A DISCONTINUOUS CAPACITY

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Abstract. We introduce the spherical capacity and show that it is not continuous on a smoothly bounded smooth family of open sets in dimension four.

1. Introduction

In [2] K. Cieliebak, H. Hofer, J. Latschev, and F. Schlenk asked the following question:

Question ([2, Problem 7])**.** Are capacities continuous on all smooth families of domains bounded by smooth hypersurfaces? Here a family of domains is called **smooth** if their boundaries fit in a smooth isotopy of embeddings.

The answer is *no* in dimension 4. Below we define a capacity, which we call the *spherical capacity* and prove the following theorem.

Theorem 1.1. *There is a smooth family U_ε, $\varepsilon \in (0,1)$, of ellipsoidal shells in $\mathbb{R}^4$ such that for the spherical capacity s the function*

$$\varepsilon \longmapsto s(U_\varepsilon)$$

is not continuous.

The proof involves a non-embedding result for ellipsoids by F. Schlenk [11] and and the symplectic 4-ball theorem by M. Gromov [6]. We begin with the definition of the spherical capacity:

Definition 1.2. For symplectic manifolds (V,ω) of dimension ≥ 4 we call

$$s(V,\omega) := \sup\{\pi r^2 > 0 \,|\, \exists \text{ symplectic embedding } S_r^{2n-1} \hookrightarrow (V,\omega)\}$$

the **spherical capacity**. By a **symplectic embedding of the sphere** S_r^{2n-1} we mean a symplectic embedding of a neighbourhood of $S_r^{2n-1} \subset \mathbb{R}^{2n}$.

In [6] Gromov proved the non-squeezing theorem which says that the open ball $B_r = B_r^{2n}$ of radius $r > 0$ embeds symplectically into the symplectic cylinder $Z_R = B_R^2 \times \mathbb{R}^{2n-2}$ if and only if $r \leq R$. It is natural to ask whether a similar result holds for symplectic embeddings of the sphere $S_r^{2n-1} = \partial B_r$ into Z_R, i.e. embeddings of neighbourhoods of $S_r^{2n-1} \subset \mathbb{R}^{2n}$. If the dimension $2n \geq 4$ a positive answer was given in [12, 5], where it was shown, that such an embedding exists precisely if $r < R$. As Gromov's non-squeezing leads to a symplectic capacity

$$w(V,\omega) = \sup\{\pi r^2 > 0 \,|\, \exists \text{ symplectic embedding } B_r \hookrightarrow (V,\omega)\},$$

the Gromov width, the spherical non-squeezing theorem from [12, 5] is related to the spherical capacity. That this is a normalized symplectic capacity in dimension ≥ 4 follows from [12, 5]. Recall that this means the following.

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Definition 1.3. A **normalized symplectic capacity** is an assignment of a real number $c(V,\omega) \in [0,\infty]$ to a symplectic manifold (V,ω) of fixed dimension satisfying the following conditions:

Monotonicity: If there exists a symplectic embedding $(V,\omega) \hookrightarrow (V',\omega')$, then $c(V,\omega) \leq c(V',\omega')$.
Conformality: For any $a > 0$ we have $c(V, a\omega) = a\, c(V,\omega)$.
Normalization: $c(B_1) = \pi = c(Z_1)$.

Given a symplectic manifold (V,ω) an **extrinsic capacity** on subsets $U \subset V$ is a real number $c(U,\omega) \in [0,\infty]$ satisfying the above conditions with monotonicity replaced by:

Relative monotonicity: If there exists a symplectomorphism of (V,ω) which maps U_1 into U_2, then $c(U_1,\omega) \leq c(U_2,\omega)$.

We suppress the standard symplectic structure $\mathrm{d}\mathbf{x} \wedge \mathrm{d}\mathbf{y}$ on $\mathbb{R}^{2n}$ in the notation.

2. Motivation

The spherical capacity is a variant of the regular coisotropic capacity of hypersurfaces introduced in [12]. Consider a hypersurfaces M in a symplectic manifold (V,ω) such that the characteristics are all closed, form a smooth fibration over the leaf space with fibre S^1, and are contractible in V. Let $\inf(M)$ denote the least positive symplectic area a smooth disc in V with boundary on a closed characteristic of M can have and set

$$a(V,\omega) := \sup\{\inf(M) \mid M \subset (V,\omega)\},$$

where the supremum runs over all hypersurfaces as described. This defines a normalized capacity on all symplectically aspherical symplectic manifolds, see [12], the **regular coisotropic hypersurface capacity**. The restriction to spheres is denoted by a_S and we get

$$w \leq s \leq a_S \leq a.$$

The **contact type embedding capacity**

$$c(V,\omega) := \sup\{\inf(\alpha) \mid \exists \text{ contact type embedding } (M,\alpha) \hookrightarrow (V,\omega)\},$$

see [4, 5], yields a second approach to the spherical capacity. Here the supremum is taken over all closed contact manifolds (M,α) of dimension $(2n-1)$, where $\inf(\alpha)$ is the infimum of all positive periods of closed Reeb orbits w.r.t. the contact from α. By a contact type embedding $j\colon (M,\alpha) \hookrightarrow (V,\omega)$ we mean that there is a Liouville vector field Y for ω defined near $j(M)$ such that $j^*(i_Y\omega) = \alpha$. If one restricts in the definition of c to contact manifolds diffeomorphic to the $(2n-1)$-sphere, one obtains a normalized capacity c_S as well. These capacities yield a second proof of the spherical non-squeezing theorem and we have

$$w \leq s \leq c_S \leq c.$$

Definition 2.1. For symplectic manifolds (V,ω) of dimension ≥ 4 we call

$$e(V,\omega) := \sup\{\pi r_1^2 > 0 \mid \exists \text{ symplectic embedding } \partial E \hookrightarrow (V,\omega)\}.$$

the **ellipsoidal capacity**. By a symplectic embedding of the boundary of E we mean a symplectic embedding of a neighbourhood of

$$\partial E := \left\{ \frac{x_1^2 + y_1^2}{r_1^2} + \ldots + \frac{x_n^2 + y_n^2}{r_n^2} = 1 \right\} \subset \mathbb{R}^{2n}$$

with positive symplectic half axes $r_1 \leq \ldots \leq r_n$.

Notice that

$$s \leq e \leq c_S.$$

The question which now appears is whether the two capacities s and e coincide.

Theorem 2.2. *The boundary of each 4-dimensional ellipsoid with different symplectic main axes has a tubular neighbourhood U, such that $s(U) < e(U)$.*

Remark 2.3. Both quantities s and c_S do not define capacities in dimension 2. Because they satisfy the monotonicity axiom, they would otherwise measure the area of the annuli $B_{1+\varepsilon} \setminus \overline{B}_{1-\varepsilon}$ in $\mathbb{R}^2$, see [7]. Alternatively, for the first observe that $(r, \theta) \mapsto (\sqrt{r^2 + a}, \theta)$ maps $S^1 = \partial B$ symplectically to the circle of radius $\sqrt{1+a}$ for all $a \in (-1, \infty)$. For the second, consider the contact form $\frac{1}{2}(r^2 + a)d\theta$ on S^1. Its smallest action equals $(1 + a)\pi$.

In contrary, if one measures the largest minimal action an embedding of restricted contact type (with image in a certain open subset) has, this results in a extrinsic normalized capacities also in dimension 2, cf. [4]. In this case the monotonicity axiom is only valid in the weaker sense requiring all symplectomorphisms defined on the ambient space, cf. [9, p. 375].

3. The Boundary Gromov width

For open subsets U of a symplectic manifold (V, ω) there is version of the Gromov width which interpolates between w and s. Consider symplectic embeddings of the closed ball $\overline{B}_r$ into (V, ω) which map the boundary sphere ∂B_r into U. The **boundary Gromov width** $w_\partial(U)$ is then defined to be the supremum of $\pi r^2 > 0$ taken over all such embeddings and is a normalized extrinsic capacity.

As a first step in the proofs of Theorems 1.1 and 2.2 we estimate $w_\partial(U_\varepsilon)$ for open subsets U_ε of $\mathbb{R}^{2n}$. For that we consider an ellipsoid

$$E := E(r_1, \ldots, r_n) = \Big\{ \frac{x_1^2 + y_1^2}{r_1^2} + \ldots + \frac{x_n^2 + y_n^2}{r_n^2} < 1 \Big\}$$

with positive symplectic half axes $r_1 \leq \ldots \leq r_n$. We define an ellipsoidal shell via

$$U_\varepsilon := (1 + \varepsilon)E \setminus \overline{(1 - \varepsilon)E}$$

provided $\varepsilon > 0$ is sufficiently small.

Lemma 3.1. *If two of the symplectic radii of E are different we have*

$$w_\partial(U_\varepsilon) \longrightarrow 0$$

as ε tends to 0.

Proof. We consider a symplectic embedding φ of $\overline{B}_r$ into $\mathbb{R}^{2n}$ such that $\varphi(S_r^{2n-1})$ is contained in U_ε. Then there are two cases which we need to consider: either $\varphi(B_r)$ is contained in U_ε or not. The latter implies that the bounded component of the complement of $\varphi(S_r^{2n-1})$ contains $(1 - \varepsilon)\partial E$ and, hence, $(1 - \varepsilon)E \subset \varphi(B_r)$. We claim that for $\varepsilon > 0$ small enough the second case can not appear, so that necessarily $\varphi(B_r) \subset U_\varepsilon$. The lemma follows then by comparing the volume.

Assume now that $\varphi(B_r) \not\subset U_\varepsilon$ for some $\varepsilon > 0$. Then $(1 - \varepsilon)E \subset \varphi(B_r)$, as we remarked above. Again comparing the volume we get a lower bound

$$(1 - \varepsilon)^n r_1 \cdot \ldots \cdot r_n < r^n.$$

For an upper bound observe that $\varphi(\overline{B}_r) \subset (1+\varepsilon)Z_{r_1}$. Invoking Gromov's non-squeezing theorem we get

$$r < (1+\varepsilon)r_1.$$

Combining both inequalities yields

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^n < \frac{r_1^n}{r_1 \cdot \ldots \cdot r_n}.$$

Because the r_j are not all the same the right hand side is < 1. Therefore, there exists a positive number ε_0, which only depends on the r_j, such that $\varepsilon > \varepsilon_0$. The lemma follows now by taking $\varepsilon \leq \varepsilon_0$ which excludes the second case. □

On the other hand:

Lemma 3.2. *We have* $e(U_\varepsilon) \longrightarrow \pi r_1^2$ *and* $c(U_\varepsilon) \longrightarrow \pi r_1^2$ *as* ε *tends to* 0.

Proof. Indeed,

$$\pi r_1^2 = \inf(\lambda_{\mathrm{st}}|_{T\partial E}) \leq e(U_\varepsilon) \leq e\big((1+\varepsilon)E\big) = (1+\varepsilon)^2 \pi r_1^2,$$

where λ_{st} denotes the radial Liouville form $\frac{1}{2}(\mathbf{x}\mathrm{d}\mathbf{y} - \mathbf{y}\mathrm{d}\mathbf{x})$ on $\mathbb{R}^{2n}$. For c the argument is the same. □

4. Discontinuity

A refinement of the proof of Lemma 3.1 shows that the boundary Gromov width w_∂ is discontinuous. We consider the ellipsoid E with radii $r_1 = \ldots = r_{n-1} = 1$ and $r_n = R$ for a real number $R \in (1, \sqrt[2n]{2})$. The corresponding ellipsoidal shell is again denoted by U_ε.

Proposition 4.1. *For* $U_\varepsilon \subset \mathbb{R}^{2n}$, $n \geq 2$, *as described above the function*

$$\varepsilon \longmapsto w_\partial(U_\varepsilon)$$

is not continuous.

Proof. We will show that the function jumps at

$$\varepsilon_0 := \frac{R-1}{R+1}.$$

For this we consider two cases.

We claim that for all $\varepsilon \in (0, \varepsilon_0]$ we have $w_\partial(U_\varepsilon) = w(U_\varepsilon)$. For this we need to exclude symplectic embeddings φ of B_r into $\mathbb{R}^{2n}$, such that $(1-\varepsilon)E \subset \varphi(B_r)$, similarly to Lemma 3.1. We use a result of F. Schlenk [11, Theorem 1], which is based on the Ekeland-Hofer capacities [3]. By this result, since $R \in (1, \sqrt[2n]{2})$, there exists a symplectic embedding of $(1-\varepsilon)E$ into B_r only if $(1-\varepsilon)R \leq r$. Moreover, Gromov's non-squeezing yields the inequality $r < (1+\varepsilon)$. Combining both we get

$$\frac{1-\varepsilon}{1+\varepsilon} < \frac{1}{R},$$

so that the symplectic embeddings under considerations can be excluded by our choice of ε_0. Hence $w_\partial(U_\varepsilon) = w(U_\varepsilon)$. Because the Gromov width of U_ε is bounded in terms of its volume from above, we obtain

$$w_\partial(U_\varepsilon) < \sqrt[n]{(1+\varepsilon)^{2n} - (1-\varepsilon)^{2n}}\, \pi R^2.$$

On the other hand, for $\varepsilon > \varepsilon_0$, where the spheres

$$S^{2n-1}_{\frac{2R}{R+1}} \subset \overline{U}_{\varepsilon_0}$$

start to appear, we have the lower bound

$$\frac{4\pi R^2}{(R+1)^2} \leq w_\partial(U_\varepsilon).$$

Consequently, we get for all $R \in (1, \sqrt[2n]{2})$ and for all $\varepsilon > \varepsilon_0$ the following estimate:

$$w_\partial(U_{\varepsilon_0}) < \sqrt[n]{R^{2n}-1}\,\frac{4\pi R^2}{(R+1)^2} < w_\partial(U_\varepsilon).$$

In other words the function $\varepsilon \mapsto w_\partial(U_\varepsilon)$ is not continuous at ε_0. □

5. Proof of the Theorems

The second ingredient of the proofs of Theorems 1.1 and 2.2 are the following considerations:

Proposition 5.1. *The capacities w and s coincide on closed minimal symplectic 4-manifolds.*

Proof. As the Gromov width is the smallest capacity we get $w \leq s$ on all symplectic manifolds. For the converse consider a symplectic embedding of S^3_r into a closed minimal symplectic 4-manifold (V, ω). By [1] or [5, Proposition 4.10] its image S_r separates, so that S_r cuts out a strong symplectic filling (W, ω). Since (W, ω) is minimal it is symplectomorphic to the standard 4-ball $\overline{B}_r$ by a theorem of Gromov [6, p. 311], c.f. [4, Remark 2.3] or [10, Theorem 9.4.2]. Hence, $s \leq w$. □

Notice that the minimality assumption is essential. The volume (and hence the Gromov width) of the symplectic blow up of $\mathbb{C}P^2$ obtained by cutting out a ball of radius $1-\varepsilon$ in $B_1 \subset \mathbb{C}P^2$ can be made arbitrary small. In contrast, the spherical capacity stays $\geq \pi$.

Proposition 5.2. *For all open subsets U in $\mathbb{R}^4$ we have $s(U) = w_\partial(U)$.*

Proof. The argument is almost the one from Proposition 5.1. Just observe that any Liouville vector field defined near and determined by the symplectic embedding S_r of S^3_r points out of the interior of S_r. Otherwise, we could use a sufficiently large ball to cut out a connected symplectic manifold with convex boundary consisting of two standard spheres. This would violate [8, Theorem 1.2] or [5, Theorem 3.4]. □

Question 5.3. The proceeding proposition is valid in greater generality, e.g. for all subcritical Stein surfaces, see [5, Theorem 3.4]. The critical case seems to be not known. Therefore, we ask:

Does the Liouville vector field defined by a closed hypersurface of contact type (M, α) in a critical Stein manifold of dimension ≥ 4 point out of the interior of M, if (M, α) is not of restricted contact type?

Proof of Theorem 2.2. By choosing $\varepsilon > 0$ small enough we can make $e(U_\varepsilon)$ as close to πr_1^2 and $w_\partial(U_\varepsilon)$ as small as we wish, see Lemmata 3.1 and 3.2. Moreover, by the proceeding proposition $s(U_\varepsilon)$ and $w_\partial(U_\varepsilon)$ are equal. □

Proof of Theorem 1.1. By Proposition 5.2 w_∂ equals the spherical capacity on $\mathbb{R}^4$ - defines there an intrinsic capacity. The claim follows from Proposition 4.1. □

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