpha\beta})_{\alpha E\beta}, (\alpha_i, z_i)_{i=1,2,3}\big),$$

where $(\bar{T}, E)$ is a finite tree, the $\bar{u}_\alpha : S^2 \to \bar{M}$ are $\bar{J}$-holomorphic maps, the $z_{\alpha\beta} \in S^2$ are the nodal points and for $i = 1, 2, 3$ $z_i \in S^2$ is a marked point thought of as lying on the component $\alpha_i \in \bar{T}$. Convergence of the sequence (u_ν, A_ν) against $(\bar{\mathbf{u}}, \mathbf{z})$ should imply that

$$\big(\bar{u}_{\alpha_1}(z_1), \ldots, \bar{u}_{\alpha_3}(z_3)\big) \in \bar{X}. \tag{7.19}$$

Furthermore, the tree $\bar{T}$ consists of more than one vertex, since otherwise for ν large enough $W_\nu = \mathcal{T}^{\frac{1}{\nu}}_{B,X}([u, A])$, where $[u, A] \in \widetilde{\mathcal{M}}^{0}_{B,X}$ is the image of $\bar{u}_{\alpha_1}$ under the inverse map to (7.18). By Proposition 6.1.2. in the book [MS3] there exists a simple stable map with three marked points

$$(\bar{\mathbf{u}}', \mathbf{z}') := \big(\bar{T}', E', (\bar{u}_\alpha)_{\alpha\in\bar{T}'}, (z'_{\alpha\beta})_{\alpha E'\beta}, (\alpha'_i)_{i=1,2,3}\big),$$

such that

$$\begin{aligned} \overline{\mathrm{ev}}_{\bar{T}',(\alpha_i),(\bar{B}_\alpha)}(\bar{\mathbf{u}}', \mathbf{z}') &= \big(\bar{u}_{\alpha_1}(z_1), \ldots, \bar{u}_{\alpha_3}(z_3)\big), \\ \bar{B} &= \sum_{\alpha\in\bar{T}'} m_\alpha \bar{B}'_\alpha. \end{aligned} \tag{7.20}$$

Here m_α are positive integers and $\bar{B}'_\alpha := [\bar{u}'_\alpha] \in H_2(\bar{M}, \mathbb{Z})$. We abbreviate $k := \sum_{i=1}^{3} \deg \alpha_i$. Since by assumption $(\bar{M}, \bar{\omega})$ is semipositive and $\bar{B}$ is good, it follows as in the proof of Theorem 6.6.1. in the book [MS3] that

$$\dim \mathcal{M}^{*}_{\bar{T}',(\alpha_i)}((\bar{B}'_\alpha)_{\alpha\in\bar{T}'}, \bar{J}) \le k - 2. \tag{7.21}$$

By our choice of the X_i's the evaluation map $\overline{\mathrm{ev}}_{\bar{T}',(\alpha_i),(\bar{B}'_\alpha)}$ is transverse to the submanifold $\bar{X} \subseteq \bar{M}^3$. Furthermore, by (7.19) and (7.20) we have

$$\overline{\mathrm{ev}}_{\bar{T}',(\alpha_i),(\bar{B}'_\alpha)}(\bar{\mathbf{u}}', \mathbf{z}') \in \bar{X}.$$

Since the codimension of $\bar{X}$ in $\bar{M}^3$ is k, and by (7.21) the domain of the evaluation map $\overline{\mathrm{ev}}_{\bar{T}',(\alpha_i),(\bar{B}'_\alpha)}$ has dimension at most $k-2$, this is a contradiction. Hence there exists a vortex on $\mathbb{C}$ with positive energy. □

A Group actions

A.1 Convergence in the quotient

Let X be a topological space, $\sim$ be an equivalence relation on X and $\pi : X \to X/\sim$ be the canonical projection. Recall that a subset $V \subseteq X/\sim$ is called open in the quotient topology iff $\pi^{-1}(V) \subseteq X$ is open. Let G be a group and $\rho : \mathrm{G} \times X \to X$ be an action. For every $g \in \mathrm{G}$ we define

$$\rho_g : X \to X, \qquad \rho_g(x) := g \cdot x := \rho(g, x).$$

The next proposition reformulates convergence of a sequence of points y_ν in the quotient X/G against a point $y_0 \in X/\mathrm{G}$. Recall that the space X is said to satisfy the first axiom of countability iff for every point $x \in X$ there are neighbourhoods $U_i \subseteq X$ of x, for $i \in \mathbb{N}$, such that for every neighbourhood U of x there is an $i \in \mathbb{N}$ such that $U_i \subseteq U$.

Proposition A.1 *Assume that* X *satisfies the first axiom of countability and that for every* $g \in \mathrm{G}$ *the map* $\rho_g : X \to X$ *is continuous. Let* y_0, $y_\nu \in X/\mathrm{G}$, $\nu \in \mathbb{N}$ *be points. Then the following conditions are equivalent.*

(i) The sequence y_ν *converges to* y_0 *in the quotient topology.*

(ii) For every representative x_0 *of* y_0 *there is a sequence of representatives* x_ν *of* y_ν *such that* $x_\nu \to x_0$.

(iii) There is a representative x_0 *of the orbit* y_0 *and for every* $\nu \in \mathbb{N}$ *there is a representative* $x_\nu \in X$ *of* y_ν *such that* $x_\nu \to x_0$.

The proof this proposition is based on the next lemma.

Lemma A.2 *Let* G *be a group,* X *be a topological space and* $\rho : \mathrm{G} \times X \to X$ *be an action. Assume that for every* $g \in \mathrm{G}$ *the map* $\rho_g : X \to X$ *is continuous. Let* $U \subseteq X$ *be an open subset. Then the image* $\pi(U) \subseteq X/\mathrm{G}$ *is open.*

Proof of Lemma A.2: It suffices to show that $\pi^{-1}(\pi(U)) \subseteq X$ is open. To see this note that

$$\pi^{-1}(\pi(U)) = \{x \in X \mid \exists y \in U : x \sim y\} = \bigcup_{g \in \mathrm{G}} g \cdot U. \tag{A.1}$$

Let $g \in \mathrm{G}$. Then the subset

$$g \cdot U = \rho_{g^{-1}}^{-1}(U) \subseteq X$$

is open, by our assumption that $\rho_{g^{-1}} : X \to X$ is continuous. It follows from (A.1) that $\pi^{-1}(\pi(U)) \subseteq X$ is open. This proves Lemma A.2. □

Proof of Proposition A.1: We show that (i) implies (ii). Let x_0 be a representative of y_0. By assumption there is a sequence $U_i \subseteq X$ of neighbourhoods of x_0 such that for every neighbourhood U of x_0 there is $i \in \mathbb{N}$ such that $U_i \subseteq U$. By replacing U_i by $U_1 \cap \ldots \cap U_i$ for every $i \in \mathbb{N}$ we may assume w.l.o.g. that $U_1 \supseteq U_2 \ldots$. Let $i \in \mathbb{N}$. By Lemma A.2 the set $\pi(U_i) \subseteq X/\mathrm{G}$ is an open neighbourhood of y_0. Therefore, there is an integer $\nu_i \in \mathbb{N}$ such that for every $\nu \geq \nu_i$ we have $y_\nu \in \pi(U_i)$. W.l.o.g. we assume that $\nu_1 < \nu_2 < \ldots$. It follows that if $\nu \geq \nu_i$ then there is a representative $x_\nu \in U_i$ of y_ν, for every $i \in \mathbb{N}$. We choose $x_\nu \in X$ for $\nu \in \mathbb{N}$ as follows. If $\nu < \nu_1$ then let $x_\nu \in X$ be arbitrary. Otherwise, let $i \in \mathbb{N}$ be the unique integer such that $\nu_i \leq \nu < \nu_{i+1}$ and let $x_\nu \in U_i$ be a representative of y_ν. We claim that x_ν converges to x_0, as $\nu \to \infty$. To see this let $U \subseteq X$ be an open neighbourhood of x_0. Let $i \in \mathbb{N}$ be so large that $U_i \subseteq U$. Since $U_1 \supseteq U_2 \supseteq \ldots$ it follows that for $\nu \geq \nu_i$ we have $x_\nu \in U_i \subseteq U$. This proves the claim and therefore (ii).

Condition (ii) immediately implies condition (iii). We are left to show that (iii) implies (i). Assume (iii) and let x_0 be a representative of y_0 and x_ν be a representative of y_ν for $\nu \in \mathbb{N}$ such that x_ν converges to x_0. Let $V \subseteq X/\mathrm{G}$ be an open neighbourhood of y_0. Since $\pi^{-1}(V) \subseteq X$ is an open neighbourhood of x_0, for large enough ν we have $x_\nu \in \pi^{-1}(V)$, i.e. $y_\nu = \pi(x_\nu) \in V$. This proves (i) and concludes the proof of Proposition A.1. □

We use the next lemma in the proof of Proposition D.23.

Lemma A.3 *Let* G *and* X *be as in Lemma A.2,* $x \in X$ *be a point and* $\{U_i\}_{i\in I}$ *be a (not necessarily countable) basis of open neighbourhoods of* x. *Then the collection* $\{\pi(U_i)\}_{i\in I}$ *is a basis of open neighbourhoods of* $\pi(x) \in X/\mathrm{G}$.

Proof of Lemma A.3: By Lemma A.2 the subsets $\pi(U_i) \subseteq X/\mathrm{G}$, $i \in I$, are open. Let $V \subseteq X/\mathrm{G}$ be an open neighbourhood of $\pi(x)$. Then the subset $\pi^{-1}(V) \subseteq X$ is an open neighbourhood of x. Hence there exists an index $i \in I$ such that $\pi^{-1}(V) \subseteq U_i$. It follows that $V = \pi(\pi^{-1}(V)) \subseteq \pi(U_i)$. This proves Lemma A.3. □

A.2 Smooth Lie group actions

Let M be a real n-dimensional manifold and G be a k-dimensional Lie group that acts smoothly on M. We denote by $\mathfrak{g}$ the Lie algebra of G, by G_x the stabilizer of a point $x \in M$ and by M^* the set of all points $x \in M$ such that if $g \in \mathrm{G}$ is such that $gx = x$ then $g = e$.

Lemma A.4 *The Lie algebra of the stabilizer of a point $x \in M$ is*

$$\mathrm{Lie}(\mathrm{G}_x) = \{\xi \in \mathfrak{g} \,|\, X_\xi(x) = 0\}.$$

Proof of Lemma A.4: This is Lemma B.15 in the book by V. Ginzburg, V. Guillemin and Y. Karshon [GGK]. □

Recall that the action of G on M is called proper iff the map

$$\mathrm{G} \times M \ni (g, x) \mapsto (x, gx) \in M \times M$$

is proper.

Theorem A.5 (Existence of local slices) *Assume that the action of* G *on M is proper. Then for every point $x \in M^*$ there is an equivariant diffeomorphism from $\mathrm{G} \times \mathbb{R}^{n-k}$ to a* G*-invariant open neighborhood of x.*

Proof: This is a special case of Theorem B.24 in the book by V. Ginzburg et al. [GGK]. □

Corollary A.6 *The subsets $M^* \subseteq M$ and $M^*/\mathrm{G} \subseteq M/\mathrm{G}$ are open. Furthermore, the canonical projection $\pi : M^* \to M^*/\mathrm{G}$ defines a principal* G*-bundle.*

Proof of Corollary A.6: The set M^* is the union of the open sets $\psi_x(\mathrm{G} \times \mathbb{R}^{n-k})$, where ψ_x is an equivariant diffeomorphism from $\mathrm{G} \times \mathbb{R}^{n-k}$ to an invariant open neighborhood of x as in Theorem A.5, for $x \in M^*$. By Lemma A.2 the subset $M^*/\mathrm{G} = \pi(M^*) \subseteq M/\mathrm{G}$ is also open. Theorem A.5 implies that the projection $\pi : M^* \to M^*/\mathrm{G}$ defines a principal G-bundle follows. This proves Corollary A.6. □

If G is a group and $\mathrm{H} \subseteq \mathrm{G}$ is a subgroup, then H acts on G by multiplication from the left. The quotient is the set of right cosets

$$\mathrm{G}/\mathrm{H} := \{\mathrm{H}g \,|\, g \in \mathrm{G}\}.$$

Corollary A.7 *Let* G *be a Lie group and $\mathrm{H} \subseteq \mathrm{G}$ be a closed subgroup. Then the canonical projection $\pi : \mathrm{G} \to \mathrm{G}/\mathrm{H}$ defines a principal* G*-bundle.*

Proof of Corollary A.7: We claim that the action of H on G is proper. To see this observe that the map

$$\mathrm{H} \times \mathrm{G} \to \mathrm{G} \times \mathrm{G}, \qquad (h, x) \mapsto (x, hx)$$

is the composition of the inclusion

$$\iota : \mathrm{H} \times \mathrm{G} \to \mathrm{G} \times \mathrm{G}$$

with the map

$$\varphi : \mathrm{G} \times \mathrm{G} \to \mathrm{G} \times \mathrm{G}, \qquad \varphi(g, x) \mapsto (x, gx).$$

The map φ is a diffeomorphism, its inverse is given by

$$(x, y) \mapsto (yx^{-1}, x).$$

Since $\mathrm{H} \subseteq \mathrm{G}$ is a closed subset, the inclusion ι is proper. It follows that φ is proper. Corollary A.7 follows now from Theorem A.5. □

The next Lemma says that given a torsion-free and G-invariant connection the covariant derivative of a vector field along a curve in M, twisted by a curve in the Lie algebra, behaves equivariantly under gauge transformations. We use this Lemma in the proof of Proposition 2.11. Fix a torsion-free and G-invariant connection ∇ on M. By the latter condition we mean that

$$\nabla_{gv} gX = g\nabla_v X,$$

for every $g \in \mathrm{G}$, $v \in T_x M$, $x \in M$, and every vector field X on M. Fix an element $\Phi \in \mathfrak{g}$. We define the twisted connection ∇^Φ on M by

$$\nabla^\Phi_v X := \nabla_v X + \nabla_{X(x)} X_\Phi,$$

for every $v \in T_x M$, $x \in M$, and every vector field X on M. Let $\mathbb{R} \ni t \mapsto \gamma(t) \in M$ be a smooth curve and X be a vector field along γ. For $t \in \mathrm{G}$ we denote by $\nabla_t X(t)$ the covariant derivative of X at the point t and by

$$\nabla^\Phi_t X(t) := \nabla_t X(t) + \nabla_{X \circ \gamma(t)} X_\Phi \tag{A.2}$$

the twisted covariant derivative at t. Note that if Φ also depends on t then the second term in (A.2) is to be understood as fixing $t \in \mathbb{R}$ and taking the covariant derivative of the vector field $X_{\Phi(t)}$. For every smooth map $\psi : \mathbb{R} \to \mathfrak{g}$ we denote

$$\partial^\Phi_t \psi := \partial_t \psi + [\Phi, \psi].$$

Lemma A.8 *Let ∇ be a torsion-free and* G*-invariant connection on M, let $\Phi : \mathbb{R} \to \mathfrak{g}$, $g : \mathbb{R} \to \mathrm{G}$ and $\gamma : \mathbb{R} \to M$ be smooth curves, and X be a vector field along γ. Then*

$$\nabla^{g_*\Phi}_t (gX) = g\nabla^\Phi_t X. \tag{A.3}$$

Furthermore, let $\psi : \mathbb{R} \to \mathfrak{g}$ be another smooth map. Then

$$\partial^{g_*\Phi}_t (g\psi g^{-1}) = g(\partial^\Phi_t \psi)g^{-1}. \tag{A.4}$$

The Lemma is a "coordinate" version of Lemma B.3. in [GS]. The map $\Phi : \mathbb{R} \to \mathfrak{g}$ corresponds to a component of the connection one form A, appearing in that lemma. See also Proposition 3.2.5 in R. Gaio's PhD thesis [Ga]. For the proof of Lemma A.8, we need the following lemma.

Lemma A.9 *Let ∇ be a torsion-free connection on a manifold M, let $\varphi : \mathbb{R} \times M \to M$ be a smooth map such that $\varphi_t := \varphi(t, \cdot) : M \to M$ is a diffeomorphism for every $t \in \mathbb{R}$, let $x \in M$ be a point and $v \in T_xM$ be a vector. Then the covariant derivative of the vector field $d\varphi_t(x)v$ along the curve $\mathbb{R} \ni t \mapsto \varphi_t(x) \in M$ is given by*

$$\nabla_t d\varphi_t(x)v = \nabla_{d\varphi_t(x)v} X_t, \tag{A.5}$$

where X_t is the vector field on M defined by

$$X_t := \dot{\varphi}_t \circ \varphi_t^{-1}.$$

Proof of Lemma A.9: We denote the dimension of M by n. Let $t_0 \in \mathbb{R}$. We choose local parametrizations $\psi : \mathbb{R}^n \to U$ and $\widetilde{\psi} : \mathbb{R}^n \to \widetilde{U}$, where $U \subseteq M$ is a neighbourhood of x and $\widetilde{U} \subseteq M$ is a neighbourhood of $\varphi_{t_0}(x)$. For $i = 1, \ldots, n$ we denote by

$$e_i := \partial_i \widetilde{\psi} \circ \widetilde{\psi}^{-1} : U \to TM$$

the i-th coordinate vector field. We denote by $\mathbb{R} \ni t \mapsto Y(t) := d\varphi_t(x)v \in T_{\varphi_t(x)}M$ the vector field along the curve $t \mapsto \varphi_t(x)$. We choose a number $\varepsilon > 0$ so small that for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ we have $\varphi_t(x) \in \widetilde{U}$. We denote by $X_t^1, \ldots, X_t^n : \widetilde{U} \to \mathbb{R}$ and $Y^1, \ldots, Y^n : (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R}$ the components of X_t and Y w.r.t. the frame $e_1, \ldots, e_n$. This means that for $y \in \widetilde{U}$, $t \in \mathbb{R}$ we have

$$X_t(y) = \sum_{i=1}^n X_t^i(y) e_i(y),$$

and for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ we have

$$Y(t) = \sum_{i=1}^n Y^i(t) e_i(\varphi_t(x)).$$

Then for every i and for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, abbreviating $w := d(\psi^{-1})_x v$,

$$\begin{aligned}
\dot{Y}^i(t) &= \frac{d}{dt}\Big(d(\widetilde{\psi}^{-1})_{\varphi_t(x)} Y(t)\Big)^i \\
&= \frac{d}{dt}\Big(d(\widetilde{\psi}^{-1} \circ \varphi_t \circ \psi)_{\psi^{-1}(x)} d(\psi^{-1})_x v\Big)^i \\
&= \frac{d}{dt}\big(d(\widetilde{\psi}^{-1} \circ \varphi_t \circ \psi)_{\psi^{-1}(x)}\big)^i w \\
&= d\Big(\frac{d}{dt}(\widetilde{\psi}^{-1} \circ \varphi_t \circ \psi)\Big)^i_{\psi^{-1}(x)} w \\
&= d\big(d(\widetilde{\psi}^{-1})_{\varphi_t(x)} X_t \circ \varphi_t \circ \psi\big)^i_{\psi^{-1}(x)} w \\
&= d(X^i_t \circ \varphi_t \circ \psi)_{\psi^{-1}(x)} w \\
&= dX^i_t(\varphi_t(x)) d\varphi_t(x) v. \qquad \text{(A.6)}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\sum_i Y^i(t) \nabla_{X_t \circ \varphi_t(x)} e_i &= \sum_{i,j} Y^i(t) X^j_t \circ \varphi_t(x) \nabla_{e_j(\varphi_t(x))} e_i \\
&= \sum_{i,j} Y^i(t) X^j_t \circ \varphi_t(x) \nabla_{e_i(\varphi_t(x))} e_j \\
&= \sum_j X^j_t \circ \varphi_t(x) \nabla_{d\varphi_t(x)v} e_j. \qquad \text{(A.7)}
\end{aligned}$$

Here in the second equality we have used that ∇ is torsion-free and that $[e_i, e_j] = 0$, since e_i is the i-th coordinate vector field. Equalities (A.6) and (A.7) imply that

$$\begin{aligned}
\nabla_t(d\varphi_t(x)v) &= \sum_i \nabla_t(Y^i(t) e_i) \\
&= \sum_i \dot{Y}^i(t) e_i + Y^i(t) \nabla_{X_t \circ \varphi_t(x)} e_i \\
&= \sum_i \big(dX^i_t(\varphi_t(x)) d\varphi_t(x) v\big) e_i + X^i_t \circ \varphi_t(x) \nabla_{d\varphi_t(x)v} e_i \\
&= \sum_i \nabla_{d\varphi_t(x)v}(X^i_t e_i) \\
&= \nabla_{d\varphi_t(x)v} X_t.
\end{aligned}$$

This proves equality (A.5). This proves Lemma A.9. □

Proof of Lemma A.8: We define the map $u : \mathbb{R}^2 \to M$ by $u(s,t) := g(s)\gamma(t)$, and the vector field Y along u by $Y(s,t) := g(s)X(t)$. Then for every point $t_0 \in \mathbb{R}$

$$\begin{aligned}\nabla_t|_{t=t_0}(gX) &= \nabla_t|_{t=t_0} Y(t,t)\\ &= \nabla_s Y(t_0,t_0) + \nabla_t Y(t_0,t_0)\\ &= \nabla_s|_{s=t_0} g(s)X(t_0) + \nabla_t|_{t_0} g(t_0)X(t)\\ &= \nabla_{(gX)(t_0)} X_{(\dot{g}g^{-1})(t_0)} + g(t_0)\nabla_t|_{t_0} X.\end{aligned}$$

Here in the last equality we have used Lemma A.9 and the G-invariance of ∇. It follows that

$$\begin{aligned}\nabla_t^{g_*\Phi}(gX) &= \nabla_t(gX) + \nabla_{gX} X_{g_*\Phi}\\ &= \nabla_{gX} X_{\dot{g}g^{-1}} + g\nabla_t X + \nabla_{gX} X_{g\Phi g^{-1} - \dot{g}g^{-1}}\\ &= \nabla_{gX} X_{\dot{g}g^{-1}} + g\nabla_t X + \nabla_{gX}(gX_\Phi) - \nabla_{gX} X_{\dot{g}g^{-1}}\\ &= g(\nabla_t X + \nabla_X X_\Phi)\\ &= g\nabla_t^\Phi X.\end{aligned}$$

Here in the forth step we have used the G-invariance of ∇. This proves equality (A.3). Equality (A.4) follows from a straight-forward computation. This proves Lemma A.8. □

A.3 Induced metric on the quotient

Let (X, d) be a metric space, G be a topological group, and let $\rho : \mathrm{G} \times X \to X$ be a continuous action by isometries, which means that $d(gx, gy) = d(x, y)$ for every $g \in \mathrm{G}$, $x, y \in X$. By $\pi : X \to X/\mathrm{G}$ we denote the canonical projection. The topology on X, determined by d, induces a topology on the quotient X/G.

Lemma A.10 *Assume that* G *is compact. Then the map* $\bar{d} : X/\mathrm{G} \times X/\mathrm{G} \to [0, \infty)$ *defined by*

$$\bar{d}(\mathrm{G}x, \mathrm{G}y) := \min_{x' \in \mathrm{G}x,\, y' \in \mathrm{G}y} d(x', y')$$

is a metric on X/G *that induces the quotient topology on* X/G.

Proof of Lemma A.10: To see that the map $\bar{d}$ is welldefined, i.e. that the minimum is attained, observe that for every $x \in X$ the orbit $\mathrm{G}x$ is the image under ρ of the compact set $\mathrm{G} \times \{x\}$ and therefore it is compact. In order to see that $\bar{d}$ is positive definite, assume that $\bar{x}, \bar{y} \in X/\mathrm{G}$ are such that $\bar{d}(\bar{x}, \bar{y}) = 0$. This means that there are points $x, y \in X$ such that $\mathrm{G}x = \bar{x}$, $\mathrm{G}y = \bar{y}$ and $d(x, y) = 0$. It follows that $x = y$ and therefore $\bar{x} = \bar{y}$. Symmetry of $\bar{d}$ follows immediately from the symmetry of d. To see that the triangle inequality holds, let $\bar{x}, \bar{y}, \bar{z} \in X/\mathrm{G}$. We choose $x, y, y', z \in X$ such that $\mathrm{G}x = \bar{x}$, $\mathrm{G}y = \mathrm{G}y' = \bar{y}$, $\mathrm{G}z = \bar{z}$, $d(x,y) = \bar{d}(\bar{x},\bar{y})$ and $d(y',z) = \bar{d}(\bar{y},\bar{z})$. Let $g \in \mathrm{G}$ be such that $y' = gy$. Then

$$\begin{aligned}\bar{d}(\bar{x},\bar{z}) &\leq d(x, g^{-1}z) \\ &\leq d(x,y) + d(y, g^{-1}z) \\ &= d(x,y) + d(y',z) \\ &= \bar{d}(\bar{x},\bar{y}) + \bar{d}(\bar{y},\bar{z}).\end{aligned}$$

This proves the triangle inequality.

For every $x \in X$ and $r > 0$ we denote by $B^d_r(x)$ the open ball w.r.t. d with center x and radius r, and analogously for $B^{\bar{d}}_r(\bar{x})$. We prove that $\bar{d}$ induces the quotient topology on X/G, i.e. that a subset $U \subseteq X/\mathrm{G}$ is open w.r.t. $\bar{d}$ if and only if $\pi^{-1}(U) \subseteq X$ is open w.r.t. d. Assume that $U \subseteq X/\mathrm{G}$ is open w.r.t. $\bar{d}$ and let $x \in \pi^{-1}(U)$. We choose an $r > 0$ so small that $B^{\bar{d}}_r(\mathrm{G}x) \subseteq U$. It follows from the definition of $\bar{d}$ that $B^d_r(x) \subseteq \pi^{-1}(U)$. So $\pi^{-1}(U)$ is open w.r.t. d. Conversely, let $U \subseteq X/\mathrm{G}$ be a subset such that $\pi^{-1}(U) \subseteq X$ is open w.r.t. d. We show that U is open w.r.t. $\bar{d}$. To see this let $\bar{x} \in U$. We choose any $x \in X$ such that $\mathrm{G}x = \bar{x}$ and any $r > 0$ such that $B^d_r(x) \subseteq \pi^{-1}(U)$. We claim that $B^{\bar{d}}_r(\bar{x}) \subseteq U$. To see this, let $\bar{y} \in B^{\bar{d}}_r(\bar{x})$, and choose any $x', y \in X$ such that $\mathrm{G}x' = \bar{x}$, $\mathrm{G}y = \bar{y}$ and $d(x',y) = \bar{d}(\bar{x},\bar{y})$. Let $g \in \mathrm{G}$ be such that $x' = gx$. Then

$$d(x, g^{-1}y) = d(x', y) = \bar{d}(\bar{x},\bar{y}) < r,$$

therefore $g^{-1}y \in B^d_r(x) \subseteq \pi^{-1}(U)$, and thus $\bar{y} \in U$. This proves that U is open w.r.t. $\bar{d}$. So we have proved that $\bar{d}$ induces the quotient topology on X/G. This concludes the proof of Lemma A.10. □

Let (M, g) be a Riemannian manifold. We denote by $\ell(\gamma)$ the length w.r.t. g of a smooth curve $\gamma : [0,1] \to M$. Assume that M is connected, so that g induces a distance function d on M. Suppose also that the action is by isometries. Recall that M^* denotes the set of all points $x \in M$ such that $ax = x$ implies that $a = e$, for every $a \in \mathrm{G}$. We define the Riemannian metric $\bar{g}$ on M^*/G by

$$\bar{g}_{\bar{x}}(\bar{v}, \bar{w}) := g_x(v, w),$$

for $\bar{v}, \bar{w} \in T_{\bar{x}}M^*/\mathrm{G}$ and every point $\bar{x} \in M^*/\mathrm{G}$, where $x \in M$ is a point in the orbit $\bar{x}$ and $v, w \in (\mathrm{im}L_x)^\perp \subseteq T_xM$ are chosen such that $d\pi(x)v = \bar{v}$, $d\pi(x)w = \bar{w}$. We denote the length of a smooth curve $\bar{\gamma} : [0,1] \to M^*/\mathrm{G}$ by $\bar{\ell}(\bar{\gamma})$ and by $d^{\bar{g}}$ the distance function on the connected components of M^*/G induced by $\bar{g}$. Let $\bar{d}$ be as in Lemma A.10.

Lemma A.11 *Assume that* G *is compact. Then*

$$d^{\bar{g}}(\bar{x},\bar{y}) \geq \bar{d}(\bar{x},\bar{y}), \tag{A.8}$$

for every two points $\bar{x}, \bar{y} \in M^*/\mathrm{G}$. Furthermore, for each compact connected subset $K \subseteq M^*/\mathrm{G}$ there is a constant $r > 0$ with the following property. If $\bar{x}, \bar{y} \in K$ satisfy $\bar{d}(\bar{x}, \bar{y}) < r$ then

$$d^{\bar{g}}(\bar{x}, \bar{y}) = \bar{d}(\bar{x}, \bar{y}). \tag{A.9}$$

Proof of Lemma A.11: Let $\bar{x}, \bar{y} \in M^*/\mathrm{G}$. It suffices to show that for every smooth path $\bar{\gamma} : [0,1] \to M^*/\mathrm{G}$ such that $\bar{\gamma}(0) = \bar{x}$, $\bar{\gamma}(1) = \bar{y}$ there is a smooth path $\gamma : [0,1] \to M^*$ such that

$$\pi \circ \gamma = \bar{\gamma}, \tag{A.10}$$
$$\ell(\gamma) = \bar{\ell}(\bar{\gamma}) \tag{A.11}$$

Fix a smooth path $\bar{\gamma} : [0,1] \to M^*/\mathrm{G}$. We choose a $\widetilde{\gamma} : [0,1] \to M^*$ such that $\pi \circ \widetilde{\gamma} = \bar{\gamma}$ and define the map $\xi : [0,1] \to \mathfrak{g}$ as follows. Fix $t \in [0,1]$. Since $\widetilde{\gamma}(t) \in M^*$ it follows from Lemma A.4 that

$$\{\xi \in \mathfrak{g} \,|\, X_\xi(\widetilde{\gamma}(t)) = 0\} = \mathrm{Lie}(\mathrm{G}_{\widetilde{\gamma}(t)}) = 0.$$

This means that the map $L_{\widetilde{\gamma}(t)} : \mathfrak{g} \to T_{\widetilde{\gamma}(t)}M$ is injective. Therefore, we can define $\xi(t) \in \mathfrak{g}$ to be the unique element such that

$$\dot{\widetilde{\gamma}} + L_{\widetilde{\gamma}(t)}\xi(t) \perp \mathrm{im} L_{\widetilde{\gamma}(t)}. \tag{A.12}$$

Let $a : [0,1] \to \mathrm{G}$ be the solution of the ordinary differential equation

$$a^{-1}\dot{a} = \xi, \qquad a(0) = e.$$

Abbreviating $\xi x := X_\xi(x)$ for any $\xi \in \mathfrak{g}$, $x \in M$, the map $\gamma := a\widetilde{\gamma} : [0,1] \to M$ satisfies

$$\begin{aligned} a^{-1}\dot{\gamma} &= a^{-1}\frac{d}{dt}(a\widetilde{\gamma}) \\ &= (a^{-1}\dot{a})\widetilde{\gamma} + \dot{\widetilde{\gamma}} \\ &= \xi\widetilde{\gamma} + \dot{\widetilde{\gamma}} \perp \mathrm{im} L_{\widetilde{\gamma}}, \end{aligned}$$

where the last equality follows from (A.12). By the definition of the metric $\bar{g}$ we have

$$|\dot{\gamma}| = |\dot{\widetilde{\gamma}} + L_{\widetilde{\gamma}}\xi| = |\dot{\bar{\gamma}}|,$$

and therefore

$$\ell(\gamma) = \bar{\ell}(\gamma).$$

This implies (A.11). This proves the first part of the Lemma.

To prove the second part, let $K \subseteq M^*/\mathrm{G}$ be a connected compact subset. Since by Corollary A.6 the subset $M^*/\mathrm{G} \subseteq M/\mathrm{G}$ is open, for each $\bar{x} \in K$

there is a number $r_{\bar{x}} > 0$ such that $B^{\bar{d}}_{2r_{\bar{x}}}(\bar{x}) \subseteq M^*/\mathrm{G}$. The open balls $B^{\bar{d}}_{r_{\bar{x}}}(\bar{x})$ with $\bar{x} \in K$ cover the set K. Since K is compact, we can pick finitely many points $\bar{x}_i \in K$, $i = 1, \dots, N$ such that abbreviating $r_i := r_{\bar{x}_i}$ the balls $B_{r_i}(\bar{x}_i)$, $i = 1, \dots, N$ still cover K. We set

$$r := \min_{i=1,\dots,N} r_i.$$

Let $\bar{x} \in K$ and $\bar{y} \in (M \setminus M^*)/\mathrm{G}$ be any two points. Choosing $i \in \{1, \dots, N\}$ such that $\bar{x} \in B^{\bar{d}}_{\bar{r}_i}(\bar{x}_i)$, we have

$$\begin{aligned} \bar{d}(\bar{x}, \bar{y}) &\geq \bar{d}(\bar{x}_i, \bar{y}) - \bar{d}(\bar{x}, \bar{x}_i) \\ &> 2r_i - r_i \\ &= r_i > r. \end{aligned} \tag{A.13}$$

Let $\bar{x}, \bar{y} \in K$ be points such that $\bar{d}(\bar{x}, \bar{y}) < r$. Since

$$\begin{aligned} \bar{d}(\bar{x}, \bar{y}) &= \inf_{\mathrm{G}x = \bar{x}, \mathrm{G}y = \bar{y}} d(x, y) \\ &= \inf\{\ell(\gamma) \,|\, \gamma \in C^\infty([0,1], M) : \ \mathrm{G}\gamma(0) = \bar{x}, \mathrm{G}\gamma(1) = \bar{y}\}, \end{aligned}$$

there is a smooth path $\gamma : [0,1] \to M$ such that $\gamma(0) = \bar{x}$, $\gamma(1) = \bar{y}$ and $\ell(\gamma) \leq r$. Let $\gamma : [0,1] \to M$ be any such path. We claim that $\gamma([0,1]) \subseteq M^*$. To see this, assume by contradiction that there was a point $t_0 \in [0,1]$ such that $\gamma(t_0) \in M \setminus M^*$. Then

$$\begin{aligned} \ell(\gamma) &\geq d(x, \gamma(t_0)) + d(y, \gamma(t_0)) \\ &\geq \bar{d}(\bar{x}, \mathrm{G}\gamma(t_0)) + \bar{d}(\bar{y}, \mathrm{G}\gamma(t_0)) \\ &\geq r + r. \end{aligned} \tag{A.14}$$

Here the last inequality follows from (A.13). This contradiction shows that indeed $\gamma([0,1]) \subseteq M^*$. It follows that

$$\begin{aligned} \bar{d}(\bar{x}, \bar{y}) &= \inf\{\ell(\gamma) \,|\, \gamma \in C^\infty([0,1], M^*) : \ \mathrm{G}\gamma(0) = \bar{x}, \mathrm{G}\gamma(1) = \bar{y}\} \\ &\geq \inf\{\bar{\ell}(\bar{\gamma}) \,|\, \bar{\gamma} \in C^\infty([0,1], M^*/\mathrm{G}) : \ \bar{\gamma}(0) = \bar{x}, \bar{\gamma}(1) = \bar{y}\} \\ &= d^{\bar{g}}(\bar{x}, \bar{y}). \end{aligned}$$

Together with (A.8) this implies (A.9). This completes the proof of Lemma A.11. □

A.4 Hamiltonian Lie group actions

A.4.1 Regular values of the moment map

Let G be a connected Lie group acting in a Hamiltonian way on a symplectic manifold (M, ω), with moment map μ. An element $\tau \in \mathfrak{g}$ is called central iff $g\tau g^{-1} = \tau$ for every $g \in \mathrm{G}$. The next lemma guarantees that for every central value $\tau \in \mathfrak{g}$ such that G acts with discrete isotropy groups on $\mu^{-1}(\tau)$ the subset $\mu^{-1}(\tau) \subseteq M$ is a submanifold. For its proof see also Proposition 2.3.2 [Ga].

Lemma A.12 (Regular value of μ) *Let $\tau \in \mathfrak{g}$ be a central element. Assume that the stabilizer G_x of every $x \in \mu^{-1}(\tau)$ is discrete, i.e. consists of isolated points. Then τ is a regular value of μ.*

Proof: Fix $x \in \mu^{-1}(\tau)$. We have to show that $d\mu(x)$ is surjective. For this it suffices to show that $(\mathrm{im} d\mu(x))^\perp = 0$. Fix any $\xi \in (\mathrm{im} d\mu(x))^\perp = 0$. Then for every $v \in T_xM$

$$0 = \langle d\mu(x)v, \xi\rangle = \omega(L_x\xi, v),$$

and therefore $L_x\xi = 0$ by the nondegeneracy of ω. By Lemma A.4 we have

$$0 = \mathrm{Lie}(\mathrm{G}_x) = \ker L_x,$$

and thus $\xi = 0$. This proves that $(\mathrm{im} d\mu(x))^\perp = 0$ and thus the Lemma.

□

A.4.2 Local equivariant symplectic action

Assume now that G is compact. Let J be a G-invariant ω-compatible almost complex structure on M. Assume that the hypothesis (H1) holds.

Lemma A.13 *There are constants $\delta, C > 0$ such that the following holds. If $x : S^1 \cong \mathbb{R}/(2\pi\mathbb{Z}) \to M$ and $\eta : S^1 \to \mathfrak{g}$ are smooth loops such that $\max_{S^1} |\mu \circ x| \leq \delta$, then there is a point $x_0 \in \mu^{-1}(0)$ and a smooth loop $g_0 : S^1 \to \mathrm{G}$ such that*

$$C^{-1} \sup_{S^1} |\eta + \dot{g}_0 g_0^{-1}| \leq \ell(x, \eta) := \int_0^{2\pi} |\dot{x} + X_\eta(x)| \, d\varphi, \tag{A.15}$$

$$d(x(\varphi), g_0(\varphi)x_0) \leq C(|\mu(x(\varphi))| + \ell(x, \eta)), \quad \forall \varphi \in S^1. \tag{A.16}$$

Furthermore, let $\tau_0 < \tau_1 \in \mathbb{R}$ and let $u \in C^\infty([\tau_0, \tau_1] \times S^1, M)$, $\eta \in C^\infty([\tau_0, \tau_1] \times S^1, \mathfrak{g})$ be such that $\max_{[\tau_0,\tau_1]\times S^1} |\mu(u)| \leq \delta$. Then there are maps $x_0 \in C^\infty([\tau_0, \tau_1], \mu^{-1}(0))$ and $g_0 \in C^\infty([\tau_0, \tau_1] \times S^1, \mathrm{G})$ such that for every $\tau \in [\tau_0, \tau_1]$ (A.15) and (A.16) hold with $x := u(\tau, .)$, η replaced by $\eta(\tau, .)$ and g_0 replaced by $g_0(\tau, .)$.

Proof: The first part of this Lemma is Lemma 11.2. in [Ga]. The second part follows as in the proof of that Lemma by choosing $\delta > 0$ enough such that $c_3\delta/2\pi$ is less than the injectivity radius of the exponential map of G, where c_3 is a constant as in the proof of that Lemma. □

We define $\iota > 0$ to be the largest number such that if $x \in \mu^{-1}(0)$ then the exponential map of $(M, g_{\omega,J})$ restricted to the open ball $B_\iota \subseteq T_x M$ is injective. We fix constants $\delta, C > 0$ as in Lemma A.13 and set

$$\delta_0 := \min\left\{\delta, \frac{\iota}{2C}\right\}. \tag{A.17}$$

Definition A.14 *Let* $(x, \eta) : S^1 \to M \times \mathfrak{g}$ *be a smooth loop such that* $\max_{S^1} |\mu(x)| < \delta_0$ *and* $\ell(x, \eta) < \delta_0$*. The* local equivariant symplectic action *of the pair* (x, η) *is defined to be*

$$\mathcal{A}(x, \eta) := -\int u^*\omega + \int_0^{2\pi} \langle \mu(x(\varphi)), \eta(\varphi) \rangle \, d\varphi,$$

where $u : [0,1] \times S^1 \to M$ *is given as follows. Let* $x_0 \in \mu^{-1}(0)$ *and* $g_0 : S^1 \to$ G *be as in Lemma A.13, and for* $\varphi \in S^1$ *let* $v(\varphi) \in B_\iota \subseteq T_{g_0(\varphi)x_0} M$ *be the unique tangent vector such that*

$$x(\varphi) = \exp_{g_0(\varphi)x_0} v(\varphi).$$

It follows from Lemma A.13 that the vector $v(\varphi)$ exists. That it is unique follows from the definition of ι. So $\mathcal{A}(x, \eta)$ is well-defined.

Lemma A.15 (Isoperimetric inequality) *There exists a constant* $c > 0$ *such that the following holds. If* $(x, \eta) : S^1 \to M$ *is a smooth loop such that* $\max_{S^1} |\mu \circ x| \leq \delta_0$ *and* $\ell(x, \eta) \leq \delta_0$*, where* δ_0 *is as in (A.17), then*

$$|\mathcal{A}(x, \eta)| \leq c \int_0^{2\pi} (|\dot{x} + X_\eta(x)|^2 + |\mu(x)|^2) \, d\varphi.$$

Proof: This is Lemma 11.3. in [GS]. □

B Gauge and reparametrizations

In this section $p > 2$ is a fixed number.

B.1 Gauge

Recall that for every open subset $\Omega \subseteq \mathbb{C}$ the gauge group $W^{2,p}_{\mathrm{loc}}(\Omega, \mathrm{G})$ acts on $W^{1,p}_{\mathrm{loc}}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ by the formula

$$g^*(u, \Phi, \Psi) := (g^{-1}u, g^{-1}(\Phi + \partial_s)g), g^{-1}(\Psi + \partial_t)g)). \tag{B.1}$$

Lemma B.1 *Let $\Omega \subseteq \mathbb{C}$ be an open subset, $w \in W^{1,p}_{\mathrm{loc}}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ be a map and $g \in W^{2,p}_{\mathrm{loc}}(\Omega, \mathrm{G})$ be a gauge transformation. Then defining $w' := (u', \Phi', \Psi') := g^*(u, \Phi, \Psi)$ we have*

$$\begin{aligned} \partial_s u' + X_{\Phi'}(u') &= g^{-1}\big(\partial_s u + X_\Phi(u)\big), &\text{(B.2)}\\ \partial_t u' + X_{\Psi'}(u') &= g^{-1}\big(\partial_t u + X_\Psi(u)\big), &\text{(B.3)}\\ \partial_s \Psi' - \partial_t \Phi' + [\Phi', \Psi'] &= g^{-1}(\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi])g, &\text{(B.4)}\\ e_{w'} &= e_w. &\text{(B.5)} \end{aligned}$$

Proof of Lemma B.1: These are straight forward computations, using that $\mu : M \to \mathfrak{g}$ is equivariant and that the metric $g_{\omega,J}$ is invariant under the action of G. □

By the Lemma, the energy is invariant under gauge transformations, i.e.

$$E(g^*w, \Omega) = E(w, \Omega),$$

for every open subset $\Omega \subseteq \mathbb{C}$, for every map $w \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ and every gauge transformation $g \in \mathcal{G}^{2,p}$.

The next Lemma is used in the proof of Proposition (D.2) (Regularity modulo gauge). As a corollary to the Theorem of Peter-Weyl, we can embed the Lie group G as a closed subgroup of the group of orthogonal matrices $\mathrm{O}(\ell)$ for every large enough integer ℓ. So w.l.o.g. we may assume that G is a closed subgroup of $\mathrm{O}(\ell) \subseteq \mathbb{R}^{\ell \times \ell}$. Every norm on a space of maps from $\Omega \subseteq \mathbb{C}$ to G is to be understood w.r.t. this embedding. For the proof of the next lemma see also [Weh], Lemma A.8.

Lemma B.2 (Regularity of the gauge transformation) *Let $k \in \mathbb{N} \cup \{0\}$ and $\Omega \subseteq \mathbb{C}$ be an open subset. Then the following holds.*

(i) Let $g \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathrm{G})$ and let $\Phi, \Psi \in C^k(\Omega, \mathfrak{g})$ be such that $g^*(\Phi, \Psi) \in C^k(\Omega, \mathfrak{g})$. Then $g \in C^{k+1}(\Omega, \mathrm{G})$.

(ii) Assume that Ω *is bounded and has a smooth boundary. Then there exists a constant* C *such that for every* $g \in W^{1,p}(\Omega, \mathrm{G})$ *and every* $(\Phi, \Psi) \in W^{k,p}(\Omega, \mathfrak{g} \times \mathfrak{g})$ *the following holds. If* $(\Phi', \Psi') := g^*(\Phi, \Psi) \in W^{k,p}(\Omega, \mathfrak{g} \times \mathfrak{g})$ *then* $g \in W^{k+1,p}(\Omega, \mathrm{G})$ *and*

$$||D^{k+1}g||_{L^p(\Omega)} \leq C\Big(\big|\big|(\Phi, \Psi, \Phi', \Psi')\big|\big|_{W^{k,p}(\Omega)} + \big|\big|(\Phi, \Psi, \Phi', \Psi')\big|\big|_{W^{k,p}(\Omega)}^{k+1}\Big). \tag{B.6}$$

Proof: If $g \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathrm{G})$ and $(\Phi, \Psi) \in L^p_{\mathrm{loc}}(\Omega, \mathfrak{g} \times \mathfrak{g})$ are maps, then defining $(\Phi', \Psi') := g^*(\Phi, \Psi)$, we have

$$\partial_s g = g\Phi' - \Phi g, \qquad \partial_t g = g\Psi' - \Psi g. \tag{B.7}$$

Part (i) of the lemma follows from this equality by induction on k.

We prove (ii) by induction on k. For $k := 0$ the assertion follows from (B.7). We fix $k \geq 1$ and assume that the assertion holds for $k - 1$. In the following C denotes a constant depending only on G, Ω, k, p and the embedding of G in $\mathrm{O}(\ell)$. It changes from estimate to estimate. Let $g \in W^{1,p}(\Omega, \mathrm{G})$ and $(\Phi, \Psi) \in W^{k,p}(\Omega, \mathfrak{g} \times \mathfrak{g})$ be such that $(\Phi', \Psi') := g^*(\Phi, \Psi) \in W^{k,p}(\Omega, \mathfrak{g} \times \mathfrak{g})$. We abbreviate

$$|| \cdot ||_{i,p} := || \cdot ||_{W^{i,p}(\Omega)}, \qquad || \cdot ||_p := || \cdot ||_{L^p(\Omega)}, \qquad || \cdot ||_{C^0} := || \cdot ||_{C^0(\bar{\Omega})},$$
$$u := (\Phi, \Psi, \Phi', \Psi') : \Omega \to \mathfrak{g}^4.$$

Equality (B.7), Morrey's inequality and the induction hypothesis imply that $g \in W^{k+1,p}(\Omega, \mathrm{G})$ and

$$\begin{aligned} ||D^{k+1}g||_p &\leq C\left(||g||_{C^0}||D^k u||_p + \sum_{i=1}^{k} ||D^i g||_p ||D^{k-i}u||_{C^0}\right) \\ &\leq C\left(||u||_{k,p} + \sum_{i=1}^{k} \big(||u||_{i-1,p} + ||u||_{i-1,p}^i\big)||u||_{k-i+1,p}\right) \\ &\leq C\left(||u||_{k,p} + ||u||_{k,p}^2 + \sum_{i=1}^{k} ||u||_{k,p}^{i+1}\right) \\ &\leq C\big(||u||_{k,p} + ||u||_{k,p}^{k+1}\big). \end{aligned}$$

Hence the assertion holds for k. This proves **(ii)** and concludes the proof of Lemma B.2. □

B.2 Reparametrizations

We can generalize the notion of a vortex and of energy as follows. Let $\Omega, \widetilde{\Omega} \subseteq \mathbb{C}$ be open subsets and $\varphi : \widetilde{\Omega} \to \Omega$ be a holomorphic map. Let $\widetilde{w} \in W^{1,p}_{\mathrm{loc}}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$. The *$\varphi$-vortex equations* for $\widetilde{w}$ are the equations

$$\begin{aligned} \partial_s \widetilde{u} + X_{\widetilde{\Phi}}(\widetilde{u}) + J(\widetilde{u})(\partial_t \widetilde{u} + X_{\widetilde{\Psi}}(\widetilde{u})) &= 0, && \text{(B.8)} \\ \partial_s \widetilde{\Psi} - \partial_t \widetilde{\Phi} + [\widetilde{\Phi}, \widetilde{\Psi}] + |\varphi'|^2 \mu \circ \widetilde{u} &= 0. && \text{(B.9)} \end{aligned}$$

We abbreviate

$$\widetilde{\kappa} := \partial_s \widetilde{\Psi} - \partial_t \widetilde{\Phi} + [\widetilde{\Phi}, \widetilde{\Psi}],$$

and define the *φ-energy density of $\widetilde{w}$* to be the map

$$e^{\varphi}_{\widetilde{w}} := \frac{1}{2}\Big(|\partial_s \widetilde{u} + X_{\widetilde{\Phi}}(\widetilde{u})|^2 + |\partial_t \widetilde{u} + X_{\widetilde{\Psi}}(\widetilde{u})|^2 + |\varphi'|^{-2}|\widetilde{\kappa}|^2 + |\varphi'|^2|\mu \circ \widetilde{u}|^2\Big) : \widetilde{\Omega} \to [0, \infty), \tag{B.10}$$

and the *φ-energy of $\widetilde{w}$ on $\widetilde{\Omega}$* by

$$E^{\varphi}(\widetilde{w}, \widetilde{\Omega}) := \int_{\widetilde{\Omega}} e^{\varphi}_{\widetilde{w}} ds dt \in [0, \infty].$$

In the following we identify $\mathfrak{g} \times \mathfrak{g}$ with the complexification $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We fix a map $w := (u, \Phi + i\Psi) \in W^{1,p}_{\mathrm{loc}}(\Omega, M \times \mathfrak{g}^{\mathbb{C}})$ and define $\widetilde{w} := (\widetilde{u}, \widetilde{\Phi} + i\widetilde{\Psi}) := \varphi^* w$ to be

$$\varphi^* w := \big(u \circ \varphi, \overline{\varphi'} \cdot (\Phi + i\Psi) \circ \varphi\big) : \widetilde{\Omega} \to M \times \mathfrak{g}^{\mathbb{C}}. \tag{B.11}$$

For $x \in M$, $v \in T_x M$ and $\lambda := a + bi \in \mathbb{C}$ we use the notation

$$\lambda v := av + bJ(x)v \in T_x M.$$

Proposition B.3 *Assume that $\Omega, \widetilde{\Omega}$ and $w, \widetilde{w}$ are as above and that $\varphi : \widetilde{\Omega} \to \Omega$ is a biholomorphic map. Then the first vortex equation (0.1) for w is equivalent to the equation (B.8) for $\widetilde{w}$ and (0.2) is equivalent to (B.9).*

Furthermore,

$$e^{\varphi}_{\widetilde{w}} = |\varphi'|^2 e_w \circ \varphi \tag{B.12}$$

and

$$E^{\varphi}(\widetilde{w}, \widetilde{\Omega}) = E(w, \Omega). \tag{B.13}$$

For the proof of Proposition B.3 we need the following lemma.

Lemma B.4 *Let $\Omega, \widetilde{\Omega}, w, \widetilde{w}$ be as in Proposition B.3 and let $\varphi : \widetilde{\Omega} \to \Omega$ be a holomorphic map. Then*

$$\partial_s\widetilde{u}+X_{\widetilde{\Phi}}(\widetilde{u})+J(\widetilde{u})(\partial_t\widetilde{u}+X_{\widetilde{\Psi}}(\widetilde{u})) = \overline{\varphi'}\cdot\big(\partial_s u+X_\Phi(u)+J(u)(\partial_t u+X_\Psi(u))\big)\circ\varphi, \tag{B.14}$$

$$\widetilde{\kappa} = |\varphi'|^2\kappa\circ\varphi, \tag{B.15}$$

$$|\partial_s\widetilde{u}+X_{\widetilde{\Phi}}(\widetilde{u})|^2+|\partial_t\widetilde{u}+X_{\widetilde{\Psi}}(\widetilde{u})|^2 = |\varphi'|^2\big(|\partial_s u+X_\Phi(u)|^2+|\partial_t u+X_\Psi(u)|^2\big)\circ\varphi. \tag{B.16}$$

For the proof of Lemma B.4 we introduce the following notation. For every 1-form $A\in\Omega^1(\mathbb{C},\mathfrak{g})$ we define

$$F_A := dA+\frac{1}{2}[A\wedge A].$$

Note that defining P to be the trivial principal G-bundle $\mathbb{C}\times\mathrm{G}$, A can be viewed as a connection one form on P and F_A corresponds to its curvature. For $k=0,1,2$, $\Omega\subseteq\mathbb{C}$ an open subset and g a metric on Ω we denote by $*_g:\Omega^k(\mathbb{C},\mathfrak{g})\to\Omega^{2-k}(\mathbb{C},\mathfrak{g})$ the *Hodge $*$-operator* w.r.t. g and the standard orientation on Ω. For every $x\in M$ and every linear map $\alpha:\mathbb{C}\to T_xM$ we define

$$\alpha^{(0,1)} := \frac{1}{2}(\alpha+J\alpha i).$$

Furthermore, for every $u\in C^\infty(\Omega,M)$ and $A\in\Omega^1(\Omega,\mathfrak{g})$ we set

$$\begin{aligned} d_Au &:= du+X_A(u)\in\Omega^1(\Omega,u^*TM),\\ \bar{\partial}_{J,A}(u) &:= (d_Au)^{(0,1)}\in\Omega^{(0,1)}(\Omega,u^*TM). \end{aligned}$$

Lemma B.5 *Let $\Omega,\widetilde{\Omega}\subseteq\mathbb{C}$ be open subsets, g be a metric on Ω, $A\in\Omega^1(\Omega,\mathfrak{g})$, $k\in\{0,1,2\}$, $\alpha\in\Omega^k(\Omega,\mathfrak{g})$ and $\varphi:\widetilde{\Omega}\to\Omega$ be an orientation preserving diffeomorphism. Then*

$$F_{\varphi^*A} = \varphi^*F_A, \tag{B.17}$$

$$*_{\varphi^*g}(\varphi^*\alpha) = \varphi^*(*_g\alpha). \tag{B.18}$$

Let $u\in C^\infty(\Omega,M)$. Assume that $\varphi:\widetilde{\Omega}\to\Omega$ is holomorphic. Then

$$\bar{\partial}_{J,\varphi^*A}(u\circ\varphi) = \varphi^*\bar{\partial}_{J,A}(u)\in\Omega^{(0,1)}(\Omega,(u\circ\varphi)^*TM). \tag{B.19}$$

Furthermore, let $\alpha\in\Omega^1(\Omega,u^*TM)$. Then

$$|\varphi^*\alpha|^2 = |\varphi'|^2|\alpha|^2\circ\varphi. \tag{B.20}$$

Proof: (B.17), (B.18) and (B.20) follow from short calculations. To prove (B.19), observe that for every $\alpha\in\Omega^1(\Omega,u^*TM)$ we have

$$(\varphi^*\alpha)^{(0,1)} = \varphi^*(\alpha^{(0,1)}).$$

Applying this to $\alpha := d_A u$ yields (B.19). This completes the proof of Lemma B.5. □

Proof of Lemma B.4: By a density argument we may assume w.l.o.g. that w and $\widetilde{w}$ are smooth. We define

$$\begin{aligned} A &:= \Phi ds + \Psi dt \in \Omega^1(\Omega, \mathfrak{g}), \\ \widetilde{A} &:= \widetilde{\Phi} ds + \widetilde{\Psi} dt \in \Omega^1(\widetilde{\Omega}, \mathfrak{g}). \end{aligned}$$

Denoting by $\Re(z)$ the real part of a number $z \in \mathbb{C}$, we have

$$\begin{aligned} \widetilde{A} &= \Re\big(\overline{\varphi'} \cdot (\Phi + i\Psi) \circ \varphi \, d\bar{z}\big) \\ &= \Re\Big(\varphi^*\big((\Phi + i\Psi) d\bar{z}\big)\Big) \\ &= \varphi^* \Re\big((\Phi + i\Psi) d\bar{z}\big) \\ &= \varphi^* A. \end{aligned} \tag{B.21}$$

To prove equation **(B.14)** note that

$$\begin{aligned} &(ds - J(\widetilde{u})dt)\big(\partial_s \widetilde{u} + X_{\widetilde{\Phi}}(\widetilde{u}) + J(\widetilde{u})(\partial_t \widetilde{u} + X_{\widetilde{\Psi}}(\widetilde{u}))\big) \\ &= 2\bar{\partial}_{J,\widetilde{A}}(\widetilde{u}) \\ &= 2\varphi^* \bar{\partial}_{J,A}(u) \\ &= \varphi^*\Big((ds - J(u)dt)\big(\partial_s u + X_\Phi(u) + J(u)(\partial_t u + X_\Psi(u))\big)\Big) \\ &= (ds - J(\widetilde{u})dt)\overline{\varphi'} \cdot \big(\partial_s u + X_\Phi(u) + J(u)(\partial_t u + X_\Psi(u))\big) \circ \varphi. \end{aligned}$$

Here in the second equality we have used (B.19) and (B.21). This implies **(B.14)**.

To prove **(B.15)** note that

$$F_A = \kappa ds \wedge dt, \qquad F_{\widetilde{A}} = \widetilde{\kappa} ds \wedge dt.$$

Therefore, Lemma B.5(B.17) and (B.21) imply

$$\begin{aligned} \widetilde{\kappa} ds \wedge dt &= F_{\widetilde{A}} \\ &= F_{\varphi^* A} \\ &= \varphi^* F_A \\ &= (\kappa \circ \varphi)\varphi^*(ds \wedge dt) \\ &= |\varphi'|^2 (\kappa \circ \varphi) ds \wedge dt. \end{aligned}$$

This implies **(B.15)**.

To prove **(B.16)** we set $\widetilde{\alpha} := d_{\widetilde{A}} \widetilde{u} \in \Omega^1(\widetilde{\Omega}, \widetilde{u}^* TM)$. Then by Lemma B.5 we have

$$\begin{aligned} |\partial_s \widetilde{u} + X_{\widetilde{\Phi}}(\widetilde{u})|^2 + |\partial_t \widetilde{u} + X_{\widetilde{\Psi}}(\widetilde{u})|^2 &= |d_{\widetilde{A}} \widetilde{u}|^2 \\ &= |\varphi^* d_A u|^2 \\ &= |\varphi'|^2 |d_A u|^2 \circ \varphi \\ &= |\varphi'|^2 (|\partial_s u + X_\Phi(u)|^2 + |\partial_t u + X_\Psi(u)|^2) \circ \varphi. \end{aligned}$$

This proves **(B.16)** and completes the proof of Lemma B.4. □

Proof of Proposition B.3: By a density argument we may assume w.l.o.g. that w and $\widetilde{w}$ are smooth. That (0.1) is equivalent to (B.8) follows from Lemma B.4(B.14). That (0.2) is equivalent to (B.9) follows from Lemma B.4(B.15). Equation (B.12) follows from B.4(B.15) and (B.16). To prove (B.13) note that

$$\begin{aligned} E^{\varphi}(\widetilde{w},\widetilde{\Omega}) &= \int_{\widetilde{\Omega}} e^{\varphi}_{\widetilde{w}}\, ds\, dt \\ &= \int_{\widetilde{\Omega}} |\varphi'|^2 e_w\, ds\, dt \\ &= \int_{\Omega} e_w\, ds\, dt = E(w,\Omega). \end{aligned}$$

This proves Proposition B.3. □

B.3 Reparametrization by an automorphism of $\mathbb{C}$

Consider now the special case $\Omega := \widetilde{\Omega} := \mathbb{C}$, and let $\varphi : \mathbb{C} \to \mathbb{C}$ be a complex automorphism. Then there exist $\lambda = Re^{i\theta} \in \mathbb{C} \setminus \{0\}$ and $z_0 \in \mathbb{C}$ such that $\varphi(z) = \lambda z + z_0$. We fix a map $w := (u, \Phi + i\Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g}^{\mathbb{C}})$. We have

$$\begin{aligned} \varphi^*(u,\Phi+i\Psi) &= \big(u, \bar{\lambda}(\Phi+i\Psi)\big)\circ\varphi \\ &= \Big(u, R\Big(\big(\cos(\theta)\Phi+\sin(\theta)\Psi\big) \\ &\qquad +i\big(-\sin(\theta)\Phi+\cos(\theta)\Psi\big)\Big)\Big)\circ\varphi. \end{aligned} \tag{B.22}$$

The φ-vortex equations are now given by

$$\partial_s u + X_\Phi(u) + J(u)(\partial_t u + X_\Psi(u)) = 0, \tag{B.23}$$

$$\partial_s \Psi - \partial_t \Phi + [\Phi,\Psi] + R^2 \mu\circ u = 0. \tag{B.24}$$

We refer to these equations also as the *R-vortex equations*.

It follows from Proposition B.3 that w solves the vortex equations (0.1),(0.2) if and only if $\varphi^* w$ solves the R-vortex equations (4.1),(4.2). If φ is an orientation preserving rigid motion of the plane then it is of the form $\varphi(z) = e^{i\theta}z+z_0$. It follows that equations (0.1),(0.2) are invariant under the action of such a φ.

For every $R > 0$ we define the *R-energy density of* $w : \Omega \to M \times \mathfrak{g} \times \mathfrak{g}$ to be

$$e^R_w := \frac{1}{2}\Big(|\partial_s u+X_\Phi\circ u|^2+|\partial_t u+X_\Psi\circ u|^2+R^{-2}|\partial_s\Psi-\partial_t\Phi+[\Phi,\Psi]|^2+R^2|\mu\circ u|^2\Big)ds\,dt$$

and the *R-energy of w on Ω* by

$$E^R(w,\Omega) := \int_\Omega e^R_w \, ds\, dt.$$

We denote $E^R(w) := E^R(w,\mathbb{C})$. Let $\varphi(z) := Re^{i\theta}z + z_0$. Then $e^R_w = e^\varphi_w$, $E^R(w,\Omega) = E^\varphi(w,\Omega)$ and it follows from Proposition B.3 that $E^R(\varphi^* w,\Omega) = E^1(w,\varphi(\Omega))$. Note also that if w solves (4.1),(4.2) then

$$e^R_w = |\partial_s u + X_\Phi(u)|^2 + R^2|\mu\circ u|^2.$$

B.4 Gauging and reparametrization

The action of the gauge group $\mathcal{G}^{2,p} = W^{2,p}_{\mathrm{loc}}(\mathbb{C},\mathrm{G})$ on $W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M\times\mathfrak{g}^\mathbb{C})$ given by (B.1) does not commute with the action of the group $\mathrm{Aut}(\mathbb{C})$ of complex automorphisms of $\mathbb{C}$ on $W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M\times\mathfrak{g}^\mathbb{C})$ given by (B.22). This would mean that $\varphi^*(g^*w)$ equaled $g^*\varphi^*w$ for every $\varphi\in\mathrm{Aut}(\mathbb{C})$, every gauge transformation $g\in\mathcal{G}^{2,p}$ and every map $w\in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M\times\mathfrak{g}^\mathbb{C})$. If this was the case then there would be an action of the product group $\mathrm{Aut}(\mathbb{C})\times\mathcal{G}^{2,p}$ on $W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M\times\mathfrak{g}^\mathbb{C})$ defined by $(\varphi,g)^*w := g^*\varphi^*w$. As a substitute, we will show that this formula defines an action of the *semi-direct product* $\mathrm{Aut}(\mathbb{C})\ltimes\mathcal{G}^{2,p}$ on $W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M\times\mathfrak{g}^\mathbb{C})$. This action descends to an action of $\mathrm{Aut}(\mathbb{C})$ on the set of gauge equivalence classes $W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M\times\mathfrak{g}^\mathbb{C})/\mathcal{G}^{2,p}$.

Proposition B.6 *Let $\Omega,\widetilde{\Omega}\subseteq\mathbb{C}$ be open subsets, $(u,\Phi+i\Psi)\in W^{1,p}_{\mathrm{loc}}(\Omega, M\times\mathfrak{g}^\mathbb{C})$ be a map, $\varphi:\widetilde{\Omega}\to\Omega$ be a holomorphic map and $g\in\mathcal{G}^{2,p}$ be a gauge transformation. Then*

$$\varphi^*(g^*(u,\Phi+i\Psi)) = (g\circ\varphi)^*\varphi^*(u,\Phi+i\Psi). \tag{B.25}$$

Proof of Proposition B.6: To simplify the proof, we work with the complexified tangent spaces $T^\mathbb{C}_g\mathrm{G} := T_g\mathrm{G}\otimes_\mathbb{R}\mathbb{C}$ for $g\in\mathrm{G}$ and write

$$g(\xi+i\eta) := g\xi + ig\eta, \qquad (\xi+i\eta)g := \xi g + i\eta g,$$

for $g\in\mathrm{G}$, $\xi,\eta\in\mathfrak{g}$. Since φ is holomorphic, we have

$$\partial_t(g\circ\varphi) = dg(\varphi)\partial_t\varphi = dg(\varphi)(i\partial_s\varphi).$$

Therefore, writing $\varphi = \varphi_1 + i\varphi_2$ we have

$$\begin{aligned}\partial_s(g\circ\varphi) + i\partial_t(g\circ\varphi) &= dg(\varphi)\partial_s\varphi + i dg(\varphi)(i\partial_s\varphi)\\ &= (\partial_s\varphi_1 - i\partial_s\varphi_2)\cdot(\partial_s g + i\partial_t g)\circ\varphi\\ &= \overline{\varphi'}\cdot(\partial_s g + i\partial_t g)\circ\varphi.\end{aligned} \tag{B.26}$$

It follows that for every map $\Phi + i\Psi : \mathbb{C} \to \mathfrak{g}^{\mathbb{C}}$ we have

$$\begin{aligned}
&(g \circ \varphi)^* \varphi^* (\Phi + i\Psi) \\
&= (g^{-1} \circ \varphi)\big(\overline{\varphi'} \cdot (\Phi + i\Psi) \circ \varphi\big) g \circ \varphi + (g^{-1} \circ \varphi)\big(\partial_s (g \circ \varphi) + i\partial_t (g \circ \varphi)\big) \\
&= \overline{\varphi'} \cdot (g^{-1}(\Phi + i\Psi) g) \circ \varphi + (g^{-1} \circ \varphi)\big(\overline{\varphi'} \cdot (\partial_s g + i\partial_t g) \circ \varphi\big) \\
&= \overline{\varphi'} \cdot \big(g^*(\Phi + i\Psi)\big) \circ \varphi \\
&= \varphi^*(g^*(\Phi + i\Psi)). \qquad \text{(B.27)}
\end{aligned}$$

Here we have used (B.26) in the third line. Equality (B.25) follows from (B.27) and from the equality

$$\begin{aligned}
\varphi^*(g^* u) &= (g^{-1} u) \circ \varphi \\
&= (g^{-1} \circ \varphi) u \circ \varphi \\
&= (g \circ \varphi)^* \varphi^* u.
\end{aligned}$$

This proves Proposition B.6. □

We recall the following definition. Let G and H be two groups with neutral elements e_{G}, e_{H}. We denote by Aut(G) the group of automorphisms of G. Let $\mathrm{G} \ni g \mapsto \rho_g \in \mathrm{Aut}(\mathrm{H})$ be an anti-homomorphism, in the sense that $\rho_{gg'} = \rho_{g'}\rho_g$. The semi-direct $\mathrm{G} \ltimes \mathrm{H}$ is defined to be the product $\mathrm{G} \times \mathrm{H}$ together with the multiplication given by

$$(g, h) \cdot (g', h') := (gg', \rho_{g'}(h) h')$$

and neutral element given by $(e_{\mathrm{G}}, e_{\mathrm{H}})$. Consider the case $\mathrm{G} := \mathrm{Aut}(\mathbb{C})$, $\mathrm{H} := \mathcal{G}^{2,p}$, and let $\mathrm{Aut}(\mathbb{C}) \ni \varphi \mapsto \rho_\varphi \in \mathrm{Aut}(\mathcal{G}^{2,p})$ be the pullback action given by

$$\rho_\varphi(g) := g \circ \varphi.$$

Corollary B.7 *The map*

$$\begin{aligned}
(\mathrm{Aut}(\mathbb{C}) \ltimes \mathcal{G}^{2,p}) \times W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g}^{\mathbb{C}}) &\to W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g}^{\mathbb{C}}), \\
(\varphi, g)^* w &:= g^*(\varphi^* w) \qquad \text{(B.28)}
\end{aligned}$$

is an action.

Proof of Corollary B.7: Let $w \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g}^{\mathbb{C}})$. It follows immediately from the definitions that $(\mathrm{id}, e)^* w = w$. Furthermore, for every (φ, g), $(\varphi', g') \in \mathrm{Aut}(\mathbb{C}) \times \mathcal{G}^{2,p}$ we have

$$\begin{aligned}
\big((\varphi, g) \cdot (\varphi', g')\big)^* w &= (\varphi \circ \varphi', (g \circ \varphi') g')^* w \\
&= ((g \circ \varphi') g')^* (\varphi \circ \varphi')^* w \\
&= g'^* (g \circ \varphi')^* \varphi'^* \varphi^* w \\
&= g'^* \varphi'^* g^* \varphi^* w \\
&= (\varphi', g')^* (\varphi, g)^* w.
\end{aligned}$$

Here the forth line follows from Proposition B.6. This proves Corollary B.7. □

Corollary B.8 *The group* $\mathrm{Aut}(\mathbb{C})$ *acts on the set of gauge equivalence classes* $W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g}^{\mathbb{C}})/\mathcal{G}^{2,p}$ *by the formula*

$$\varphi^*[w] := [\varphi^* w]. \tag{B.29}$$

Proof of Corollary B.8: We show that (B.29) is welldefined, i.e. $[\varphi^* w]$ does not depend on the choice of the representative of $[w]$. Let $w \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g}^{\mathbb{C}})$, $\varphi \in \mathrm{Aut}(\mathbb{C})$ and $g \in \mathcal{G}^{2,p}$. By Proposition B.6 the map $\varphi^*(g^* w)$ is gauge equivalent to $\varphi^* w$ via the gauge transformation $g \circ \varphi$, hence $[\varphi^* g^* w] = [\varphi^* w]$. This proves Corollary B.8. □

B.5 Sequences of reparametrizations

The next lemma reformulates conditions (i) and (ii) of Definition 3.6 (Convergence) in an equivalent way assuming that we are in some special case.

Lemma B.9 *Let* (T, E) *be a tree, with* $T = V \sqcup \bar{T}$, *let* $\alpha_0 \in T$, $z_0 := \infty$ *and let* $z_{\alpha\beta} \in S^2$ *for* $\alpha E \beta$ *be points such that condition (i) of Definition 3.2 with* $k := 0$ *holds. Let* φ^ν_α *be a Möbius transformation for* $\alpha \in T$, $\nu \in \mathbb{N}$. *Furthermore, assume that* $z_{\alpha,0} = \infty$ *for every* $\alpha \in \bar{T}$, *where* $z_{\alpha,0}$ *is defined as in (3.7) and (3.8) with* $i := 0$. *Then the following holds.*

(*) If $\alpha, \beta \in T$ *are such that* $\alpha E \beta$ *then either* $z_{\alpha\beta} = \infty$ *or* $z_{\beta\alpha} = \infty$ *(but not both).*

Furthermore, condition (i) of Definition 3.6 is equivalent to

(i') For every $\alpha \in T$, $\nu \in \mathbb{N}$ *there are numbers* $0 \neq \lambda^\nu_\alpha \in \mathbb{C}$ *and* $z^\nu_\alpha \in \mathbb{C}$ *such that* $\varphi^\nu_\alpha(z) = \lambda^\nu_\alpha z + z^\nu_\alpha$. *If* $\alpha \in V$ *then* $\lambda^\nu_\alpha = 1$. *Furthermore, if* $\alpha \in \bar{T}$ *then* $\lambda^\nu_\alpha \to \infty$ *for* $\nu \to \infty$.

Assume now that (i') is satisfied. Then condition (ii) of Definition 3.6 is equivalent to

(ii') If $\alpha, \beta \in T$ *are such that* $\alpha E \beta$ *and* $z_{\alpha\beta} \neq \infty$ *then*

$$\frac{\lambda^\nu_\beta}{\lambda^\nu_\alpha} \to 0, \qquad \frac{z^\nu_\beta - z^\nu_\alpha}{\lambda^\nu_\alpha} \to z_{\alpha\beta},$$

where $\lambda^\nu_\alpha, \lambda^\nu_\beta, z^\nu_\alpha, z^\nu_\beta$ *are as in (i').*

Proof: We prove **(*)**. Assume that $\alpha, \beta \in T$ are such that $\alpha E \beta$. From section 1 recall the definition of the subtree $T_{\alpha\beta} \subseteq T$, which contains the vertex β. Suppose that $z_{\alpha\beta} = \infty$. Since by hypothesis $z_{\alpha,0} = \infty$, distinctness of the nodal points at β (condition 3.2(i)) implies that $\alpha_0 \in T_{\alpha\beta}$. It follows that $z_{\beta\alpha} \neq z_{\beta,0} = \infty$.

Suppose now that $z_{\alpha\beta} \neq \infty$. By distinctness of the nodal points at α we have $\alpha_0 \in T_{\beta\alpha}$. It follows that $z_{\beta\alpha} = z_{\beta,0} = \infty$. This proves **(*)**.

We prove that 3.6(i) implies (i'). Let $\alpha \in T$ be a vertex. If $\alpha \in V$ then the assertion of (i') follows immediately. Assume that $\alpha \in \bar{T}$. Our hypothesis that $z_{\alpha,0} = \infty$ and condition 3.6(i) imply that there are numbers $\lambda_\alpha^\nu \in \mathbb{C} \setminus \{0\}$ and $z_\alpha^\nu \in \mathbb{C}$ such that $\varphi_\alpha^\nu(z) = \lambda_\alpha^\nu z + z_\alpha^\nu$. Furthermore, setting $\psi_\alpha := \mathrm{id} : S^2 \to S^2$, the last part of 3.6(i) implies that

$$\lambda_\alpha^\nu = (\varphi_\alpha^\nu)'(z) \to \infty,$$

for $\nu \to \infty$, for every $z \in \mathbb{C}$. That **(i')$\Longrightarrow$3.6(i)** follows analogously.

Assume now that (i') is satisfied. In order to prove that 3.6(ii) and (ii') are equivalent, we fix numbers λ_α^ν and z_α^ν as in (i'). Observe that if $\alpha E \beta$ then

$$\varphi_{\alpha\beta}^\nu(z) := (\varphi_\alpha^\nu)^{-1} \circ \varphi_\beta^\nu(z) = \frac{\lambda_\beta^\nu}{\lambda_\alpha^\nu} z + \frac{z_\beta^\nu - z_\alpha^\nu}{\lambda_\alpha^\nu}. \tag{B.30}$$

We show that **3.6(ii) implies (ii')**. Assume that $\alpha E \beta$ and that $z_{\alpha\beta} \neq \infty$. Then by (*) we have $z_{\beta\alpha} = \infty$. Therefore, condition 3.6(ii) and (B.30) imply that

$$\varphi_{\alpha\beta}^\nu(0) = \frac{z_\beta^\nu - z_\alpha^\nu}{\lambda_\alpha^\nu} \to z_{\alpha\beta}.$$

Furthermore, using again (B.30) and 3.6(ii), we have

$$\begin{aligned} \frac{\lambda_\beta^\nu}{\lambda_\alpha^\nu} &= \frac{\lambda_\beta^\nu}{\lambda_\alpha^\nu} + \frac{z_\beta^\nu - z_\alpha^\nu}{\lambda_\alpha^\nu} - \frac{z_\beta^\nu - z_\alpha^\nu}{\lambda_\alpha^\nu} \\ &= \varphi_{\alpha\beta}^\nu(1) - \frac{z_\beta^\nu - z_\alpha^\nu}{\lambda_\alpha^\nu} \\ &\to z_{\alpha\beta} - z_{\alpha\beta} = 0. \end{aligned}$$

Hence (ii') holds.

We show that **(ii') implies 3.6(ii)**. Assume that $\alpha E \beta$. In the **case** $z_{\alpha\beta} \neq \infty$ the assertion of 3.6(ii) follows from (B.30) and condition (ii'). Assume that $z_{\alpha\beta} = \infty$. Then (*) implies that $z_{\beta\alpha} \neq \infty$. Hence (ii') implies that

$$\frac{\lambda_\alpha^\nu}{\lambda_\beta^\nu} \to 0, \qquad \frac{z_\alpha^\nu - z_\beta^\nu}{\lambda_\beta^\nu} \to z_{\beta\alpha}.$$

It follows that

$$\varphi_{\alpha\beta}^\nu = \frac{\lambda_\beta^\nu}{\lambda_\alpha^\nu} \left(\cdot - \frac{z_\alpha^\nu - z_\beta^\nu}{\lambda_\beta^\nu} \right) \to \infty,$$

uniformly on every compact subset $Q \subseteq S^2 \setminus \{z_{\beta\alpha}\}$. Thus the assertion of 3.6(ii) holds in every case. This completes the proof of Lemma B.9. □

C Vortices on bounded subsets of $\mathbb{C}$

We fix a number $p > 2$.

C.1 An a priori Lemma

For $r > 0$ and $z_0 \in \mathbb{C}$ we denote by $B_r(z_0)$ the open ball in $\mathbb{C}$, and we abbreviate $B_r := B_r(z_0)$.

Lemma C.1 (A priori Lemma) *Let $K \subseteq M$ be a compact subset. Then there exists a constant $E_0 > 0$ such that the following holds. For every $z_0 \in \mathbb{C}$, $r > 0$ and every solution $w := (u, \Phi, \Psi) \in W^{1,p}(B_r(z_0), M \times \mathfrak{g} \times \mathfrak{g})$ of the vortex equations (0.1),(0.2) satisfying*

$$u(B_r(z_0)) \subseteq K, \tag{C.1}$$

$$E(w, B_r(z_0)) = \int_{B_r(z_0)} e_w ds\, dt \leq E_0, \tag{C.2}$$

the estimate

$$e_w(z_0) \leq \frac{8}{\pi r^2} E(w, B_r(z_0)) \tag{C.3}$$

holds.

For the proof of Lemma C.1 we need the following lemma.

Lemma C.2 (Heinz) *Let $r > 0$ and $c \geq 0$. Then for every function $f \in C^2(B_r, \mathbb{R})$ satisfying the inequalities*

$$f \geq 0, \qquad \Delta f \geq -cf^2, \qquad \int_{B_r} f < \frac{\pi}{8c} \tag{C.4}$$

we have

$$f(0) \leq \frac{8}{\pi r^2} \int_{B_r} f. \tag{C.5}$$

Proof of Lemma C.2: This is Lemma 4.3.2 in the book [MS3]. □

Proof of Lemma C.1: Let $K \subseteq M$ be a compact subset. Consider the set

$$K' := \mathrm{G} \cdot K := \{gx \,|\, g \in \mathrm{G},\, x \in K\}.$$

Since K' is the image of the compact sets G and K under the continuous map $\mathrm{G} \times M \ni (g, x) \mapsto gx \in M$, it is compact. By the calculation in Step 1 of the proof of Proposition 11.1. in [GS], there is a constant $c(K') > 0$ with the following property. If $\Omega \subseteq \mathbb{C}$ is an open subset and $w' := (u', \Phi', \Psi') \in C^2(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ is a solution of (0.1), (0.2) satisfying $u'(\Omega) \subseteq K'$ then

$$\Delta e_{w'} \geq -c(K') e_{w'}^2. \tag{C.6}$$

We set

$$E_0 := \frac{\pi}{9c(K')}.$$

Let $r > 0$ be a number, $z_0 \in \mathbb{C}$ be a point and $w = (u, \Phi, \Psi) \in W^{1,p}(B_r(z_0), M \times \mathfrak{g} \times \mathfrak{g})$ be a solution of (0.1), (0.2) such that (C.1), (C.2) are satisfied. By replacing the map w by $\widehat{w} := w(\cdot + z_0)$ we may assume w.l.o.g. that $z_0 = 0$. By Theorem C.6 there exists a gauge transformation $g \in W^{2,p}(B_r(z_0))$ such that

$$w' := (u', \Phi', \Psi') := g^* w \in C^\infty(\bar{B}_r(z_0)).$$

By (C.1) we have

$$u'(B_r(z_0)) = (g^{-1}u)(B_r(z_0)) \subseteq \mathrm{G} \cdot K = K',$$

and hence inequality (C.6) is satisfied. Furthermore, Lemma B.1 implies that $e_{w'} = e_w$ and hence by (C.2)

$$E(w', B_r(z_0)) = E(w, B_r(z_0)) \leq E_0 = \frac{\pi}{9c(K')}.$$

It follows that condition (C.4) of Lemma C.2 is satisfied with $f := e_w$ and $c := c(K')$. Applying Lemma C.2 we conclude that inequality (C.3) holds. This proves Lemma C.1. □

The next lemma follows from Lemma 9.1 in [GS].

Lemma C.3 *Let $c > 0$, $p > 2$, $\Omega \subseteq \mathbb{C}$ be an open set and $Q \subseteq \Omega$ and $K \subseteq M$ be compact subsets. Then there are positive constants R_0 and $C_p := C(c, p, \Omega, Q, K)$ such that the following holds. Let $R \geq R_0$ and let $w = (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ be a solution of the R-vortex equations (4.1), (4.2) such that*

$$\begin{aligned} u(\Omega) &\subseteq K, && \text{(C.7)}\\ \|\partial_s u + X_\Phi \circ u\|_{L^\infty(\Omega)} &\leq c, && \text{(C.8)}\\ |\xi| &\leq c|L_{u(z)}\xi|, \forall z \in \Omega, \forall \xi \in \mathfrak{g}. && \text{(C.9)} \end{aligned}$$

Then

$$\int_Q |\mu \circ u|^p \leq C_p R^{-2p}, \qquad \sup_Q |\mu \circ u| \leq C_p R^{2/p-2}.$$

Proof of Lemma C.3: Let $\Omega' \subseteq \Omega$ be an open bounded subset with smooth boundary such that $Q \subseteq \Omega'$. We define $K' := \mathrm{G} \cdot K \subseteq M$. Since K and G are compact, K' is also compact. Let $R > 0$ be a number and consider the Riemann surface $\Sigma := \bar{\Omega}'$ with volume form $\mathrm{dvol}_R := R^2 ds \wedge dt$ and the principal G-bundle $P := \bar{\Omega}' \times \mathrm{G}$. Denote by $\mathcal{A}^{1,p}(P)$ the affine space of $W^{1,p}$-connections on P and by $W^{1,p}_{\mathrm{G}}(P,M)$ the set of equivariant $W^{1,p}$-maps $\hat{u}:P\to M$. For every map $w:=(u,\Phi,\Psi)\in W^{1,p}(\Omega',M\times\mathfrak{g}\times\mathfrak{g})$ we define $(\hat{u},A)\in W^{1,p}_{\mathrm{G}}(P,M)\times\mathcal{A}^{1,p}(P)$ by

$$\hat{u}(z,g):=g^{-1}u(z),\qquad A_{(z,g)}(\zeta,g\xi):=\xi+g^{-1}\big(\zeta_1\Phi(z)+\zeta_2\Psi(z)\big)g,$$

for $(z,g)\in\bar{\Omega}\times\mathrm{G}$ and $(\zeta,g\xi)\in T_{(z,g)}(\bar{\Omega}\times\mathrm{G})=\mathbb{C}\times g\mathfrak{g}$. Then w solves the R-vortex equations (4.1), (4.2) on Ω' if and only if the pair $(\hat{u},A)$ solves the vortex equations on P,

$$\bar{\partial}_{J,A}(u) = 0, \tag{C.10}$$
$$*F_A+\mu(u) = 0. \tag{C.11}$$

Here $*:\Omega^2(\bar{\Omega}',\mathfrak{g}_P)\to\Omega^0(\bar{\Omega}',\mathfrak{g}_P)$ denotes the Hodge star operator w.r.t. dvol_R. It follows from the proof of Theorem 3.1 in [CGMS] that for every R-vortex $w\in W^{1,p}(\Omega',M\times\mathfrak{g}\times\mathfrak{g})$ there is a gauge transformation $g\in W^{2,p}(\Omega',\mathrm{G})$ such that $g^*w\in C^\infty(\Omega',M\times\mathfrak{g}\times\mathfrak{g})$. To see this we have to use a version of the Local Slice Theorem that allows the base manifold of the principal G-bundle P to have a nonempty smooth boundary, see Theorem 8.1 in the book [Weh]. Furthermore, if $u(\Omega')\subseteq K$ then $g^*u(\Omega')\subseteq K'$. Lemma C.3 follows now as in the proof of Lemma 9.1 in [GS]. □

Lemma C.4 *Let $C_0>0$, $\Omega\subseteq\mathbb{C}$ be a bounded open subset, $Q\subset\Omega$ be a compact subset, and $\lambda:\bar{\Omega}\to(0,\infty)$ be a smooth function. Then there exist constants $R_0>0$ and $C>0$ such that the following holds. If $R\geq R_0$ is a number and $w:=(u,\Phi,\Psi):\Omega\to M\times\mathfrak{g}\times\mathfrak{g}$ is a smooth solution of the equations*

$$\partial_s u+X_\Phi(u)+J(\partial_t u+X_\Psi(u))=0,\qquad \partial_s\Psi-\partial_t\Phi+[\Phi,\Psi]+R^2\lambda^2\mu\circ u=0,$$

such that

$$\sup_{\Omega}\big(|\partial_s u+X_\Phi(u)|+R|\mu\circ u|\big)\leq C_0, \tag{C.12}$$

then

$$||\mu\circ u||_{L^p(Q)}\leq CR^{-1-\frac{2}{p}}\big(||\partial_s u+X_\Phi(u)||_{L^2(\Omega)}+R||\mu\circ u||_{L^2(\Omega)}\big), \tag{C.13}$$

for $2\leq p\leq\infty$.

Proof of Lemma C.4: This is part of the statement of Lemma 9.3 in the paper by R. Gaio and D. A. Salamon [GS]. □

C.2 Regularity modulo gauge

Lemma C.5 *Let $\Omega \subseteq \mathbb{C}$ be an open subset and $(u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\Omega, M\times\mathfrak{g}\times\mathfrak{g})$ be a solution of the first vortex equations (0.1). Then $u \in W^{2,p}_{\mathrm{loc}}(\Omega, M)$.*

Proof of Lemma C.5: Let $z_0 \in \Omega$ be a point, $\varphi : U \subseteq M \to \mathbb{R}^{2n}$ be a local coordinate chart such that $u(z_0) \in U$, and let $\Omega' := u^{-1}(U) \subseteq \Omega$. We define $u^{\mathrm{loc}} := \varphi \circ u : \Omega' \to \mathbb{R}^{2n}$ and $J^{\mathrm{loc}} : \Omega' \to \mathbb{R}^{2n\times 2n}$ by

$$J^{\mathrm{loc}}(z) := d\varphi(u(z))J(u(z))d\varphi(u(z))^{-1},$$

for $z \in \Omega'$. Then $u^{\mathrm{loc}} \in W^{1,p}_{\mathrm{loc}}(\Omega', \mathbb{R}^{2n})$ and $J^{\mathrm{loc}} \in W^{1,p}_{\mathrm{loc}}(\Omega', \mathbb{R}^{2n\times 2n})$. Proposition E.10 applied with Ω replaced by Ω' implies that $u^{\mathrm{loc}} \in W^{2,p}_{\mathrm{loc}}(\Omega', \mathbb{R}^{2n})$ and therefore $u|_{\Omega'} = \varphi^{-1} \circ u^{\mathrm{loc}} \in W^{2,p}_{\mathrm{loc}}(\Omega', M)$. Since $z_0 \in \Omega$ is arbitrary, it follows that $u \in W^{2,p}_{\mathrm{loc}}(\Omega, M)$. This proves Lemma C.5. □

Theorem C.6 *Let k be a positive integer or be equal to ∞, let J be a G-invariant ω-compatible almost complex structure on M, of class C^k, let $p > 2$ and $R \geq 0$ be numbers and $\Omega \subseteq \mathbb{C}$ be an open bounded subset with smooth boundary. Then for every solution $w = (u, \Phi, \Psi) \in W^{1,p}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ of the R-vortex equations (4.1), (4.2) there exists a gauge transformation $g \in W^{2,p}(\bar{\Omega}, \mathrm{G})$ such that $g^*w \in W^{k+1,p}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ in the case $k < \infty$ and $g^*w \in C^\infty(\bar{\Omega})$ in the case $k = \infty$.*

Proof of Theorem C.6: This follows from the proof of Theorem 3.1 in [CGMS] with $\ell := k$, $P := \bar{B}_r(z_0) \times \mathrm{G}$ and the connection 1-form A on P given by

$$A_{(z,g)}(\zeta, g\xi) := \xi + g^{-1}\big(\zeta_1\Phi(z) + \zeta_2\Psi(z)\big)g,$$

for $(z, g) \in P$ and $(\zeta, g\xi) \in T_{(z,g)}P = \mathbb{C} \times g\mathfrak{g}$. Observe that that Theorem holds also in the case in which the base Σ of the principal G-bundle P has boundary. To see this, we use a version of the local slice theorem for the case $\partial\Sigma \neq \emptyset$, for example Theorem 8.1. in the book [Weh]. □

By the Whitney embedding theorem (see the book by Hirsch [Hi], Theorem 3.2.14 p. 55) there exists an embedding $\iota : M \to \mathbb{R}^{2\dim M+1}$. For every open subset $\Omega \subseteq \mathbb{C}$, every nonnegative integer k, every number $p > 2$ and every k-times weakly differentiable map $w := (u, \Phi, \Psi) : \Omega \to M \times \mathfrak{g} \times \mathfrak{g}$ we denote

$$||w||_{W^{k,p}(\Omega)} := ||\iota \circ u||_{W^{k,p}(\Omega)} + ||(\Phi, \Psi)||_{W^{k,p}(\Omega)} \in [0, \infty].$$

Note that this norm depends on the embedding ι.

Theorem C.7 (**$W^{k,p}$-bound modulo gauge for vortices on a bounded set**) *Let $c > 0$ be a constant, $\Omega \subseteq \mathbb{C}$ be a bounded open subset with smooth boundary, and $K \subseteq M$ be a compact subset. Then for every $k \in \mathbb{N}$ and every $p > 2$ there is a constant $C_{k,p} := C(c, \Omega, K, k, p)$ such that for every solution $w = (u, \Phi, \Psi) \in W^{1,p}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ of the vortex equations (0.1), (0.2) the following holds. If*

$$u(\Omega) \subseteq K, \qquad ||\partial_s u + X_\Phi(u)||_{L^\infty(\Omega)} \leq c$$

then there is a gauge transformation $g \in W^{2,p}(\Omega, \mathrm{G})$ such that

$$||g^* w||_{W^{k,p}(\Omega)} \leq C_{k,p}.$$

Proof of Theorem C.7: Assume by contradiction that there are a constant $c > 0$, an open subset $\Omega \subseteq \mathbb{C}$ with smooth boundary, and numbers $k \in \mathbb{N}$ and $p > 2$ such that there is no such constant $C_{k,p}$. This means that there is a sequence $w_\nu = (u_\nu, \Phi_\nu, \Psi_\nu) \in W^{1,p}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ of solutions of the vortex equations (0.1), (0.2) such that

$$u_\nu(\Omega) \subseteq K, \qquad ||\partial_s u_\nu + X_{\Phi_\nu}(u_\nu)||_{L^\infty(\Omega)} \leq c$$

and for every sequence $g_\nu \in W^{2,p}(\Omega, \mathbb{C})$ we have

$$||g_\nu^* w_\nu||_{W^{k,p}(\Omega)} \to \infty,$$

and therefore $g_\nu^* w_\nu$ has no $C^k(\bar{\Omega})$-convergent subsequence. This contradicts the assertion of Theorem 3.2. in [CGMS], with $\Sigma := \bar{\Omega}$, $\mathrm{dvol}_\Sigma := ds \wedge dt$, $P := \bar{\Omega} \times \mathrm{G}$, $\tau_\nu := \tau := 0$, $(\omega_\nu, \mu_\nu, J_\nu, H_\nu) := (\omega, \mu, J, 0)$. Note that the proof of that theorem also works in our case, in which the base $\bar{\Omega}$ of the principal G-bundle $P = \Omega \times \mathrm{G}$ has boundary. For this one has to use a version of Uhlenbeck compactness for a compact base with boundary, see Theorem E.7, and a version of the local slice theorem allowing boundary, see Theorem 8.1 in the book [Weh]. This proves Theorem C.7. □

C.3 The energy-action identity

In this subsection we state and prove a lemma which says that for $\tau_- \leq \tau_+$ the energy of an $\exp_{\mathbb{C}}$-vortex on $[\tau_-, \tau_+] \times S^1$ equals the difference of the actions of its end loops. Since every such $\exp_{\mathbb{C}}$-vortex corresponds to an ordinary vortex on the annulus $A(e^{\tau_-}, e^{\tau_+})$, it follows that the energy of a vortex on an annulus equals the difference of the actions of its suitably reparametrized end loops. The lemma is used in the proof of the Annulus Lemma 4.11.

Consider the equations for a map $(\widetilde{u},\widetilde{\Phi},\widetilde{\Psi})\in W^{1,p}_{\mathrm{loc}}(\mathbb{C},M\times\mathfrak{g}\times\mathfrak{g})$

$$\begin{aligned}\partial_s\widetilde{u}+X_{\widetilde{\Phi}}(\widetilde{u})+J(\widetilde{u})(\partial_t\widetilde{u}+X_{\widetilde{\Psi}}(\widetilde{u})) &= 0, &\text{(C.14)}\\ \partial_s\widetilde{\Psi}-\partial_t\widetilde{\Phi}+[\widetilde{\Phi},\widetilde{\Psi}]+e^{2\tau}\mu\circ\widetilde{u} &= 0. &\text{(C.15)}\end{aligned}$$

In the language of section B.2 these are the $\exp_{\mathbb{C}}$-vortex equations, where

$$\exp_{\mathbb{C}}:\mathbb{R}\times S^1\cong\mathbb{R}\times(\mathbb{R}/(2\pi\mathbb{Z}))\to\mathbb{C}$$

is the exponential map given by $\exp_{\mathbb{C}}(\tau,\varphi):=e^{\tau+i\varphi}$. The $\exp_{\mathbb{C}}$-energy density of a map $\widetilde{w}=(\widetilde{u},\widetilde{\Phi},\widetilde{\Psi})\in W^{1,p}_{\mathrm{loc}}(\Omega,M\times\mathfrak{g}\times\mathfrak{g})$ defined on an open subset $\Omega\subseteq\mathbb{R}\times S^1$ is given by

$$\widetilde{e}_{\widetilde{w}}=\frac{1}{2}\Big(|\partial_\tau\widetilde{u}+X_{\widetilde{\Phi}}(\widetilde{u})|^2+|\partial_\varphi\widetilde{u}+X_{\widetilde{\Psi}}(\widetilde{u})|^2+e^{-2\tau}|\widetilde{\kappa}|^2+e^{2\tau}|\mu\circ\widetilde{u}|^2\Big),$$

where

$$\widetilde{\kappa}:=\partial_\tau\widetilde{\Psi}-\partial_\varphi\widetilde{\Phi}+[\widetilde{\Phi},\widetilde{\Psi}].$$

Furthermore, the $\exp_{\mathbb{C}}$-energy of such a map $\widetilde{w}$ on a measurable subset $X\subseteq\Omega$ is given by

$$\widetilde{E}(\widetilde{w},X)=\int_X\widetilde{e}_{\widetilde{w}}\,ds\,dt.$$

Lemma C.8 *Let δ_0 be as in (A.17). Let $\tau_-\le\tau_+$ be real numbers and $\widetilde{w}:=(\widetilde{u},\widetilde{\Phi},\widetilde{\Psi})\in C^\infty([\tau_-,\tau_+]\times S^1,M\times\mathfrak{g}\times\mathfrak{g})$ be a solution of the $\exp_{\mathbb{C}}$-vortex equations (C.14), (C.15). Assume that*

$$\begin{aligned}|\mu(\widetilde{u}(\tau,\varphi))|&<\delta_0,\ \forall\tau\in[\tau_-,\tau_+],\ \varphi\in S^1, &\text{(C.16)}\\ \ell((\widetilde{u},\widetilde{\Psi})(\tau,\cdot))=\int_0^{2\pi}\big|\partial_\varphi\widetilde{u}(\tau,\varphi)+X_{\widetilde{\Psi}(\tau,\varphi)}(\widetilde{u}(\tau,\varphi))\big|d\varphi&<\delta_0,\ \forall\tau. &\text{(C.17)}\end{aligned}$$

Then

$$\widetilde{E}(\widetilde{w},[\tau_-,\tau_+]\times S^1)=-\mathcal{A}((\widetilde{u},\widetilde{\Psi})(\tau_+,.))+\mathcal{A}((\widetilde{u},\widetilde{\Psi})(\tau_-,.)).$$

Proof of Lemma C.8: We define $\iota>0$ to be the largest number such that if $x\in\mu^{-1}(0)$ then the exponential map of $(M,g_{\omega,J})$ restricted to the open ball $B_\iota\subseteq T_xM$ is injective. We fix constants C and δ as in Lemma A.13. Applying that lemma with $u:=\widetilde{u}$ and $\eta:=\widetilde{\Psi}$ there exists $x_0\in C^\infty([\tau_-,\tau_+],\mu^{-1}(0))$ and $g_0\in C^\infty([\tau_-,\tau_+]\times S^1,\mathrm{G})$ such that (A.15) and (A.16) with $x:=\widetilde{u}(\tau,\cdot)$ and $g_0(\varphi):=g_0(\tau,\varphi)$ are satisfied, for every $\tau\in[\tau_-,\tau_+]$. We define $w':=g_0^*\widetilde{w}$. Since $2C\delta_0<\iota$, (A.16) and the hypotheses (C.16), (C.17) imply that for every $\tau\in\mathbb{R}$ and $\varphi\in S^1\cong\mathbb{R}/(2\pi\mathbb{Z})$

$$\begin{aligned}d(u'(\tau,\varphi),x_0(\tau)) &= d(\widetilde{u}(\tau,\varphi),g_0(\tau,\varphi)x_0(\tau))\\ &\le C\big(|\mu(\widetilde{u}(\tau,\varphi))|+\ell(\widetilde{u}(\tau,\cdot),\widetilde{\Psi}(\tau,\cdot))\big)\\ &\le C(\delta_0+\delta_0)<\iota.\end{aligned}$$

Therefore for each $(\tau, \varphi) \in [\tau_-, \tau_+] \times S^1$ we can define $v(\tau, \varphi) \in T_{x_0(\tau)}M$ to be the unique vector of norm less than ι such that

$$\exp_{x_0(\tau)} v(\tau, \varphi) = u'(\tau, \varphi).$$

The map $v : [\tau_-, \tau_+] \times S^1 \to TM$ is smooth. We define

$$h : [\tau_-, \tau_+] \times [0,1] \times S^1 \to M, \qquad h_\tau(\lambda, \varphi) := h(\tau, \lambda, \varphi) := \exp_{x_0(\tau)} \lambda v(\tau, \varphi).$$

The calculation in the proof of Proposition 3.1 in [CGS] shows that

$$\widetilde{e}_{w'} = \omega(\partial_\tau u', \partial_\varphi u') - \partial_\tau \langle \mu(u'), \Psi' \rangle + \partial_\varphi \langle \mu(u'), \Phi' \rangle.$$

Therefore,

$$\begin{aligned} \widetilde{E}(w', [\tau_-, \tau_+] \times S^1) &= \int_{[\tau_-, \tau_+] \times S^1} \big(\omega(\partial_\tau u', \partial_\varphi u') - \partial_\tau \langle \mu(u'), \Psi' \rangle \\ &\qquad + \partial_\varphi \langle \mu(u'), \Phi' \rangle \big)\, d\tau\, d\varphi \\ &= \int_{[\tau_-, \tau_+] \times S^1} (u')^* \omega - \int_{\tau_-}^{\tau_+} \partial_\tau \int_0^{2\pi} \langle \mu(u'), \Psi' \rangle \, d\varphi \, d\tau + 0 \\ &= \int_{[\tau_-, \tau_+] \times S^1} (u')^* \omega - \int_0^{2\pi} \langle \mu(u'(\tau_+, \varphi)), \Psi'(\tau_+, \varphi) \rangle \, d\varphi \\ &\qquad + \int_0^{2\pi} \langle \mu(u'(\tau_-, \varphi)), \Psi'(\tau_-, \varphi) \rangle \, d\varphi. \end{aligned} \tag{C.18}$$

Furthermore, since $dh^*\omega = h^* d\omega = 0$, we have

$$\begin{aligned} 0 &= \int_{[\tau_-, \tau_+] \times [0,1] \times S^1} dh^* \omega \\ &= \int_{\partial\left([\tau_-, \tau_+] \times [0,1] \times S^1\right)} h^* \omega \\ &= \int_{[0,1] \times S^1} h_{\tau_+}^* \omega - \int_{[0,1] \times S^1} h_{\tau_-}^* \omega \\ &\quad - \int_{[\tau_-, \tau_+] \times S^1} h(\cdot, 1, \cdot)^* \omega + \int_{[\tau_-, \tau_+] \times S^1} h(\cdot, 0, \cdot)^* \omega. \end{aligned} \tag{C.19}$$

Since for $\lambda := 0$ the expression $h(\tau, 0, \varphi)$ does not depend on φ, we have

$$\int_{[\tau_-, \tau_+] \times S^1} h(\cdot, 0, \cdot)^* \omega = 0.$$

Furthermore, for $\lambda := 1$ we have $h(\tau, 1, \varphi) = u'(\tau, \varphi)$. Hence equality (C.19) together with (C.18) implies that

$$\begin{aligned}\widetilde{E}(w', [\tau_-, \tau_+] \times S^1) &= -\int_{[0,1]\times S^1} h_{\tau_-}^* \omega + \int_{[0,1]\times S^1} h_{\tau_+}^* \omega \\ &\quad - \int_0^{2\pi} \langle \mu(u'(\tau_+, \varphi)), \Psi'(\tau_+, \varphi) \rangle \, d\varphi \\ &\quad + \int_0^{2\pi} \langle \mu(u'(\tau_-, \varphi)), \Psi'(\tau_-, \varphi) \rangle \, d\varphi \\ &= -\mathcal{A}((u', \Psi')(\tau_+, .)) + \mathcal{A}((u', \Psi')(\tau_-, .)) \\ &= -\mathcal{A}((\widetilde{u}, \widetilde{\Psi})(\tau_+, .)) + \mathcal{A}((\widetilde{u}, \widetilde{\Psi})(\tau_-, .)).\end{aligned}$$

This proves Lemma C.8. □

D Vortices on $\mathbb{C}$

D.1 Quantization of energy

For every compact symplectic manifold (M, ω) and every ω-compatible almost structure J on M the infimum of the energies of all nonconstant J-holomorphic spheres in M is positive, see Proposition 4.1.4 in [MS3]. The next lemma implies that the same holds for vortices.

Lemma D.1 (Quantization of energy) *We have*

$$\inf\left\{E(w)\,|\,w \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M\times\mathfrak{g}\times\mathfrak{g}) : (0.1), (0.2), E(w) > 0, \overline{u(\mathbb{C})} \textit{ compact}\right\} \geq E_0,$$

where $E_0 > 0$ is the constant from Lemma C.1.

Proof of Lemma D.1: Let $E_0 > 0$ be as in Lemma C.1. Let $w = (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ be a solution of (0.1),(0.2) such that $\overline{u(\mathbb{C})}$ is compact and $E(w) < E_0$. Let $z_0 \in \mathbb{C}$. Since the estimate (C.3) holds for every $r > 0$ it follows that $e_w(z_0) = 0$. This proves Lemma D.1. □

D.2 Existence of good gauges

For $k \in \mathbb{N} \cup \{0, \infty\}$ we denote by $\mathcal{J}^k_{\mathrm{G}}(M, \omega)$ the set of ω-compatible G-invariant almost complex structures of class C^k.

Proposition D.2 (Regularity modulo gauge) *Let $k \in \mathbb{N} \sqcup \{\infty\}$, $J \in \mathcal{J}^k_{\mathrm{G}}(M, \omega)$, $p > 2$, $R \geq 0$ be numbers, and $w \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M\times\mathfrak{g}\times\mathfrak{g})$ be a solution of the R-vortex equations (4.1),(4.2). Then there is a gauge transformation $g \in \mathcal{G}^{2,p}$ such that $g^*w \in W^{k+1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ in the case $k < \infty$ and $g^*w \in C^\infty(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ in the case $k = \infty$.*

The proof of Proposition D.2 follows the lines of the proofs of Theorem 3.6 and Theorem A.3 in [Fr1].

Proof of Proposition D.2: Consider the **case** $k < \infty$.

Claim 1 *There exists a collection of gauge transformations $g_j \in W^{2,p}(B_{j+1}, \mathrm{G})$, $j \in \mathbb{N}$, such that for every $j \in \mathbb{N}$ we have*

$$g_j^* w \in W^{k+1,p}(B_{j+1}, M \times \mathfrak{g} \times \mathfrak{g}). \tag{D.1}$$

$$g_{j+1}|_{B_j} = g_j|_{B_j}. \tag{D.2}$$

Proof: By Theorem C.6 there exists a gauge transformation $g_1 \in W^{2,p}(B_2, \mathrm{G})$ such that $g_1^* w \in W^{k+1,p}(B_2)$. Let $\ell \in \mathbb{N}$ be an integer and assume by induction that there exist gauge transformations $g_j \in W^{2,p}(B_{j+1}, \mathrm{G})$, for

$j = 1, \dots, \ell$, such that (D.1) holds for $j = 1, \dots, \ell$, and (D.2) holds for $j = 1, \dots, \ell - 1$. We show that there exists a gauge transformation $g_{\ell+1} \in W^{2,p}(B_{\ell+2}, \mathrm{G})$ such that

$$g_{\ell+1}^* w \in W^{k+1,p}(B_{\ell+2}), \tag{D.3}$$
$$g_{\ell+1}|_{B_\ell} = g_\ell|_{B_\ell}. \tag{D.4}$$

We choose a smooth function $\rho : \bar{B}_{\ell+2} \to B_{\ell+1}$ such that $\rho(z) = z$ for $z \in B_\ell$. By Theorem C.6 there exists a gauge transformation $h \in W^{2,p}(B_{\ell+2}, \mathrm{G})$ such that

$$h^* w \in W^{k+1,p}(\bar{B}_{\ell+2}). \tag{D.5}$$

We define

$$g_{\ell+1} := h\big((h^{-1} g_\ell) \circ \rho\big).$$

Then $g_{\ell+1} \in W^{2,p}(B_{\ell+2}, \mathrm{G})$ and (D.4) is satisfied. Furthermore, since $h^* w \in W^{k+1,p}(B_{\ell+2})$ and

$$g_\ell^* w = (h^{-1} g_\ell)^* h^* w \in W^{k+1,p}(B_{\ell+1}),$$

Lemma B.2(ii) implies that $h^{-1} g_\ell \in W^{k+2,p}(B_{\ell+1})$. It follows that $(h^{-1} g_\ell) \circ \rho \in W^{k+2,p}(B_{\ell+2})$, and therefore

$$g_{\ell+1}^* w = \big((h^{-1} g_\ell) \circ \rho\big)^* h^* w \in W^{k+1,p}(B_{\ell+2}).$$

So (D.3) is also satisfied. This terminates the induction and concludes the proof of Claim 1. □

We choose now a collection of gauge transformations g_j, $j \in \mathbb{N}$ as in Claim 1 and define $g \in \mathcal{G}^{2,p}$ by

$$g(z) := g_j(z),$$

if $z \in B_j$. By (D.2) this map is well-defined. Furthermore, (D.1) implies that $g^* w \in W^{k+1,p}_{\mathrm{loc}}(\mathbb{C})$. This proves the statement of the proposition in the case $k < \infty$. In the **case** $k = \infty$ the proof is analogous. This proves Proposition D.2. □

We denote by

$$\exp_{\mathbb{C}} : \mathbb{C} \to \mathbb{C} \setminus \{0\}$$

the exponential map on $\mathbb{C}$. We identify $\mathfrak{g} \times \mathfrak{g}$ with the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. For every function $(u, \Phi + i\Psi) : \mathbb{C} \setminus 0 \to M \times \mathfrak{g}^{\mathbb{C}}$ we define $\widetilde{w} := (\widetilde{u}, \widetilde{\Phi} + i\widetilde{\Psi}) := \exp_{\mathbb{C}}^* w$, as in (B.11). This means that

$$\begin{aligned} \widetilde{u}(e^{\tau,\varphi}) &:= u(e^{\tau+i\varphi}), \\ \widetilde{\Phi}(\tau,\varphi) &:= e^\tau\big(\cos(\varphi)\Phi(e^{\tau+i\varphi}) + \sin(\varphi)\Psi(e^{\tau+i\varphi})\big), \\ \widetilde{\Psi}(\tau,\varphi) &:= e^\tau\big(-\sin(\varphi)\Phi(e^{\tau+i\varphi}) + \cos(\varphi)\Psi(e^{\tau+i\varphi})\big). \end{aligned} \tag{D.6}$$

For every $g : \mathbb{C} \setminus 0 \to \mathrm{G}$ we define

$$\widetilde{g} := g \circ \exp_{\mathbb{C}} : \mathbb{C} \to \mathrm{G}.$$

It follows that for every $w \in W^{1,p}_{\mathrm{loc}}(\mathbb{C} \setminus 0, M \times \mathfrak{g}^{\mathbb{C}})$ and $g \in W^{2,p}_{\mathrm{loc}}(\mathbb{C} \setminus 0, \mathrm{G})$ we have

$$(g^* w)\widetilde{\ } = \widetilde{g}^* \widetilde{w} := \Big(\widetilde{g}^{-1}\widetilde{u}, \widetilde{g}^{-1}\big(\widetilde{\Phi} + \partial_\tau + i(\widetilde{\Psi} + \partial_\varphi)\big)\widetilde{g}\Big). \quad \text{(D.7)}$$

Recall the definitions (1.15) and (1.16) of the sets $\widetilde{\mathcal{M}}^{k,p} := \widetilde{\mathcal{M}}^{k,p}_J$ and $\widetilde{\mathcal{M}} := \widetilde{\mathcal{M}}_J$ of vortices on $\mathbb{C}$, and the definitions (1.17) and (1.19) of the gauge groups $\mathcal{G}^{k+1,p}$ and $\mathcal{G}$. For $\tau_0 \in \mathbb{R}$ we define

$$\widetilde{\mathcal{M}}_{\tau_0} := \big\{(u, \Phi, \Psi) \in \widetilde{\mathcal{M}} \,|\, \widetilde{\Phi}(\tau + i\varphi) = 0,\ \forall \tau \geq \tau_0, \varphi \in \mathbb{R}\big\}$$

and

$$\mathcal{G}_{\tau_0} := \big\{g \in C^\infty(\mathbb{C}, \mathrm{G}) \,|\, \partial_\tau \widetilde{g}(\tau + i\varphi) = 0\, \forall \tau \geq \tau_0, \varphi \in \mathbb{R}\big\}.$$

Note that the action of $\mathcal{G}$ on $\widetilde{\mathcal{M}}$ restricts to an action of $\mathcal{G}_{\tau_0}$ on $\widetilde{\mathcal{M}}_{\tau_0}$.

Proposition D.3 *Let $k \in \mathbb{N}$, $p > 2$ and let $\tau_0 \in \mathbb{R}$. Then the map*

$$\widetilde{\mathcal{M}}_{\tau_0}/\mathcal{G}_{\tau_0} \to \widetilde{\mathcal{M}}^{k,p}/\mathcal{G}^{k+1,p}, \qquad \mathcal{G}^*_{\tau_0}(u, \Phi + i\Psi) \mapsto (\mathcal{G}^{k+1,p})^*(u, \Phi + i\Psi) \quad \text{(D.8)}$$

is a bijection.

Proof of Proposition D.3: We prove that the map is injective. Assume that $\mathcal{G}^*_{\tau_0} w, \mathcal{G}^*_{\tau_0} w' \in \widetilde{\mathcal{M}}_{\tau_0}/\mathcal{G}_{\tau_0}$ are such that $(\mathcal{G}^{k+1,p})^* w = (\mathcal{G}^{k+1,p})^* w'$. This means that there is gauge transformation $g \in W^{k+1,p}_{\mathrm{loc}}(\mathbb{C}, \mathrm{G})$ such that $g^* w = w'$. Lemma B.2 implies that $g \in C^\infty(\mathbb{C}, \mathrm{G})$. Furthermore, for $\tau \geq \tau_0$ we have $\widetilde{\Phi} = \widetilde{\Phi}' = 0$. Since

$$\widetilde{\Phi}' = \widetilde{g}^* \widetilde{\Phi} = \widetilde{g}^{-1}(\widetilde{\Phi} + \partial_\tau)\widetilde{g}$$

it follows that $\partial_\tau \widetilde{g} = 0$ for $\tau \geq \tau_0$. Therefore $g \in \mathcal{G}_{\tau_0}$ and thus $\mathcal{G}^*_{\tau_0} w = \mathcal{G}^*_{\tau_0} w'$. This implies injectivity.

To prove that the map (D.8) is surjective let $(\mathcal{G}^{k+1,p})^*(u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{k,p}/\mathcal{G}^{k+1,p}$. By Proposition D.2 there is $g \in \mathcal{G}^{2,p}$ such that $w' := g^* w \in C^\infty(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$. We define $h : \mathbb{C} \to \mathrm{G}$ as follows. On $B_{e^{\tau_0 - 2}}$ we set $h \equiv e$. Let now $\rho \in C^\infty([\tau_0 - 2, \infty), [0, 1])$ be such that $\rho(\tau) = 0$ for $\tau \leq \tau_0 - 1$ and $\rho(\tau) = 1$ for $\tau \geq \tau_0$. For $\tau \geq \tau_0 - 2$, $\varphi \in \mathbb{R}$ we define $h(e^{\tau + i\varphi}) := \widetilde{h}(\tau, \varphi)$, where $\widetilde{h}(., \varphi) : [\tau_0 - 2, \infty) \to \mathrm{G}$ is the unique solution of the ordinary differential equation

$$\partial_\tau \widetilde{h}(., \varphi) = -\rho \widetilde{\Phi}'(., \varphi)\widetilde{h}(., \varphi)$$

with initial condition $\widetilde{h}(\tau_0 - 2, \varphi) = e$. We define $w'' := h^* w'$. Then for $\tau \geq \tau_0$ we have $\partial_\tau \widetilde{h} = -\widetilde{\Phi}' \widetilde{h}$ and therefore

$$\widetilde{\Phi''} = \widetilde{h}^* \widetilde{\Phi}' = \widetilde{h}^{-1}(\widetilde{\Phi}' + \partial_\tau)\widetilde{h} = 0.$$

Furthermore, since the data $\rho\widetilde{\Phi}'$ of the differential equation depends smoothly on τ, φ it follows that $h \in C^\infty(\mathbb{C}, \mathrm{G})$ and therefore $w'' \in C^\infty(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$. Thus $w'' \in \widetilde{\mathcal{M}}_{\tau_0}$. Under the map (D.8) the equivalence class $\mathcal{G}^*_{\tau_0} w''$ gets mapped to $(\mathcal{G}^{k+1,p})^* w'' = (\mathcal{G}^{k+1,p})^* w$. This proves surjectivity and completes the proof of Proposition D.3. □

That the map (D.8) is surjective, implies that for every $r > 0$ and every vortex $w \in \widetilde{\mathcal{M}}^{1,p}$ there exists a gauge transformation $g \in \mathcal{G}^{2,p}$ such that $g^* w$ is smooth and in radial gauge outside the ball B_r.

The next proposition is used in the proof of Proposition 4.6 (Hard rescaling) and in the proof of Lemma (D.14) (Vortices of 0 energy).

Proposition D.4 *Let $(\Phi, \Psi) \in C^\infty(\mathbb{C}, \mathfrak{g} \times \mathfrak{g})$ be such that $\kappa := \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] = 0$. Then there exists a gauge transformation $g \in \mathcal{G}$ such that $g^*(\Phi, \Psi) = 0$.*

The proof of this proposition uses the following lemma. For its proof see also the dissertation by U. Frauenfelder [Fr1], Corollary 3.7.

Lemma D.5 (s-gauge) *Let $(\Phi, \Psi) \in C^\infty(\mathbb{R}^2, \mathfrak{g} \times \mathfrak{g})$. Then there is a gauge transformation $g \in \mathcal{G}$ such that $(\Phi', \Psi') := g^*(\Phi, \Psi)$ satisfies*

$$\Phi' \equiv 0, \qquad \Psi'(0, t) = 0,$$

for every $t \in \mathbb{R}$.

Proof: We define $g_1 : \mathbb{R}^2 \to \mathrm{G}$ as follows. For every $t_0 \in \mathbb{R}$ we define $g_1(., t_0)$ to be the unique smooth solution of the ordinary differential equation

$$\frac{d}{ds} g_1(., t_0) = -\Phi(., t_0) g_1(., t_0)$$

with initial condition $g_1(0, t_0) := e$. Then the map $g_1 : \mathbb{R}^2 \to \mathrm{G}$ is smooth. Furthermore, we define $g_2 : \mathbb{R} \to \mathrm{G}$ to be the unique smooth solution of the ordinary differential equation

$$\frac{d}{dt} g_2 = -g_1^* \Psi(0, .) g_2 = -\big(g_1^{-1}(\Psi + \partial_t) g_1\big)(0, \cdot) g_2 \tag{D.9}$$

with initial condition $g_2(0) := e$. Then g_2 is also smooth. We define now $g : \mathbb{R}^2 \to \mathrm{G}$ by $g(s,t) := g_1(s,t) g_2(t)$. We check that g satisfies the requirements of the lemma. It is smooth. Furthermore,

$$g_1^* \Phi = g_1^{-1}(\partial_s + \Phi) g_1 = 0.$$

Therefore, considering g_2 as a map from $\mathbb{R}^2$ to G by setting $g_2(s,t) := g_2(t)$, we have

$$\Phi' := g^* \Phi = g_2^* g_1^* \Phi = g_2^{-1} \partial_s g_2 + 0 = 0.$$

Moreover, by (D.9) we have

$$\Psi'(0,\cdot) = g^*\Psi(0,\cdot) = g_2^* g_1^* \Psi(0,\cdot) = \big(g_2^{-1}(\partial_t + g_1^*\Psi)g_2\big)(0,\cdot) = 0.$$

This proves the lemma. □

Proof of Proposition D.4: By Lemma D.5 there is $g \in \mathcal{G}$ such that $(\Phi', \Psi') := g^*(\Phi, \Psi)$ satisfies $\Phi' = 0$ and $\Psi'(0,t) = 0$ for every $t \in \mathbb{R}$. Furthermore,

$$\partial_s \Psi' = \partial_s \Psi' - \partial_t \Phi' + [\Phi', \Psi'] =: \kappa' = g^{-1}\kappa g = 0.$$

It follows that $\Psi' \equiv 0$. This proves Proposition D.4. □

D.3 Asymptotic behaviour of finite energy vortices on $\mathbb{C}$

Given a map $w = (u, \Phi + i\Psi) : \mathbb{C} \to M \times \mathfrak{g}^{\mathbb{C}}$ we denote by

$$\widetilde{w} := (\widetilde{u}, \widetilde{\Phi} + i\widetilde{\Psi}) := \exp_{\mathbb{C}}^* w$$

the coordinate transformed map, as in (D.6). Here we identify S^1 with $\mathbb{R}/(2\pi\mathbb{Z})$. Given a number $\tau_0 \in \mathbb{R}$ we say that w is in radial gauge outside the ball $B_{e^{\tau_0}}$ iff

$$\widetilde{\Phi}(\tau, \varphi) = 0, \qquad \forall \tau \geq \tau_0,\ \varphi \in \mathbb{R}.$$

Recall that by (1.11) with $R = 1$ for every open subset $\Omega \subseteq \mathbb{C}$ the energy density of a map $w = (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ is defined by

$$e_w := \frac{1}{2}\Big(|\partial_s u + X_\Phi \circ u|^2 + |\partial_t u + X_\Psi \circ u|^2 + |\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi]|^2 + |\mu \circ u|^2\Big),$$

and recall from (1.13) with $R = 1$ that

$$E(w, X) := \int_X e_w \, ds\, dt$$

denotes the energy of w on the measurable subset $X \subseteq \Omega$. Recall that $\widetilde{\mathcal{M}}^{1,p}$ denotes the set of all solutions $w = (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ of the vortex equations (0.1), (0.2) such that $E(w) < \infty$ and the subset $\overline{u(\mathbb{C})} \subseteq M$ is compact. The next proposition makes it possible to define an evaluation map $\overline{\mathrm{ev}}_\infty : \mathcal{M} := \widetilde{\mathcal{M}}^{1,p}/\mathcal{G}^{2,p} \to \bar{M}$, see Definition 3.1. Its proof is based on Proposition 11.1. in [GS] and Proposition D.2. Part (C) is a refinement of an estimate in Proposition 11.1 in [GS].

Proposition D.6 (Asymptotic behaviour) *Let (M,ω) be a symplectic manifold, G be a compact connected Lie group that acts on M in a Hamiltonian way with moment map μ, and let J be a G-invariant ω-compatible almost complex structure on M. Assume that hypothesis (H1) is satisfied, i.e. that G acts freely on $\mu^{-1}(0)$ and μ is proper. Then the following statements hold.*

(A) For every finite energy vortex $w=(u,\Phi,\Psi)\in\widetilde{\mathcal{M}}^{1,p}$ the map $\mathrm{G}u:\mathbb{C}\to M/\mathrm{G}$ extends continuously to a map $\bar{u}:S^2=\mathbb{C}\cup\{\infty\}\to M/\mathrm{G}$, such that $\bar{u}(\infty)\in\bar{M}=\mu^{-1}(0)/\mathrm{G}$.

(B) Assume that (H2) (convexity at ∞) holds. Then there is a G-invariant compact subset $K_0\subseteq M$ such that for every vortex $w=(u,\Phi,\Psi)\in\widetilde{\mathcal{M}}^{1,p}$ we have $u(\mathbb{C})\subseteq K_0$.

(C) Assume that (H2) holds. Then there are constants $E>0$, $C>0$ and $\delta>0$ such that the following holds. For every vortex $w\in\widetilde{\mathcal{M}}^{1,p}$, every $R\geq 1$ such that

$$E(w,\mathbb{C}\setminus B_R)<E \tag{D.10}$$

and every $z\in\mathbb{C}\setminus B_{2R}$ we have

$$e_w(z)\leq CR^{\delta}|z|^{-2-\delta}. \tag{D.11}$$

For the proof of Proposition D.6 we need the following. We identify $S^1\cong\mathbb{R}/(2\pi\mathbb{Z})$.

Proposition D.7 *(A) Let $w:=(u,\Phi,\Psi):\mathbb{C}\to M\times\mathfrak{g}\times\mathfrak{g}$ be a smooth finite energy vortex in radial gauge outside B_{r_0} for some number $r_0>0$, and assume that $\overline{u(\mathbb{C})}$ is compact. Then the function $u^*\omega:\mathbb{C}\to\mathbb{R}$ is Lebesgue integrable and*

$$E(w)=\int_{\mathbb{C}}u^*\omega. \tag{D.12}$$

(B) There exists a number $\delta>0$ such that the following holds. Let $2\leq p<4/(2-\delta)$ be a real number. Let $w:=(u,\Phi,\Psi):\mathbb{C}\to M\times\mathfrak{g}\times\mathfrak{g}$ be a smooth finite energy vortex in radial gauge outside B_{r_0} for some number $r_0>0$ and assume that $\overline{u(\mathbb{C})}$ is compact. Then the limit

$$x_0:=\lim_{r\to\infty}u(r) \tag{D.13}$$

exists and lies in $\mu^{-1}(0)$, and there exists a map $g_0\in W^{1,p}(S^1,\mathrm{G})$ such that

$$\lim_{r\to\infty}\max_{\varphi\in S^1} d(u(re^{i\varphi}), g_0(\varphi)x_0) \;=\; 0, \tag{D.14}$$

$$\sup_{\tau_0\geq 0} ||\dot g_0 g_0^{-1} + \widetilde\Psi(\tau_0,\cdot)||_{L^p(S^1)} e^{\left(-1+\frac{2}{p}+\frac{\delta}{2}\right)\tau_0} \;<\; \infty. \tag{D.15}$$

Proof of Proposition D.7: Statement **(A)** is part of the statement of Proposition 11.1 in [GS].

Proof of (B): We fix $\delta_0 > 0$ as in (A.17) and choose $\delta > 0$ to be a number less than the constant $1/c$, where $c > 0$ is as in Lemma A.15 (Isoperimetric inequality), depending on δ_0. Let $2 \leq p < 4/(2-\delta)$ and let $w := (u,\Phi,\Psi)$ be a vortex as in the hypothesis, and let $r_0 > 0$ be so large that w is in radial gauge outside B_{r_0}. As in Step 6 in the proof of Proposition 11.1 in [GS] we have

$$C_0 := \sup_{(\tau,\varphi)\in[0,\infty)\times S^1} \big(|\partial_\tau \widetilde u(\tau,\varphi)| + e^\tau|\mu\circ\widetilde u(\tau,\varphi)|\big)e^{\frac{\delta}{2}\tau} < \infty. \tag{D.16}$$

Claim 1

$$C_1 := \sup_{\tau_0\geq\log r_0} e^{\left(p+2+\frac{\delta p}{2}\right)(\tau_0+1)} \int_{\tau_0}^{\tau_0+1}\int_0^{2\pi} |\mu\circ\widetilde u|^p d\varphi\, d\tau < \infty. \tag{D.17}$$

The proof of Claim 1 is analogous to part of the proof of the second estimate in Step 6 in the proof of Proposition 11.1 in [GS]. For the convenience of the reader I repeat it here.

Proof of Claim 1: By Lemma C.4 with C_0, $\Omega := (-1,2)\times(-\pi,3\pi)$, $Q := [0,1]\times[0,2\pi]$ and $\lambda(\tau,\varphi) := e^\tau$ there exist constants $R_0 > 0$ and $C > 0$ such that the following holds. If $R \geq R_0$ and $\widehat w := (\widehat u,\widehat\Phi,\widehat\Psi)$ is a smooth solution of the equations

$$\partial_\tau\widehat u + X_{\widehat\Phi}(\widehat u) + J(\partial_\varphi\widehat u + X_{\widehat\Psi}(\widehat u)) \;=\; 0, \tag{D.18}$$

$$\partial_\tau\widehat\Psi - \partial_\varphi\widehat\Phi + [\widehat\Phi,\widehat\Psi] + R^2e^{2\tau}\mu\circ\widehat u \;=\; 0 \tag{D.19}$$

on $(-1,2)\times(-\pi,3\pi)$, such that

$$\sup_{(-1,2)\times(-\pi,3\pi)} \big(|\partial_\tau\widehat u + X_{\widehat\Phi}(\widehat u)| + R|\mu\circ\widehat u|\big) \leq C_0, \tag{D.20}$$

then

$$||\mu\circ\widehat u||_{L^p([0,1]\times[0,2\pi])} \;\leq\; CR^{-1-\frac{2}{p}}\Big(||\partial_\tau\widehat u + X_{\widehat\Phi}(\widehat u)||_{L^2\left((-1,2)\times(-\pi,3\pi)\right)} + R||\mu\circ\widehat u||_{L^2\left((-1,2)\times(-\pi,3\pi)\right)}\Big). \tag{D.21}$$

Proposition B.3 implies that for $\tau_0 \in \mathbb{R}$ the shifted map $\widehat{w} := \widetilde{w}(\tau_0 + \cdot, \cdot)$ satisfies (D.18), (D.19) with $R := e^{\tau_0}$. Furthermore, (D.20) holds with $R := e^{\tau_0}$, for $\tau_0 \geq 1$. Since $\widetilde{\Phi}(\tau_0, \varphi) = 0$ for $\tau_0 \geq \log(r_0)$, it follows that for τ_0 large enough

$$\begin{aligned} & e^{(p+2)\tau_0} \int_{\tau_0}^{\tau_0+1} \int_0^{2\pi} |\mu \circ \widetilde{u}|^p d\varphi \, d\tau \\ \leq\ & 2^{\frac{p}{2}} C^p \left(\int_{\tau_0-1}^{\tau_0+2} \int_{-\pi}^{3\pi} \left(|\partial_\tau \widetilde{u}|^2 + e^{2\tau_0} |\mu \circ \widetilde{u}|^2 \right) d\varphi \, d\tau \right)^{\frac{p}{2}}. \end{aligned} \tag{D.22}$$

On the other hand, as in Step 5 of the proof of Proposition 11.1 in [GS], we have

$$\sup_{\tau_0 \geq \log(r_0)} e^{\delta \tau_0} \int_{\tau_0}^{\infty} \int_0^{2\pi} \left(|\partial_\tau \widetilde{u}|^2 + e^{2\tau} |\mu \circ \widetilde{u}|^2 \right) d\varphi \, d\tau < \infty.$$

Estimate (D.17) follows from this and (D.22). This proves Claim 1. □

For $\tau \geq \log r_0$ we have

$$\partial_\tau \widetilde{\Psi} = \widetilde{\kappa} := \partial_\tau \widetilde{\Psi} - \partial_\varphi \widetilde{\Phi} + [\widetilde{\Phi}, \widetilde{\Psi}] = -e^{2\tau} \mu \circ \widetilde{u}, \tag{D.23}$$

where in the last equality we have used the second vortex equation (0.2) and Proposition B.3. We fix real numbers $\tau_0 \geq \log r_0$ and $\tau_0 \leq \tau_1 \leq \tau_0 + 1$, and define $C_1 > 0$ as in Claim 1. Equality (D.23) and Hölder's inequality imply that

$$\begin{aligned} |\widetilde{\Psi}(\tau_1, \varphi) - \widetilde{\Psi}(\tau_0, \varphi)|^p &= \left| \int_{\tau_0}^{\tau_1} \partial_\tau \widetilde{\Psi}(\tau, \varphi) d\tau \right|^p \\ &\leq \int_{\tau_0}^{\tau_1} e^{2p\tau} |\mu \circ \widetilde{u}(\tau, \varphi)|^p \, d\tau \\ &\leq e^{2p\tau_1} \int_{\tau_0}^{\tau_1} |\mu \circ \widetilde{u}(\tau, \varphi)|^p \, d\tau. \end{aligned}$$

Defining $\varepsilon := -1 + 2/p + \delta/2$, it follows that

$$\begin{aligned} \int_0^{2\pi} |\widetilde{\Psi}(\tau_1, \varphi) - \widetilde{\Psi}(\tau_0, \varphi)|^p d\varphi &\leq e^{-p\varepsilon\tau_1} e^{\left(p+2+\frac{\delta p}{2}\right)\tau_1} \int_{\tau_0}^{\tau_1} \int_0^{2\pi} |\mu \circ \widetilde{u}(\tau, \varphi)|^p d\varphi, d\tau \\ &\leq C_1 e^{-p\varepsilon\tau_1}. \end{aligned} \tag{D.24}$$

Here in the second estimate we have used (D.17). Since $p < 4/(2-\delta)$, the number ε is positive. Hence (D.24) implies that for every number $\tau_0 \geq \log r_0$ and every positive integer k

$$\begin{aligned} ||\widetilde{\Psi}(\tau_0, \cdot) - \widetilde{\Psi}(\tau_0 + k, \cdot)||_{L^p(S^1)} &\leq \sum_{j=0}^{k-1} ||\widetilde{\Psi}(\tau_0 + j, \cdot) - \widetilde{\Psi}(\tau_0 + j + 1, \cdot)||_{L^p(S^1)} \\ &\leq C_1^{\frac{1}{p}} \sum_{j=0}^{k-1} e^{-\varepsilon(\tau_0 + j)} \\ &< \frac{C_1^{\frac{1}{p}}}{1 - e^{-\varepsilon}} e^{-\varepsilon\tau_0}. \end{aligned} \tag{D.25}$$

Hence for every $\tau_0 \geq \log r_0$ the sequence of maps $\widetilde{\Psi}(\tau_0 + k, \cdot) : S^1 \to \mathfrak{g}$ converges in $L^p(S^1, \mathfrak{g})$ to some map $\xi \in L^p(S^1, \mathfrak{g})$. Inequality (D.24) implies that the limit ξ does not depend on the choice of τ_0. Estimate (D.16) implies that for $\tau_1 \geq \tau_0 \geq \log(r_0)$ and $\varphi \in S^1$ we have

$$d\big(\widetilde{u}(\tau_1, \varphi), \widetilde{u}(\tau_0, \varphi)\big) \leq \int_{\tau_0}^{\tau_1} |\partial_\tau \widetilde{u}(\tau, \varphi)| \, d\tau \leq \frac{2C_0}{\delta} e^{-\frac{\delta}{2}\tau_0}.$$

It follows that for $\tau_0 \to \infty$ the sequence $\widetilde{u}(\tau_0, \cdot)$ converges uniformly to some continuous map $x : S^1 \to M$. Again by estimate (D.16) the limit x takes values in $\mu^{-1}(0)$. Furthermore, by the Whitney embedding theorem (see the book by Hirsch [Hi], Theorem 3.2.14 p. 55), there exists an embedding $\iota : M \to \mathbb{R}^{2\dim M + 1}$. Since w is in radial gauge outside B_{r_0}, it follows that on $[\log(r_0), \infty) \times S^1$

$$\begin{aligned} \partial_\varphi(\iota \circ \widetilde{u}) &= d\iota(\widetilde{u})\partial_\varphi \widetilde{u} \\ &= d\iota(\widetilde{u})\big(\partial_\varphi \widetilde{u} + X_{\widetilde{\Psi}}(\widetilde{u}) - X_{\widetilde{\Psi}}(\widetilde{u})\big) \\ &= d\iota(\widetilde{u})(J\partial_\tau \widetilde{u} - X_{\widetilde{\Psi}}(\widetilde{u})). \end{aligned} \tag{D.26}$$

Since as $\tau_0 \to \infty$ the maps $\widetilde{u}(\tau_0, \cdot)$ converge uniformly to x and the maps $\widetilde{\Psi}(\tau_0, \cdot)$ converge to ξ in $L^p(S^1)$, it follows that the maps $\big(d\iota(\widetilde{u})X_{\widetilde{\Psi}}(\widetilde{u})\big)(\tau_0, \cdot)$ converge to $d\iota(x)X_\xi(x)$ in $L^p(S^1, \mathfrak{g})$. Furthermore, estimate (D.16) implies that $\partial_\tau \widetilde{u}(\tau_0, \cdot)$ converges to 0 in $L^\infty(S^1)$, as $\tau_0 \to \infty$. Hence (D.26) implies that $\partial_\varphi(\iota \circ \widetilde{u})(\tau_0, \cdot)$ converges to $-X_\xi(x)$ in $L^p(S^1)$. It follows that $x \in W^{1,p}(S^1)$, $\widetilde{u}(\tau_0, \cdot)$ converges to x in $W^{1,p}(S^1)$, as $\tau_0 \to \infty$, and that

$$\dot{x} = -X_\xi(x), \tag{D.27}$$

almost everywhere on S^1. We denote by $\pi : \mu^{-1}(0) \to \bar{M} = \mu^{-1}(0)/\mathrm{G}$ the projection to the quotient. Equality (D.27) implies that the map $\pi \circ x : S^1 \to \bar{M}$ satisfies

$$(\pi \circ x)^{\cdot} = d\pi(x)\dot{x} = -d\pi(x)X_\xi(x) = 0$$

almost everywhere on S^1. It follows that $\pi \circ x \equiv \mathrm{G}x(0)$, so the image of x is contained in the orbit $\mathrm{G}x(0)$. For $\varphi \in S^1$ we define $g_0(\varphi) \in \mathrm{G}$ to be the unique element such that

$$g_0(\varphi)x(0) = x(\varphi).$$

Since G acts freely on $\mu^{-1}(0)$ the map $g_0 : S^1 \to \mathrm{G}$ is welldefined. Furthermore, $g_0 \in W^{1,p}(S^1, \mathrm{G})$, since by the Local Slice Theorem A.5 the map

$$\mathrm{G} \to \mathrm{G}x(0), \qquad g \mapsto gx(0)$$

is an embedding, and g_0 is the composition of x with the inverse of this map.

We check the conditions of statement (B). We have already proved that the limit

$$x_0 := \lim_{r\to\infty} u(r) = \lim_{\tau_0\to\infty} \widetilde{u}(\tau_0, 0) = x(0)$$

exists and lies in $\mu^{-1}(0)$. Furthermore, condition (D.14) follows from the uniform convergence of the map $\widetilde{u}(\tau_0,\cdot)$ against the map $x = g_0 x_0$. We prove (D.15). Since the map $\widetilde{\Psi}(\tau_0,\cdot)$ converges to ξ in $L^p(S^1,\mathfrak{g})$, for $\tau_0 \to \infty$, inequality (D.25) implies that

$$\sup_{\tau_0\geq 0} ||\widetilde{\Psi}(\tau_0,\cdot) - \xi||_{L^p(S^1)} e^{\varepsilon\tau_0} < \infty.$$

Therefore, it suffices to prove that

$$\dot{g}_0 g_0^{-1} = -\xi, \tag{D.28}$$

almost everywhere on S^1. To see this, observe that by (D.27) we have

$$\begin{aligned} -X_\xi \circ x|_{\varphi_0} &= \dot{x}(\varphi_0) \\ &= \frac{d}{d\varphi}\Big|_{\varphi=\varphi_0} \big(g_0(\varphi)g_0(\varphi_0)^{-1}x(\varphi_0)\big) \\ &= X_{\dot{g}_0 g_0^{-1}}(x)|_{\varphi_0}, \end{aligned} \tag{D.29}$$

for almost every $\varphi_0 \in S^1$. Since G acts freely on $\mu^{-1}(0)$, it follows from Lemma A.4 that the map $L_{x(\varphi)} : \mathfrak{g} \to T_{x(\varphi)}M$ is injective for every $\varphi \in S^1$. Hence (D.28) follows from (D.29). This proves statement **(B)** and completes the proof of Proposition D.7. □

Proof of Proposition D.6: Let $M, \omega, \mathrm{G}, \mu, J$ be as in the hypothesis.

Proof of Part (A): Let $w = (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$ be a vortex. By Proposition D.3 there is a gauge transformation $g \in \mathcal{G}^{2,p}$ such that $w' := (u', \Phi', \Psi') := g^*w$ is smooth and in radial gauge outside B_1. Hence by Proposition B the limit $x_0 := \lim_{r\to\infty} u'(r)$ exists and lies in $\mu^{-1}(0)$, and there exists a map $g_0 \in W^{1,2}(S^1, \mathrm{G})$ such that

$$\lim_{r\to\infty} \max_{z\in S^1} d(u'(rz), g_0(z)x_0) = 0. \tag{D.30}$$

Recall the definition (4.42) of the distance function $\bar{d}$ on the connected components of M/G. We have

$$\bar{d}(\mathrm{G}u(rz), \mathrm{G}x_0) = \bar{d}(\mathrm{G}u'(rz), \mathrm{G}g_0(z)x_0) \leq d(u'(rz), g_0(z)x_0),$$

for every $r > 0$, $z \in S^1$. Hence (D.30) implies that

$$\lim_{r\to\infty} \max_{z\in S^1} \bar{d}(\mathrm{G}u(rz), \mathrm{G}x_0) = 0,$$

and therefore we can extend the map $\mathrm{G}u : \mathbb{C} \to M/\mathrm{G}$ continuously to a map $\bar{u} : S^2 \cong \mathbb{C} \cup \{\infty\} \to M/\mathrm{G}$. This proves Part (A).

Proof of Part (B): We choose a function $f : M \to [0, \infty)$ and a constant $c > 0$ as in hypothesis (H2). By hypothesis (H1) the map μ is proper and therefore the set $\mu^{-1}(0)$ is compact. Thus $f(\mu^{-1}(0)) \subseteq [0, \infty)$ is compact, and increasing c if necessary, we may assume w.l.o.g. that $f(\mu^{-1}(0)) \subseteq [0, c]$. We define $K_0 := f^{-1}([0, c]) \subseteq M$. Since f is proper, the set K_0 is compact, and since f is G-invariant, K_0 is G-invariant. Let $w = (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$ be a vortex. We choose a gauge transformation $g \in \mathcal{G}^{2,p}$ such that $w' := (u', \Phi', \Psi') := g^*w$ is smooth and in radial gauge outside B_1, as in Proposition D.3. By Proposition 11.1. in [GS] we have for every $z \in \mathbb{C}$

$$f \circ u(z) = f \circ u'(z) \le c,$$

and hence $u(\mathbb{C}) \subseteq K_0 = f^{-1}([0, c])$.

Proof of Part (C): We fix a G-invariant compact subset $K_0 \subseteq M$ as in Part (B), and constants $E_0 > 0$ as in Lemma C.1 and $E_1, C_1, a > 0$ as in Lemma 4.11, corresponding to $K := K_0$ and $r_0 := 1/2$. We set

$$E := \min\{E_0, E_1\}, \qquad C := \frac{32C_1}{\pi}, \qquad \delta := a.$$

Let $w \in \widetilde{\mathcal{M}}^{1,p}$ be a vortex and $R \ge 1$ be a number such that (D.10) is satisfied, and let $z \in \mathbb{C} \setminus B_{2R}$ be a point. Since $B_{|z|/2}(z) \subseteq \mathbb{C} \setminus B_R$, hypothesis (D.10) implies that

$$E(w, B_{|z|/2}(z)) \le E(w, \mathbb{C} \setminus B_R) < E \le E_0.$$

Hence Lemma C.1 implies that

$$e_w(z) \le \frac{8E(w, B_{|z|/2}(z))}{\pi(|z|/2)^2} \le \frac{32E(w, \mathbb{C} \setminus B_{|z|/2}(0))}{\pi|z|^2}. \tag{D.31}$$

By Lemma 4.11 with $r := R/2$, $\lambda := |z|/R$ and R replaced by ∞ we have

$$E(w, \mathbb{C} \setminus B_{|z|/2}) \le \frac{C_1 R^\delta E(w, \mathbb{C} \setminus B_{R/2})}{|z|^\delta}.$$

This together with (D.31) and hypothesis (D.10) implies that

$$e_w(z) \le \frac{CR^\delta E}{|z|^{2+\delta}}.$$

This proves inequality (D.11) and hence assertion (C).

This completes the proof of Proposition D.6. □

D.4 The action of $\mathcal{G}^{2,p}$ and of $\mathrm{Isom}^+(\mathbb{R}^2)$

Recall that the gauge group $\mathcal{G}^{2,p}$ acts on the set $\widetilde{\mathcal{M}}^{1,p}$ of finite energy vortices (u, Φ, Ψ) on $\mathbb{C}$ such that $\overline{u(\mathbb{C})}$ is compact, by the formula

$$g^*(u, \Phi, \Psi) := \big(g^{-1}u, g^{-1}(\Phi + \partial_s)g, g^{-1}(\Psi + \partial_t)g\big). \tag{D.32}$$

Lemma D.8 *The action (D.32) is free.*

Proof: Assume that $w := (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$ and $g \in \mathcal{G}^{2,p}$ are such that $g^*w = w$. We show that $g \equiv e$. By Proposition D.3 there is $h \in \mathcal{G}^{2,p}$ such that $w_1 := (u_1, \Phi_1, \Psi_1) := h^*w$ is smooth. We define $g_1 := h^{-1}gh \in \mathcal{G}^{2,p}$. Then $g_1^* w_1 = w_1$, i.e.

$$g_1^{-1}u_1 = u_1 \tag{D.33}$$

$$\partial_s g_1 = g_1\Phi_1 - \Phi_1 g_1, \qquad \partial_t g_1 = g_1\Psi_1 - \Psi_1 g_1. \tag{D.34}$$

Let $\delta_0 > 0$ be so small that G acts freely on $M_{\delta_0} := \{x \in M \,|\, |\mu(x)| \leq \delta_0\}$. Proposition D.6(C) implies that there is a number $R > 0$ such that $u_1(z) \in M_{\delta_0}$ if $|z| \geq R$. By (D.33) it follows that $g_1(z) = e$ if $|z| \geq R$. Now let $s_0 + it_0 \in B_R$. By (D.34) the map $g_1(s_0 + \cdot, t_0)$ is an integral curve of the smooth vector field X on G defined by $X(g) := g\Phi_1 - \Phi_1 g \in T_g\mathrm{G}$. Since $|z_0 - 2R| > R$ the map $g_1(s_0 + \cdot, t_0)$ satisfies the initial condition $g_1(s_0 - 2R, t_0) = e$. Since the constant map e is an integral curve of X on $[-2R, 0]$ with the same initial condition if follows that $g_1(s_0 + \cdot, t_0) \equiv e$. Since this holds for every $s_0 + it_0 \in B_R$ it follows that $g_1 \equiv e$ on all of $\mathbb{C}$ and therefore $g = hg_1h^{-1} \equiv e$. This proves Lemma D.8. □

A rigid motion of $\mathbb{C} = \mathbb{R}^2$ equipped with the Euclidian metric is a map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $\varphi(z) = e^{i\theta}z + z_0$, where $\theta \in \mathbb{R}$ and $z_0 \in \mathbb{C}$. We denote the group of rigid motions by $\mathrm{Isom}^+(\mathbb{R}^2)$. It acts on $\widetilde{\mathcal{M}}^{1,p}$ by the formula

$$\begin{aligned}\varphi^*(u, \Phi, \Psi)(z) \;:=\; &\big(u \circ \varphi(z), \cos(\theta)\Phi \circ \varphi(z) + \sin(\theta)\Psi \circ \varphi(z),\\ &- \sin(\theta)\Phi \circ \varphi(z) + \cos(\theta)\Psi \circ \varphi(z)\big)\end{aligned} \tag{D.35}$$

By Corollary B.7 this induces an action of $\mathrm{Isom}^+(\mathbb{R}^2)$ on $\mathcal{M} = \widetilde{\mathcal{M}}^{1,p}/\mathcal{G}^{2,p}$ defined by

$$\varphi^*[w] := [\varphi^* w]. \tag{D.36}$$

Consider the restriction of this action to the subgroup of $\mathrm{Isom}^+(\mathbb{R}^2)$ consisting of the translations on $\mathbb{C}$. If $T : \mathbb{C} \to \mathbb{C}$ is a translation and $[w] \in \mathcal{M}$ then the action of T on $[w]$ is given by

$$T^*[w] = [w \circ T]. \tag{D.37}$$

Lemma D.9 *The restriction of the action of the group of translations on* $\mathbb{C}$ *to the subset*

$$\mathcal{M}_{>0} := \big\{ w \in \widetilde{\mathcal{M}}^{1,p} \,\big|\, E(w) > 0 \big\} / \mathcal{G}^{2,p} \subseteq \mathcal{M}$$

is free.

Proof of Lemma D.9: Let $0 \neq \zeta \in \mathbb{C}$ be a vector and $T : \mathbb{C} \to \mathbb{C}$ be the translation by ζ defined by $T(z) := z + \zeta$ and let $w := (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ be a vortex of positive energy such that $\overline{u(\mathbb{C})}$ is compact. Assume that $T^* w = w \circ T$ is gauge equivalent to w. The lemma is a consequence of the following claim.

Claim 1 *We have*

$$E(w) = \infty.$$

Proof.: By our assumption there is a gauge transformation $g \in \mathcal{G}^{2,p}$ such that $T^* w = w \circ T = g^* w$. We fix $k \in \mathbb{N}$ and write $T^k := T \circ \ldots \circ T$. It follows from Proposition B.6 that

$$\begin{aligned}
(T^k)^* w &= (T^*)^k w \\
&= (T^*)^{k-1} (g^* w) \\
&= (T^*)^{k-2} (g \circ T)^* T^* w \\
&= (T^*)^{k-2} (g \circ T)^* g^* w \\
&= \ldots \\
&= (g \circ T^{k-1})^* \cdots g^* w = h^* w,
\end{aligned}$$

where $h := g \cdots (g \circ T^{k-1}) \in \mathcal{G}^{2,p}$. This implies for the energy density

$$e_{w \circ T^k} = e_{(T^k)^* w} = e_{h^* w} = e_w.$$

Here the last equality follows from Lemma B.1. On the other hand, it follows from the definition of the energy density that $e_w \circ T^k = e_{w \circ T^k}$. Thus for every number $k \in \mathbb{N}$ and every measurable subset $S \subseteq \mathbb{C}$, writing $S - k\zeta := \{ z - k\zeta \,|\, z \in S \}$, we have

$$\begin{aligned}
E(w, S - k\zeta) &= \int_{S - k\zeta} e_w \, ds \, dt \\
&= \int_{S - k\zeta} e_w \circ T^k \, ds \, dt \\
&= \int_S e_w \, ds \, dt = E(w, S). \qquad \text{(D.38)}
\end{aligned}$$

Since $E(w) > 0$ there is a point $z_0 \in \mathbb{C}$ such that $E(w, B_{|\zeta|/2}(z_0)) > 0$. Since the balls $B_{|\zeta|/2}(z_0 - k\zeta)$ with $k \in \mathbb{N}$ are all disjoint, by (D.38) applied with $S := B_{|\zeta|/2}(z_0)$ we get

$$\begin{aligned} E(w) &\geq \sum_{k \in \mathbb{N}} \int_{B_{|\zeta|/2}(z_0 - k\zeta)} e_w \, ds \, dt \\ &= \sum_{k \in \mathbb{N}} E(w, B_{|\zeta|/2}(z_0)) = \infty. \end{aligned}$$

This proves Claim 1 and concludes the proof of Lemma D.9. □

The action of the whole group $\mathrm{Isom}^+(\mathbb{R}^2)$ on $\mathcal{M}_{>0}$ is in general not free, as the following example shows.

Example D.10 Consider the action of $\mathrm{G} := S^1 \subseteq \mathbb{C}$ on $M := \mathbb{C}$ by multiplication. Let d be a nonnegative integer. By Proposition D.22 there is a finite energy vortex $w_0 = (u_0, \Phi_0, \Psi_0)$ on $\mathbb{C}$ such that

$$u_0(z) \neq 0, \qquad \forall z \neq 0 \tag{D.39}$$

$$\deg\left(\tfrac{u_0}{|u_0|} : S^1 \to S^1\right) = d. \tag{D.40}$$

Furthermore, every other w with these properties is gauge equivalent to w_0. Note that it follows that if $d \geq 1$ then $u_0(0) = 0$. On the other hand, if $d = 0$ then w_0 is gauge equivalent to the constant map $(1, 0, 0) : \mathbb{C} \to \mu^{-1}(0) \times \mathrm{Lie}(S^1) \times \mathrm{Lie}(S^1) = S^1 \times i\mathbb{R} \times i\mathbb{R}$. Let $R \in \mathrm{SO}(2)$ be a rotation. Then the vortex $w := R^* w_0$ also solves the conditions (D.39), (D.40) and therefore it is gauge equivalent to w_0. Thus the restricted action of the subgroup $\mathrm{SO}(2) \subseteq \mathrm{Isom}^+(\mathbb{R}^2)$ on $\mathcal{M}_{>0}$ is not free and thus the action of $\mathrm{Isom}^+(\mathbb{R}^2)$ on $\mathcal{M}_{>0}$ is not free.
More generally, for every $z_0 \in \mathbb{C}$ and $d \in \mathbb{N} \sqcup \{0\}$ let $w_0 := (u_0, \Phi_0, \Psi_0)$ be a finite energy vortex such that $u(z) \neq 0$ for every $z \neq z_0$ and

$$\deg\left(\frac{u_0}{|u_0|} : S^1(z_0) \to S^1\right) = d.$$

For every $\zeta \in \mathbb{C}$ we denote by $T_\zeta : \mathbb{C} \to \mathbb{C}$ the translation by ζ. Then by an analogous argument as above, for every $R \in \mathrm{SO}(2)$ the isometries $T_{z_0} R T_{z_0}^{-1} \in \mathrm{Isom}^+(\mathbb{R}^2)$ fix the map w_0.

If $d \geq 2$ then the choice of the vortex $w_0 = (u_0, \Phi_0, \Psi_0)$ above is very special. The map u_0 vanishes only at the point z_0 and the "multiplicity" or local degree of z_0 equals d. As opposed to that, for a "generic" vortex (u, Φ, Ψ) the map u vanishes at d distinct points $z_1, \ldots, z_d$, each of "multiplicity" one. The choice of w_0 in the cases $d = 0$ and $d = 1$ is however the most general one.

It would be interesting to know in the general case under what conditions on $M, \omega, \mathrm{G}, \mu, J$ and the homology class $B \in H_2^{\mathrm{G}}(M, \mathbb{Z})$ the group $\mathrm{Isom}^+(\mathbb{R}^2)$ acts freely on the set $\mathcal{M}_B \subseteq \mathcal{M}$ of gauge equivalence classes of vortices representing B.

D.5 Vortices lie in $\mathcal{B}^p_\lambda$

In section 2 I introduced an abstract setting for the vortex equations. We interpreted them as a section of some "infinite dimensional vector bundle" over some "infinite dimensional manifold" $\mathcal{B}^p_\lambda$. The purpose of the Proposition below is to justify the definition of $\mathcal{B}^p_\lambda$ by showing that for good choices of p and λ the finite energy vortices (u, Φ, Ψ) for which $\overline{u(\mathbb{C})}$ is compact indeed lie in $\mathcal{B}^p_\lambda$.

For every pair of numbers $p > 2$ and $\lambda > -2/p + 1$ recall the definition of the set

$$\begin{aligned} \mathcal{B}^p_\lambda \;:=\; & \Big\{(u, \Phi + i\Psi) \in C^\infty(\mathbb{C}, M \times \mathfrak{g}^{\mathbb{C}}) \,\Big|\, \overline{u(\mathbb{C})} \text{ is compact}, \\ & |\mu \circ u(re^{i\varphi})| \to 0,\ r \to \infty\ \forall \varphi,\ ||d^{\Phi,\Psi} u|| \in L^p_\lambda, \\ & \exists g \in C^\infty(\mathbb{C} \setminus B_1, \mathrm{G}) :\ g^*(\Phi + i\Psi) \in L^p_\lambda(\mathbb{C} \setminus B_1, \mathfrak{g}^{\mathbb{C}})\Big\}. \end{aligned} \tag{D.41}$$

Recall that $\widetilde{\mathcal{M}}$ denotes the set of all smooth finite energy solution $w := (u, \Phi + i\Psi) : \mathbb{C} \to M \times \mathfrak{g}^{\mathbb{C}}$ of the vortex equations (0.1) and (0.2) for which $\overline{u(\mathbb{C})}$ is compact.

Proposition D.11 *There exists a real number $p_0 > 2$ such that if $2 < p < p_0$ and $-2/p + 1 < \lambda < -2/p_0 + 1$ then*

$$\widetilde{\mathcal{M}} \subseteq \mathcal{B}^p_\lambda.$$

Remark D.12 If $2 < p \le q$ and λ and μ are real numbers such that $\lambda + 2/p < \mu + 2/q$ then $\mathcal{B}^q_\mu \subseteq \mathcal{B}^p_\lambda$. This follows from part (i) of Proposition E.6, which states that there exists a constant $C > 0$ such that for every measurable function $f : \mathbb{C} \to \mathbb{C}$

$$||f||_{L^p_\lambda(\mathbb{C})} \le C||f||_{L^q_\mu(\mathbb{C})}.$$

It follows that if $q > 2$ and $\mu > -2/q+1$ are such that $\widetilde{\mathcal{M}} \subseteq \mathcal{B}^q_\mu$ then $\widetilde{\mathcal{M}}$ is also contained in $\mathcal{B}^p_\lambda$, for every pair (p, λ) with $2 < p \le q$ and $\lambda + 2/p < \mu + 2/q$.

Remark D.13 Using the Uhlenbeck gauge Theorem, it should be possible to prove a stronger version of Proposition D.11 saying that there exists a number $\varepsilon > 0$ such that for every $p > 2$ and every $-2/p + 1 < \lambda < -2/p + 1 + \varepsilon$ we have $\widetilde{\mathcal{M}} \subseteq \mathcal{B}^p_\lambda$. (Setting $p_0 := 2/(1 - \varepsilon)$ this would imply the proposition.)

Proof of Proposition D.11: Let $E > 0$, $C_1 := C > 0$ and $\delta > 0$ be as in part (C) of Proposition D.6 (Asymptotic behaviour). Decreasing δ we may assume that $\delta < 2$ and that it satisfies also the condition of part (B) of Proposition D.7. We define

$$p_0 := \frac{4}{2-\delta}.$$

Let

$$2 < p < p_0, \qquad -\frac{2}{p} + 1 < \lambda < -\frac{2}{p_0} + 1,$$

and let $w := (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}$. We abbreviate

$$\kappa := \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi].$$

Recall the definition of the energy density of w

$$e_w := \frac{1}{2}\big(|d^{\Phi,\Psi}u|^2 + |\mu \circ u|^2 + |\kappa|^2\big).$$

Let $R \geq 1$ be so large that $E(w, \mathbb{C} \setminus B_R) < E$. Then by the statement in part (C) of Proposition D.6 we have for every $z \in \mathbb{C} \setminus B_{2R}$

$$e_w(z) \leq C_1 R^\delta |z|^{-2-\delta}.$$

This implies that $|\mu \circ u(re^{i\varphi})| \to 0$ for $r \to \infty$, uniformly in $\varphi \in \mathbb{R}$. Furthermore, since $\lambda < -2/p_0 + 1 = \delta/2$ and $p > 2$, we have

$$||d^{\Phi,\Psi}u||^p_{p,\lambda} \leq C_1^{\frac{p}{2}} R^{\frac{\delta p}{2}} \int_{\mathbb{C}} \big(\langle\cdot\rangle^{-1-\frac{\delta}{2}} \langle\cdot\rangle^{\frac{\delta}{2}}\big)^p \, ds\, dt < \infty.$$

We prove that there exists a $g \in C^\infty(\mathbb{C} \setminus B_1, \mathrm{G})$ such that $g^*(\Phi + i\Psi) \in L^p_\lambda(\mathbb{C} \setminus B_1)$. By Proposition D.3 there exists a gauge transformation $g_0 \in \mathcal{G}^{2,p} = W^{2,p}_{\mathrm{loc}}(\mathbb{C}, \mathrm{G})$ such that $w_0 := (u_0, \Phi_0 + \Psi_0) := g_0^* w$ is smooth and in radial gauge outside the ball B_1. Recall that the radial gauge condition means that defining $\widetilde{\Phi}_0 + i\widetilde{\Psi}_0 : \mathbb{C} \to \mathfrak{g}^{\mathbb{C}}$ by

$$(\widetilde{\Phi}_0 + i\widetilde{\Psi}_0)(\tau + i\varphi) := e^{\tau - i\varphi}(\Phi_0 + i\Psi_0)(e^{\tau + i\varphi}),$$

we have $\widetilde{\Phi}_0 = 0$ on $[0, \infty) + i\mathbb{R} \subseteq \mathbb{C}$. Since w and $w_0 = g_0^* w$ are smooth, by Lemma B.2(i) g_0 is smooth as well. We identify $S^1 \cong \mathbb{R}/(2\pi\mathbb{Z})$. By the statement of part (B) of Proposition D.7 the limit $x_\infty := \lim_{r\to\infty} u(r)$ exists and lies in $\mu^{-1}(0)$, and there exist a map $g_\infty \in W^{1,p}(S^1, \mathrm{G})$ and a number $C_0 > 0$ such that

$$\lim_{r\to\infty} \max_{\varphi \in S^1} d(u(re^{i\varphi}), g_\infty(\varphi)x_\infty) \;=\; 0, \tag{D.42}$$

$$\int_0^{2\pi} |\dot{g}_\infty(\varphi) g_\infty(\varphi)^{-1} + \widetilde{\Psi}_0(\tau, \varphi)|^p \, d\varphi \;\leq\; C_0 e^{\left(p - 2 - \frac{p\delta}{2}\right)\tau}, \; \forall \tau \geq 0. \tag{D.43}$$

We define the map $\widetilde{g}_\infty \in W^{1,p}_{\mathrm{loc}}((0,\infty)\times S^1, \mathrm{G})$ by

$$\widetilde{g}_\infty(\tau + i\varphi) := g_\infty(\varphi).$$

By Whitney's embedding theorem, setting $N := 2\dim \mathrm{G} + 1$, there exists an embedding $\iota : \mathrm{G} \to \mathbb{R}^N$, see the book by M. W. Hirsch [Hi], Theorem 3.2.14 p. 55. For $1 \leq q \leq \infty$ we abbreviate

$$||\cdot||_{q;k} := ||\cdot||_{L^q((k,k+1)\times S^1)}, \qquad ||\cdot||_{1,q;k} := ||\cdot||_{W^{1,q}((k,k+1)\times S^1)}.$$

Claim 1 *For every sequence of positive numbers ε_k, $k \geq 0$ there exists a gauge transformation $\widetilde{g}_1 \in C^\infty([0,\infty)\times S^1, \mathrm{G})$ such that*

$$||\iota\circ\widetilde{g}_1 - \iota\circ\widetilde{g}_\infty||_{\infty,k} + \Big|\Big|\, |\widetilde{g}_1^{-1}\partial_\tau \widetilde{g}_1| + |\widetilde{g}_1^{-1}\partial_\varphi \widetilde{g}_1 - \widetilde{g}_\infty^{-1}\partial_\varphi \widetilde{g}_\infty|\, \Big|\Big|_{p;k} < \varepsilon_k, \quad \text{(D.44)}$$

for every nonnegative integer k.

Proof of Claim 1: For every subset $X \subseteq \mathbb{R}^N$ and every $\delta > 0$ we denote by $U_\delta(X) := X + B_\delta \subseteq \mathbb{R}^N$ the open δ-neighbourhood of X. By the tubular neighbourhood theorem (see Theorem 5.1 in the book by M. W. Hirsch [Hi]) there exists a number $\delta > 0$ and a smooth map

$$r : U_\delta(\iota(\mathrm{G})) \to \iota(\mathrm{G}),$$

such that $r(x) = x$ for every $x \in \iota(\mathrm{G})$ and $|r(x) - x| < \delta$ for every $x \in U_\delta(\iota(\mathrm{G}))$. We choose a smooth map $\rho : [0,1] \to [0,1]$ such that $\rho(\tau) = 1$ for $\tau \leq 1/3$ and $\rho(\tau) = 0$ for $\tau \geq 2/3$. Let $\varepsilon_k < \delta$, $k \geq 0$ be a sequence of positive numbers. We fix a nonnegative integer k. The usual argument with mollifiers implies that $C^\infty([k-1,k+1]\times S^1, \mathbb{R}^N)$ is dense in $W^{1,p}((k-1,k+1)\times S^1, \mathbb{R}^N)$. Hence there exists a smooth map $h_k : [k-1,k+1]\times S^1 \to \mathbb{R}^N$ such that

$$||\iota\circ\widetilde{g}_\infty - h_k||_{L^\infty([k-1,k+1]\times S^1)} < \varepsilon_k, \quad \text{(D.45)}$$

$$||\iota\circ\widetilde{g}_\infty - h_k||_{W^{1,p}((k-1,k+1)\times S^1)} < \varepsilon_k. \quad \text{(D.46)}$$

We define

$$\rho_k : [k,k+1]\times S^1 \to [0,1], \quad \rho_k(\tau + i\varphi) := \rho(\tau - k).$$

Furthermore, we define

$$f_k := \rho_k h_k + (1-\rho_k) h_{k+1} \in C^\infty([k-1,k+1]\times S^1, \mathbb{R}^N),$$

and we define $\widetilde{g}_1 : [0,\infty)\times S^1 \to \mathrm{G}$ to be the map whose restriction to $[k,k+1)\times S^1$ is given by

$$\iota\circ\widetilde{g}_1 := r\circ f_k,$$

for every k. Note that the right hand side makes sense, since inequality (D.45) and $\varepsilon_k < \delta$ imply that f_k takes values in $U_\delta(\iota(\mathrm{G}))$. In the following we denote by $C > 0$ a constant depending only on the embedding $\iota : \mathrm{G} \to \mathbb{R}^N$ and on the retraction $r : U_\delta(\iota(\mathrm{G})) \to \iota(\mathrm{G})$, but not on $\widetilde{g}_\infty$ and $\widetilde{g}_1$. It may change from estimate to estimate. We fix a nonnegative integer k. Inequality (D.45) implies that

$$\begin{aligned}
||\iota \circ \widetilde{g}_1 - \iota \circ \widetilde{g}_\infty||_{\infty;k} &= ||r \circ f_k - r \circ \iota \circ \widetilde{g}_\infty||_{\infty;k} \\
&\leq C||f_k - \iota \circ \widetilde{g}_\infty||_{\infty;k} \\
&\leq C\Big(||\rho_k(h_k - \iota \circ \widetilde{g}_\infty)||_{\infty;k} \\
&\quad +||(1-\rho_k)(h_{k+1} - \iota \circ \widetilde{g}_\infty)||_{\infty;k}\Big) \\
&\leq C(\varepsilon_k + \varepsilon_{k+1}). \qquad \text{(D.47)}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
||\widetilde{g}_1^{-1} d\widetilde{g}_1 - \widetilde{g}_\infty^{-1} d\widetilde{g}_\infty||_{p;k} &= ||\widetilde{g}_\infty \widetilde{g}_1^{-1} d\widetilde{g}_1 - d\widetilde{g}_\infty||_{p;k} \\
&\leq C\big|\big|d\iota(\widetilde{g}_\infty)(\widetilde{g}_\infty \widetilde{g}_1^{-1} d\widetilde{g}_1 - d\widetilde{g}_\infty)\big|\big|_{p;k} \\
&\leq C\Big(\big|\big|\big(d\iota(\widetilde{g}_\infty)\widetilde{g}_\infty \widetilde{g}_1^{-1} - d\iota(\widetilde{g}_1)\big)d\widetilde{g}_1\big|\big|_{p;k} \\
&\quad +||d\iota(\widetilde{g}_1)d\widetilde{g}_1 - d\iota(\widetilde{g}_\infty)d\widetilde{g}_\infty||_{p;k}\Big) \\
&\leq C\Big(||\iota \circ \widetilde{g}_1 - \iota \circ \widetilde{g}_\infty||_{\infty;k}||d\widetilde{g}_1||_{p;k} \\
&\quad +||d(\iota \circ \widetilde{g}_1 - \iota \circ \widetilde{g}_\infty)||_{p;k}\Big) \\
&\leq C\Big((\varepsilon_k + \varepsilon_{k+1}) \cdot \\
&\qquad \cdot \big(||d(\iota \circ \widetilde{g}_1 - \iota \circ \widetilde{g}_\infty)||_{p;k} + ||d(\iota \circ \widetilde{g}_\infty)||_{p;k}\big) \\
&\quad +||d(\iota \circ \widetilde{g}_1 - \iota \circ \widetilde{g}_\infty)||_{p;k}\Big) \\
&\leq C\Big((\varepsilon_k + \varepsilon_{k+1} + 1)||d(\iota \circ \widetilde{g}_1 - \iota \circ \widetilde{g}_\infty)||_{p;k} \qquad \text{(D.48)} \\
&\quad +(\varepsilon_k + \varepsilon_{k+1})||dg_\infty||_{L^p(S^1)}\Big).
\end{aligned}$$

Here in the forth and in the last step we have used (D.47). Moreover,

$$\begin{aligned}
||d(\iota\circ\widetilde{g}_1-\iota\circ\widetilde{g}_\infty)||_{p;k} &= ||d(r\circ f_k - r\circ\iota\circ\widetilde{g}_\infty)||_{p;k} \\
&\le ||(dr(f_k)-dr(\iota\circ\widetilde{g}_\infty))df_k||_{p;k} \\
&\quad +||dr(\iota\circ\widetilde{g}_\infty)(df_k-d(\iota\circ\widetilde{g}_\infty))||_{p;k} \\
&\le C\Big(||f_k-\iota\circ\widetilde{g}_\infty||_{\infty;k}||df_k||_{p;k}+||df_k-d(\iota\circ\widetilde{g}_\infty)||_{p;k}\Big) \\
&\le C\Big((\varepsilon_k+\varepsilon_{k+1})||df_k||_{p;k} \\
&\quad +\Big|\Big|d\big(\rho_k(h_k-\iota\circ\widetilde{g}_\infty)+(1-\rho_k)(h_{k+1}-\iota\circ\widetilde{g}_\infty)\big)\Big|\Big|_{p;k}\Big) \\
&\le C\Big((\varepsilon_k+\varepsilon_{k+1})||df_k||_{p;k} \\
&\qquad +\Big|\Big|(d\rho_k)(h_k-\iota\circ\widetilde{g}_\infty)+\rho_k d(h_k-\iota\circ\widetilde{g}_\infty)\Big|\Big|_{p;k} \\
&\qquad +\Big|\Big|-d\rho_k(h_{k+1}-\iota\circ\widetilde{g}_\infty) \\
&\qquad\quad +(1-\rho_k)d(h_{k+1}-\iota\circ\widetilde{g}_\infty)\Big|\Big|_{p;k}\Big) \\
&\le C\Big((\varepsilon_k+\varepsilon_{k+1})\big(||h_k||_{1,p;k}+||h_{k+1}||_{1,p;k}\big) \\
&\qquad +||h_k-\iota\circ\widetilde{g}_\infty||_{1,p;k}+||h_{k+1}-\iota\circ\widetilde{g}_\infty||_{1,p;k}\Big) \\
&\le C\Big((\varepsilon_k+\varepsilon_{k+1}+1)\cdot \\
&\qquad \cdot\big(||h_k-\iota\circ\widetilde{g}_\infty||_{1,p;k}+||h_{k+1}-\iota\circ\widetilde{g}_\infty||_{1,p;k}\big) \\
&\qquad +2(\varepsilon_k+\varepsilon_{k+1})||\iota\circ\widetilde{g}_\infty||_{1,p;k}\Big) \\
&\le C(\varepsilon_k+\varepsilon_{k+1})\big(\varepsilon_k+\varepsilon_{k+1}+1+2||dg_\infty||_{L^p(S^1)}\big). \qquad \text{(D.49)}
\end{aligned}$$

Here in the forth step we have used (D.47), and in the last step we have used (D.46). Inserting estimate (D.49) into (D.48) yields, using $\partial_\tau\widetilde{g}_\infty=0$

$$\begin{aligned}
\Big|\Big|\,|\widetilde{g}_1^{-1}\partial_\tau\widetilde{g}_1|+|\widetilde{g}_1^{-1}\partial_\varphi\widetilde{g}_1-\widetilde{g}_\infty^{-1}\partial_\varphi\widetilde{g}_\infty|\,\Big|\Big|_{p;k} &\le \sqrt{2}||\widetilde{g}_1^{-1}d\widetilde{g}_1-\widetilde{g}_\infty^{-1}d\widetilde{g}_\infty||_{p;k} \\
&\le C(\varepsilon_k+\varepsilon_{k+1})(1+||dg_\infty||_{L^p(S^1)}).
\end{aligned} \qquad \text{(D.50)}$$

We have shown that for every sequence $\varepsilon_k>0$, $k\ge 0$, there exists a gauge transformation $\widetilde{g}_1\in C^\infty([0,\infty)\times S^1,\mathrm{G})$ such that inequalities (D.47) and (D.50) hold. Claim 1 follows from this by replacing the given sequence ε_k by a sequence of numbers $\widetilde{\varepsilon}_k$ such that

$$C(\widetilde{\varepsilon}_k+\widetilde{\varepsilon}_{k+1})(2+||dg_\infty||_{L^p(S^1)})<\varepsilon_k.$$

□

By Claim 1 there exists a gauge transformation $\widetilde{g}_1 \in C^\infty([0,\infty)\times S^1, \mathrm{G})$ such that inequality (D.44) with $\varepsilon_k := e^{\left(1-\frac{2}{p}-\frac{\delta}{2}\right)k}$ holds. We define $g \in C^\infty(\mathbb{C}\setminus B_1, \mathrm{G})$ by

$$g(e^{\tau+i\varphi}) := g_0(e^{\tau+i\varphi})\widetilde{g}_1(\tau+i\varphi).$$

Claim 2 *We have*

$$g^*(\Phi+i\Psi) \in L^p_\lambda(\mathbb{C}\setminus B_1). \tag{D.51}$$

Proof of Claim 2: We fix a nonnegative integer k. We denote by C a constant that depends only on the embedding $\iota : \mathrm{G} \to \mathbb{R}^N$ and on the retraction $r : U_\delta(\iota(\mathrm{G})) \to \iota(\mathrm{G})$. It may change from estimate to estimate. We have

$$\begin{aligned}
||\widetilde{g}_\infty^{-1}\widetilde{\Psi}_0\widetilde{g}_\infty - \widetilde{g}_1^{-1}\widetilde{\Psi}_0\widetilde{g}_1||_{p;k} &\leq C||\iota\circ\widetilde{g}_\infty - \iota\circ\widetilde{g}_1||_{\infty;k}||\widetilde{\Psi}_0||_{p;k} \\
&\leq Ce^{\left(1-\frac{2}{p}-\frac{\delta}{2}\right)k}\cdot \\
&\quad \cdot\Big(||(\partial_\varphi\widetilde{g}_\infty)\widetilde{g}_\infty^{-1} + \widetilde{\Psi}_0||_{p;k} + ||\dot{g}_\infty g_\infty^{-1}||_{L^p(S^1)}\Big) \\
&\leq Ce^{\left(1-\frac{2}{p}-\frac{\delta}{2}\right)k}\Big(C_0^{\frac{1}{p}} + ||\dot{g}_\infty g_\infty^{-1}||_{L^p(S^1)}\Big) \\
&=: C'e^{\left(1-\frac{2}{p}-\frac{\delta}{2}\right)k}.
\end{aligned}$$

Here in the second inequality we have used (D.44), and in the third inequality we have used (D.43) and the fact that $p-2-p\delta/2<0$. It follows that

$$\begin{aligned}
||\widetilde{g}_1^*(i\widetilde{\Psi}_0)||_{p;k} &= \big|\big|\widetilde{g}_1^{-1}(\partial_\tau\widetilde{g}_1 + i(\widetilde{\Psi}_0+\partial_\varphi)\widetilde{g}_1)\big|\big|_{p;k} \\
&\leq \big|\big|\,|\widetilde{g}_1^{-1}\partial_\tau\widetilde{g}_1| + |\widetilde{g}_1^{-1}\partial_\varphi\widetilde{g}_1 - \widetilde{g}_\infty^{-1}\partial_\varphi\widetilde{g}_\infty|\,\big|\big|_{p;k} \\
&\quad + ||\widetilde{g}_\infty^{-1}\partial_\varphi\widetilde{g}_\infty + \widetilde{g}_\infty^{-1}\widetilde{\Psi}_0\widetilde{g}_\infty||_{p;k} + ||\widetilde{g}_\infty^{-1}\widetilde{\Psi}_0\widetilde{g}_\infty - \widetilde{g}_1^{-1}\widetilde{\Psi}_0\widetilde{g}_1||_{p;k} \\
&< (1+C_0^{\frac{1}{p}}+C')e^{\left(1-\frac{2}{p}-\frac{\delta}{2}\right)k} \\
&=: C''e^{\left(1-\frac{2}{p}-\frac{\delta}{2}\right)k}.
\end{aligned} \tag{D.52}$$

Here in the last inequality we have used (D.43) and (D.44). We define $\widetilde{g}_0 \in C^\infty(\mathbb{R}\times S^1, \mathrm{G})$ by

$$\widetilde{g}_0(\tau+i\varphi) := g_0(e^{\tau+i\varphi}).$$

Then it follows from Proposition B.6 that for every $\tau+i\varphi \in \mathbb{R}\times S^1$

$$(\widetilde{g}_0\widetilde{g}_1)^*(\widetilde{\Phi}+i\widetilde{\Psi})(\tau+i\varphi) = e^{\tau-i\varphi}\big(g^*(\Phi+i\Psi)\big)(e^{\tau+i\varphi}),$$

and therefore

$$\big|(\widetilde{g}_0\widetilde{g}_1)^*(\widetilde{\Phi}+i\widetilde{\Psi})(\tau+i\varphi)\big| = e^\tau|g^*(\Phi+i\Psi)(e^{\tau+i\varphi})|.$$

It follows that for every nonnegative integer k

$$\begin{aligned}\int_{A(e^k,e^{k+1})}|g^*(\Phi+i\Psi)|^p ds\,dt &= \int_k^{k+1}\int_0^{2\pi}|(\widetilde{g}_0\widetilde{g}_1)^*(\widetilde{\Phi}+i\widetilde{\Psi})|^p e^{(2-p)\tau}\,d\varphi\,d\tau \\ &= \int_k^{k+1}\int_0^{2\pi}|\widetilde{g}_1^*(i\widetilde{\Psi}_0)|^p e^{(2-p)\tau}\,d\varphi\,d\tau \\ &\le (C'')^p e^{-\frac{\delta pk}{2}}. \qquad (D.53)\end{aligned}$$

Here in the second step we have used the radial gauge condition $\widetilde{\Phi}_0=0$, and in the last step we have used (D.52). Since the number $\lambda-\frac{\delta}{2}$ is negative, it follows that

$$\begin{aligned}||g^*(\Phi+i\Psi)||^p_{L^p_\lambda(\mathbb{C}\setminus B_1)} &= \sum_{k=0}^\infty ||g^*(\Phi+i\Psi)||^p_{L^p_\lambda(A(e^k,e^{k+1}))} \\ &\le \sum_{k=0}^\infty (C'')^p e^{(\lambda)p}e^{pk(\lambda-\frac{\delta}{2})} \\ &\le \frac{(C'')^p e^{(\lambda)p}}{1-e^{p(\lambda-\frac{\delta}{2})}}<\infty.\end{aligned}$$

Here in the second step we have used (D.53). This proves Claim 2. □

This concludes the proof of Proposition D.11. □

D.6 Vortices of 0 energy

Let (M,ω) be a symplectic manifold, G be a Lie group with Lie algebra $\mathfrak{g}$, and $\langle\cdot,\cdot\rangle$ be an invariant inner product on $\mathfrak{g}$. Suppose that G acts on M in a Hamiltonian way with moment map μ, let J be an ω-compatible G-invariant almost complex structure on M and let $p>2$ be a number. The next lemma characterizes the vortices with 0 energy.

Lemma D.14 (Vortices with 0 energy) *Let $w\in W^{1,p}_{\mathrm{loc}}(\mathbb{C},M\times\mathfrak{g}\times\mathfrak{g})$ be a solution of the vortex equations (0.1) and (0.2) on the plane $\mathbb{C}$. Then the following two conditions are equivalent.*

(i) $E(w)=0$.

(ii) There is a point $x\in\mu^{-1}(0)$ and a gauge transformation $g\in\mathcal{G}^{2,p}=W^{2,p}_{\mathrm{loc}}(\mathbb{C},\mathrm{G})$ such that

$$w=g^*(x,0,0)=(g^{-1}x,g^{-1}\partial_s g,g^{-1}\partial_t g). \qquad (D.54)$$

Proof: Assume condition (i). Then it follows from the definition of $E(w)$ ((1.13) with $R = 1$) that

$$\kappa := \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] = 0.$$

By Proposition D.2 there is a gauge transformation $h_1 \in \mathcal{G}^{2,p}$ such that $(u_1, \Phi_1, \Psi_1) := h_1^*(u, \Phi, \Psi)$ is smooth. Furthermore,

$$\kappa_1 := \partial_s \Psi_1 - \partial_t \Phi_1 + [\Phi_1, \Psi_1] = h_1^{-1} \kappa h_1 = 0.$$

Therefore, by Proposition D.4 there is a gauge transformation $h_2 \in \mathcal{G}$ such that $(\Phi_2, \Psi_2) := h_2^*(\Phi_1, \Psi_1) = (0, 0)$. We define $u_2 := h_2^{-1} h_1^{-1} u$ and $g := h_2^{-1} h_1^{-1}$. Since the energy is invariant under gauge transformations, we have

$$\begin{aligned} 0 &= E(u, \Phi, \Psi) \\ &= E(u_2, 0, 0) \\ &= \frac{1}{2} \int_{\mathbb{C}} \left(|\partial_s u_2|^2 + |\partial_t u_2|^2 + |\mu(u_2)|^2 \right) ds\, dt. \end{aligned} \tag{D.55}$$

Therefore, $\partial_s u_2 \equiv 0$, $\partial_t u_2 \equiv 0$ and thus u_2 equals some constant point $x \in M$. Furthermore, equality (D.55) implies that $x \in \mu^{-1}(0)$. Moreover, (D.54) is satisfied, and therefore condition (ii) holds.

Conversely, assume that condition (ii) holds. Then $E(w) = E(x, 0, 0) = 0$, and therefore, condition (i) holds. This proves Lemma D.14. □

D.7 A topology on $\mathcal{M}$

By the Whitney embedding theorem ([Hi], Theorem 3.2.14 p. 55), setting $N := 2 \dim M + 1$, there exists an embedding $\iota : M \to \mathbb{R}^N$. Hence, we may assume w.l.o.g. that $M \subseteq \mathbb{R}^N$. Consider the distance function on $X := W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ defined by

$$d_X(w, w') := \sum_{i \in \mathbb{N}} 2^{-i} \frac{d_{\bar{B}_i}(w, w')}{d_{\bar{B}_i}(w, w') + 1} \in [0, 1], \tag{D.56}$$

where for every compact subset $Q \subseteq \mathbb{C}$ we have set

$$d_Q\big((u, \Phi, \Psi), (u', \Phi', \Psi')\big) := ||u - u'||_{C^1(Q)} + ||(\Phi, \Psi) - (\Phi', \Psi')||_{C^0(Q)}.$$

The distance function d_X induces a topology on X, which does not depend on ι. We endow the quotient $Y := X / \mathcal{G}^{2,p}$ with the quotient topology. This means that a subset $U \subseteq Y$ is open iff $\pi^{-1}(U) \subseteq X$ is open, where $\pi : X \to Y = X / \mathcal{G}^{2,p}$ denotes the canonical projection. We endow the subset $\mathcal{M} = \widetilde{\mathcal{M}}^{1,p} / \mathcal{G}^{2,p} \subseteq Y$ of gauge equivalence classes of vortices with the relative topology.

Lemma D.15 *Let $w_\nu := (u_\nu, \Phi_\nu, \Psi_\nu)$ and $w := (u, \Phi, \Psi) \in X$ be maps. Then the following two conditions are equivalent.*

(i) The maps w_ν converge to w w.r.t. to d_X.

(ii) For every compact subset $Q \subseteq \mathbb{C}$ the maps u_ν converge to u in $C^1(Q)$ and the maps (Φ_ν, Ψ_ν) converge to (Φ, Ψ) in $C^0(Q)$.

Proof of Lemma D.15: Assume that (i) holds. We fix an index $i \in \mathbb{N}$. If ν is so large that $d_X(w_\nu, w) < 2^{-i}$, then follows from (D.56) that

$$||u_\nu - u||_{C^1(\bar{B}_i)} + ||(\Phi_\nu, \Psi_\nu) - (\Phi, \Psi)||_{C^0(\bar{B}_i)} \leq \left(2^{-i} d_X(w_\nu, w)^{-1} - 1\right)^{-1}.$$

This expression converges to 0 as $\nu \to \infty$, and therefore (ii) holds.

Assume now in turn that (ii) holds. Let $\varepsilon > 0$. We choose an integer i_0 so large that $2^{-i_0} < \varepsilon$ and ν_0 so large that for $\nu \geq \nu_0$ and $i = 1, \ldots, i_0$ we have

$$||u_\nu - u||_{C^1(\bar{B}_i)} + ||(\Phi_\nu, \Psi_\nu) - (\Phi, \Psi)||_{C^0(\bar{B}_i)} < \varepsilon.$$

It follows that for $\nu \geq \nu_0$

$$\begin{aligned} d_X(w_\nu, w) &= \sum_{i=1}^{i_0} 2^{-i} \frac{d_{\bar{B}_i}(w_\nu, w)}{d_{\bar{B}_i}(w_\nu, w) + 1} + \sum_{i \geq i_0+1} 2^{-i} \frac{d_{\bar{B}_i}(w_\nu, w)}{d_{\bar{B}_i}(w_\nu, w) + 1} \\ &< \sum_{i=1}^{i_0} 2^{-i} \varepsilon + \sum_{i \geq i_0+1} 2^{-i} \\ &< \varepsilon + 2^{-i_0} < 2\varepsilon. \end{aligned}$$

It follows that (i) holds. This proves Lemma D.15. □

Lemma D.16 *Let $w_\nu := (u_\nu, \Phi_\nu, \Psi_\nu)$ and $w := (u, \Phi, \Psi) \in X$ be maps. Then the following two conditions are equivalent.*

(i) There exist gauge transformations $g_\nu \in \mathcal{G}^{2,p}$ such that for every compact subset $Q \subseteq \mathbb{C}$ the maps $g_\nu^{-1} u_\nu$ converge to u in $C^1(Q)$ and the maps $g_\nu^*(\Phi_\nu, \Psi_\nu)$ converge to (Φ, Ψ) in $C^0(Q)$.

(ii) The gauge equivalence classes $[w_\nu]$ converge to $[w]$ in the quotient topology on Y.

Proof of Lemma D.16: We check the hypotheses of Proposition A.1. Since the topology on X is induced by the distance function d, it satisfies the first axiom of countability. Furthermore, let $g \in \mathcal{G}^{2,p}$ be a gauge transformation. In order to see that $\rho_g : X \to X$ is continuous, let $w_\nu := (u_\nu, \Phi_\nu, \Psi_\nu) \in X$ be a sequence that converges to some map $w := (u, \Phi, \Psi) \in X$ w.r.t. d_X. Fix a compact subset $Q \subseteq \mathbb{C}$. By Lemma D.15 the maps u_ν converge to u in $C^1(Q)$ and the maps (Φ_ν, Ψ_ν) converge to (Φ, Ψ) in $C^0(Q)$. It follows that $g^{-1}u_\nu$ converges to $g^{-1}u$ in $C^1(Q)$ and $g^*(\Phi_\nu, \Psi_\nu)$ converges to $g^*(\Phi, \Psi)$ in $C^0(Q)$. Since this holds for every compact subset $Q \subseteq \mathbb{C}$, Lemma D.15 implies that $g^* w_\nu$ converges to $g^* w$. Hence ρ_g is continuous. So by Proposition A.1 the statement (ii) is equivalent to

(i') There exist gauge transformations $g_\nu \in \mathcal{G}^{2,p}$ such that $g_\nu^* w_\nu$ converges to w w.r.t. d_X.

By Lemma D.15 this condition is equivalent to condition (i). This proves Lemma D.16. □

D.8 Finite energy $(S^1, \mathbb{C})$-vortices on $\mathbb{C}$

Let $\mathrm{G} := S^1$ act on $M := \mathbb{C}$ by multiplication with moment map $\mu : \mathbb{C} \to i\mathbb{R}$ given by $\mu(z) := \frac{i}{2}(1 - |z|^2)$, let the almost complex structure $J : T\mathbb{C} \to T\mathbb{C}$ be given by multiplication by i and let $p > 2$ be a number. Recall that $\widetilde{\mathcal{M}}^{1,p}$ denotes the set of all solutions $w = (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ of (0.1), (0.2) such that $E(w) < \infty$ and the set $\overline{u(\mathbb{C})} \subseteq M$ is compact. In this case, the last condition on w is superfluous, as the following shows.

Remark D.17 Let $w = (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ be a solution of (0.1), (0.2) such that $E(w) < \infty$. Then the set $\overline{u(\mathbb{C})}$ is compact. To see this observe that in our case in the a-priori-Lemma C.1 we can dispose of the assumption that $u(B_r(z_0)) \subseteq K$. The crucial point of the proof of that Lemma is that our solution w satisfies the estimate $\Delta e_w \geq -C(e_w + e_w^2)$, where $C > 0$ is a constant independent of w. This is true in our case, as can be seen from the proof of L.9.3 in [GS]. It follows as in Lemma C.1 that there exists a constant $E_0 > 0$ such that for every $z_0 \in \mathbb{C}$, $r > 0$ and every solution $w = (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(B_r(z_0), M \times \mathfrak{g} \times \mathfrak{g})$ of (0.1), (0.2) such that $E(w, B_r(z_0)) < E_0$ we have

$$e_w(z_0) = \big(|\mu(u)|^2 + |\partial_s u + X_\Phi(u)|^2\big)(z_0) \leq \frac{8}{\pi r^2} E(w, B_r(z_0)). \tag{D.57}$$

Let $w := (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ be a solution of (0.1), (0.2) such that $E(w) < \infty$. Let $R > 0$ be so large that $E(w, \mathbb{C} \setminus B_{R+1}) < E_0$. Then by (D.57) with $r := 1$ we have $|\mu(u(z))| \leq \sqrt{(8E_0)/\pi}$ for every $z \in \mathbb{C} \setminus B_{R+1}$. This implies that $a := \sup_{z \in \mathbb{C}} |\mu(u(z))| < \infty$, i.e. $u(\mathbb{C}) \subseteq \mu^{-1}(\bar{B}_a)$. Since μ is proper the set $\mu^{-1}(\bar{B}_a)$ is compact and therefore the same holds for the closure $\overline{u(\mathbb{C})} \subseteq \mu^{-1}(\bar{B}_a)$.

By Proposition D.6(A) for every vortex $w = (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$ the point $\mathrm{G}u(re^{i\varphi}) \in \mathbb{C}/S^1$ converges to the unique point in $\bar{M} = S^1/S^1$, for $r \to \infty$, uniformly in $\varphi \in \mathbb{R}$. Hence for r large enough we have $u(re^{i\varphi}) \neq 0$ for every $\varphi \in \mathbb{R}$. The makes the following definition meaningful.

Definition D.18 *Let $w = (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$ be a finite energy vortex. Let $R > 0$ be so large that $u(z) \neq 0$ if $|z| \geq R$. We define the* degree of w *by*

$$\deg(w) := \deg\left(\frac{u}{|u|} : S^1_R \to S^1\right).$$

Remark D.19 If (M, ω) is a general symplectic manifold and G is a general compact connected Lie group acting on M in a Hamiltonian way, then we can not associate a degree to w in an analogous way. However, denoting by $[X, Y]$ the set of (free) homotopy classes of maps between two topological spaces X and Y, a vortex w determines a class $a_w \in [S^1, \mathrm{G}]$ as follows. Let $w' \in \widetilde{\mathcal{M}}$ be a smooth vortex that is in radial gauge outside B_1 and gauge equivalent to w, as in Proposition D.3 and let $g \in W^{1,2}(S^1, \mathrm{G})$ and $x \in \mu^{-1}(0)$ be such that $u'(re^{i\varphi})$ converges to $g(e^{i\varphi})x_0$, uniformly in $\varphi \in \mathbb{R}$, as in Proposition D.7(B), corresponding to w'. We define a_w to be the free homotopy class of g. It is welldefined, i.e. does not depend on the choice w'. Note that in the case $\mathrm{G} := S^1$ acting on $M := \mathbb{C}$ by multiplication, the map

$$[S^1, S^1] \ni [\gamma] \mapsto \deg(\gamma) \in \mathbb{Z}$$

is a bijection and a_w gets mapped to the degree $\deg(w)$ under this map.

Proposition D.20 *Let $w \in \widetilde{\mathcal{M}}^{1,p}$ be a vortex. Then*

$$E(w) = \deg(w)\pi. \tag{D.58}$$

Proof: This is part of Theorem III.1.1 in the book by Jaffe and Taubes [JT]. □

Corollary D.21 *The degree of a $(S^1, \mathbb{C})$-vortex on $\mathbb{C}$ is always nonnegative.*

Proof: This follows immediately from Proposition D.20, since $E(w) \geq 0$. □

Classification of finite energy $(S^1, \mathbb{C})$-vortices on $\mathbb{C}$

Let $\mathrm{G} := S^1 \subseteq \mathbb{C}$ act on $M := \mathbb{C}$ be multiplication with moment map $\mu : \mathbb{C} \to i\mathbb{R}$ given by $\mu(z) := \frac{i}{2}(1 - |z|^2)$. As shown in the book by A. Jaffe and C. Taubes [JT], the $(S^1, \mathbb{C})$-vortices on $\mathbb{C}$ can be classified as follows. For every $d \in \mathbb{N}$ we denote by $\mathrm{Sym}^d(\mathbb{C}) := \mathbb{C}^d/S_d$ the d-fold symmetric product. Here the group S_d of permutations of $\{1, \ldots, d\}$ acts on $\mathbb{C}^d$ by

$$\sigma \cdot (z_1, \ldots, z_d) := (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(d)}).$$

Consider the set $\widetilde{\mathrm{Sym}}^d(\mathbb{C})$ of all maps $m : \mathbb{C} \to \mathbb{N} \sqcup \{0\}$ such that $m(z) \neq 0$ only for finitely many points $z \in \mathbb{C}$ and

$$\sum_{z \in \mathbb{C}} m(z) = d. \tag{D.59}$$

We can identify the symmetric product with $\widetilde{\mathrm{Sym}}^d(\mathbb{C})$ by assigning to $\mathbf{z} := [z_1, \ldots, z_d] \in \mathrm{Sym}^d(\mathbb{C})$ the *multiplicity* map $m_{\mathbf{z}} : \mathbb{C} \to \mathbb{N} \sqcup \{0\}$ defined by

$$m_{\mathbf{z}}(z) := \#\{i \in \{1, \ldots, d\} \mid z_i = z\}.$$

We fix a vortex $w = (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}$. By Proposition 2.2 in the book by A. Jaffe and C. Taubes [JT] the zero set $u^{-1}(0) \subseteq \mathbb{C}$ is finite. We define the *local degree map* $\deg_w : \mathbb{C} \to \mathbb{Z}$ by

$$\deg_w(z) := \deg\left(\frac{u}{|u|} : S_\varepsilon(z) \to S^1\right), \tag{D.60}$$

where $\varepsilon > 0$ is a number smaller than the distance of each two points in $u^{-1}(0)$. Again by Proposition 2.2 in the book [JT] $\deg_w(z) \geq 0$ for every $z \in \mathbb{C}$, and if $u(z) = 0$ then $\deg_w(z) > 0$. Furthermore, the local degree only depends on the gauge equivalence class of w. Let d be a nonnegative integer. We define

$$\widetilde{\mathcal{M}}_d^{1,p} := \{w \in \widetilde{\mathcal{M}}^{1,p} \mid \deg(w) = d\}$$

The next proposition is proved in [JT].

Proposition D.22 *The map*

$$\mathcal{M}_d := \widetilde{\mathcal{M}}_d^{1,p}/\mathcal{G}^{2,p} \ni [w] \mapsto \deg_w \in \mathrm{Sym}^d(\mathbb{C}). \tag{D.61}$$

is a bijection.

D.9 Convergence of $(S^1, \mathbb{C})$-vortices on $\mathbb{C}$

For every topological space X and every $d \in \mathbb{N} \cup \{0\}$ the set $\mathrm{Sym}^d(X) = X^d/S_d$ is endowed with the quotient topology. For $0 \leq d' \leq d$ we define the inclusion

$$\iota : \mathrm{Sym}^{d'}(\mathbb{C}) \to \mathrm{Sym}^d(S^2), \qquad \iota([z_1, \ldots, z_{d'}]) := [z_1, \ldots, z_{d'}, \infty, \ldots, \infty],$$

where we identify $S^2 := \mathbb{C} \sqcup \{\infty\}$.

Proposition D.23 *Let* $\mathrm{G} := S^1 \subseteq \mathbb{C}$ *act on* $M := \mathbb{C}$ *by multiplication with moment map* $\mu : \mathbb{C} \to i\mathbb{R}$ *given by* $\mu(z) := \frac{i}{2}(1 - |z|^2)$. *Let* $0 \le d' \le d$ *be integers,* $w = (u, \Phi, \Psi) \in \widetilde{\mathcal{M}}^{1,p}_{d'}$ *be a finite energy vortex of degree* d' *and let* $w_\nu = (u_\nu, \Phi_\nu, \Psi_\nu) \in \widetilde{\mathcal{M}}^{1,p}_{d}$ *be a sequence of vortices of degree* d. *Then the following conditions are equivalent.*

(i) There exist gauge transformations $g_\nu \in \mathcal{G}^{2,p}$ *such that* $g_\nu^{-1} u_\nu$ *converges to* u *in* $C^1(Q)$ *and* $g_\nu^*(\Phi_\nu, \Psi_\nu)$ *converges to* (Φ, Ψ) *in* $C^0(Q)$ *for every compact subset* $Q \subseteq \mathbb{C}$.

(ii) The point in the symmetric product $\deg_{w_\nu} \in \mathrm{Sym}^d(\mathbb{C}) \subseteq \mathrm{Sym}^d(S^2)$ *converges to* $\iota(\deg_w) \in \mathrm{Sym}^d(S^2)$.

Lemma D.24 *Let* X *be a topological space,* $x \in X$ *and* $x_\nu \in X$ *be a sequence. Then* x_ν *converges to* x *for* $\nu \to \infty$ *if and only if for every subsequence* $(\nu_i)_{i\in\mathbb{N}}$ *there is a further subsequence* $(i_j)_{j\in\mathbb{N}}$ *such that* $x_{\nu_{i_j}}$ *converges to* x *for* $i \to \infty$.

Proof of Lemma D.24: The “only if” part follows immediately. We prove the “if” part by showing the contraposition. Let $x \in X$ and $x_\nu \in X$ be a sequence and assume that x_ν does not converge to x for $\nu \to \infty$. We claim that there is a subsequence (ν_i) such that for every further subsequence (i_j) the points $x_{\nu_{i_j}}$ do not converge to x for $j \to \infty$. To see this note that by assumption there is an open neighbourhood $U \subseteq X$ of x such that for every $\nu_0 \in \mathbb{N}$ there is a $\nu \ge \nu_0$ such that $x_\nu \notin U$. Thus there is a subsequence (ν_i) such that $x_{\nu_i} \notin U$, for every i. Let (i_j) be a further subsequence. Then $x_{\nu_{i_j}} \notin U$ and therefore $x_{\nu_{i_j}}$ does not converge to x. This proves the claim and concludes the proof of Lemma D.24. □

Proof of Proposition D.23: In this proof, for every $r > 0$ and $z \in S^2$ we denote by $B_r(z)$ and $S^1_r(z)$ the ball and the circle on S^2 w.r.t. the standard metric d^{S^2}, with center z and radius r. We show that **(i) implies (ii)**. Let $g_\nu \in \mathcal{G}^{2,p}$ be a sequence of gauge transformations as in (i). Let $x_1, \dots, x_k \in \mathbb{C}$ be the distinct zeros of u, with multiplicities $d_i := \deg_w(x_i)$. Then the point $\mathbf{x} := (x_1, \dots, x_1, \dots, x_k, \dots, x_k, \infty, \dots, \infty) \in (S^2)^d$ (x_i occuring d_i times) represents the equivalence class $\iota(\deg_w) \in \mathrm{Sym}^d(S^2) = (S^2)^d / S_d$. Consider the basis of open neighbourhoods of $\mathbf{x}$ given by

$$\left\{U_\varepsilon := \big(B_\varepsilon(x_1)\big)^{d_1} \times \ldots \times \big(B_\varepsilon(x_k)\big)^{d_k} \times \big(B_\varepsilon(\infty)\big)^{d-d'}\right\}_{\varepsilon>0}.$$

By Lemma A.3 the collection

$$\{V_\varepsilon := \pi(U_\varepsilon)\}_{\varepsilon>0}$$

is a basis of open neighbourhoods of $\iota(\deg_w) \in \mathrm{Sym}^d(S^2)$, where $\pi : (S^2)^d \to \mathrm{Sym}^d(S^2) = (S^2)^d / S_d$ denotes the canonical projection. Hence it suffices to prove that for every $\varepsilon > 0$ there exists an index ν_0 such that for every $\nu \geq \nu_0$ we have $\deg_{w_\nu} \in V_\varepsilon$.

We denote $x_0 := \infty \in S^2$. Let $\varepsilon > 0$ be so small that $\varepsilon < d^{S^2}(x_i, x_j)/2$ for $i, j = 0, \ldots, k$ and $i \neq j$. We denote $u'_\nu := g_\nu^{-1} u_\nu$. Then u'_ν converges to u, uniformly on compact subsets of $\mathbb{C}$, and therefore there exists an integer ν_0 so large that for $\nu \geq \nu_0$ and every

$$z \in X_\varepsilon := S^2 \setminus \bigcup_{i=0,\ldots,k} B_\varepsilon(x_i)$$

we have

$$|u'_\nu(z) - u(z)| < \frac{|u(z)|}{2}. \tag{D.62}$$

Note that here we use that u does not vanish on X_ε. Fix indices $\nu \geq \nu_0$ and $j = 1, \ldots, k$. Consider the map

$$h : [0,1] \times S^1_\varepsilon(x_j) \to \mathbb{C}, \qquad h(\lambda, z) := \lambda u'_\nu(z) + (1-\lambda) u(z).$$

Inequality (D.62) implies that h does not vanish anywhere. Hence we can define the map

$$\frac{h}{|h|} : S^1_\varepsilon(x_j) \to S^1.$$

This is a homotopy connecting the restricted maps

$$\frac{u}{|u|}, \frac{u'_\nu}{|u'_\nu|} : S^1_\varepsilon(x_j) \to S^1.$$

It follows that

$$\deg\left(\frac{u'_\nu}{|u'_\nu|} : S^1_\varepsilon(x_j) \to S^1\right) = \deg\left(\frac{u}{|u|} : S^1_\varepsilon(x_j) \to S^1\right) = \deg_w(x_j) = d_j.$$

On the other hand,

$$\deg\left(\frac{u'_\nu}{|u'_\nu|} : S^1_\varepsilon(x_j) \to S^1\right) = \sum_{z \in B_\varepsilon(x_j)} \deg_{w_\nu}(z).$$

Hence for every $j = 1, \ldots, k$

$$\sum_{z \in B_\varepsilon(x_j)} \deg_{w_\nu}(z) = d_j.$$

We choose a representative $(x_1^\nu, \ldots, x_d^\nu) \in \mathbb{C}^d$ of the point $\deg_{w_\nu} \in \mathrm{Sym}^d(\mathbb{C})$, such that

$$x_1^\nu, \ldots, x_{d_1}^\nu \in B_\varepsilon(x_1), \quad \ldots, \quad x^\nu_{d_1+\cdots+d_{k-1}+1}, \ldots, x^\nu_{d'} \in B_\varepsilon(x_k).$$

Inequality (D.62) implies that the map u'_ν, and hence the map $u_\nu = g_\nu u'_\nu$ does not vanish on X_ε. Therefore, the remaining zeros of u_ν, $x^\nu_{d'+1}, \ldots, x^\nu_d$ lie in the ball $B_\varepsilon(\infty)$. It follows that $(x^\nu_1, \ldots, x^\nu_d) \in U_\varepsilon$ and hence $\deg_{w_\nu} \in V_\varepsilon = \pi(U_\varepsilon)$. This proves that **(i) implies (ii)**.

To see that **(ii) implies (i)**, assume that (ii) is satisfied. Recall from subsection D.7 the notation $X := W^{1,p}_{\mathrm{loc}}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ and the definition (D.56) of the distance function d_X. By Lemma D.16 condition (i) is equivalent to

(i') The gauge equivalence classes $[w_\nu] \in \mathcal{M} \subseteq X/\mathcal{G}^{2,p}$ converge to $[w]$ in the quotient topology.

Hence by Lemma D.24, it suffices to show for every subsequence $(\nu_i)_{i\in\mathbb{N}}$ there exists a further subsequence $(i_j)_{j\in\mathbb{N}}$ such that the classes $[w_{\nu_{i_j}}]$ converge to $[w]$. Let (ν_i) be a subsequence. By Proposition 4.3 (Compactness modulo bubbling) there exists a vortex $w_0 := (u_0, \Phi_0, \Psi_0)$, a subsequence (i_j) and gauge transformations $g_j \in \mathcal{G}^{2,p}$ such that for every compact subset $Q \subseteq \mathbb{C}$ the maps $g_j^{-1} u_{\nu_{i_j}}$ converge to u_0 in $C^1(Q)$ and the maps $g_j^*(\Phi_{\nu_{i_j}}, \Psi_{\nu_{i_j}})$ converge to (Φ_0, Ψ_0) in $C^0(Q)$. It follows that $u_0(\mathbb{C}) \subseteq K_0$, where K_0 is as in Proposition D.6(B), and $\deg(w_0)\pi = E(w_0) \leq \sup_\nu E(w_\nu) = d\pi < \infty$, where we have used by Proposition D.20. Hence $w_0 \in \widetilde{\mathcal{M}}^{1,p}$. By Lemma D.16 the gauge equivalence classes $[w_{\nu_{i_j}}]$ converge to $[w_0]$, as $j \to \infty$. Hence it remains to show that $[w_0] = [w]$. Applying the implication (i)$\Longrightarrow$(ii) with $(w_\nu), w, d'$ replaced by $(w_{\nu_{i_j}}), w_0, \deg(w_0)$, it follows that the point $\deg_{w_{i_j}} \in \mathrm{Sym}^d(S^2)$ converges to $\iota(\deg_{w_0})$. Since by assumption $\deg_{w_{i_j}}$ also converges to $\iota(\deg_w)$, it follows that $\iota(\deg_{w_0}) = \iota(\deg_w)$ and therefore $\deg_{w_0} = \deg_w$. Proposition D.22 (Classification) implies that $[w_0] = [w]$. Hence (i) is satisfied. This proves that **(ii) implies (i)** and concludes the proof of Proposition D.23. □

Remark D.25 The proof of Proposition D.23 shows that (ii) already holds, if we assume a weaker condition than (i), namely that there is a sequence of gauge transformations $g_\nu \in \mathcal{G}^{2,p}$ such that $g_\nu^* w_\nu$ converges to w in $C^0(K)$ for every compact subset $K \subseteq \mathbb{C}$.

E Additional topics

E.1 A little bit of functional analysis

Fix a normed vector space X and a linear subspace $V \subseteq X$. We denote by X^* the dual space of X, and by

$$V^\perp := \{\varphi \in X^* \mid \varphi(x) = 0, \forall x \in V\}$$

the *annihilator* of V. Likewise, for a linear subspace $W \subseteq X^*$ we define

$$W_\perp := \{x \in X \mid \varphi(x) = 0, \forall \varphi \in W\}.$$

Note that $V^\perp$ is a closed linear subspace of X^* and $W_\perp$ is a closed linear subspace of X. If $V \subseteq X$ is a closed linear subspace then we endow X/V with the quotient norm.

Lemma E.1 *Let X be a normed vector space and $V \subseteq X$ be a closed linear subspace. Then the map*

$$V^\perp \to (X/V)^*, \qquad \varphi \mapsto \big(x + V \mapsto \varphi(x)\big) \tag{E.1}$$

is well-defined and an isometric isomorphism.

For the proof of this lemma see also Satz III.1.10 in the book by D. Werner [Wer].

Proof of Lemma E.1: The proof that the map is well-defined is straightforward. The inverse map is given by

$$(X/V)^* \to V^\perp, \qquad \psi \mapsto \psi \circ \pi.$$

This map is bounded with norm ≤ 1. It remains to show that the map (E.1) is bounded with norm ≤ 1. In order to see this, let $\varphi \in V^\perp$ be such that $||\varphi|| \leq 1$. It suffices to prove that the map

$$(X/V)^* \ni x + V \mapsto \varphi(x) \in \mathbb{R} \tag{E.2}$$

has norm ≤ 1. To see this, let $x \in X$ be a vector and $\varepsilon > 0$ be a number. We choose a vector $v \in V$ such that $||x + v||_X \leq ||x + V||_{X/V} + \varepsilon$. It follows that

$$|\varphi(x)| = |\varphi(x + v)| \leq ||\varphi||\big(||x + V||_{X/V} + \varepsilon\big).$$

Hence

$$|\varphi(x)| \leq ||\varphi|| \cdot ||x + V||,$$

for every $x \in X$, and therefore the norm of the map (E.2) is bounded by 1. This proves Lemma E.1. □

Proposition E.2 (Closed image) *Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear map. Then denoting by $T^* : Y^* \to X^*$ the adjoint operator, the following statements are equivalent.*

(i) $\mathrm{im} T$ *is closed.*

(ii) $\mathrm{im} T = (\ker T^*)_\perp$

(iii) $\mathrm{im} T^*$ *is closed.*

(iv) $\mathrm{im} T^* = (\ker T)^\perp$

Proof of Proposition E.2: This is Theorem IV.5.1 in the book [Wer]. □

E.2 Weighted Sobolev spaces and a Hardy-type inequality

The following lemma is used in section 2, in order to define a norm on the vector space $\mathcal{X}_d$. We choose a smooth map $\rho_0 : \mathbb{C} \to [0,1]$ such that $\rho_0(z) = 0$ for $|z| \le 1/2$ and $\rho_0(z) = 1$ for $|z| \ge 1$. For every integer d we define $p_d(z) := z^d$.

Lemma E.3 *Let $1 < p < \infty$ and $\lambda > -2/p$ be real numbers, let d be an integer, and let $\rho_0 : \mathbb{C} \to [0,1]$ be as above. Then the maps*

$$\mathbb{C} \times L^{1,p}_\lambda(\mathbb{C},\mathbb{C}) \to \mathbb{C} + L^{1,p}_\lambda(\mathbb{C},\mathbb{C}), \qquad (v_\infty, v) \mapsto v_\infty + v, \tag{E.3}$$

$$\mathbb{C} \times L^{1,p}_{\lambda-d}(\mathbb{C},\mathbb{C}) \to \mathbb{C} \cdot \rho_0 p_d + L^{1,p}_{\lambda-d}(\mathbb{C},\mathbb{C}), \qquad (v_\infty, v) \mapsto v_\infty \rho_0 p_d + v \tag{E.4}$$

are isomorphisms of vector spaces.

Proof of Lemma E.3: The maps are linear and surjective. In order to see that the map (E.3) is injective, assume that $(v_\infty, v) \in \mathbb{C} \times L^{1,p}_\lambda(\mathbb{C},\mathbb{C})$ is such that $v_\infty + v = 0$. Since $\lambda > -2/p$, a calculation in polar coordinates shows that

$$||1||^p_{L^p_\lambda(\mathbb{C})} = \int_{\mathbb{C}} \langle \cdot \rangle^{\lambda p} = \infty.$$

Therefore, if v_∞ did not vanish then $||v||_{L^p_\lambda(\mathbb{C})} = ||v_\infty||_{L^p_\lambda(\mathbb{C})} = \infty$, a contradiction. It follows that $(v_\infty, vv) = (0,0)$, and hence the map (E.3) is injective. Injectivity of (E.4) follows analogously. This proves Lemma E.3. □

Proposition E.4 (Hardy-type inequality) *Let n and k be positive integers and $p > n$ and $\lambda > -n/p$ be real numbers. Then there exists a constant $C > 0$ with the following property. If $u : \mathbb{R}^n \to \mathbb{R}^k$ is a weakly differentiable* *map such that* $||Du|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n)} < \infty$ *then* $u(rx)$ *converges to some vector* $y_\infty \in \mathbb{R}^k$ *for* $r \to \infty$*, uniformly in* $x \in S^{n-1}$*. Furthermore, the inequality*

$$||(u - y_\infty)|\cdot|^\lambda||_{L^p(\mathbb{R}^n)} \leq C||Du|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n)} \tag{E.5}$$

holds.

Lemma E.5 (Hardy's inequality) *Let* n *be a positive integer and* $1 < p < \infty$ *and* $\lambda > -n/p$ *be numbers. Then for every weakly differentiable function* $u : \mathbb{R}^n \to \mathbb{R}$ *vanishing almost everywhere outside some large enough ball, the following inequality holds.*

$$||u|\cdot|^\lambda||_{L^p(\mathbb{R}^n)} \leq \frac{p}{\lambda + \frac{n}{p}}||Du|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n)} \, (\in [0,\infty]) \tag{E.6}$$

Proof of Lemma E.5: Let φ be a mollifier, i.e. a smooth function on $\mathbb{R}^n$ with compact support such that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. For $\varepsilon > 0$ we define $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$, and $u_\varepsilon := u * \varphi_\varepsilon$. Then u_ε is smooth and has compact support, hence by Exercise 21, Chapter 6, in the book by O. Kavian [Ka], inequality (E.6) holds for u_ε. Taking the limit $\varepsilon \to 0$, it follows that

$$\begin{aligned} ||u|\cdot|^\lambda||_{L^p(\mathbb{R}^n)} &= \lim_{\varepsilon\to 0}||u_\varepsilon|\cdot|^\lambda||_{L^p(\mathbb{R}^n)} \\ &\leq \lim_{\varepsilon\to 0}\frac{p}{\lambda + \frac{n}{p}}||Du_\varepsilon|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n)} \\ &= \frac{p}{\lambda + \frac{n}{p}}||Du|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{E.7}$$

This proves the lemma. □

Proof of Proposition E.4: It suffices to consider the case $k := 1$. Let n, p, λ as in the hypothesis. We set $\varepsilon := \lambda + \frac{n}{p}$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a weakly differentiable map, such that $||Du||_{L^p_{\lambda+1}(\mathbb{R}^n)} < \infty$.

Claim 1 *There exists a constant* C *independent of* u *such that if* $x, y \in \mathbb{R}^n$ *are points with* $0 < |x| \leq |y|$ *then*

$$|u(x) - u(y)| \leq C|x|^{-\varepsilon}||Du|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n\setminus B_{|x|})}. \tag{E.8}$$

Proof of Claim 1: Let $N \in \mathbb{N}$ be such that $2^{N-1}|x| \leq |y| \leq 2^N|x|$. For $i = 0, \dots, N$ we define $x_i := 2^i x \in \mathbb{R}^n$. Furthermore, for $i = N+1, \dots, N+7$ we choose points $x_i \in S^{n-1}_{2^N|x|}$ such that

$$\frac{x_{N+7}}{|x_{N+7}|} = \frac{y}{|y|},$$

and $|x_i - x_{i-1}| \leq 2^{N-1}|x|$, for $i = N+1, \ldots, N+7$, and we set $X_{N+8} := y$. Then for $i = 0, \ldots, N-1$ we have $x_i \in \bar{B}_{2^i|x|}(2^{i+1}x)$, and therefore by Morrey's theorem

$$\begin{aligned}
|u(x_{i+1}) - u(x_i)| &\leq C(2^i|x|)^{1-\frac{n}{p}}||Du||_{L^p\left(B_{2^i|x|}(2^{i+1}x)\right)} \\
&\leq C(2^i|x|)^{1-\frac{n}{p}}(2^i|x|)^{-\lambda-1}||Du|\cdot|^{\lambda+1}||_{L^p\left(B_{2^i|x|}(2^{i+1}x)\right)} \\
&\leq C(2^i|x|)^{-\varepsilon}||Du|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n\setminus B_{|x|})},
\end{aligned}$$

where C is a constant that depends only on n and p. Moreover, for $i = N, \ldots, N+7$ we have $x_{i+1} \in \bar{B}_{2^{N-1}|x|}(x_i)$, and hence analogously,

$$|u(x_{i+1}) - u(x_i)| \leq C(2^{N-1}|x|)^{-\varepsilon}||Du|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n\setminus B_{|x|})}.$$

It follows that

$$\begin{aligned}
|u(y) - u(x)| &\leq \sum_{i=0}^{N+7}|u(x_{i+1}) - u(x_i)| \\
&\leq C|x|^{-\varepsilon}||Du|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n\setminus B_{|x|})}\left(\sum_{i=0}^{N-1}2^{-i\varepsilon} + 8\cdot 2^{(1-N)\varepsilon}\right) \\
&\leq C'|x|^{-\varepsilon}||Du|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n\setminus B_{|x|})},
\end{aligned}$$

where the constant C' only depends on n, p and λ. This proves Claim 1. □

The claim implies that for every $\delta > 0$ there exists a number $R > 0$ such that $|u(x) - u(y)| < \delta$ if $|x|, |y| \geq R$. Hence defining $y_\infty := \lim_{\nu\to\infty} u(\nu)$, we have

$$\lim_{r\to\infty}\sup_{\varphi\in\mathbb{R}}|u(re^{i\varphi}) - y_\infty| = 0,$$

as claimed. To prove inequality (E.5), we choose a smooth map $\rho : [0,\infty) \to [0,1]$ such that $\rho(t) = 1$ for $0 \leq t \leq 1$, $\rho(t) = 0$ for $t \geq 2$ and $|\rho'(t)| \leq 2$. We fix a number $R > 0$ and define $\rho_R : \mathbb{R} \to [0,1]$ by $\rho_R(x) := \rho(x/R)$. Then

$$\begin{aligned}
||(u - y_\infty)|\cdot|^\lambda||_{L^p(B_R)} &\leq ||\rho_R\cdot(u - y_\infty)|\cdot|^\lambda||_{L^p(\mathbb{R}^n)} \\
&\leq C||D(\rho_R\cdot(u - y_\infty))|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n)} \\
&\leq C\Big(||(D\rho_R)(u - y_\infty)|\cdot|^{\lambda+1}||_{L^p(B_{2R}\setminus B_R)} \\
&\quad +||\rho_R Du|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n)}\Big) \\
&\leq C\Big(||2R^{-1}(u - y_\infty)2R|\cdot|^\lambda||_{L^p(B_{2R}\setminus B_R)} \\
&\quad +||Du|\cdot|^{\lambda+1}||_{L^p(\mathbb{R}^n)}\Big), \qquad \text{(E.9)}
\end{aligned}$$

where C is a constant independent of u. Here in the second line, we have used lemma E.5 for the function $\rho_R(u - y_\infty)$. Claim 1 implies that for $x \in \mathbb{R}^n \setminus B_R$

$$|u(x) - y_\infty| \le C|x|^{-\lambda - \frac{n}{p}} ||Du| \cdot |^{\lambda+1}||_{L^p(\mathbb{R}^n \setminus B_R)},$$

and hence

$$\begin{aligned} ||(u - y_\infty)| \cdot |^\lambda||^p_{L^p(B_{2R}\setminus B_R)} &\le C||Du| \cdot |^{\lambda+1}||^p_{L^p(\mathbb{R}^n\setminus B_R)} \int_{B_{2R}\setminus B_R} |x|^{-n} dx \\ &= C|S^{n-1}| \log 2 ||Du| \cdot |^{\lambda+1}||^p_{L^p(\mathbb{R}^n\setminus B_R)}, \end{aligned}$$

where in the last step we have used a calculation in polar coordinates. Inserting this into (E.9) yields

$$||(u - y_\infty)| \cdot |^\lambda||_{L^p(B_R)} \le C||Du| \cdot |^{\lambda+1}||_{L^p(\mathbb{R}^n)},$$

where C does not depend on u and R. Since this holds for every $R > 0$, inequality (E.5) follows. This proves Proposition E.4. □

The following Proposition is used in the proof of Proposition 2.12. We abbreviate

$$||u||_{p,\lambda} := ||u||_{L^p_\lambda(\mathbb{R}^N)},$$

and use the convention $t/\infty := 0$, for every real number t.

Proposition E.6 (Weighted Sobolev spaces) *Let N be a positive integer. Then the following statements hold.*

(i) Let $1 \le p \le q \le \infty$ and λ and μ be real numbers such that $\lambda + N/p < \mu + N/q$. Then there exists a constant $C > 0$ such that for every measurable function $u : \mathbb{R}^N \to \mathbb{C}$ we have

$$||u||_{p,\lambda} \le C||u||_{q,\mu}. \tag{E.10}$$

(ii) Let λ and $1 < p < \infty$ be real numbers. Then for every integer d the map

$$L^{1,p}_\lambda(\mathbb{R}^N \setminus B_1) \ni u \mapsto p_d u \in L^{1,p}_{\lambda-d}(\mathbb{R}^N \setminus B_1) \tag{E.11}$$

is well-defined and an isomorphism of Banach spaces.

(iii) Let λ and $N < p < \infty$ be real numbers. Then there exists a constant C such that for every $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ we have

$$||u\langle\cdot\rangle^{\lambda+\frac{N}{p}}||_{L^\infty(\mathbb{R}^N)} \le C||u||_{L^{1,p}_\lambda(\mathbb{R}^N)}. \tag{E.12}$$

Furthermore, if $\lambda > -N/p$ then there is a compact inclusion

$$L^{1,p}_\lambda(\mathbb{R}^N) \hookrightarrow C_b(\mathbb{R}^N) := \{u \in C(\mathbb{R}^N) \mid \sup_{x\in\mathbb{R}^N} |u(x)| < \infty\}. \tag{E.13}$$

(iv) *Let λ and $1 < p < \infty$ be real numbers. Then the map*

$$W^{1,p}_\lambda(\mathbb{R}^N) \ni u \mapsto \langle\cdot\rangle^\lambda u \in W^{1,p}(\mathbb{R}^N) \tag{E.14}$$

is well-defined and an isomorphism of Banach spaces.

(v) *Let λ and $p > 1$ be real numbers, and let $f \in L^\infty(\mathbb{R}^N)$ be a function such that*

$$||f||_{L^\infty(\mathbb{R}^N \setminus B_i)} \to 0, \tag{E.15}$$

for $i \to \infty$. Then the map

$$W^{1,p}_\lambda(\mathbb{R}^N) \ni u \mapsto fu \in L^p_\lambda(\mathbb{R}^N) \tag{E.16}$$

is compact.

Proof of Proposition E.6:

Proof of (i): This is part (i) of Theorem 1.2 in the paper by R. Bartnik [Ba]. For the convenience of the reader I give a proof here. In the case $p = q$ the estimate (E.10) follows from the definition of $||\cdot||_{p,\lambda}$. Assume that $p < q < \infty$. Then by Hölder's inequality we have

$$\begin{aligned} ||u||_{p,\lambda} &= ||(u\langle\cdot\rangle^\mu)\langle\cdot\rangle^{\lambda-\mu}||_p \\ &\leq ||u\langle\cdot\rangle^\mu||_q ||\langle\cdot\rangle^{\lambda-\mu}||_{\frac{pq}{q-p}} \\ &= C||u||_{q,\mu}, \end{aligned}$$

where we have set $C := ||\langle\cdot\rangle^{\lambda-\mu}||_{\frac{pq}{q-p}}$. Since $\lambda + N/p < \mu + N/q$, a short calculation using polar coordinates shows that $C < \infty$. The case $p < q = \infty$ is analogous. This proves **(i)**.

Proof of (ii): We claim that the map (E.11) is welldefined and bounded. We fix a map $u \in L^{1,p}_\lambda(\mathbb{R}^N \setminus B_1)$. Then

$$||p_d u||_{L^p_{\lambda-d}(\mathbb{R}^N \setminus B_1)} \leq C||u||_{L^p_\lambda(\mathbb{R}^N \setminus B_1)}, \tag{E.17}$$

where C is a constant independent of u. Moreover, we have on $\mathbb{C} \setminus B_1$

$$\begin{aligned} |D(p_d u)|^2 &= 2\big(|\partial_z(p_d u)|^2 + |\partial_{\bar z}(p_d u)|^2\big) \\ &= 2\big(|d \cdot p_{d-1}u + p_d \partial_z u|^2 + |p_d \partial_{\bar z} u|^2\big) \\ &\leq 4\Big(|d|^2 |\cdot|^{2(d-1)}|u|^2 + |\cdot|^{2d}\big(|\partial_z u|^2 + |\partial_{\bar z} u|^2\big)\Big) \\ &= 4|d|^2 |\cdot|^{2(d-1)}|u|^2 + 2|\cdot|^{2d}|Du|^2. \end{aligned}$$

Combining this with (E.17), it follows that

$$||p_d u||_{L^{1,p}_{\lambda-d}(\mathbb{R}^N \setminus B_1)} \leq C||u||_{L^{1,p}_\lambda(\mathbb{R}^N \setminus B_1)},$$

for some constant $C > 0$ independent of u, i.e. the map (E.11) is welldefined and bounded, for every $d \in \mathbb{Z}$. Replacing d by $-d$, it follows that the map

$$L^{1,p}_{\lambda-d}(\mathbb{R}^N \setminus B_1) \ni v \mapsto p_{-d}v \in L^{1,p}_{\lambda}(\mathbb{R}^N \setminus B_1)$$

is a bounded inverse of (E.11). This proves **(ii)**.

Proof of (iii): Inequality (E.12) follows from inequality (1.11) in Theorem 1.2 in the paper by R. Bartnik [Ba]. Assume now that $\lambda > -N/p$. We fix a function $u \in L^{1,p}_\lambda(\mathbb{R}^N)$. Then by Morreys embedding theorem u has a continuous representative, also denoted by u. Furthermore, inequality (E.12) implies that

$$\sup_{x\in\mathbb{R}^N} |u(x)| \leq ||u\langle\cdot\rangle^{\lambda+\frac{N}{p}}||_{L^\infty(\mathbb{R}^N)} < \infty.$$

Hence $u \in C_b(\mathbb{R}^N)$. In order to show that the inclusion (E.13) is compact, let $u_\nu \in L^{1,p}_\lambda(\mathbb{R}^N)$ be a sequence such that

$$C_0 := \sup_\nu ||u_\nu||_{L^{1,p}_\lambda(\mathbb{R}^N)} < \infty. \tag{E.18}$$

Fix a number $R > 0$. By the Banach-Alaoglu compactness theorem there exists a subsequence of u_ν that weakly converges in $W^{1,p}(B_R)$ to some map $u^{(R)}$. Since $p > N$, by Kondrachov's compactness theorem there exists a further subsequence that converges in $C(\bar{B}_R)$, with the same limit $u^{(R)}$. We set $R := 1$ and choose such a subsequence $(\nu^{(1)}_j)_{j\in\mathbb{N}}$. By induction, there exists a sequence of consecutive subsequences $(\nu^{(i)}_j)_{j\in\mathbb{N}}$, $i \in \mathbb{N}$ (i.e. $(\nu^{(i)}_j)$ is a subsequence of $(\nu^{(i-1)}_j)$ for every i), such that $u_{\nu^{(i)}_j}$ converges to some map $u^{(i)} \in W^{1,p}(B_i)$, for $j \to \infty$, in $C(\bar{B}_R)$ and weakly in $W^{1,p}(B_R)$. It follows that $u^{(i')} = u^{(i)}$ on the ball $B^{(i)}$, if $i \leq i'$. So we may define $u : \mathbb{R}^N \to \mathbb{R}$ by $u(x) := u^{(i)}(x)$, where we choose i (depending on x) so large that $x \in B_i$. Fix a positive integer i. Since $u_{\nu^{(i)}_j}$ converges to u, weakly in $W^{1,p}(B_i)$, it follows that

$$||u||_{L^{1,p}_\lambda(B_i)} \leq \limsup_j ||u_{\nu_j}||_{L^{1,p}_\lambda(B_i)} \leq C_0.$$

It follows that

$$||u||_{L^{1,p}_\lambda(\mathbb{R}^N)} \leq C_0, \tag{E.19}$$

and hence $u \in C_b(\mathbb{R}^N)$. We define the diagonal subsequence $\nu_j := \nu^{(j)}_j$. We prove that

$$||u_{\nu_j} - u||_{C_b(\mathbb{R}^N)} := \sup_{x\in\mathbb{R}^N} |u_{\nu_j}(x) - u(x)| \to 0, \tag{E.20}$$

for $j \to \infty$. To see this, we choose a smooth function $\rho : \mathbb{R}^N \to [0,1]$ such that $\rho(x) = 0$ for $x \in B_1$ and $\rho(x) = 1$ for $x \in \mathbb{R}^N \setminus B_2$. Let $\varepsilon > 0$. Since $\lambda > -N/p$ we may choose a number $R \geq 1$ so large that $R^{-\lambda-N/p} < \varepsilon$. We define $\rho_R := \rho(\cdot/R) : \mathbb{R}^N \to [0,1]$. We choose $j_0 \in \mathbb{N}$ so large that for $j \geq j_0$ we have $||u_{\nu_j} - u||_{L^\infty(B_{2R})} < \varepsilon$. It follows that for $j \geq j_0$

$$\begin{aligned} ||u_{\nu_j} - u||_{C_b(\mathbb{R}^N)} &\leq ||(u_{\nu_j} - u)(1-\rho_R)||_{C_b(\mathbb{R}^N)} + ||(u_{\nu_j} - u)\rho_R||_{C_b(\mathbb{R}^N)} \\ &< \varepsilon + CR^{-\lambda-\frac{N}{p}}||(u_{\nu_j} - u)\rho_R||_{L^{1,p}_\lambda(\mathbb{R}^N)} \\ &\leq \varepsilon + C'\varepsilon||u_{\nu_j} - u||_{L^{1,p}_\lambda(\mathbb{R}^N)} \\ &\leq \varepsilon + C'\varepsilon\big(||u_{\nu_j}||_{L^{1,p}_\lambda(\mathbb{R}^N)} + ||u||_{L^{1,p}_\lambda(\mathbb{R}^N)}\big) \\ &\leq \varepsilon + 2C'C_0\varepsilon, \end{aligned}$$

where C and C' are constants depending only on N, p, λ. Here we have used inequality (E.12) in the second line. Hence u_{ν_j} converges to u in $C_b(\mathbb{R}^N)$ for $j \to \infty$. This proves that the inclusion (E.13) is compact and terminates the proof of statement **(iii)**.

Proof of (iv): We show that the map (E.14) is well-defined and bounded. A short calculation using polar coordinates shows that

$$|D(\langle\cdot\rangle^\lambda)| \leq |\lambda|\langle\cdot\rangle^{\lambda-1}.$$

Abbreviating $||\cdot||_p := ||\cdot||_{L^p(\mathbb{R}^N)}$ and $||\cdot||_{p,\lambda} := ||_{L^p_\lambda(\mathbb{R}^N)}$, it follows that for every $u \in W^{1,p}_\lambda(\mathbb{R}^N)$

$$\begin{aligned} ||\langle\cdot\rangle^\lambda u||_{W^{1,p}(\mathbb{R}^N)} &= ||\langle\cdot\rangle^\lambda u||_p + ||D(\langle\cdot\rangle^\lambda u)||_p \\ &\leq ||u||_{p,\lambda} + ||\langle\cdot\rangle^\lambda Du||_p + |\lambda| \cdot ||\langle\cdot\rangle^{\lambda-1}u||_p \\ &\leq (1+|\lambda|)||u||_{W^{1,p}_\lambda(\mathbb{R}^N)}. \end{aligned}$$

This proves that the map (E.14) is well-defined and bounded. Replacing λ by $-\lambda$ it follows that the map

$$W^{1,p}(\mathbb{R}^N) \ni v \mapsto \langle\cdot\rangle^{-\lambda}v \in W^{1,p}_\lambda(\mathbb{R}^N)$$

is well-defined and a bounded inverse of (E.14). This proves **(iv)**.

Proof of (v): Let $f \in L^\infty(\mathbb{R}^N)$ be such that (E.15) holds. Let $u_\nu \in W^{1,p}_\lambda(\mathbb{R}^N)$ be a sequence such that

$$C_1 := \sup_\nu ||u_\nu||_{W^{1,p}_\lambda(\mathbb{R}^N)} < \infty. \tag{E.21}$$

Fix a number $R > 0$. By (E.21) the sequence fu_ν is uniformly bounded in $W^{1,p}(B_R)$. Therefore, it follows from Kondrachov's compactness theorem and from Hölder's inequality that there exists a subsequence of fu_ν that converges in $L^p(B_R)$ to some map $v^{(R)}$. We set $R := 1$ and choose such a subsequence $(\nu_j^{(1)})_{j\in\mathbb{N}}$. By induction, there exists a sequence of consecutive subsequences

$(\nu_j^{(i)})$, such that $fu_{\nu_j^{(i)}}$ converges in $L^p(B_i)$ to some map $fu^{(i)} \in L^p(B_i)$, for $j \to \infty$. We define $v : \mathbb{R}^N \to \mathbb{R}$ by $v(x) := v^{(i)}(x)$, where we choose i (depending on x) so large that $x \in B_i$. Then for every i

$$||v||_{L^p_\lambda(B_i)} \leq \limsup_j ||fu_{\nu_j}||_{L^p_\lambda(B_i)} \leq ||f||_{L^\infty(\mathbb{R}^N)} C_1.$$

Hence

$$||v||_{L^p_\lambda(\mathbb{R}^N)} \leq C_1 ||f||_{L^\infty(\mathbb{R}^N)},$$

and thus $v \in L^p_\lambda(\mathbb{R}^N)$. We define the diagonal subsequence $\nu_j := \nu_j^{(j)}$. We claim that fu_{ν_j} converges to v in $L^p_\lambda(\mathbb{R}^N)$. To see this let $\varepsilon > 0$. We choose i so large that

$$||f||_{L^\infty(\mathbb{R}^N \setminus B_i)} < \varepsilon, \tag{E.22}$$

$$||v||_{L^p_\lambda(\mathbb{R}^N \setminus B_i)} < \varepsilon. \tag{E.23}$$

Since fu_{ν_j} converges to v in $L^p(B_i)$, there exists an integer j_0 such that for $j \geq j_0$ we have

$$||fu_{\nu_j} - v||_{L^p_\lambda(B_i)} < \varepsilon.$$

It follows that for $j \geq j_0$

$$\begin{aligned} ||fu_{\nu_j} - v||_{L^p_\lambda(\mathbb{R}^N)} &\leq ||fu_{\nu_j} - v||_{L^p_\lambda(B_i)} + ||fu_{\nu_j} - v||_{L^p_\lambda(\mathbb{R}^N \setminus B_i)} \\ &\leq \varepsilon + ||fu_{\nu_j}||_{L^p_\lambda(\mathbb{R}^N \setminus B_i)} + ||v||_{L^p_\lambda(\mathbb{R}^N \setminus B_i)} \\ &\leq \varepsilon + \varepsilon C_1 + \varepsilon. \end{aligned}$$

Here in the last inequality we have used (E.22), (E.21) and (E.23). Hence fu_{ν_j} converges to v in $L^p_\lambda(\mathbb{R}^N)$. This proves **(v)** and completes the proof of Proposition E.6. □

E.3 Uhlenbeck compactness

Let n be a positive integer, X be a smooth n-manifold (possibly with boundary), $k \in \mathbb{N} \sqcup \{0\}$, $\ell \in \mathbb{N}$ and $p \in [1, \infty]$. We denote by $\mathrm{H}^n \subseteq \mathbb{R}^n$ the closed upper half space consisting of all $x \in \mathbb{R}^n$ such that $x_n \geq 0$. We define $W^{k,p}_{\mathrm{loc}}(X, \mathbb{R}^\ell)$ to be the vector space of all maps $u : X \to \mathbb{R}^\ell$ such that for every local parametrization $\psi : U \subseteq \mathrm{H}^n \to X$ we have $u \circ \psi \in W^{k,p}_{\mathrm{loc}}(U, \mathbb{R}^\ell)$, i.e. for every compact subset $Q \subseteq U$ we have $u \circ \psi \in W^{k,p}(Q, \mathbb{R}^\ell)$. Note that such a subset Q can have nonempty intersection with the boundary of H^n. Assume now that $k \geq 1$ and $kp > n$ and let X' be a manifold of dimension n'. We define $W^{k,p}(X, X')$ to be the set of all maps $u : X \to X'$ such that for every pair of local parametrizations $\psi : U \subseteq \mathrm{H}^n \to X$ and $\psi' : U' \subseteq \mathrm{H}^{n'} \to X'$ we have $\psi'^{-1} \circ u \circ \psi \in W^{k,p}_{\mathrm{loc}}(\psi^{-1}(\psi'(U')), \mathbb{R}^{n'})$. That this definition makes sense follows from our assumption $kp > n$.

Let E be a real vector bundle on X of rank ℓ with projection $\pi : E \to X$. We define $W^{k,p}_{\mathrm{loc}}(X, E)$ to be the vector space of all maps $s : X \to E$ such that $\pi \circ s = \mathrm{id}$ and for every local trivialization $\Psi : U \times \mathbb{R}^\ell \to E$ we have $\mathrm{pr}_2 \circ \Psi^{-1} \circ s \in W^{k,p}_{\mathrm{loc}}(U, \mathbb{R}^\ell)$, where $\mathrm{pr}_2 : U \times \mathbb{R}^\ell \to \mathbb{R}^\ell$ denotes the projection to the second factor. Let now G be a compact Lie group with Lie algebra $\mathfrak{g}$, let $\langle \cdot, \cdot \rangle$ be an invariant inner product on $\mathfrak{g}$, and let P be a principal G-bundle over X. We denote by $P_\mathfrak{g} := P \times_{\mathrm{G}} \mathfrak{g}$ the adjoint bundle. Furthermore, we define the space of $W^{k,p}_{\mathrm{loc}}$-connection one forms on P by

$$\mathcal{A}^{1,p} := A_0 + W^{k,p}_{\mathrm{loc}}(X, T^*X \otimes P_\mathfrak{g}),$$

where $A_0 \in \mathcal{A}(P)$ is a smooth connection one form. We denote

$$P_{\mathrm{G}} := (P \times \mathrm{G})/\mathrm{G},$$

where G acts on itself by the formula

$$g^*h := g^{-1}hg.$$

We define the set of $W^{k,p}_{\mathrm{loc}}$-gauge transformations by

$$\mathcal{G}^{k,p} := W^{k,p}_{\mathrm{loc}}(X, P_{\mathrm{G}}).$$

The next theorem is Theorem A in the book by K. Wehrheim [Weh]. See also the paper by K. Uhlenbeck [Uh2], Theorem 1.5.

Theorem E.7 (Weak Uhlenbeck compactness with a compact base) *Let n be a positive integer, G be a compact Lie group, X be a compact smooth Riemannian n-manifold (possibly with boundary), P be a principal G-bundle over X and $1 < p < \infty$ be a real number such that $p > n/2$. Let $A^\nu \in \mathcal{A}^{1,p}(P)$, $\nu \in \mathbb{N}$ be a sequence of connection 1-forms and assume that*

$$\sup_{\nu \in \mathbb{N}} ||F_{A^\nu}||_{L^p(X)} < \infty. \tag{E.24}$$

Then passing to some subsequence there are gauge transformations $g_\nu \in \mathcal{G}^{2,p}(P)$ such that $g_\nu^* A^\nu - \hat{A}$ converges weakly in $W^{1,p}(X, \mathfrak{g}_P)$, where $\hat{A} \in \mathcal{A}(P)$ is an arbitrary smooth connection one form.

Theorem E.8 (Weak Uhlenbeck compactness with noncompact base) *Let n be a positive integer, G be a compact Lie group, X be a Riemannian n-manifold (possibly with boundary), P be a principal G-bundle over X and $1 < p < \infty$ be a real number such that $p > n/2$. Assume that there are* *compact submanifolds $X_\nu \subseteq X$ for $\nu \in \mathbb{N}$ (possibly with nonempty boundary ∂X_ν) such that*

$$X = \bigcup_{\nu \in \mathbb{N}} X_\nu, \tag{E.25}$$

$X_\nu \subseteq X_{\nu+1} \setminus \partial X_{\nu+1}$ and X_ν is a deformation retract of X for every $\nu \in \mathbb{N}$. Let $A^\nu \in \mathcal{A}^{1,p}(P|_{X_\nu})$ be a sequence of connection one forms such that for every $\nu_0 \in \mathbb{N}$ we have

$$\sup_{\nu \geq \nu_0} ||F_{A^\nu}||_{L^p(X_{\nu_0})} < \infty, \tag{E.26}$$

Then passing to some subsequence there are gauge transformations $g_\nu \in \mathcal{G}^{2,p}(P)$ such that the following holds. If $\hat{A} \in \mathcal{A}(P)$ is a smooth connection one form and $\nu_0 \in \mathbb{N}$ then $g_\nu^* A^\nu - \hat{A}$ converges weakly in $W^{1,p}(X_{\nu_0}, T^*X_{\nu_0} \otimes \mathfrak{g}_P)$.

Proof of Theorem E.8: Let $\widehat{A} \in \mathcal{A}(P)$ be a smooth connection 1-form. For $\nu \in \mathbb{N}$ we choose a connection 1-form $\widetilde{A}_\nu \in \mathcal{A}^{1,p}(P)$ whose restriction to X_ν agrees with A_ν. By Theorem A' in the book [Weh], passing to some subsequence, there exist gauge transformations $g \in \mathcal{G}^{2,p}$ such that for every $\nu_0 \in \mathbb{N}$ the sequence $g_\nu^* \widetilde{A}_\nu - \widehat{A}$ converges weakly in $W^{1,p}(X_{\nu_0}, T^*X_{\nu_0} \otimes \mathfrak{g}_P)$. Since for $\nu \geq \nu_0$ we have $A_\nu|_{X_{\nu_0}} = \widetilde{A}_\nu|_{X_{\nu_0}}$, the sequence $g_\nu^* A_\nu - \widehat{A}$ converges weakly in $W^{1,p}(X_{\nu_0}, T^*X_{\nu_0} \otimes \mathfrak{g}_P)$, for every $\nu_0 \in \mathbb{N}$. This proves Theorem E.8. □

E.4 Compactness for $\bar{\partial}_J$

We denote by $\mathcal{J}^k(M)$ the set of C^k-almost complex structures on M. The proof of the next proposition is almost the same as the proof of Proposition B.4.2 in the book by D. McDuff and D. A. Salamon [MS3]. For two open subsets $\Omega, \Omega' \subseteq \mathbb{R}^m$ the notation $\Omega \subset\subset \Omega'$ means that $\bar{\Omega} \subseteq \Omega'$ and $\bar{\Omega}$ is compact.

Proposition E.9 (Compactness for $\bar{\partial}_J$) *Let M be a manifold without boundary, $k \in \mathbb{N}$, $p > 2$, $J \in \mathcal{J}^k(M)$, $\Omega_1 \subseteq \Omega_2 \subseteq \ldots \subseteq \mathbb{C}$ be a sequence of open subsets, $u_\nu \in W^{1,p}_{\text{loc}}(\Omega_\nu)$ be a sequence and $\Omega := \bigcup_\nu \Omega_\nu$. Assume that $\partial_s u_\nu + J(u_\nu)\partial_t u_\nu \in W^{k,p}_{\text{loc}}(\Omega_\nu)$ for every ν and that for every compact set $Q \subseteq \Omega$ the following holds. If $\nu_0 \in \mathbb{N}$ is so large that $Q \subseteq \Omega_{\nu_0}$ then there is a compact set $K \subseteq M$ such that*

$$u_\nu(Q) \subseteq K, \qquad \forall \nu \geq \nu_0, \tag{E.27}$$

$$\sup_{\nu \geq \nu_0} ||du_\nu||_{L^p(Q)} < \infty, \tag{E.28}$$

$$\sup_{\nu \geq \nu_0} ||\partial_s u_\nu + J(u_\nu)\partial_t u_\nu||_{W^{k,p}(Q)} < \infty. \tag{E.29}$$

Then there is a subsequence of u_ν that converges weakly in $W^{k+1,p}(Q)$ and in $C^k(Q)$ for every compact subset $Q \subseteq \Omega$.

Proof of Proposition E.9: Let $M, K, p, J, \Omega_\nu, \Omega$ and u_ν be as in the hypothesis. By the Whitney embedding theorem (Theorem 3.2.14 p. 55 in the book by M. W. Hirsch [Hi]), we may assume w.l.o.g. that $M \subseteq \mathbb{R}^{2\dim M+1}$. We choose a sequence of open subsets $\Omega^{(i)} \subseteq \Omega$, $i \in \mathbb{N}$, such that $\Omega^{(i)} \subset\subset \Omega^{(i+1)}$ and

$$\bigcup_{i\in\mathbb{N}} \Omega^{(i)} = \Omega.$$

We define $\nu_j^0 := j$, for $j \in \mathbb{N}$, and for $i \in \mathbb{N}$ we construct a subsequence $(\nu_j^i)_{j\in\mathbb{N}}$ of the given sequence inductively as follows. Let $i \in \mathbb{N}$ and assume that $(\nu_j^{i-1})_{j\in\mathbb{N}}$ has already been constructed. It follows from the hypotheses that there is a compact subset $K_i \subseteq M$ such that $u_\nu(\bar{\Omega}^{(i)}) \subseteq K_i$, for every ν. Together with assumption (E.28) this implies that the $W^{1,p}$-norm of $u_{\nu_j^{i-1}}$ on $\bar{\Omega}^{(i)}$ is bounded, uniformly in j. Since the inclusion $W^{1,p}(\bar{\Omega}^{(i)}) \subseteq C^0(\bar{\Omega}^{(i)})$ is compact, there is a subsequence, denoted by u_μ, that converges weakly in $W^{1,p}(\bar{\Omega}^{(i)})$ and uniformly on $\bar{\Omega}^{(i)}$ to some $u_0^i \in W^{1,p}(\bar{\Omega}^{(i)})$. Note that u_0^i is an extension of u_0^{i-1}.

Claim 1

$$\sup_\mu ||u_\mu||_{W^{k+1,p}(\bar{\Omega}^{(i)})} < \infty. \tag{E.30}$$

Assume that the claim is true. Since the inclusion $W^{k+1,p}(\bar{\Omega}^{(i)}) \subseteq C^k(\bar{\Omega}^{(i)})$ is compact, there is subsequence, denoted by (ν_j^i), such that $u_{\nu_j^i}$ converges to u_0^i in $C^k(\bar{\Omega}^{(i)})$. This terminates the inductive construction of the subsequence. We define now the subsequence ν_j to be the diagonal subsequence $\nu_j := \nu_j^j$. Since the sets $\Omega^{(i)}$ exhaust Ω this subsequence satisfies the conclusion of the theorem, and thus the theorem is proved, once the claim is proved.

Proof of the Claim: To simplify notation we write $K := K_i$ and $u_0 := u_0^i$. We choose local parametrizations $\psi_j : \mathbb{R}^{2n} \to M$, $j = 1, \dots, N$, such that $K \subseteq \bigcup_{j=1}^N \psi_j(B_1)$. Then the open subsets $U_j^0 := u_0^{-1}(\psi_j(B_1)) \cap \Omega^{(i+1)}$ cover $\bar{\Omega}^{(i)}$. We fix open sets $U_j^{k+1} \subset\subset \dots \subset\subset U_j^0$ such that

$$\bar{\Omega}^{(i)} \subseteq \bigcup_{j=1}^N U_j^{k+1}. \tag{E.31}$$

Furthermore, we choose an integer μ_0 so large that for $\mu \geq \mu_0$ and $j = 1, \dots, N$ we have

$$u_\mu(U_j^1) \subseteq \psi_j(B_1).$$

W.l.o.g. we assume that $\mu_0 = 1$. We define

$$v_j^\nu : \psi_j^{-1} \circ u_\mu : U_j^0 \to \mathbb{R}^{2n}.$$

We prove by induction that for $\ell = 1, \ldots, k+1$

$$\sup_\mu ||v_j^\mu||_{W^{\ell,p}(U_j^\ell)} \;<\; \infty. \tag{E.32}$$

Using (E.31) and setting $\ell = k+1$ this will imply the claim. For $\ell = 1$ (E.32) is true because of the hypotheses (E.27) and (E.28) and since $U_j^1 \subseteq \Omega^{(i+1)} \subset\subset \Omega$. Let $\ell \in \{1, \ldots, k\}$ and assume that we have already proved (E.32) for ℓ. We define $J_j^\mu := (d\psi_j)^{-1} J(u_\mu) d\psi_j : U_j^\ell \to \mathbb{R}^{2n\times 2n}$. In order to simplify notation we abbreviate $U^\ell := U_j^\ell$, $\psi := \psi_j$, $v^\mu := v_j^\mu$ and $J^\mu := J_j^\mu$. The derivatives of J^μ up to order ℓ involve derivatives of v^μ up to the same order. Therefore, it follows from (E.32) that

$$\sup_\mu ||J^\mu||_{W^{\ell,p}(U^\ell)} < \infty.$$

Thus by Proposition E.10 there is a constant C that does not depend on μ such that

$$||v^\mu||_{W^{\ell+1,p}(U^{\ell+1})} \le C\Big(||\partial_s u_j^\mu + J^\mu \partial_t u_j^\mu||_{W^{\ell,p}(U^\ell)} + ||u_j^\mu||_{L^p(U^\ell)}\Big). \tag{E.33}$$

On U^ℓ we have

$$\begin{aligned} \partial_s v^\mu + J^\mu \partial_t v^\mu &= d(\psi^{-1})\partial_s u_\mu + d(\psi^{-1}) J(u_\mu) d\psi\, \partial_t v^\mu \\ &= d(\psi^{-1})(\partial_s u_\mu + J(u_\mu)\partial_t u_\mu). \end{aligned}$$

Using that $\bar{U}^\ell$ is compact, it follows that there is a constant C independent of μ such that

$$||\partial_s v^\mu + J^\mu \partial_t v^\mu||_{W^{\ell,p}(U^\ell)} \le C\Big(||\partial_s u_\mu + J(u_\mu)\partial_t u_\mu||_{W^{\ell,p}(U^\ell)} + 1\Big).$$

Inserting this into (E.33), together with the hypothesis (E.29) implies that, increasing the constant C,

$$\sup_\mu ||v^\mu||_{W^{\ell+1,p}(U^{\ell+1})} \le C \sup_\mu \Big(||\partial_s u_\mu + J(u_\mu)\partial_t u_\mu||_{W^{\ell,p}(U^\ell)} + 1\Big) < \infty.$$

This proves (E.32) with ℓ replaced by $\ell+1$ and completes the induction. So (E.32) holds for every $\ell \in \{1, \ldots, k+1\}$. This completes the proof of the claim and of Proposition (E.9). □

The next proposition follows by repeatedly applying Proposition B.4.9 in [MS3].

Proposition E.10 (Regularity for $\bar{\partial}_J$ on $\mathbb{R}^{2n}$) *Let $\Omega \subseteq \mathbb{C}$ be an open subset, $k \in \mathbb{N}$, $p > 2$, $J \in W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n\times 2n})$ be such that $J^2 = -\mathbf{1}$. Then the following holds. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ is such that $\partial_s u + J\partial_t u \in W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$*

then $u \in W^{k+1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$. *Furthermore, for every open set* Ω' *such that* $\overline{\Omega'} \subseteq \Omega$ *and every constant* c *there is a constant* C *with the following significance. Assume that* $J \in W^{k,p}(\Omega, \mathbb{R}^{2n\times 2n})$,

$$||J||_{W^{k,p}(\Omega,\mathbb{R}^{2n\times 2n})} \leq c, \tag{E.34}$$

$$u \in W^{1,p}(\Omega, \mathbb{R}^{2n}), \tag{E.35}$$

$$\partial_s u + J\partial_t u \in W^{k,p}(\Omega, \mathbb{R}^{2n}). \tag{E.36}$$

Then $u \in W^{k+1,p}(\Omega', \mathbb{R}^{2n})$ *and*

$$||u||_{W^{k+1,p}(\Omega',\mathbb{R}^{2n})} \leq C\Big(||\partial_s u + J\partial_t u||_{W^{k,p}(\Omega,\mathbb{R}^{2n})} + ||u||_{L^p(\Omega,\mathbb{R}^{2n})}\Big).$$

E.5 Pseudo-holomorphic curves to the symplectic quotient

This section follows the lines of the overview article [Zi2] and of section 1.5 ("Lifting the $\bar{\partial}$-equation") in [Ga]. There the results are formulated in an intrinsic way, involving principal G-bundles. However, here we use the local description, since that is what is needed in this Ph.D. thesis.

Let (M, ω) be a symplectic manifold, G a compact connected Lie group with Lie algebra $\mathfrak{g}$, and $\langle\cdot,\cdot\rangle$ be an invariant inner product on $\mathfrak{g}$. Assume that G acts on M in a Hamiltonian way, with moment map μ, and let J be a G-invariant ω-compatible almost complex structure. Recall the following definitions. For every $x \in M$ we denote by Gx the G-orbit of x. The Riemannian metric $g_{\omega,J}$ on M is defined by $(g_{\omega,J})_x(v, w) := \omega_x(v, Jw)$, for $x \in M$, $v, w \in T_xM$. For $x \in \mu^{-1}(0)$ we define the horizontal subspace

$$H_x := \ker d\mu(x) \cap \mathrm{im} L_x^{\perp} \subseteq T_xM.$$

By $\pi : \mu^{-1}(0) \to \bar{M} := \mu^{-1}(0)/\mathrm{G}$ we denote the canonical projection. Moreover, the almost complex structure $\bar{J}$ on $\bar{M} := \mu^{-1}(0)/\mathrm{G}$ is defined by

$$\bar{J}\bar{v} := d\pi(x)Jv,$$

for every vector $\bar{v} \in T_{\bar{x}}\bar{M}$ and every point $\bar{x} \in \bar{M}$, where $x \in \mu^{-1}(0)$ is a point such that $\pi(x) = \bar{x}$ and $v \in H_x$ is the unique vector such that $d\pi(x)v = \bar{v}$. For every open subset $\Omega \subseteq S^2 \cong \mathbb{C} \cup \{\infty\}$ the energy density of a map $\bar{u} \in W^{1,p}_{\mathrm{loc}}(\Omega, \bar{M})$ is given by

$$e_{\bar{u}}(z) := \frac{1}{2}(|\partial_s \bar{u}|^2 + |\partial_t \bar{u}|^2).$$

Recall also that for every map $w := (u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ we define

$$e^{\infty}_w(z) := \frac{1}{2}(|\partial_s u + X_\Phi(u)|^2 + |\partial_t u + X_\Psi(u)|^2).$$

Proposition E.11 *Let $\Omega \subseteq \mathbb{C}$ be an open subset and $(u, \Phi, \Psi) \in W^{1,p}_{\mathrm{loc}}(\Omega, M \times \mathfrak{g} \times \mathfrak{g})$ be a solution of the first vortex equation (0.1) such that $\mu \circ u = 0$. Then*

$$\partial_s \mathrm{G}u + \bar{J}(\mathrm{G}u)\partial_t \mathrm{G}u = 0. \tag{E.37}$$

Furthermore,

$$e_{\mathrm{G}u} = e^{\infty}_{(u,\Phi,\Psi)}. \tag{E.38}$$

We will use the next lemma in the proof of Proposition E.11. Here, for every point $x \in M$ the map $L_x^* : T_xM \to \mathfrak{g}$ denotes the adjoint of $L_x : \mathfrak{g} \to T_xM$ w.r.t. the inner product on $\mathfrak{g}$ and $g_{\omega,J}$.

Lemma E.12 *For every $x \in M$ and $\xi, \eta \in \mathfrak{g}$ we have*

$$\begin{aligned} \langle d\mu(x)X_\xi(x), \eta\rangle &= \langle \mu(x), [\eta, \xi]\rangle, & \text{(E.39)}\\ d\mu(x)J(x) &= L_x^*. & \text{(E.40)} \end{aligned}$$

Proof: Let $x \in M$ and $\xi, \eta \in \mathfrak{g}$. To see (E.39), let $\mathbb{R} \ni t \mapsto g(t) \to \mathrm{G}$ be a smooth curve such that $g(0) = e$ and $\dot{g}(0) = \xi$. Then

$$\begin{aligned} \langle d\mu(x)X_\xi(x), \eta\rangle &= \langle \frac{d}{dt}\Big|_{t=0} \mu(gx), \eta\rangle \\ &= \langle \frac{d}{dt}\Big|_{t=0} (g\mu(x)g^{-1}), \eta\rangle \\ &= \langle [\xi, \mu(x)], \eta\rangle \\ &= \langle \mu(x), [\eta, \xi]\rangle. \end{aligned}$$

Here the last equality uses the fact

$$\langle [\xi, \zeta], \eta\rangle + \langle \zeta, [\xi, \eta]\rangle = 0,$$

for every $\xi, \eta, \zeta \in \mathfrak{g}$. This follows from the invariance of the inner product $\langle \cdot, \cdot \rangle$ under the adjoint action of G. This proves (E.39).

To see (E.40) observe that for every $x \in M$, $v \in T_xM$ and $\xi \in \mathfrak{g}$ we have

$$\begin{aligned} \langle d\mu(x)Jv, \xi\rangle &= \omega(X_\xi(x), Jv) \\ &= g_{\omega,J}(X_\xi(x), v) \\ &= g_{\omega,J}(L_x\xi, v) \\ &= \langle \xi, L_x^* v\rangle. \end{aligned}$$

This proves Lemma E.12. □

Proof of Proposition E.11: Claim: We have on Ω

$$d\mu(u)(\partial_s u + X_\Phi(u)) = 0, \qquad d\mu(u)(\partial_t u + X_\Psi(u)) = 0.$$

Proof of the Claim: We have by assumption

$$d\mu(u)\partial_s u = \partial_s(\mu \circ u) = 0, \qquad d\mu(u)\partial_t u = \partial_t(\mu \circ u) = 0. \tag{E.41}$$

Furthermore, by Lemma E.12 we have for every $\xi \in \mathfrak{g}$

$$\langle d\mu(u)X_\Phi(u), \xi\rangle = \langle \mu(u), [\xi, \Phi]\rangle = 0.$$

Here the last equality uses $\mu \circ u = 0$. It follows that

$$d\mu(u)X_\Phi(u) = 0.$$

Similarly we have

$$d\mu(u)X_\Psi(u) = 0.$$

Together with (E.41) this implies the Claim. □

We prove (E.37). By the first vortex equation (0.1) and the Claim we have

$$\begin{aligned} d\mu(u)J(u)(\partial_s u + X_\Phi(u)) &= d\mu(u)(\partial_t u + X_\Phi(u)) = 0, \\ d\mu(u)J(u)(\partial_t u + X_\Psi(u)) &= -d\mu(u)(\partial_s u + X_\Phi(u)) = 0. \end{aligned}$$

By Lemma E.12 this means that

$$\partial_s u + X_\Phi(u) \in \ker(d\mu(u)J(u)) = \ker L_u^* = \mathrm{im} L_u^\perp,$$

and similarly $\partial_t u + X_\Psi(u) \in \mathrm{im} L_u^\perp$. Using again the Claim it follows that

$$\partial_s u + X_\Phi(u) \in H_u, \qquad \partial_t u + X_\Psi(u) \in H_u. \tag{E.42}$$

The equality (E.37) follows now from the first vortex equation (0.1) and the definition of $\bar{J}$.

To see that (E.38) holds, note that (E.42) and the definition of the metric on $\bar{M}$ imply that

$$\begin{aligned} e_{\mathrm{G}u} &= \frac{1}{2}\big(|\partial_s \mathrm{G}u|^2 + |\partial_t \mathrm{G}u|^2\big) \\ &= \frac{1}{2}\big(|\partial_s u + X_\Phi(u)|^2 + |\partial_t u + X_\Psi(u)|^2\big) \\ &= e^\infty_{(u,\Phi,\Psi)}. \end{aligned}$$

Here the second equality uses (E.42). This proves (E.38) and concludes the proof of Proposition E.11. □

E.6 Ordinary linear integral equations

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$. Then $\langle\cdot,\cdot\rangle$ induces a left invariant metric on G, again denoted by $\langle\cdot,\cdot\rangle$, by

$$\langle v, w\rangle := \langle g^{-1}v, g^{-1}w\rangle,$$

for every pair $v, w \in T_g\mathrm{G}$ and every point $g \in \mathrm{G}$. We denote by $|\cdot|_{\langle\cdot,\cdot\rangle}$ the norm on each tangent space of G and by $d^{\langle\cdot,\cdot\rangle}$ the distance function on the connected components of G induced by the metric $\langle\cdot,\cdot\rangle$. The next result is used in the proof of Proposition 6.3.

Proposition E.13 (Ordinary linear integral equations) *Let n be a positive integer, $T \geq 0$ be a number and $\xi : (0,T) \to \mathbb{R}^{n\times n}$ be a Lebesgue integrable map. Then there exists a unique continuous solution $g : [0,T] \to \mathrm{GL}(n,\mathbb{R})$ of the linear integral equation*

$$g(t) = \mathbf{1} + \int_0^t \xi(s)g(s)ds, \qquad g(0) = \mathbf{1}. \tag{E.43}$$

Furthermore, let $\mathrm{G} \subseteq \mathrm{GL}(n,\mathbb{R})$ be a closed subgroup. If ξ takes values in $\mathfrak{g}$ then $g(t) \in \mathrm{G}$ for every $t \in [0,T]$. Moreover, assume that there exists an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ that is invariant under the adjoint action of G. *Suppose that $\xi, \eta : [0,T] \to \mathfrak{g}$ are two Lebesgue integrable maps. Then the corresponding solutions $g, h : [0,T] \to \mathrm{G}$ of (E.43) satisfy*

$$d^{\langle\cdot,\cdot\rangle}(g(t), h(t)) \leq ||\xi - \eta||_{L^1((0,t))} := \int_0^t |\xi(s) - \eta(s)|_{\langle\cdot,\cdot\rangle}\, ds. \tag{E.44}$$

Remark E.14 If G is compact, then there exists an invariant inner product on $\mathfrak{g}$, hence the last part of the Proposition applies.

Proof of Proposition E.13: Let n, T and ξ be as in the hypothesis. In Chapter III.1 of the book [DK], Ju. L. Daleckii and M. G. Krein prove that there exists a unique continuous solution $g : [0,T] \to \mathrm{GL}(n,\mathbb{R})$ of (E.43). Assume now that ξ takes values in the Lie algebra $\mathfrak{g}$ of some closed subgroup $\mathrm{G} \subseteq \mathrm{GL}(n,\mathbb{R})$. Consider the subset $X \subseteq [0,T]$ of all points t such that $g(t) \in \mathrm{G}$. The point 0 lies in X, since $g(0) = \mathbf{1} \in \mathrm{G}$. Since g is continuous and $\mathrm{G} \subseteq \mathrm{GL}(n,\mathbb{R})$ is a closed subset, X is a closed in $[0,T]$. We show that X is open. To see this, we fix a point $t_0 \in [0,T]$ such that $g(t_0) \in \mathrm{G}$. By the Corollary A.7 of the Local Slice Theorem the projection $\pi : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})/\mathrm{G}$ defines a principal G-bundle. So there exists a local equivariant coordinate chart around the point $g(t_0)$. Abbreviating $m := n^2 - \dim(\mathrm{G})$, this is a pair (U, ψ), where $U \subseteq \mathrm{GL}(n,\mathbb{R})$ is a G-invariant open neighbourhood of $g(t_0)$ and

$$\varphi := (\widehat{\varphi}, \mathrm{g}) : U \to \mathbb{R}^m \times \mathrm{G}$$

is a diffeomorphism, such that

$$\begin{aligned} \varphi(ax) = (\widehat{\varphi}(x), a\mathrm{g}(x)), &\qquad \forall a \in \mathrm{G},\, x \in U, \\ \widehat{\varphi}(x) = 0, &\qquad \forall x \in \mathrm{G}. \end{aligned}$$

We choose a number $\varepsilon > 0$ so small that $g([t_0 - \varepsilon, t_0 + \varepsilon]) \subseteq U$. By equation (E.43), the map g and hence the map $\widehat{\varphi} \circ g : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathbb{R}^m$ is absolutely continuous. Hence $\widehat{\varphi} \circ g$ is differentiable almost everywhere, and for every $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ we have

$$\begin{aligned} \widehat{\varphi} \circ g(t) &= \widehat{\varphi} \circ g(t_0) + \int_{t_0}^{t} (\widehat{\varphi} \circ g)^{\cdot}(s)\, ds \\ &= \widehat{\varphi} \circ g(t_0) + \int_{t_0}^{t} d\widehat{\varphi}(g(s))\dot{g}(s)\, ds \\ &= \widehat{\varphi} \circ g(t_0) + \int_{t_0}^{t} d\widehat{\varphi}(g(s))\xi(s)g(s)\, ds \\ &= \widehat{\varphi} \circ g(t_0) = 0. \end{aligned}$$

Here in the third equality we have used (E.43). It follows that $g(t) \in \mathrm{G}$. This proves that X is open. Hence $X = [0, T]$, i.e. $g(t) \in \mathrm{G}$ for every $t \in [0, T]$, as claimed.

To see that the last statement holds, assume that there exists an invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, and let $\xi, \eta : [0, T] \to \mathfrak{g}$ be two Lebesgue integrable maps. We denote the corresponding solutions of (E.43) by $g, h : [0, T] \to \mathrm{G}$. Using the left-invariance of the metric $\langle \cdot, \cdot \rangle$, we have for every $t \in [0, T]$, abbreviating $d := d^{\langle \cdot, \cdot \rangle}$,

$$d(g(t), h(t)) = d(\mathbf{1}, g(t)^{-1}h(t)).$$

This is bounded by the length of the path $[0, t] \ni s \mapsto g(s)^{-1}h(s) \in \mathrm{G}$, hence, abbreviating $|\cdot| := |\cdot|_{\langle \cdot, \cdot \rangle}$,

$$\begin{aligned} d(g(t), h(t)) &\leq \int_0^t \left|\frac{d}{ds}(g(s)^{-1}h(s))\right| ds \\ &= \int_0^t \big| - g(s)^{-1}\dot{g}(s)g(s)^{-1}h(s) + g(s)^{-1}\dot{h}(s)\big|\, ds \\ &= \int_0^t \big|g(s)^{-1}\big(- \dot{g}(s)g(s)^{-1} + \dot{h}(s)h(s)^{-1}\big)h(s)\big|\, ds \\ &= \int_0^t \big| - \dot{g}(s)g(s)^{-1} + \dot{h}(s)h(s)^{-1}\big|\, ds \\ &= \int_0^t | - \xi(s) + \eta(s)|\, ds. \end{aligned}$$

Here in the forth line we have used the bi-invariance of the metric on G. This proves the last statement and completes the proof of Proposition E.13. □

E.7 Homology and cohomology

Let X be a topological space, and k be a nonnegative integer. We denote by $C_k(X,\mathbb{R})$ and $C^k(X,\mathbb{R}) := C_k(X,\mathbb{R})^*$ the vector spaces of singular k-chains and k-cochains with coefficients in $\mathbb{R}$, and by $\partial : C_k(X,\mathbb{R}) \to C_{k-1}(X,\mathbb{R})$ and $d : C^k(X,\mathbb{R}) \to C^{k+1}(X,\mathbb{R})$ the boundary and the coboundary operator. Furthermore, we denote by

$$\begin{aligned} H_k(X,\mathbb{R}) &:= \frac{\ker\big(\partial : C_k(X,\mathbb{R}) \to C_{k-1}(X,\mathbb{R})\big)}{\partial C_{k+1}(X,\mathbb{R})}, \\ H^k(X,\mathbb{R}) &:= \frac{\ker\big(d : C^k(X,\mathbb{R}) \to C^{k+1}(X,\mathbb{R})\big)}{dC^{k-1}(X,\mathbb{R})} \end{aligned}$$

the k-th singular homology and cohomology groups. By the universal coefficients theorem (see 13.4.8 Satz in the book [SZ]) the map

$$H^k(X,\mathbb{R}) \to H_k(X,\mathbb{R})^*, [\varphi] \mapsto ([z] \mapsto \varphi(z)) \tag{E.45}$$

is a linear isomorphism.

Let M be a (smooth) manifold, possibly noncompact and with boundary. By a smooth k-chain in M we mean a finite formal linear combination

$$c := \sum_i \lambda_i c_i,$$

where $c_i : [0,1]^k \to M$ is a smooth map and $\lambda_i \in \mathbb{R}$. We denote by $C_k^\infty(M,\mathbb{R})$ the vector space of smooth k-chains. Restriction of a smooth chain to the components of the boundary $\partial[0,1]^k$ induces a boundary map $\partial : C_k^\infty(M,\mathbb{R}) \to C_{k-1}^\infty(M,\mathbb{R})$. We define the C^∞-homology groups of M to be

$$H_k^\infty(M,\mathbb{R}) := \frac{\ker\big(d : C_k^\infty(M,\mathbb{R}) \to C_{k-1}^\infty(M,\mathbb{R})\big)}{dC_{k+1}^\infty(M,\mathbb{R})}.$$

Fixing a partition of the cube $[0,1]^k$ into simplices, we get a linear map

$$C_k^\infty(M,\mathbb{R}) \to C_k(M,\mathbb{R}). \tag{E.46}$$

Proposition E.15 *For every nonnegative k and every partition of the cube $[0,1]^k$ into simplices, the induced map (E.46) descends to an isomorphism*

$$H_k^\infty(M,\mathbb{R}) \to H_k(M,\mathbb{R}). \tag{E.47}$$

Proof: This is Theorem 24.12 in the book by Fulton [Fu]. □

By the previous proposition the composed map

$$H^k(M,\mathbb{R}) \to H_k(M,\mathbb{R})^* \to H_k^\infty(M,\mathbb{R})^* \tag{E.48}$$

is an isomorphism. Here the first map is given by (E.45) and the second map is induced by the map (E.47).

We denote the de Rham cohomology of M in degree k by $H^k(M)$. Consider the linear map

$$\mathrm{dR} := \mathrm{dR}_M : H^k(M) \to H^k(M, \mathbb{R})$$

given by composing the map

$$H^k(M) \to H_k^\infty(M, \mathbb{R})^*, \quad [\omega] \mapsto \left(\left[\sum_i \lambda_i c_i \right] \mapsto \sum_i \lambda_i \int_{[0,1]^k} c_i^* \omega \right) \tag{E.49}$$

with the inverse of the map (E.48).

Theorem E.16 (de Rham's theorem) *Let M be a manifold, possibly noncompact and with boundary. Then the map*

$$\mathrm{dR} : H^k(M) \to H^k(M, \mathbb{R})$$

is a linear isomorphism.

Proof: This is equivalent to saying that the map (E.49) itself is an isomorphism, which is Theorem 24.13 in the book [Fu]. Note that the proof of that theorem goes through in the case that M has nonempty boundary. In the local considerations one has to replace $\mathbb{R}^n$ by the closed half-space $\mathrm{H}^n := \{x \in \mathbb{R}^n \,|\, x_n \geq 0\}$. □

Assume now that M is orientable and endowed with an orientation. Let n be the dimension of M and $0 \leq k \leq n$. We denote by $\Omega_\mathrm{c}^k(M)$ the vector space of differential k-forms on M with compact support in the interior $\mathring{M} := M \setminus \partial M$, and by

$$H_\mathrm{c}^k(M) := \frac{\ker(d : \Omega_\mathrm{c}^k(M) \to \Omega_\mathrm{c}^{k+1}(M))}{d\Omega_\mathrm{c}^{k-1}(M)}$$

the *de Rham cohomology of M with compact supports*. Consider the linear map

$$\Omega^k(M) \to \Omega_\mathrm{c}^{n-k}(M)^*, \qquad \omega \mapsto \left(\eta \to \int_M \omega \wedge \eta \right).$$

It follows from Stokes' theorem that this map induces a map on cohomology,

$$\mathrm{PD} : H^k(M) \to H_\mathrm{c}^{n-k}(M)^*.$$

Theorem E.17 (Poincaré duality) *Let M be an oriented n-manifold, possibly with boundary. Then the map*

$$\mathrm{PD}_M : H^k(M) \to H^{n-k}_{\mathrm{c}}(M)^*$$

is a linear isomorphism for every $0 \le k \le n$.

Proof: This is Theorem I, Chap. V.4 in Vol. I of the book by W. Greub, St. Halperin and R. Vanstone [GHV]. Note that their proof goes through in the case that M has nonempty boundary. For this, in the local considerations one has to replace $\mathbb{R}^n$ by the closed half-space $\mathrm{H}^n \subseteq \mathbb{R}^n$. In the case that M admits a finite good cover see also the book by R. Bott and L. W. Tu [BT], formula (5.4) p.44. □

By theorem E.17, every k-dimensional oriented submanifold $N \subseteq M$ that is closed as a subset, defines a cohomology class

$$\alpha_N := \alpha_N^M := \mathrm{PD}_M^{-1}\left([\omega] \mapsto \int_N \omega\right) \in H^{n-k}(M). \tag{E.50}$$

For details see the book by R. Bott and L. W. Tu [BT]. We call a de Rham cohomology class $\alpha \in H^k(M)$ an integer class iff $\mathrm{dR}(\alpha)$ lies in the image of the map

$$H^k(M, \mathbb{Z}) \to H^k(M, \mathbb{R})$$

induced by the inclusion $C_k(M, \mathbb{Z}) \to C_k(M, \mathbb{R})$, where $C_k(M, \mathbb{Z})$ and $H^k(M, \mathbb{Z})$ denote the singular k-chain group and k-th singular cohomology group with integer coefficients.

Theorem E.18 (Representing a cohomology class) *Let M be a compact oriented manifold, possibly with boundary ∂M, let k be a nonnegative integer and let $\alpha \in H^k(M)$ be an integer de Rham cohomology class. Then there exist a rational number $\lambda \neq 0$ and a compact oriented submanifold N of dimension $n-k$, possibly with boundary $\partial N \subseteq \partial M$, such that*

$$\lambda \alpha_N^M = \alpha. \tag{E.51}$$

Proof: This is a consequence of Theorem II.29 in the paper [Th] by R. Thom. Consider the double of M,

$$\widetilde{M} := M \# \bar{M},$$

defined by identifying the two boundaries, where $\bar{M}$ denotes the manifold M with the reversed orientation. Note that $\widetilde{M}$ is an oriented manifold. We fix a collar neighbourhood of the boundary ∂M in M, i.e. a pair (U, ψ), where $U \subseteq M$ is an open neighbourhood of ∂M and

$$\psi : (-1, 0] \times \partial M \to U$$

is a diffeomorphism, such that $\psi(\{0\} \times \partial M) = \partial M$. We choose a smooth map $\rho : (-1, 0] \to (-1, 0]$ such that $\rho(t) \equiv t$ for t smaller than some number $t_0 > -1$, and $\rho \equiv 0$ in some neighbourhood of $0 \in (-1, 0]$. We define the map $r : M \to M$ by

$$r(x) := \begin{cases} \psi \circ r^{\mathrm{loc}} \circ \psi^{-1}(x), & \text{if } x \in U, \\ x & \text{otherwise,} \end{cases}$$

where

$$r^{\mathrm{loc}} : (-1, 0] \times \partial M \to (-1, 0] \times \partial M, \quad r^{\mathrm{loc}}(t, x) := (\rho(t), x).$$

Let $\alpha \in H^k(M)$. We choose a representative $\omega \in \Omega^k(M)$ of α. Consider the form $\widetilde{\omega}$ defined by

$$\widetilde{\omega} := \begin{cases} r^*\omega, & \text{on the first component of } \widetilde{M} = M \# \bar{M}, \\ S^* r^* \omega, & \text{on the second component of } \widetilde{M}, \end{cases}$$

where $S : \bar{M} \to M$ is the map that identifies the two components of $\widetilde{M} = M \# \bar{M}$. By definition of the smooth structure on $\widetilde{M}$, we can extend the colar neighbourhood ψ of ∂M in M to a colar neighbourhood

$$\widetilde{\psi} : (-1, 1) \times \partial M \to \widetilde{U} := U \# \bar{U},$$

by setting $\widetilde{\psi} := \psi \circ S$ on $\bar{U}$. Since $\rho \equiv 0$ in some neighbourhood of $0 \in (-1, 0]$, it follows that

$$\widetilde{\psi}^* \widetilde{\omega} = \pi^* \iota^* \omega,$$

in some neighbourhood of $\{0\} \times \partial M \subseteq (-1, 1) \times \partial M$, where $\pi : (-1, 1) \times \partial M \to \partial M$ denotes the projection to the first component and $\iota : \partial M \to M$ denotes the inclusion. Hence the form $\widetilde{\omega}$ is smooth along ∂M. Furthermore, it is closed and hence represents a cohomology class $[\widetilde{\omega}] \in H^*(\widetilde{M})$. Since $\widetilde{M}$ is compact and $\partial \widetilde{M} = \emptyset$, we have $H^{n-k}_{\mathrm{c}}(\widetilde{M}) = H^{n-k}(\widetilde{M})$. Consider the adjoint map of $\mathrm{dR}_{\widetilde{M}}$,

$$\mathrm{dR}^*_{\widetilde{M}} : H_{n-k}(\widetilde{M}, \mathbb{R}) \cong H^{n-k}(\widetilde{M}, \mathbb{R})^* \to H^{n-k}(\widetilde{M})^*.$$

By de Rham's theorem E.16, this map is an isomorphism. We define $\widetilde{\varphi} \in H^{n-k}(\widetilde{M})^*$ by

$$\widetilde{\varphi}([\widetilde{\eta}]) := \int_{\widetilde{M}} \widetilde{\omega} \wedge \widetilde{\eta},$$

and we set

$$\widetilde{a} := (\mathrm{dR}^*_{\widetilde{M}})^{-1}(\widetilde{\varphi}) \in H_{n-k}(\widetilde{M}, \mathbb{R}).$$

Our assumption that α is an integer de Rham cohomology class implies that $\widetilde{a}$ lies in the image of the natural map

$$H_{n-k}(\widetilde{M}, \mathbb{Z}) \to H_{n-k}(\widetilde{M}, \mathbb{R}).$$

Therefore, by Theorem II.29 in the paper [Th] by R. Thom, there exists a closed oriented submanifold $\widetilde{N} \subseteq \widetilde{M}$ of dimension $n-k$ and a rational number λ such that

$$\widetilde{a} = \lambda[\widetilde{N}],$$

where $[\widetilde{N}]$ denotes the fundamental class of $\widetilde{N}$. This means that for every de Rham cohomology class $[\widetilde{\eta}] \in H^{n-k}(\widetilde{M})$ we have

$$\begin{aligned}\int_{\widetilde{M}} \widetilde{\omega} \wedge \widetilde{\eta} &= \widetilde{\varphi}([\widetilde{\eta}]) \\ &= \mathrm{dR}^*_{\widetilde{M}} \widetilde{a}([\widetilde{\eta}]) \\ &= \langle \mathrm{dR}_{\widetilde{M}}([\widetilde{\eta}]), \widetilde{a} \rangle \\ &= \lambda \langle \mathrm{dR}_{\widetilde{M}}([\widetilde{\eta}]), [\widetilde{N}] \rangle \\ &= \lambda \int_{\widetilde{N}} \widetilde{\eta}. \end{aligned} \tag{E.52}$$

Perturbing $\widetilde{N}$ slightly in the colar neighbourhood of ∂M in $\widetilde{M}$, we may assume w.l.o.g. that $\widetilde{N}$ intersects ∂M transversely. It follows that the intersection

$$N := \widetilde{N} \cap M$$

is a submanifold of M with boundary $\partial N \subseteq \partial M$.

We show (E.51), i.e. that $\lambda \alpha_N^M = \alpha$. Let $[\eta] \in H_c^{n-k}(M)$ be a cohomology class, where η is a closed $n-k$-form with compact support in the interior $\mathring{M}$. Extending η by 0 on the second component of $\widetilde{M}$, yields a closed form $\widetilde{\eta} \in \Omega^{n-k}(\widetilde{M})$. It follows from (E.52) that

$$\begin{aligned}\lambda \, \mathrm{PD}_M \left(\alpha_N^M\right)([\eta]) &= \lambda \int_N \eta \\ &= \lambda \int_{\widetilde{N}} \widetilde{\eta} \\ &= \int_{\widetilde{M}} \widetilde{\omega} \wedge \widetilde{\eta} \\ &= \int_M r^*\omega \wedge \eta \\ &= \mathrm{PD}_M(r^*\alpha)([\eta])\end{aligned}$$

It follows that

$$\lambda \alpha_N^M = r^*\alpha = \alpha.$$

Here we have used that r is smoothly homotopic to the identity on M. This completes the proof of Theorem E.18. □

Lemma E.19 *Let M be an n-dimensional oriented manifold, possibly non-compact and with boundary, $k \geq 0$ be an integer, $N \subseteq M$ be an oriented $(n-k)$-dimensional submanifold without boundary that is closed as a subset, and $b \in H_k(M, \mathbb{R})$ be a singular homology class. Assume that there exists a cycle $c := \sum_i \lambda_i c_i$ that represents b such that*

$$\operatorname{im}(c_i) \cap N = \emptyset. \tag{E.53}$$

Then

$$\langle \mathrm{dR}(\alpha_N), b \rangle = 0. \tag{E.54}$$

Proof of Lemma E.19: By Proposition E.15 there exists a smooth $(n-k)$-cycle $c' := \sum_i \lambda_i' c_i'$ in the open subset $M \setminus N \subseteq M$ that is homologous to c in $M \setminus N$. Hence c' is homologous to c in M. By Proposition 6.25 (Localization Principle) in the book [BT] there exists a representative $\omega \in \Omega^k(M)$ of α_N with support in an arbitrary small neighbourhood of N. Since $\operatorname{im}(c_i') \cap N = \emptyset$, we may choose a representative ω of α_N with support in $M \setminus \bigcup_i \operatorname{im}(c_i')$. It follows that ${c_i'}^*\omega = 0$ and therefore

$$\begin{aligned} \langle \mathrm{dR}(\alpha_N), b \rangle &= \langle \mathrm{dR}([\omega]), b \rangle \\ &= \sum_i \lambda_i' \int_{[0,1]^{n-k}} {c'}_i^* \omega = 0. \end{aligned}$$

This proves Lemma E.19. □

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