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# Hofer Geometry of a Subset of a Symplectic Manifold

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**Abstract** To every closed subset $X$ of a symplectic manifold $(M, \omega)$ we associate a natural group of Hamiltonian diffeomorphisms $\mathrm{Ham}(X, \omega)$. We equip this group with a semi-norm $\|\cdot\|^{X,\omega}$, generalizing the Hofer norm. We discuss $\mathrm{Ham}(X, \omega)$ and $\|\cdot\|^{X,\omega}$ if $X$ is a symplectic or isotropic submanifold. The main result involves the relative Hofer diameter of $X$ in $M$. Its first part states that for the unit sphere in $\mathbb{R}^{2n}$ this diameter is bounded below by $\frac{\pi}{2}$, if $n \geq 2$. Its second part states that for $n \geq 2$ and $d \geq n$ there exists a compact subset $X$ of the closed unit ball in $\mathbb{R}^{2n}$, such that $X$ has Hausdorff dimension at most $d+1$ and relative Hofer diameter bounded below by $\pi / \mathrm{k}(n, d)$, where $\mathrm{k}(n, d)$ is an explicitly defined integer.

**Keywords** Hofer norm · Hofer diameter · relative Hofer geometry · group of Hamiltonian diffeomorphisms

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## 1 Motivation and main results

The theme of this article is the following.

*Question 1* How much symplectic geometry can a small subset of a symplectic manifold carry?

To be specific, we interpret "small" as "of Hausdorff dimension bounded above by a given number". In the article [SZ1] we gave some answers to this question in terms of the displacement energy of the subset, non-squeezing, and exoticness of symplectic structures. Here we look at this question from a dynamical point of view. The goal is to lay the foundations of a Hofer geometry for a subset of a symplectic manifold, both from an absolute and relative view-point, and to explore this geometry in examples.

### *Absolute Hofer geometry of a subset*

Let $(M,\omega)$ be a symplectic manifold and $X \subseteq M$ a closed subset. (For simplicity all manifolds in this article are assumed to have empty boundary.) We define the set of *Hamiltonian diffeomorphisms of* $X$, $\mathrm{Ham}(X,M,\omega)$, as follows.

Let $V : [0,1] \times M \to TM$ be a smooth time-dependent vector field on $M$. For every $t \in [0,1]$ we denote by $\varphi_V^t$ the time-$t$ flow of $V$. Its domain is by definition the set $\mathcal{D}_V^t$ of all points $x_0 \in M$ for which the problem

$$\dot{x}(s) = V(s,x(s)), \quad x(0) = x_0$$

has a solution $x \in C^\infty([0,t],M)$. We say that $V$ is $X$*-compatible* iff $X \subseteq \mathcal{D}_V^1$, and $\varphi_V^t(X) = X$, for every $t \in [0,1]$. For a function $H \in C^\infty([0,1] \times M,\mathbb{R})$ we denote by $X_H := X_H^\omega$ its time-dependent Hamiltonian vector field, and we abbreviate $\varphi_H^t := \varphi_{H,\omega}^t := \varphi_{X_H}^t$. We define

$$\begin{aligned} \mathcal{H}(M,\omega,X) &:= \big\{H \in C^\infty([0,1] \times M,\mathbb{R}) \,\big|\, X_H \text{ is } X\text{-compatible}\big\}, \\ \mathrm{Ham}(X,\omega) := \mathrm{Ham}(X,M,\omega) &:= \big\{\varphi_H^1|_X \,\big|\, H \in \mathcal{H}(M,\omega,X)\big\}. \end{aligned} \tag{1}$$

Note that for $X = M$

$$\mathcal{H}(M,\omega) := \mathcal{H}(M,\omega,M)$$

is the set of all functions $H \in C^\infty([0,1]\times M,\mathbb{R})$ whose Hamiltonian time-$t$ flow is well-defined on $M$ and a diffeomorphism of $M$, for every $t \in [0,1]$. Furthermore, $\mathrm{Ham}(M,\omega)$ is the set of all time-one flows of functions in $\mathcal{H}(M,\omega)$. The following result shows that $\mathrm{Ham}(X,\omega)$ together with composition is a group, and that it naturally generalizes $\mathrm{Ham}(M,\omega)$.

**Proposition 1 (Hamiltonian diffeomorphisms of a subset)** *The following statements hold.*

*(i) The set* $\mathrm{Ham}(X,\omega)$ *is a subgroup of the group of homeomorphisms of* $X$*.*

*(ii) If* $X$ *is a symplectic submanifold of* $M$ *then*

$$\mathrm{Ham}(X,M,\omega) = \mathrm{Ham}(X,X,\omega|_X) \tag{2}$$

The trickiest part of the proof of this result is the inclusion "$\supseteq$" in (2). The idea is to extend a given Hamiltonian function $H : [0,1] \times X \to \mathbb{R}$ to a function $\widetilde{H} : [0,1] \times M \to \mathbb{R}$ in such a way that the restriction of the time-$t$ flow of $\widetilde{H}$ to $X$ agrees with the time-$t$ flow of $H$ (see Proposition 12 below).

We define the *Hofer semi-norm on* $\mathrm{Ham}(X, \omega)$ to be the map

$$\| \cdot \|^{X,\omega} : \mathrm{Ham}(X, \omega) \to [0, \infty]$$

given as follows. Let $H \in C^\infty([0,1] \times M, \mathbb{R})$. We define the *Hofer semi-norm of* $H$ *on* $X$ to be

$$\|H\|_X := \int_0^1 \left( \sup_X H(t, \cdot) - \inf_X H(t, \cdot) \right) dt \in [0, \infty]. \tag{3}$$

(It follows from Lemma 33 below that this integral is well-defined.) For every $\varphi \in \mathrm{Ham}(X, \omega)$ we define

$$\|\varphi\|^{X,\omega} := \inf \left\{ \|H\|_X \,\middle|\, H \in \mathcal{H}(M, \omega, X) : \varphi_H^1|_X = \varphi \right\}.$$

By the next result the map $\| \cdot \|^{X,\omega}$ is a semi-norm, which naturally generalizes $\| \cdot \|^{M,\omega}$. Furthermore, $\| \cdot \|^{M,\omega}$ is a norm. We will use the following definition. Let $G$ be a group. By a *semi-norm* on $G$ we mean a map $\| \cdot \| : G \to [0, \infty]$ such that

$$\|\mathbf{1}\| = 0, \tag{4}$$

$$\|g^{-1}\| = \|g\|, \tag{5}$$

$$\|gh\| \le \|g\| + \|h\|, \tag{6}$$

for every $g, h \in G$. We call $\| \cdot \|$ a *norm* iff also

$$\|g\| = 0 \quad \Longrightarrow \quad g = \mathbf{1}. \tag{7}$$

We call $\| \cdot \|$ *invariant* iff

$$\|ghg^{-1}\| = \|h\|, \quad \forall g, h \in G.$$

**Proposition 2 (Hofer semi-norm for a subset)** *The following statements hold.*

*(i) The map* $\| \cdot \|^{X,\omega}$ *is an invariant semi-norm.*
*(ii) Assume that* $X$ *is a symplectic submanifold of* $M$*. Then the map* $\| \cdot \|^{X,\omega}$ *is a norm and*

$$\| \cdot \|^{X,\omega} = \| \cdot \|^{X,\omega|_X}. \tag{8}$$

The proof of this result is similar to the proof of Proposition 1.

For a general closed subset $X \subseteq M$ the map $\| \cdot \|^{X,\omega}$ may be degenerate, i.e., not satisfy (7). It is maximally degenerate if $X$ is an isotropic submanifold. This is a consequence of the following result.

**Proposition 3** *If* $X$ *is a connected isotropic submanifold then*

$$\| \cdot \|_X \equiv 0 : \mathcal{H}(M, \omega, X) \to [0, \infty].$$

*Relative Hofer geometry*

Let $Y \subseteq M$ be a closed subset containing $X$. We may compare the Hofer geometries of the sets $X$ and $Y$ as follows: We define the *Hofer semi-norm on $X$ relative to $Y$* to be the map

$$\|\cdot\|_X^{Y,\omega} : \mathrm{Ham}(X,\omega) \to [0,\infty], \tag{9}$$
$$\|\varphi\|_X^{Y,\omega} := \inf\left\{\|\psi\|^{Y,\omega} \,\middle|\, \psi \in \mathrm{Ham}(Y,\omega) :\ \psi|_X = \varphi\right\}.$$

Intuitively, this map measures how short a Hamiltonian path on $X$ can be made inside $Y$. The definition (9) has the following natural properties.

**Proposition 4 (Relative Hofer semi-norm)** *The following statements hold.*

*(i) The map $\|\cdot\|_X^{Y,\omega}$ is a semi-norm.*
*(ii) If $X$ is compact and contained in the interior of $Y$ then $\|\cdot\|_X^{Y,\omega}$ is invariant (under conjugation by elements of* $\mathrm{Ham}(X,\omega)$*).*
*(iii) Let $Y' \subseteq M$ be a closed subset such that $Y$ is contained in the interior of $Y'$. If $Y$ is compact and non-empty, then we have*

$$\|\cdot\|_X^{Y,\omega} \geq \|\cdot\|_X^{Y',\omega}.$$

In the case $X = Y$ we have, by definition,

$$\|\cdot\|_X^{X,\omega} = \|\cdot\|^{X,\omega}.$$

However, in general, the semi-norms $\|\cdot\|_X^{Y,\omega}$ and $\|\cdot\|^{X,\omega}$ may differ a lot. As an example, in [Zi2, Corollary 7] the second author proved the following result.

**Theorem 5 (Relative Hofer diameter)** *Let $(M,\omega)$ and $(M',\omega')$ be connected symplectic manifolds and $X' \subseteq M'$ a finite subset. Assume that $M$ is closed and $M'$ has positive dimension. Then we have*

$$\|\cdot\|_{M\times X'}^{M\times M',\omega\oplus\omega'} \equiv 0. \tag{10}$$

In contrast with this result, under the hypotheses of Theorem 5, the absolute Hofer semi-norm $\|\cdot\|^{M\times X',\omega\oplus\omega'}$ is non-degenerate. This follows from Proposition 2(ii).

The relative Hofer semi-norm gives rise to the *Hofer diameter of $X$ relative to $Y$*, which we define as

$$\mathrm{diam}(X,Y,\omega) := \sup\left\{\|\varphi\|_X^{Y,\omega} \,\middle|\, \varphi \in \mathrm{Ham}(X,\omega)\right\}. \tag{11}$$

This quantity measures how much Hamiltonian dynamics of $Y$ is captured by the subset $X$. Our main result is motivated by the following instances of Question 1.

*Question 2 (Hofer diameter of a subset)* What is the relative Hofer diameter $\mathrm{diam}(X,M,\omega)$ for a given (small) closed subset $X \subseteq M$?

We now fix a subset $X_0 \subseteq M$ and a number $d \in [0,\infty)$.

*Question 3 (Maximal Hofer diameter)* What is the supremum of the numbers $\mathrm{diam}(X, M, \omega)$, where $X$ is a compact subset of $X_0$, of Hausdorff dimension at most $d$?

In order to state our result, we define the map

$$\mathrm{k} : \mathbb{N} \times [0, \infty) \to \mathbb{N} \cup \{\infty\} \tag{12}$$

as follows. For $(n, d) \in \mathbb{N} \times [0, \infty)$ we define $\mathrm{k}(n, d)$ to be the infimum of all sums $\sum_{i=1}^{\ell} k_i$, where $\ell \in \mathbb{N}$ is such that

$$\ell \geq 2, \tag{13}$$

and $k_1, \ldots, k_\ell \in \mathbb{N}$ are integers for which there exist numbers $n_i \in \mathbb{N}$, for $i = 1, \ldots, \ell$, such that the following conditions hold:

$$n_i \geq k_i, \tag{14}$$

$$\textstyle\sum_i k_i n_i \geq n, \quad \sum_i k_i(2n_i - k_i) \leq d, \tag{15}$$

$$2 \min\{n_1, \ldots, n_\ell\} \leq n. \tag{16}$$

Our main result provides lower bounds on the quantities in Questions 2 and 3 in the case $(M, \omega) := (\mathbb{R}^{2n}, \omega_0)$, with $X$ the unit sphere $S^{2n-1}$ (for Question 2) and $X_0$ the closed unit ball $\overline{B}^{2n} \subseteq \mathbb{R}^{2n}$:

**Theorem 6 (Relative Hofer diameters of small sets)** *Let $n \in \{2, 3, \ldots\}$ be an integer. Then the following statements hold.*

*(i) We have*

$$\mathrm{diam}\left(S^{2n-1}, \mathbb{R}^{2n}, \omega_0\right) \geq \frac{\pi}{2}. \tag{17}$$

*(ii) For every real number $d \geq n$ there exists a compact subset $X \subseteq \overline{B}^{2n}$ of Hausdorff dimension at most $d + 1$, such that*

$$\mathrm{diam}\left(X, \mathbb{R}^{2n}, \omega_0\right) \geq \frac{\pi}{\mathrm{k}(n, d)}. \tag{18}$$

The estimate (17) is sharp up to a factor of 16. This follows from the argument after Proposition 8 below. The estimate is wrong for $n = 1$. In fact, in this case we have $\mathrm{diam}(S^1, \mathbb{R}^2, \omega_0) = 0$. This follows from Propositions 3 and 4(iii) (with $X = Y = S^1$ and $Y' = M = \mathbb{R}^2$).

The proof of Theorem 6 is based on a coisotropic intersection result proved by the authors in [SZ1]. As another key ingredient, given a pair $(X_0, \alpha)$, where $X_0 \subseteq M$ is a subset and $\alpha \in \Omega^1(M)$, we will define what it means for $(X_0, \alpha)$ to be "rigidifying". Given a compact subset $X_0$, we will prove a lower bound on the Hofer norm of a certain Hamiltonian diffeomorphism, if there exists a function $f : M \to \mathbb{R}$ for which $(X_0, df)$ is rigidifying and some other conditions hold (Lemma 15 below). We will show that these conditions are satisfied if there exists a certain Hamiltonian Lie group action (Lemma 17).

The next result summarizes some properties of the map k, which occurs in part (ii) of Theorem 6. We define the function $\mathrm{K}:\mathbb{N}\to\mathbb{N}$ by

$$\mathrm{K}(n):=\min\left\{\sum_{i=1}^{\ell}k_i\,\Big|\,\ell\in\mathbb{N},\;k_1,\dots,k_\ell\in\mathbb{N}:\;n=\sum_i k_i^2\right\}.$$

The first few values of this function are

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n=$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $\mathrm{K}(n)=$ | 1 | 2 | 3 | 2 | 3 | 4 | 5 | 4 | 3 | 4 | 5 | 6 | 5 | 6 | 7 | 4 | 5 | 6 | 7 | 6 |

**Proposition 7** *For every $n\in\{2,3,\dots\}$ we have*

$$\mathrm{k}(n,d)\le 2n-d,\quad \forall d\in[n,2n-2], \tag{19}$$

$$\mathrm{k}(n,n)=\mathrm{K}(n),\quad \textit{if } n\neq k^2,\ \forall k\in\mathbb{N}, \tag{20}$$

$$\mathrm{K}(n)<\sqrt{n}+2^{\frac{3}{2}}\sqrt[4]{n}. \tag{21}$$

This proposition implies explicit lower bounds on the right-hand side of inequality (18).

To put Theorem 6 into perspective, for each open subset $U\subseteq M$, we define the *extension relative Hofer diameter of $U$* to be

$$\mathrm{Diam}(U,M,\omega):=$$

$$\sup\left\{\|\varphi_H^1\|^{M,\omega}\,\Big|\,H\in\mathcal{H}(M,\omega):\ \text{support of } H\subseteq[0,1]\times U\right\}\in[0,\infty]$$

(where $\mathcal{H}(M,\omega):=\mathcal{H}(M,\omega,M)$). This diameter measures the sizes of trivial extensions of Hamiltonian diffeomorphisms generated by functions with support in $[0,1]\times U$. Note that in contrast with this, the definition of $\mathrm{diam}(X,M,\omega)$ involves the restriction of a map $\psi:M\to M$ to $X$. The two diameters are related to each other as follows:

**Proposition 8 (Relative Hofer diameters)** *Let $(M,\omega)$ be a symplectic manifold, $U\subseteq M$ an open subset, and $X\subseteq U$ a compact subset. Then*

$$\mathrm{Diam}(U,M,\omega)\ge\mathrm{diam}(X,M,\omega). \tag{22}$$

We denote by $B^{2n}(a)\subseteq\mathbb{R}^{2n}$ the open ball of radius $\sqrt{a/\pi}$ around 0. It follows from [Zi2, Corollary 2] and a cutoff argument that

$$\mathrm{Diam}\left(B^{2n}(a),\mathbb{R}^{2n},\omega_0\right)\le 8a.$$

(The proof of this result is a variant of an argument by J.-C. Sikorav [Si].) Combining this with (22), it follows that

$$\mathrm{diam}\left(S^{2n-1},\mathbb{R}^{2n},\omega_0\right)\le 8\pi.$$

This shows that the estimate (17) in Theorem 6 is sharp up to a factor of 16.

*Remarks*

- **On Theorem 6.** A straight-forward calculation shows that

$$\operatorname{diam}\big(rX, \mathbb{R}^{2n}, \omega_0\big) = r^2 \operatorname{diam}\big(X, \mathbb{R}^{2n}, \omega_0\big),$$

  for every $X \subseteq \mathbb{R}^{2n}$ and $r > 0$. Hence Theorem 6 implies "rescaled versions" of itself, e.g., that $\operatorname{diam}\big(S^{2n-1}(a), \mathbb{R}^{2n}, \omega_0\big) \geq \frac{a}{2}$ for every $n \geq 2$ and $a > 0$. Here $S^{2n-1}(a) \subseteq \mathbb{R}^{2n}$ denotes the sphere of radius $\sqrt{a/\pi}$ around 0.
  The number $\mathrm{k}(n, d)$ occurring in this result is a modified version of a quantity defined in [SZ1].
- **On further research.** In Section 3 below we will develop a framework for finding lower bounds on relative Hofer diameters. This technique can be exploited in further examples.
- **On compact supports.** Analogously to the group $\operatorname{Ham}(X, \omega)$ (as defined in (1)), one can define the group $\operatorname{Ham}_c(X, \omega)$ of Hamiltonian diffeomorphisms on $X$ generated by a compactly supported function. In general, this group is strictly smaller than $\operatorname{Ham}(X, \omega)$, as can be seen from the example

$$M = X := \mathbb{R}^{2n}, \quad \omega := \omega_0, \quad \varphi^t : \mathbb{R}^{2n} = \mathbb{C}^n \to \mathbb{C}^n, \varphi^t(x) := e^{it}x,$$

  for some $t \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. Then $\varphi^t \in \operatorname{Ham}(\mathbb{R}^{2n}, \omega_0) \setminus \operatorname{Ham}_c(\mathbb{R}^{2n}, \omega_0)$.
  For the purpose of this article it seems more natural to work with the group $\operatorname{Ham}(X, \omega)$. One argument for this is that $\operatorname{Ham}(M, \omega) \setminus \operatorname{Ham}_c(M, \omega)$ may contain physically relevant maps, as in the above example. (There $\varphi^t$ is the time-$t$ evolution of the harmonic oscillator.)
  Furthermore, if we define the displacement energy $e(X, M, \omega)$ of a subset $X \subseteq M$ based on $\operatorname{Ham}(M, \omega)$ (see (42) below), then there exist triples $(M, \omega, X)$, for which $\overline{X}$ is non-compact and $e(M, \omega, X) < \infty$. (Take for example $(M, \omega) := (\mathbb{R}^2, \omega_0)$ and $X := \mathbb{R} \times \{0\}$.) In contrast with this, if we base the definition of $e(X, M, \omega)$ on $\operatorname{Ham}_c(M, \omega)$ instead, then we have to take special care of subsets $X \subseteq M$ for which $\overline{X}$ is non-compact.
  Note also that unlike $\operatorname{Ham}_c$, Ham has the nice product property

$$\varphi \times \mathrm{id} \in \operatorname{Ham}\big(M \times M', \omega \oplus \omega'\big), \quad \forall \varphi \in \operatorname{Ham}(M, \omega),$$

  for arbitrary symplectic manifolds $(M, \omega)$ and $(M', \omega')$. This gives rise to an estimate for the displacement energy of a product set.

*Related work*

J.-C. Sikorav proved that for every open subset $U \subseteq \mathbb{R}^{2n}$ the diameter $\operatorname{Diam}(U, \mathbb{R}^{2n}, \omega_0)$ is bounded above by 16 times the proper displacement energy of $U$. (See [Si] or Theorem 10, Section 5.6 in the book [HZ].)

On the other hand, let $(M, \omega)$ be a *closed* symplectic manifold with $\pi_2(M) = 0$ and $U \subseteq M$ a non-empty open subset. Then it follows from the proof of Theorem 1.1. in the article [Os] by Y. Ostrover that $\operatorname{Diam}(U, M, \omega) = \infty$.

The *absolute* Hofer diameter

$$\mathrm{diam}(M,\omega) := \mathrm{diam}(M,M,\omega) = \mathrm{Diam}(M,M,\omega)$$

has been calculated for many closed symplectic manifolds. In all known examples it is infinite. For a recent overview and references, see the article by D. McDuff [McD].

In [SZ1] we considered Question 1 from a different point of view, obtaining an inequality between the stable displacement energy and the Gromov width, a stable non-squeezing result for products of spheres, and existence of a stably exotic structure on $\mathbb{R}^{2n}$. These results are consequences of the key result, Theorem 14 below. They involve functions similar to k (as defined in (12)). In [SZ2] we improved the non-squeezing result of [SZ1].

*Organization of the article*

In Section 2.1 we start by proving the first parts of Propositions 1 and 2 in a parallel way. Then we do the same for the second parts. In Section 2.2 we prove Propositions 3, 4, 7, and 8.

In Section 3 we develop a framework for proving a lower bound on the relative Hofer diameter of some subset, and we prove Theorem 6. In Subsection 3.1 we state the key result about coisotropic intersections (Theorem 14), which we proved in the article [SZ1]. In Subsection 3.2 we introduce some "rigidifying property" and show how this implies a lower bound on the relative Hofer norm of a certain Hamiltonian diffeomorphism (Lemma 15). We also prove a sufficient criterion for the "rigidifying property" (Lemma 17). In Subsection 3.3 we prove Theorem 6.

The appendix contains some auxiliary results about symplectic geometry, point-set topology, and manifolds, which are used in the proofs of the results of Section 1.

## 2 Proofs of the propositions

### 2.1 Proofs of Propositions 1 and 2

We start by proving the first parts of Propositions 1 and 2 in a parallel way, postponing the proofs of the second parts to page 11.

We need the following. Let $M$ be a smooth manifold and $U \subseteq [0,1] \times M$ an open subset. We denote by $\pi : TM \to M$ the canonical projection. Let $V : U \to TM$ be a smooth map such that $\pi \circ V(t,x) = x$, for every $(t,x) \in U$. (If $U = [0,1] \times M$ then this means that $V$ is a time-dependent vector field on $M$.) We denote $V^t := V(t,\cdot)$, for $t \in [0,1]$. We define $\mathcal{D}_V$ to be the set of all pairs $(t_0,x_0) \in [0,1] \times M$ for which there exists a solution $x \in C^\infty([0,t_0],M)$ of the equations

$$x(0) = x_0, \quad (t,x(t)) \in U, \quad \dot{x}(t) = V^t \circ x(t), \quad \forall t \in [0,t_0]. \qquad (23)$$

Furthermore, we define the flow of $V$ to be the map

$$\mathcal{D}_V \ni (t_0, x_0) \mapsto \varphi_V^{t_0}(x_0) := \varphi_V(t_0, x_0) := x(1) \in M,$$

where $x \in C^\infty([0,1], M)$ the unique solution of (23).

In the following, $(M, \omega)$ is a symplectic manifold and $X \subseteq M$ a closed subset. Let $H, K \in C^\infty([0,1] \times M, \mathbb{R})$. We define

$$\overline{H} : \mathcal{D}_{X_H} \to \mathbb{R}, \quad \overline{H}^t := -H^t \circ \varphi_H^t, \tag{24}$$
$$H\#K : \left\{(t,x) \in [0,1] \times M \,\middle|\, x \in \varphi_H^t(\mathcal{D}_{X_H}^t)\right\} \to \mathbb{R},$$
$$(H\#K)^t := H^t + K^t \circ (\varphi_H^t)^{-1}.$$

It follows from Remark 28 below that the inverse $(\varphi_H^t)^{-1}$ exists and hence $H\#K$ is well-defined, and that the domains of the functions $\overline{H}$ and $H\#K$ are open subsets of $[0,1] \times M$. Their Hamiltonian vector fields are defined on the same sets.

Let $X \subseteq M$ be a closed subset.

**Lemma 9** *If $H, K \in \mathcal{H}(M, \omega, X)$ then we have*

$$X \subseteq \mathcal{D}^1_{X_{\overline{H}}}, \tag{25}$$
$$X \subseteq \mathcal{D}^1_{X_{H\#K}}, \tag{26}$$
$$\varphi^t_{\overline{H}}|_X = \varphi^t_H|_X^{-1}, \quad \forall t \in [0,1], \tag{27}$$
$$\varphi^t_{H\#K}|_X = \varphi^t_H \circ \varphi^t_K|_X, \quad \forall t \in [0,1]. \tag{28}$$

*Proof (of Lemma 9)* These assertions follow from arguments as in the proof of [HZ, Chapter 5, Proposition 1].

*Proof (of Proposition 1(i))* Let $\varphi \in \mathrm{Ham}(X, \omega)$. We show that $\varphi$ is a bijection on $X$: We choose $H \in \mathcal{H}(M, \omega, X)$ such that $\varphi_H^1|_X = \varphi$. By Remark 28 below the map $\varphi_H^1$ is injective. Furthermore, by the definition of $\mathcal{H}(M, \omega, X)$, we have $\varphi(X) = \varphi_H^1(X) = X$. It follows that $\varphi$ is a bijection from $X$ to itself.

**Claim 1** *We have $\varphi^{-1} \in \mathrm{Ham}(X, \omega)$.*

In the proof of this claim we will denote by $\mathrm{Int}\, A$ the interior of a subset $A \subseteq M$.

*Proof (of Claim 1)* We define $\overline{H}$ as in (24). By (25) (Lemma 9) we have $X \subseteq \mathcal{D}^1_{X_{\overline{H}}}$. Since $X$ is closed and $\mathcal{D}^1_{X_{\overline{H}}}$ is open, it follows that there exist closed sets $A_0, A_1 \subseteq M$ such that $X \subseteq \mathrm{Int}\, A_1$, $M \setminus \mathcal{D}^1_{X_{\overline{H}}} \subseteq \mathrm{Int}\, A_0$ and $A_0 \cap A_1 = \emptyset$. By Lemma 29 below there exists a function $f \in C^\infty(M, \mathbb{R})$ such that $f \equiv i$ on $A_i$, for $i = 0, 1$. We define $\widetilde{H} : [0,1] \times M \to \mathbb{R}$ by $\widetilde{H}^t(x) := f(x)\overline{H}^t(x)$, if $x \in A_0$, and $\widetilde{H}^t(x) := 0$, otherwise.

Note that $\widetilde{H}^t = \overline{H}^t$ on $A_1$. Using that $X \subseteq \mathrm{Int}\, A_1$, it follows that $X \subseteq \mathcal{D}^t_{X_{\widetilde{H}}}$ and $\varphi^t_{\widetilde{H}}|_X = \varphi^t_{\overline{H}}|_X$, for every $t \in [0,1]$. Combining this with equality (27) of Lemma 9, it follows that $\varphi^{-1} = \varphi^1_{\widetilde{H}}|_X$. The same equality implies that $\widetilde{H} \in \mathcal{H}(M, \omega, X)$. Hence it follows that $\varphi^{-1} \in \mathrm{Ham}(X, \omega)$. This proves Claim 1.

A similar argument, using (26,28) in Lemma 9 shows that $\mathrm{Ham}(M,\omega)$ is closed under composition. The statement of Proposition 1(i) is a consequence of this, Claim 1 and the fact $\mathrm{id}_X \in \mathrm{Ham}(X,\omega)$.

*Proof (of Proposition 2(i))* That the map $\|\cdot\|^{X,\omega}$ is a semi-norm follows from an argument similar to the proof of Proposition 1(i), using Lemma 9. Invariance follows from a straight-forward argument.

We continue by proving the second parts of Propositions 1 and 2 in a parallel way. We need the following two results.

**Lemma 10** *Assume that $X$ is a symplectic submanifold of $M$, and $H \in \mathcal{H}(M,\omega,X)$. Then we have*

$$\mathcal{D}^1_{X^{\omega|_X}_{H|_X}} = X, \quad \varphi^t_{H|_X,\omega|_X} = \varphi^t_{H,\omega}|_X,\, \forall t \in [0,1]. \tag{29}$$

For the proof of Lemma 10 we need the following result, which will also be used for the proof of Proposition 3.

**Lemma 11** *Assume that $X \subseteq M$ is a submanifold. Then for every $H \in \mathcal{H}(M,\omega,X)$, $t \in [0,1]$, and $x \in X$, we have*

$$X^\omega_{H^t}(x) \in T_xX.$$

*Proof (of Lemma 11)* Let $x_0 \in X$. For $t \in [0,1]$ we denote $x^t := \varphi^t_{H,\omega}(x_0)$. By definition, we have $\varphi^t_{H,\omega}(X) = X$ and hence $x^t \in X$, for every $t \in [0,1]$. Let $t \in [0,1]$. It follows that

$$X_{H^t}(x^t) = \frac{d}{dt}x^t \in T_{x^t}X.$$

Hence, using again $\varphi^t_{H,\omega}(X) = X$, the statement of Lemma 11 follows.

*Proof (of Lemma 10)* Let $x \in X$. Then for every $v \in T_xX$, we have

$$\omega|_X\left(X^{\omega|_X}_{H^t|_X}(x), v\right) = d(H^t|_X)v = dH^t\, v = \omega(X^\omega_{H^t}(x), v).$$

Applying Lemma 11 and using the fact that $T_xX$ is a symplectic subspace of $T_xM$, it follows that

$$X^{\omega|_X}_{H^t|_X}(x) = X^\omega_{H^t}(x).$$

The statement of Lemma 10 follows.

**Proposition 12** *If $X$ is a symplectic submanifold then for every $H \in \mathcal{H}(X,\omega|_X)$ there exists $\widetilde{H} \in \mathcal{H}(M,\omega,X)$ such that*

$$\varphi^t_{\widetilde{H},\omega}|_X = \varphi^t_{H,\omega|_X}, \quad \forall t \in [0,1], \tag{30}$$

$$\widetilde{H}|_{[0,1]\times X} = H. \tag{31}$$

In the proof of this result we will use the following notation. Let $(V,\omega)$ be a symplectic vector space and $W \subseteq V$ a linear subspace. We denote

$$W^\omega := \big\{ v \in V \,\big|\, \omega(v,w) = 0,\ \forall w \in W \big\}. \tag{32}$$

*Proof (of Proposition 12)* Let $H \in \mathcal{H}(X, \omega|_X)$. By Proposition 30 below (applied with $N := X$) there exists an embedding $\psi : E := TX^\omega := \bigcup_{x\in X} T_xX^\omega \to M$ satisfying conditions (81,82). We define $U := \psi(E)$. This is an open subset of $M$ containing $X$.

Since $M \setminus U$ and $X$ are closed and do not intersect, there exists a pair of closed subsets $A_0, A_1 \subseteq M$ such that $M \setminus U \subseteq \mathrm{Int}(A_0)$, $X \subseteq \mathrm{Int}(A_1)$, and $A_0 \cap A_1 = \emptyset$. We choose a function $f$ as in Lemma 29 below. We denote by $\pi : TX^\omega \to X$ the canonical projection, and define

$$\widetilde{H} : [0,1] \times M \to \mathbb{R}, \quad \widetilde{H}(t,x) := \begin{cases} f(x)H\big(t, \pi\circ\psi^{-1}(x)\big), & \text{if } x \in \psi(E), \\ 0, & \text{otherwise.} \end{cases}$$

This function is smooth and satisfies equality (31). We define

$$r := \pi \circ \psi^{-1} : \mathrm{Int}\, A_1 \to X.$$

It follows from (81,82) and our choice $E = TX^\omega$ that this is a smooth retraction onto $X$, satisfying $\ker dr(x) = T_xX^\omega$, for every $x \in X$. Let $t \in [0,1]$. Then we have $H^t \circ r = \widetilde{H}^t$ on $\mathrm{Int}\, A_1$. Hence Lemma 23 below implies that $X^\omega_{\widetilde{H}^t}(x) = X^{\omega|_X}_{H^t}(x)$, for every $x \in X$. It follows that $\widetilde{H} \in \mathcal{H}(M,\omega,X)$ and equality (30) holds. This proves Proposition 12.

We are now ready for the proofs of the remaining parts of Propositions 1 and 2.

*Proof (of Proposition 1(ii))* We show the inclusion "$\subseteq$" in **(2)**: Let $\varphi \in \mathrm{Ham}(X,M,\omega)$. Choosing $H \in \mathcal{H}(M,\omega,X)$ such that $\varphi^1_H|_X = \varphi$, the inclusion "$\subseteq$" is a consequence of Lemma 10.

The inclusion "$\supseteq$" in (2) is a consequence of Proposition 12. This completes the proof of Proposition 1(ii).

*Proof (of Proposition 2(ii))* We show the inequality "$\geq$" in **(8)**: Let $\varphi \in \mathrm{Ham}(X,\omega)$. Let $H \in \mathcal{H}(M,\omega,X)$ be such that $\varphi^1_H|_X = \varphi$. By Lemma 10 the conditions (29) hold. By the definition of $\|\cdot\|^{X,\omega|_X}$, it follows that

$$\|\varphi\|^{X,\omega|_X} \leq \|H|_X\|_X = \|H\|_X.$$

It follows that $\|\varphi\|^{X,\omega|_X} \leq \|\varphi\|^{X,\omega}$. This proves inequality "$\geq$" in (8).

The inequality "$\leq$" in (8) is a consequence of Proposition 12.

It remains to show that $\|\cdot\|^{X,\omega}$ is non-degenerate, i.e., **condition (7)** holds. By (8) it suffices to prove the following claim.

**Claim 1** *If $X = M$ then condition (7) holds.*

For the proof of this claim we denote by $w(U) := w(U, \omega|_U)$ the Gromov width of an open subset $U \subseteq M$.

*Proof (of Claim 1)* Let $\mathrm{id} \neq \varphi \in \mathrm{Ham}(M,\omega)$. Let $H \in \mathcal{H}(M,\omega,M)$ be such that $\varphi_H^1 = \varphi$. We choose $x_0 \in M$ such that $\varphi(x_0) \neq x_0$ and an open neighborhood $U$ of $x_0$ with compact closure, such that $\varphi(\overline{U}) \cap \overline{U} = \emptyset$.

Let $\varepsilon > 0$. By Lemma 27 below there exists $\psi \in \mathrm{Ham}_c(M,\omega)$ such that $\psi|_{\overline{U}} = \varphi|_{\overline{U}}$ and condition (80) holds. By a result by D. McDuff and F. Lalonde [LM, Theorem 1.1] we have

$$\|\psi\|_c^{M,\omega} \geq \frac{1}{2} w(U).$$

Combining this with (80), and using that $\varepsilon > 0$ is arbitrary, it follows that $\|\varphi\|^{M,\omega} \geq \frac{1}{2} w(U) > 0$. This proves Claim 1 and completes the proof of Proposition 2(ii).

### 2.2 Proofs of Propositions 3, 4, 7, and 8

*Proof (of Proposition 3)* Let $H \in \mathcal{H}(M,\omega,X)$. It suffices to show that for every path $x \in C^\infty([0,1],X)$ and $t \in [0,1]$ we have

$$H^t \circ x(0) = H^t \circ x(1). \tag{33}$$

To see this, note that for every $s \in [0,1]$, we have

$$\frac{d}{ds}(H^t \circ x)(s) = dH^t \dot{x}(s) = \omega\big(X_{H^t} \circ x(s), \dot{x}(s)\big). \tag{34}$$

By Lemma 11 we have $X_{H^t} \circ x(s) \in T_{x(s)}X$, for every $s \in [0,1]$. Since $X$ is isotropic, it follows that the last expression in (34) vanishes. Hence (34) implies (33). This proves Proposition 3.

*Proof (of Proposition 4)* We prove **statement (i)**. Conditions (4,5) (with $\|\cdot\| := \|\cdot\|_X^{Y,\omega}$) follow from straight-forward arguments. To see that condition (6) holds, let $\varphi_1, \varphi_2 \in \mathrm{Ham}(X,\omega)$. Without loss of generality assume that $\|\varphi_i\|_X^{Y,\omega} < \infty$, for $i = 1,2$. Let $\varepsilon > 0$. By definition of $\|\varphi_i\|_X^{Y,\omega}$ there exist maps $\psi_1, \psi_2 \in \mathrm{Ham}(Y,\omega)$ such that

$$\psi_i|_X = \varphi_i, \qquad \|\psi_i\|^{Y,\omega} < \|\varphi_i\|_X^{Y,\omega} + \varepsilon, \tag{35}$$

for $i = 1,2$. We have $\psi := \psi_1 \circ \psi_2 \in \mathrm{Ham}(Y,\omega)$ and $\psi|_X = \varphi_1 \circ \varphi_2$. It follows that

$$\begin{aligned} \|\varphi_1 \circ \varphi_2\|_X^{Y,\omega} &\leq \|\psi\|^{Y,\omega} \\ &\leq \|\psi_1\|^{Y,\omega} + \|\psi_2\|^{Y,\omega} \\ &< \|\varphi_1\|_X^{Y,\omega} + \|\varphi_2\|_X^{Y,\omega} + 2\varepsilon, \end{aligned}$$

where in the second inequality we used Proposition 2(i), and in the last inequality we used (35). Since $\varepsilon > 0$ is arbitrary, the triangle inequality

$$\|\varphi_1 \circ \varphi_2\|_X^{Y,\omega} \leq \|\varphi_1\|_X^{Y,\omega} + \|\varphi_2\|_X^{Y,\omega}$$

follows. This proves (6) and completes the proof of statement (i).

To prove **statement (ii)**, let $\varphi, \varphi' \in \mathrm{Ham}(X,\omega)$. Since by hypothesis $X$ is compact and contained in the interior of $Y$, there exists $\psi' \in \mathrm{Ham}(Y,\omega)$ such that $\psi'|_X = \varphi'$. (We may construct such a map by choosing a function $H' \in \mathcal{H}(M,\omega,X)$ which satisfies $\varphi^1_{H'}|_X = \varphi'$, and applying a suitable cutoff argument.) For every $\psi \in \mathrm{Ham}(Y,\omega)$ satisfying $\psi|_X = \varphi$, the map $\psi' \circ \psi \circ \psi'^{-1}$ lies in $\mathrm{Ham}(Y,\omega)$ and restricts to $\varphi' \circ \varphi \circ \varphi'^{-1}$ on $X$. It follows that

$$\begin{aligned} &\left\|\varphi' \circ \varphi \circ \varphi'^{-1}\right\|_X^{Y,\omega} \leq \\ \inf \Big\{ &\left\|\psi' \circ \psi \circ \psi'^{-1}\right\|^{Y,\omega} \,\Big|\, \psi \in \mathrm{Ham}(Y,\omega) :\, \psi|_X = \varphi \Big\}. \end{aligned} \tag{36}$$

By invariance of the semi-norm $\|\cdot\|^{Y,\omega}$ we have $\left\|\psi' \circ \psi \circ \psi'^{-1}\right\|^{Y,\omega} = \|\psi\|^{Y,\omega}$. Combining this with inequality (36), it follows that $\left\|\varphi' \circ \varphi \circ \varphi'^{-1}\right\|_X^{Y,\omega} \leq \|\varphi\|_X^{Y,\omega}$. The opposite inequality follows from an analogous argument. This proves statement (ii).

To prove **statement (iii)**, let $Y'$ as in the hypothesis and $\varphi \in \mathrm{Ham}(X,\omega)$. The statement is a consequence of the following claim.

**Claim 1** *For every $\psi \in \mathrm{Ham}(Y,\omega)$ satisfying $\psi|_X = \varphi$ and every $\varepsilon > 0$, there exists $\psi' \in \mathrm{Ham}(Y',\omega)$ such that*

$$\psi'|_X = \varphi, \tag{37}$$

$$\|\psi'\|^{Y',\omega} < \|\psi\|^{Y,\omega} + 3\varepsilon. \tag{38}$$

*Proof (of Claim 1)* Assume that $\psi$ and $\varepsilon$ are as above. We choose a function $H \in \mathcal{H}(M,\omega,Y)$ such that

$$\varphi^1_H|_Y = \psi, \quad \|H\|_Y < \|\psi\|^{Y,\omega} + \varepsilon. \tag{39}$$

Since by hypothesis, $Y$ is compact and contained in $\mathrm{Int}\, Y'$, there exists a compact neighborhood $K_1$ of $Y$ that is contained in $\mathrm{Int}\, Y'$ and satisfies

$$\max_{K_1} H^t \leq \max_Y H^t + \varepsilon, \quad \min_{K_1} H^t \geq \min_Y H^t - \varepsilon, \tag{40}$$

for every $t \in [0,1]$. We choose a compact neighborhood $K_2$ of $Y$ that is contained in $\mathrm{Int}\, K_1$. By Lemma 29 below there exists a function $f \in C^\infty(M,[0,1])$ such that $f \equiv 1$ on $K_2$ and $f \equiv 0$ on $M \setminus \mathrm{Int}\, K_1$. We choose a point $x_0 \in Y$ and define $H' : [0,1] \times M \to \mathbb{R}$ by

$$H'(t,x) := {H'}^t(x) := f(x)\big(H^t(x) - H^t(x_0)\big).$$

The support of this function is contained in $[0,1] \times K_1$ and hence compact. Hence its Hamiltonian flow exists on $M$. We define $\psi' := \varphi^1_{H'}|_{Y'}$. It follows that $\psi' \in \mathrm{Ham}(Y',\omega)$. For each $t \in [0,1]$, the functions ${H'}^t$ and $H^t$ differ on $K_2$ by the constant $H^t(x_0)$. Since $\varphi^1_H|_Y = \psi$, $\psi|_X = \varphi$ and $X \subseteq \mathrm{Int}\, K_2$, equality (37) follows.

Using that $f \leq 1$, $f \equiv 0$ on $M \setminus K_1$, and inequalities (40), we have, for every $t \in [0,1]$,

$$\begin{aligned}\sup_{Y'} {H'}^t - \inf_{Y'} {H'}^t &\leq \max_{K_1}(H^t - H^t(x_0)) - \min_{K_1}(H^t - H^t(x_0)) \\ &\leq \max_Y H^t - \min_Y H^t + 2\varepsilon.\end{aligned}$$

It follows that

$$\|H'\|_{Y'} \leq \|H\|_Y + 2\varepsilon.$$

Combining this with the inequality $\|\psi'\|^{Y',\omega} \leq \|H'\|_{Y'}$ and the inequality in (39), inequality (38) follows. Hence $\psi'$ has the required properties. This proves Claim 1 and hence statement (iii), and completes the proof of Proposition 4.

*Proof (of Proposition 7)* **Inequality (19)** follows by taking $\ell := 2n - d$, $k_i := 1$, for $i = 1, \ldots, \ell$, $n_i := 1$, for $i = 1, \ldots, \ell - 1$, and $n_\ell := d - n + 1$.

Let $n \in \mathbb{N}$ be such that $n \neq k^2$, for every $k \in \mathbb{N}$. **Inequality "$\geq$" in (20)** is a consequence of the next claim. Let $\ell \geq 2$ and $k_1, \ldots, k_\ell$ be as in the definition of $\mathrm{k}(n,n)$.

**Claim 1** *We have*

$$\sum_{i=1}^{\ell} k_i^2 = n. \tag{41}$$

*Proof (of Claim 1)* We choose integers $n_1, \ldots, n_\ell$ such that the inequalities (14,15,16) are satisfied. Subtracting the first from the second inequality in (15), we obtain $\sum_i k_i(n_i - k_i) \leq 0$. Using the inequalities (14), it follows that $n_i = k_i$, for every $i = 1, \ldots, \ell$. Combining this with (15), the equality (41) follows. This proves Claim 1.

We show **inequality "$\leq$" in (20)**: Let $\ell \in \mathbb{N}$ and $k_1, \ldots, k_\ell \in \mathbb{N}$ be as in the definition of $\mathrm{K}(n)$. This means that $\sum_{i=1}^{\ell} k_i^2 = n$. Our hypothesis that $n \neq k^2$, for every $k \in \mathbb{N}$, implies that condition (13) is satisfied.

We define $n_i := k_i$, for $i = 1, \ldots, \ell$. Conditions (14,15) are satisfied with $d = n$. Furthermore, using that $\ell \geq 2$, it follows that (16) holds. Inequality "$\leq$" in (20) follows. This proves (20).

**Inequality (21)** was proved in [SZ1] (Proposition 8, inequality (36)). This completes the proof of Proposition 7.

*Proof (of Proposition 8)* Assume that $c \in \big[0, \mathrm{diam}(X, M, \omega)\big)$. By definition there exists $\varphi \in \mathrm{Ham}(X, \omega)$ such that $\|\varphi\|_X^{M,\omega} \geq c$. We choose a function $H \in \mathcal{H}(M, \omega, X)$ such that $\varphi_H^1|_X = \varphi$. We also choose a function $\rho \in C^\infty(\mathbb{R}^{2n}, [0,1])$ with compact support contained in $U$, such that $\rho \equiv 1$ in some neighborhood $V \subseteq M$ of $X$. We define $\widetilde{H} : [0,1] \times M \to \mathbb{R}$ by $\widetilde{H}(t,x) := \rho(x) H(t,x)$.

Note that the support of $\widetilde{H}$ is compact and contained in $U$. Furthermore, we have $\varphi_{\widetilde{H}}^1|_X = \varphi$. It follows that

$$\mathrm{Diam}(U, M, \omega) \geq \|\varphi_{\widetilde{H}}^1\|_M \geq \|\varphi\|_X^{M,\omega} \geq c.$$

Since $c < \mathrm{diam}(X, M, \omega)$ is arbitrary, the inequality (22) follows. This proves Proposition 8.

## 3 Coisotropic intersections and relative Hofer diameters

This section is the core of the article. We develop a framework for proving a lower bound on the relative Hofer diameter of a set. We use this to prove the main result, Theorem 6, in Section 3.3. The method described here is of interest in its own, since it can be used to prove similar results in different settings.

### 3.1 Coisotropic intersections

The proof of Theorem 6 is based on the following result about coisotropic intersections, which we proved in [SZ1]. To state it, let $(M, \omega)$ be a symplectic manifold. We call it *(weakly geometrically) bounded* iff there exist an almost complex structure $J$ and a complete Riemannian metric $g$ on $M$ such that the following conditions hold:

- The sectional curvature of $g$ is bounded and $\inf_{x\in M} \iota^g_x > 0$, where $\iota^g_x$ denotes the injectivity radius of $g$ at the point $x \in M$.
- There exists a constant $C \in (0, \infty)$ such that

  $$|\omega(v, w)| \leq C|v|\,|w|, \quad \omega(v, Jv) \geq C^{-1}|v|^2,$$

  for every $v, w \in T_xM$ and $x \in M$. Here $|v| := \sqrt{g(v, v)}$.

This is a mild condition on $(M, \omega)$. (For examples see [SZ1].)

Recall that a submanifold $N$ of $M$ is called *coisotropic* iff for every $x \in N$ the subspace

$$T_xN^\omega = \big\{v \in T_xM \,\big|\, \omega(v, w) = 0,\ \forall w \in T_xN\big\}$$

of $T_xM$ is contained in $T_xN$. As an example, $N$ is coisotropic if it is a hypersurface. Let $N \subseteq M$ be a coisotropic submanifold. The isotropic (or characteristic) distribution $TN^\omega$ on $N$ is integrable, and hence by Frobenius' theorem, it gives rise to a foliation on $N$. We call the leaves of this foliation isotropic leaves, denote

$$N_\omega := \big\{\text{isotropic leaf of } N\big\},$$

and define the *action spectrum* and the *minimal action* of $(M, \omega, N)$ as

$$S(M, \omega, N) := \left\{ \int_{\mathbb{D}} u^*\omega \,\middle|\, u \in C^\infty(\mathbb{D}, M) : \exists F \in N_\omega : u(S^1) \subseteq F \right\},$$
$$A(M, \omega, N) := \inf\big(S(M, \omega, N) \cap (0, \infty)\big) \in [0, \infty].$$

(Here our convention is that $\inf \emptyset := \infty$.) We define the *split minimal symplectic action of $N$*, $A_\times(M, \omega, N)$ as follows. We define a *bounded splitting* *of* $(M,\omega,N)$ to be a tuple $(M_i,\omega_i,N_i)_{i=1,\ldots,k}$, where $k\in\mathbb{N}$ and for every $i=1,\ldots,k$, $(M_i,\omega_i)$ is a bounded symplectic manifold and $N_i\subseteq M_i$ a coisotropic submanifold, such that there exists a symplectomorphism $\varphi$ from $\left(\times_{i=1}^k M_i,\oplus_{i=1}^k\omega_i\right)$ to $(M,\omega)$, satisfying $\varphi\left(\times_{i=1}^k N_i\right)=N$. We define

$$A_\times(N):=A_\times(M,\omega,N):=$$
$$\sup\left\{\min_{i=1,\ldots k}A(M_i,\omega_i,N_i)\,\middle|\,(M_i,\omega_i,N_i)_i \text{ bounded splitting of } (M,\omega,N)\right\}.$$

(Here our convention is that $\sup\emptyset=0$.)

**Remark 13** *If $(M,\omega)$ is not bounded then $(M,\omega,N)$ does not admit any bounded splitting, and therefore $A_\times(M,\omega,N)=0$. This follows from the facts that a finite product of bounded symplectic manifolds is bounded, and boundedness is invariant under symplectomorphisms.*

We call a coisotropic submanifold $N\subseteq M$ *regular* iff its isotropy relation is a closed subset and a submanifold of $N\times N$. Equivalently, the symplectic quotient of $N$ is well-defined. (For more details and examples see [Zi1].) We abbreviate $\mathcal{H}(M,\omega):=\mathcal{H}(M,\omega,M)$ and define the *displacement energy* of a subset $X\subseteq M$ to be

$$e(X,M):=e(X,M,\omega):=\inf\left\{\|H\|_M\,\middle|\,H\in\mathcal{H}(M,\omega)\,\middle|\,\varphi_H^1(X)\cap X=\emptyset\right\}. \quad (42)$$

We call a manifold *closed* iff it is compact and its boundary is empty. We are now able to formulate the key result.

**Theorem 14 (Coisotropic intersections, see [SZ1] (Theorem 1))** *Let $(M,\omega)$ be a symplectic manifold and $\emptyset\neq N\subseteq M$ a closed connected regular coisotropic submanifold. Then we have*

$$e(N,M)\geq A_\times(N). \quad (43)$$

The proof of this result (see [SZ1]) is based on a certain Lagrangian embedding of $N$ and on the Main Theorem in the article [Ch] by Y. Chekanov.

The idea of proof for part (i) of Theorem 6 is to find a Hamiltonian flow $[0,1]\times\mathbb{R}^{2n}\ni(t,x)\mapsto\varphi^t(x)\in\mathbb{R}^{2n}$ that preserves $S^{2n-1}$, such that the following holds. Let $\Phi$ be a Hamiltonian diffeomorphism on $\mathbb{R}^{2n}$, such that $\Phi|_{S^{2n-1}}=\varphi^1|_{S^{2n-1}}$. Then there exists a regular closed coisotropic submanifold $\emptyset\neq N\subseteq\mathbb{R}^{2n}$ such that

$$A_\times(N)\geq\frac{\pi}{2},\quad \Phi(N)\cap N=\emptyset.$$

It then follows from Theorem 14 that

$$\|\varphi^1|_{S^{2n-1}}\|_{S^{2n-1}}^{\mathbb{R}^{2n},\omega_0}\geq\frac{\pi}{2}.$$

The claimed inequality (17) is a consequence of this.

In the following subsection we will put this idea into a more general framework, which we will use for the proof of Theorem 6.

### 3.2 Rigidifying pairs

Let $(M, \omega)$ be a symplectic manifold. In this subsection, given a compact subset $X_0 \subseteq M$ and a Hamiltonian $S^1$-action on $M$, we construct a pair $(X, \varphi)$, where $X \subseteq M$, and $\varphi \in \mathrm{Ham}(X, \omega)$, and we prove a lower bound on the Hofer norm of $\varphi$ on $X$ relative to $M$. This is a key ingredient in the proof of Theorem 6.

Let $X_0 \subseteq M$ be a subset, and $\alpha \in \Omega^1(M)$. We call the pair $(X_0, \alpha)$ *rigidifying* iff for every symplectomorphism $\varphi : M \to M$ the following holds. If $\varphi|_{X_0} = \mathrm{id}_{X_0}$ then, for every $x \in X_0$ and $v \in T_x M$, we have

$$\alpha \, d\varphi v = \alpha v. \tag{44}$$

As an example, $(X_0, \alpha)$ is rigidifying if $X_0$ is open.

Let $S^1 \times M \ni (z, x) \mapsto \varphi^z(x) \in M$ be a Hamiltonian action. We fix a compact subset $X_0 \subseteq M$, and define

$$X := \bigcup_{z \in S^1} \varphi^z(X_0). \tag{45}$$

This is a compact subset of $M$. Let $z_0 \in S^1 \subseteq \mathbb{C}$. We denote by $\bar{z}_0$ its complex conjugate. Note that $\varphi^{\bar{z}_0}|_X \in \mathrm{Ham}(X, \omega)$. The next result gives a lower bound on the relative Hofer norm $\|\varphi^{\bar{z}_0}|_X\|_X^{M,\omega}$ (defined as in (9)), if there exists a suitable one-form $\alpha$, such that the pair $(X_0, \alpha)$ is rigidifying. Recall the definition (42) of the displacement energy $e(X, M)$ of a subset $X \subseteq M$.

**Lemma 15 (Main Lemma)** *Let $[0,1] \times M \ni (t, x) \mapsto \psi^t(x) \in M$ be a smooth map satisfying*

$$\psi^0 = \mathrm{id}\,. \tag{46}$$

*Assume that there exists a function $f \in C^\infty(M, \mathbb{R})$ such that the pair $(X_0, df)$ is rigidifying, and*

$$f(X_0) \subseteq [0, \infty), \quad f \circ \varphi^{z_0}(X_0) \subseteq (-\infty, 0], \tag{47}$$

$$\left.\frac{d}{dt}\right|_{t=0} (f \circ \psi^t(x)) > 0, \quad \left.\frac{d}{dt}\right|_{t=0} \big(f \circ \varphi^{z_0} \circ \psi^t(x)\big) \le 0, \ \forall x \in X_0. \tag{48}$$

*Then we have (with $X$ as in (45))*

$$\|\varphi^{\bar{z}_0}|_X\|_X^{M,\omega} \ge \limsup_{t \searrow 0} e(\psi^t(X_0), M). \tag{49}$$

To motivate this result, note that given a "small" subset $X_0 \subseteq M$, this lemma provides a lower bound on the relative Hofer norm of some Hamiltonian diffeomorphism, defined on a certain "small" subset $X \subseteq M$ constructed from $X_0$. The point of definition (45) of this set $X$ is that for the given Hamiltonian action $(\varphi^z)$ of the circle on $M$, the map $\varphi^{e^{it}}$ preserves $X$, for every $t \in \mathbb{R}$, and hence by definition, $\varphi^{e^{it}}$ restricts to a Hamiltonian diffeomorphism on $X$, for every $t \in \mathbb{R}$.

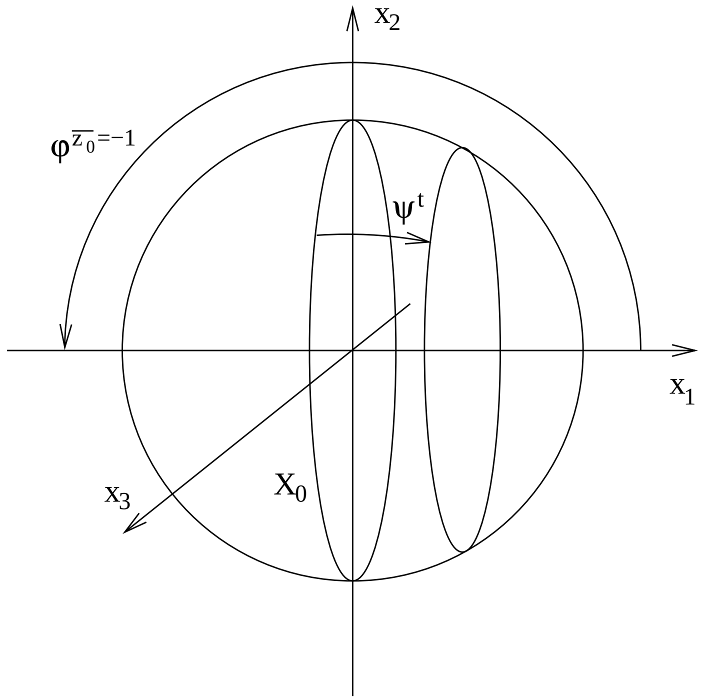


**Fig. 1** The situation of Example 16.

**Example 16** *Let $M := S^2$ be the sphere, $\omega$ the standard area form on $S^2$, $X_0 := \{x \in S^2 \subseteq \mathbb{R}^3 \,|\, x_1 = 0\}$ the equator, and the Hamiltonian circle action be given by $\varphi^z(x) := \big(R_z(x_1, x_2), x_3\big)$, for $z \in S^1$ and $x \in S^2 \subseteq \mathbb{R}^3$, where $R_z$ denotes the rotation in $\mathbb{R}^2$ by the angle $z$. Furthermore, let $[0,1]\times S^2 \ni (t,x) \mapsto \psi^t(x) \in S^2$ be a smooth map satisfying $\psi^0 = \mathrm{id}$ and $\psi^t(x) = \big(x_1+t, x_2, x_3\big)$ if $t$ and $x_1$ lie in some neighborhood of 0. (See Figure 1.) Let $z_0 := -1 \in S^1 \subseteq \mathbb{C}$. Then the hypotheses of Lemma 15 are satisfied, with $f : S^2 \subseteq \mathbb{R}^3 \to \mathbb{R}$ the height function given by $f(x) := x_1$. (That $(X_0, df)$ is rigidifying follows from an elementary argument, or from Remark 18 below.) Since $X = S^2$, the left-hand side of (49) equals the usual Hofer norm of the map $\varphi^{\bar{z}_0=-1}$. Furthermore, we have $\psi^t(X_0) = \{x \in S^2 \subseteq \mathbb{R}^3 \,|\, x_1 = t\}$, provided that $t$ is close enough to 0. The displacement energy of this circle equals the area of the "right cap" that it bounds. (The inequality $\le$ follows from an elementary argument using Moser isotopy, and the opposite estimate is e.g. a consequence of the main result in Y. Chekanov's article [Ch], or of an elementary argument.) Hence the right hand side of (49) equals $2\pi$, and therefore, the statement of the lemma is that*

$$\|\varphi^{\bar{z}_0=-1}\|^{S^2,\omega} \geq 2\pi.$$

*This is a well-known fact.*

To prove Lemma 15, we will show that any Hamiltonian diffeomorphism $\Phi$ of $M$ that restricts to $\varphi^{\bar{z}_0}$ on $X$, displaces the set $\psi^t(X_0)$ for every small enough

$t$ (where smallness depends on $\Phi$). (This set is close to $X_0$, since $\psi^0 = \mathrm{id}$ by (46).) This will be a consequence of the simple fact that two sets are disjoint if they are mapped to disjoint sets by some function. We will apply this with the sets $\psi^t(X_0), \Phi \circ \psi^t(X_0)$ and the function $f \circ \varphi^{z_0}$, where $f$ is as in the hypothesis of the lemma. Conditions (47,48) and the rigidifying property (44) for the pair $(X_0, \alpha := df)$ will ensure that for $t > 0$ small enough, we have $f \circ \varphi^{z_0}(\psi^t(X_0)) \subseteq (-\infty, 0]$ and $f \circ \varphi^{z_0}\big(\Phi \circ \psi^t(X_0)\big) \subseteq (0, \infty)$, and hence these sets are disjoint.

*Proof (of Lemma 15)* Let $\Phi \in \mathrm{Ham}(M, \omega)$ be such that

$$\Phi|_X = \varphi^{\bar{z}_0}|_X. \tag{50}$$

**Claim 1** *There exists* $t_0 \in (0, 1]$ *such that for* $t \in (0, t_0]$*, we have*

$$\Phi \circ \psi^t(X_0) \cap \psi^t(X_0) = \emptyset.$$

*Proof (of Claim 1)* We define $\widetilde{f} := f \circ \varphi^{z_0}$. We check the hypotheses of Lemma 31 below with $\varphi := \Phi$ and $f$ replaced by $\widetilde{f}$: The inclusions (83) follow from (47,50).

We prove the inequalities (84): The first inequality in (84) follows from the second inequality in (48). Let $x \in X_0$. To prove the second inequality, observe that by (46), we have $\psi^0(x) = x$. Furthermore, (50) implies that $\varphi^{z_0} \circ \Phi|_{X_0} = \mathrm{id}_{X_0}$. Therefore the hypothesis that $(X_0, df)$ is rigidifying implies that

$$df\, d(\varphi^{z_0} \circ \Phi)\, \frac{d}{dt}\bigg|_{t=0} \psi^t(x) = df\, \frac{d}{dt}\bigg|_{t=0} \psi^t(x). \tag{51}$$

The left hand side of this equality equals $\frac{d}{dt}\big|_{t=0}\big(\widetilde{f} \circ \Phi \circ \psi^t(x)\big)$. Furthermore, by the first inequality in (48), the right-hand side of (51) is positive. Hence the second inequality in (84) is satisfied. Therefore, all the hypotheses of Lemma 31 are satisfied. Applying this lemma, the statement of Claim 1 follows.

Claim 1 implies that

$$\|\Phi\|^{M,\omega} \geq \limsup_{t \searrow 0} e(\psi^t(X_0), M).$$

Since this holds for every $\Phi \in \mathrm{Ham}(M, \omega)$ satisfying (50), inequality (49) follows. This proves Lemma 15.

The next result provides a large class of examples of rigidifying pairs $(X, \alpha)$. Let $X, Y, Y'$ be smooth manifolds and $f \in C^\infty(X, Y)$ and $f' \in C^\infty(X, Y')$ maps. We say that $f$ *factors* through $f'$ iff there exists a map $g \in C^\infty(Y', Y)$ satisfying $f = g \circ f'$. Let $(M, \omega)$ be a symplectic manifold, $X_0 \subseteq M$ a subset, and $f \in C^\infty(M, \mathbb{R})$.

**Lemma 17** *The pair $(X_0, df)$ is rigidifying, provided that there exist symplectic manifolds $(\widetilde{M}, \widetilde{\omega})$ and $(M', \omega')$, a connected Lie group $G$, a Hamiltonian action of $G$ on $\widetilde{M}$, a moment map $\mu : \widetilde{M} \to \mathfrak{g}^*$ for the action, and a symplectomorphism $\psi : \widetilde{M} \to M \times M'$ such that the following holds. (Note that by definition, $\mu$ is equivariant.) We denote by $\mathrm{pr} : M \times M' \to M$ the canonical projection. Then the composition $f \circ \mathrm{pr} \circ \psi$ factors through $\mu$, and we have*

$$X_0 = \mathrm{pr} \circ \psi(\mu^{-1}(0)). \tag{52}$$

**Remark 18** *In particular, this result implies that given a Hamiltonian action of a connected Lie group $G$ on $M$, with moment map $\mu : M \to \mathfrak{g}^*$, and given an element $\xi \in \mathfrak{g}$, the pair $(X_0, \alpha) := \big(\mu^{-1}(0) \subseteq M, d\langle \mu, \xi \rangle\big)$ is rigidifying. (Here $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ denotes contraction.) This will be used to prove the first part of the main result. (The proof of the second part will rely on the full statement of Lemma 17.)*

*It follows that in the situation of Example 16, the pair $(X_0, df)$ is rigidifying. (To see this, consider the action of $G := S^1$ on $S^2$ given by rotation around the $x_1$-axis. The map $S^2 \ni x \mapsto ix_1 \in i\mathbb{R} = \mathrm{Lie}\, S^1 \cong (\mathrm{Lie}\, S^1)^*$ is a moment map for this action.)*

The proof of Lemma 17 is based on the following result. Let $(M, \omega)$ be a symplectic manifold, $\varphi$ a symplectomorphism of $M$, and $G$ a connected Lie group. We fix a Hamiltonian action of $G$ on $M$ and a moment map $\mu : M \to \mathfrak{g}^*$.

**Lemma 19** *Assume that $\varphi(\mu^{-1}(0)) = \mu^{-1}(0)$ and the restriction of $\varphi$ to $\mu^{-1}(0)$ is $G$-equivariant. Then we have*

$$d(\mu \circ \varphi)(x) = d\mu(x), \quad \forall x \in \mu^{-1}(0). \tag{53}$$

**Remark 20** *This lemma immediately implies the statement of Lemma 17 in the special case*

$$M' = \{\mathrm{pt}\}, \quad \psi = \mathrm{id} : \widetilde{M} \to M, \quad (X_0, f) = \big(\mu^{-1}(0), \langle \mu, \xi \rangle\big),$$

*where $\xi \in \mathfrak{g}$ is a fixed element.*

In the proof of this lemma, for $\xi \in \mathfrak{g}$ we denote by $X_\xi$ the vector field on $M$ generated by $\xi$.

*Proof (of Lemma 19)* Let $x \in \mu^{-1}(0)$, $v \in T_xM$ and $\xi \in \mathfrak{g}$. Then we have

$$\big\langle d\mu(\varphi(x)) d\varphi(x) v, \xi \big\rangle = \omega\big(X_\xi \circ \varphi(x), d\varphi(x) v\big). \tag{54}$$

Since, by assumption, $\varphi|_{\mu^{-1}(0)}$ is $G$-equivariant, we have $\varphi\big(\exp(t\xi)x\big) = \exp(t\xi)\varphi(x)$, for every $t \in \mathbb{R}$. Taking the derivative at $t = 0$, it follows that

$$d\varphi(x) X_\xi(x) = X_\xi \circ \varphi(x).$$

Combining this with equality (54) and using that $\varphi$ is a symplectomorphism, it follows that

$$\big\langle d\mu(\varphi(x)) d\varphi(x) v, \xi \big\rangle = \langle d\mu(x) v, \xi \rangle.$$

Equality (53) follows. This proves Lemma 19.

*Proof (of Lemma 17)* Let $\widetilde{M}$ etc. be as in the hypothesis. By a straight-forward argument, we may assume without loss of generality that $\widetilde{M} = M \times M'$ and $\psi$ is the identity map on $\widetilde{M}$. Let $\varphi : M \to M$ be a symplectomorphism satisfying

$$\varphi|_{X_0} = \mathrm{id}_{X_0} . \tag{55}$$

In order to show that equality (44) holds, we define $\widetilde{\varphi} := \varphi \times \mathrm{id}_{M'} : \widetilde{M} \to \widetilde{M}$. By hypothesis there exists a map $g \in C^\infty(\mathfrak{g}^*, \mathbb{R})$ such that

$$f \circ \mathrm{pr} = g \circ \mu. \tag{56}$$

Using that $\varphi \circ \mathrm{pr} = \mathrm{pr} \circ \widetilde{\varphi}$, it follows that

$$d(f \circ \varphi) d\, \mathrm{pr} = dg\, d(\mu \circ \widetilde{\varphi}). \tag{57}$$

The map $\widetilde{\varphi}$ is an $\omega \oplus \omega'$-symplectomorphism. Furthermore, equalities (55,52) imply that $\widetilde{\varphi}|_{\mu^{-1}(0)} = \mathrm{id}_{\mu^{-1}(0)}$. Therefore, we may apply Lemma 19 with $\varphi$ replaced by $\widetilde{\varphi}$, and conclude that

$$d(\mu \circ \widetilde{\varphi})(\widetilde{x}) = d\mu(\widetilde{x}), \quad \forall \widetilde{x} \in \mu^{-1}(0).$$

Combining this with equalities (57,56), we obtain

$$d(f \circ \varphi)\, d\, \mathrm{pr}(\widetilde{x}) = df\, d\, \mathrm{pr}(\widetilde{x}), \quad \forall \widetilde{x} \in \mu^{-1}(0).$$

Using (52) and that pr is submersive, it follows that $df\, d\varphi(x) = df(x)$, for every $x \in X_0$. It follows that $(X_0, df)$ is rigidifying. This proves Lemma 17.

## 3.3 Proof of Theorem 6 (Relative Hofer diameters of small sets)

Both parts of this result are proved along similar lines. The idea for the first part is to define $X_0$ to be the product of a circle and a sphere in $\mathbb{R}^{2n-2}$, each of radius $1/\sqrt{2}$, $(\varphi^z)$ a certain unitary linear action of $S^1$ on $\mathbb{R}^{2n}$, $X := \bigcup_{z \in S^1} \varphi^z(X_0)$, and $\psi^t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ a map that expands the circle-factor by $(1+t)$. It follows that $X \subseteq S^{2n-1}$. We may then apply Lemmas 17 and 15, obtaining inequality (49).

Since $\psi^t(X_0)$ is a regular coisotropic submanifold of $\mathbb{R}^{2n}$, we may then use the key result, Theorem 14, to estimate the right-hand side of inequality (49) from below by $\frac{\pi}{2}$. The claimed inequality (17) will be a consequence of this and the following remark.

**Remark 21** *Let $(M, \omega)$ be a symplectic manifold, $X \subseteq Y \subseteq M$ closed subsets and $H \in C^\infty\big([0,1] \times M, \mathbb{R}\big)$ a function such that $Y \subseteq \mathcal{D}^1_{X_H}$ (the domain of the flow $\varphi^1_H$) and $\varphi^t_H(X) = X$, $\varphi^t_H(Y) = Y$, for every $t \in [0,1]$. Then we have*

$$\|\varphi^1_H|_X\|^{M,\omega}_X \le \|\varphi^1_H|_Y\|^{M,\omega}_Y .$$

*This follows from a straight-forward argument.*

*Proof (of Theorem 6(i))* For $k \in \mathbb{N}$ and $a > 0$ we denote by $S^{2k-1}(a) \subseteq \mathbb{R}^{2k}$ the sphere of radius $\sqrt{a/\pi}$ around 0. We define

$$X_0 := S^1\left(\frac{\pi}{2}\right) \times S^{2n-3}\left(\frac{\pi}{2}\right),$$

(Here we use the hypothesis that $n \geq 2$.) Furthermore, we define the map

$$S^1 \times \mathbb{R}^{2n} \ni (z,x) \mapsto \varphi^z(x) \in \mathbb{R}^{2n} \tag{58}$$

as follows. Let $z \in S^1$. We denote by $R^z : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation by $z$. (Identifying $\mathbb{R}^2 = \mathbb{C}$, it is given by the formula $R^z(q_1 + iq_2) := z(q_1 + iq_2)$.) We define $\varphi^z : \mathbb{R}^{2n} = \mathbb{C}^n = \mathbb{C}^2 \times \mathbb{C}^{n-2} \to \mathbb{C}^n$ to be the unique complex linear extension of the map

$$R^z \times \mathrm{id}_{\mathbb{R}^{n-2}} : \mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^n.$$

(Note that the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ here is not compatible with the identification of $\mathbb{R}^2$ with $\mathbb{C}$ in the above formula for $R^z$.) The map (58) is a Hamiltonian $S^1$-action on $\mathbb{C}^n$. (It is generated by the function $H : \mathbb{C}^n \to \mathbb{R}$ defined by $H(q+ip) := q_1p_2 - q_2p_1$.) We define $X := \bigcup_{z\in S^1} \varphi^z(X_0)$. Since $X_0 \subseteq S^{2n-1}$ and $\varphi^z$ is orthonormal, for every $z \in S^1$, it follows that $X \subseteq S^{2n-1}$, and $\varphi^z$ preserves $X$ and $S^{2n-1}$, for every $z \in S^1$. Therefore, by Remark 21, we have

$$\|\varphi^{-i}|_{S^{2n-1}}\|^{\mathbb{R}^{2n},\omega_0}_{S^{2n-1}} \geq \|\varphi^{-i}|_X\|^{\mathbb{R}^{2n},\omega_0}_X. \tag{59}$$

We define

$$(M,\omega) := (\mathbb{C}^n, \omega_0), \quad z_0 := i,$$
$$\psi : [0,1] \times \mathbb{C}^n \to \mathbb{C}^n, \quad \psi^t(y,y') := \psi(t,y,y') := \big((1+t)y, y'\big),$$

for $t \in [0,1]$ and $(y,y') \in \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$.

**Claim 1** *The hypotheses of Lemma 15 are satisfied.*

*Proof (of Claim 1)* The **condition (46)** is clearly satisfied. We define the map

$$f : \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1} \to \mathbb{R}, \quad f(y,y') := |y|^2 - \frac{1}{2}.$$

Consider the action of $G := S^1 \times S^1$ on $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ given by $(z,z')\cdot(y,y') := (zy, z'y')$. This action is Hamiltonian, with moment map

$$\mu : \mathbb{C} \times \mathbb{C}^{n-1} \to \mathfrak{g}^* \cong \mathfrak{g} = i\mathbb{R} \times i\mathbb{R}, \quad \mu(y,y') := \frac{i}{2}\left(\frac{1}{2} - |y|^2, \frac{1}{2} - |y'|^2\right).$$

Since $X_0 = \mu^{-1}(0)$, it follows from Remark 18 that $(X_0, df)$ **is rigidifying**.

The next hypothesis of Lemma 15, the **inclusions (47)**, follow from the facts

$$f(X_0) = \{0\}, \quad f \circ \varphi^i(X_0) \subseteq \left[-\frac{1}{2}, 0\right].$$

(Here in the second condition we used that $\varphi^i(y,y'',y''')=(-y'',y,y''')$, for every $(y,y'',y''')\in\mathbb{C}\times\mathbb{C}\times\mathbb{C}^{n-2}=\mathbb{C}^n$.) We prove that the **inequalities (48)** are satisfied: Direct calculations show that

$$\frac{d}{dt}\bigg|_{t=0}\big(f\circ\psi^t(y,y')\big)=2|y|^2,\quad \frac{d}{dt}\bigg|_{t=0}\big(f\circ\varphi^i\circ\psi^t(y,y')\big)=0,$$

for every $(y,y')\in\mathbb{C}^n=\mathbb{C}\times\mathbb{C}^{n-1}$. Since every $x=(y,y')\in X_0$ satisfies $2|y|^2=1$, (48) follows. Hence all the hypotheses of Lemma 15 are satisfied. This proves Claim 1.

By Claim 1 we may apply Lemma 15, to conclude that inequality (49) holds.

Let $t\in[0,1]$. Then we have $\psi^t(X_0)=S^1\big((1+t)^2\frac{\pi}{2}\big)\times S^{2n-3}(\frac{\pi}{2})$. This is a closed regular coisotropic submanifold of $\mathbb{R}^{2n}$. Therefore, applying Theorem 14, inequality (43) holds with $N:=\psi^t(X_0)$. Proposition 25 below implies that

$$A_\times\big(\mathbb{R}^{2n},\omega_0,\psi^t(X_0)\big)\geq\min\left\{(1+t)^2\frac{\pi}{2},\frac{\pi}{2}\right\}=\frac{\pi}{2}.$$

Combining this with (59,49,43), it follows that

$$\|\varphi^{-i}|_{S^{2n-1}}\|_{S^{2n-1}}^{\mathbb{R}^{2n},\omega_0}\geq\frac{\pi}{2}.$$

Recalling the definition (11) of $\mathrm{diam}(X,Y,\omega)$, inequality (17) follows. This proves statement (i) of Theorem 6.

**Remark 22** *Theorem 14 is a refinement of [Zi1, Theorem 1]. (That result involves the minimal symplectic action $A(N)$, rather than the "split version" of it, $A_\times(N)$. Note that $A(N)$ is bounded above by $A_\times(N)$.) Since for $\psi^t$ and $X_0$ as in the above proof of Theorem 6(i), we have*

$$A(\psi^t(X_0))\to 0\quad \textit{as } t\searrow 0,$$

*[Zi1, Theorem 1] is not suitable for this proof. Instead, we need the refinement given in the present article. The same holds for the proof of Theorem 6(ii) (see below).*

**Outline of the proof of the second part of Theorem 6:** This is a refinement of the technique used in the proof of part (i). The idea is as follows: We choose $\ell\geq 2$ and $k_1,\ldots,k_\ell\in\mathbb{N}$, such that

$$\sum_{i=1}^{\ell}k_i=\mathrm{k}(n,d),$$

and there exist integers $n_1,\ldots,n_\ell$ as in the definition of $\mathrm{k}(n,d)$. Without loss of generality we may assume that $n_1=\min_i n_i$. Assume first that

$$\sum_{i=1}^{\ell}k_in_i=n. \tag{60}$$

For every pair $k, m \in \mathbb{N}$ satisfying $k \le m$, and $a > 0$, we define the *Stiefel manifold of area* $a$ to be

$$V(k,n,a) := \left\{ \Theta \in \mathbb{C}^{k \times n} \,\middle|\, \Theta\Theta^* = \frac{a}{\pi}\mathbf{1}_k \right\}.$$

We define

$$a := \frac{\pi}{\mathrm{k}(n,d)}, \quad X_0 := \times_{i=1}^{\ell} V(k_i, n_i, a) \subseteq \times_{i=1}^{\ell} \mathbb{C}^{k_i \times n_i} = \mathbb{C}^n.$$

Note that $X_0 \subseteq \overline{B}^{2n}$. The second part of condition (15) guarantees that the dimension of $X_0$ is bounded above by $d$.

We choose a unitary linear action $S^1 \times \mathbb{C}^n \ni (z,x) \mapsto \varphi^z(x) \in \mathbb{C}^n$ which for a given tuple of matrices $(\Theta_1, \dots, \Theta_\ell) \in X_0$ intertwines the first row of $\Theta_1$ with part of the first row of $\Theta_2$. (This makes sense because of our assumption that $n_1 = \min_i n_i$.) The set $X := \bigcup_{z \in S^1} \varphi^z(X_0)$ has the properties required in statement (ii): That $X \subseteq \overline{B}^{2n}$ follows from the fact $X_0 \subseteq \overline{B}^{2n}$ and the orthogonality of the action $\varphi$. Furthermore, since $\dim X_0 \le d$, the Hausdorff dimension of $X$ is bounded above by $d+1$.

The main task is to show that inequality (18) holds. We will prove this by showing that the restriction of the map $\varphi^{-i} : \mathbb{C}^n \to \mathbb{C}^n$ to $X$ has relative Hofer semi-norm bounded below by the right-hand side of (18). The proof of this bound is based on Lemmas 15 and 17. The remainder of the argument is now analogous to the argument for part (i).

If the integers $\ell, k_1, \dots, k_\ell, n_1, \dots, n_\ell$ cannot be chosen such that the equality (60) holds, then the idea is to "project away" the extra $\left(\sum_i k_i n_i\right) - n$ complex dimensions. This means that we construct a suitable surjective linear map

$$\Psi : \times_{i=1}^{\ell} \mathbb{C}^{k_i \times n_i} \to \mathbb{C}^n,$$

and define

$$X_0 := \Psi\left(\times_{i=1}^{\ell} V(k_i, n_i, a)\right) \subseteq \mathbb{C}^n.$$

We may then carry out a modified version of the above argument.

*Proof (of Theorem 6(ii))* We choose $\ell \in \{2, 3, \dots\}$ and $k_1, \dots, k_\ell \in \mathbb{N}$ as in the definition of $\mathrm{k}(n,d)$ such that

$$\sum_{i=1}^{\ell} k_i = \mathrm{k}(n,d).$$

We also choose integers $n_1, \dots, n_\ell$ such that conditions (14,15,16) are satisfied. Reordering the pairs $(k_i, n_i)$, we may assume that $n_1 = \min_i n_i$. We choose an injective map

$$(\mathrm{L}, \mathrm{K}, \mathrm{N}) : \{n_1 + n_2 + 1, \dots, n\} \to \{1, \dots, \ell\} \times \mathbb{N} \times \mathbb{N}$$

satisfying

$$\mathrm{K}(i) \le k_{\mathrm{L}(i)}, \quad \mathrm{N}(i) \le n_{\mathrm{L}(i)}, \tag{61}$$

$$(\mathrm{L},\mathrm{K})(i) \ne (1,1) \text{ or } (2,1), \tag{62}$$

for every $i \in \{n_1+n_2+1,\dots,n\}$. (Our convention is that $\{m,\dots,n\} := \emptyset$ if $m > n$.) To see that we may choose this map to be injective, note that the number of allowed choices for $(\mathrm{L},\mathrm{K},\mathrm{N})(i)$ is

$$(k_1-1)n_1 + (k_2-1)n_2 + \sum_{L=3}^{\ell} k_L n_L = \sum_{L=1}^{\ell} k_L n_L - n_1 - n_2.$$

(The first and second term are obtained by considering $L = 1, 2$, and the other terms by considering $L \ge 3$.) By the first condition in (15) the right-hand side of this equality is bounded below by

$$n - n_1 - n_2 = \big|\big\{n_1+n_2+1,\dots,n\big\}\big|.$$

It follows that we may choose the map $(\mathrm{L},\mathrm{K},\mathrm{N})$ to be injective. We extend $(\mathrm{L},\mathrm{K},\mathrm{N})$ to $\{1,\dots,n\}$ by defining

$$(\mathrm{L},\mathrm{K},\mathrm{N})(i) := (1,1,i), \quad \forall i \in \{1,\dots,n_1\}, \tag{63}$$

$$(\mathrm{L},\mathrm{K},\mathrm{N})(i) := (2,1,i-n_1), \; \forall i \in \big\{n_1+1,\dots,\min\{n_1+n_2,n\}\big\}. \tag{64}$$

(Since $2n_1 = 2\min_i n_i \le n$, (63) makes sense.) We define the map $\varPsi : \times_{i=1}^{\ell} \mathbb{C}^{k_i \times n_i} \to \mathbb{C}^n$ by

$$\varPsi^i(\varTheta_1,\dots,\varTheta_\ell) := (\varTheta_{\mathrm{L}(i)})^{\mathrm{K}(i)}_{\mathrm{N}(i)}, \; \forall i \in \{1,\dots,n\}, \tag{65}$$

where for a matrix $\varTheta$ the number $\varTheta^i_j \in \mathbb{C}$ denotes its $(i,j)$-th entry. It follows from the inequalities (61) (holding for $i \in \{n_1+n_2+1,\dots,n\}$) and (63,64) that this definition makes sense. We define

$$a := \frac{\pi}{\mathrm{k}(n,d)},$$

$$X_0 := \varPsi\big(\times_{i=1}^{\ell} V(k_i,n_i,a)\big) \subseteq \mathbb{C}^n. \tag{66}$$

Furthermore, we define the map

$$S^1 \times \mathbb{C}^n \ni (z,x) \mapsto \varphi^z(x) \in \mathbb{C}^n \tag{67}$$

as follows. Let $z \in S^1$. We denote by $R^z : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation by $z$, and define

$$T^z : \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_1}, \quad T^z(q,q') := (Q,Q'),$$

where $(Q_i,Q'_i) := R^z(q_i,q'_i)$, for every $i \in \{1,\dots,n_1\}$. We define $\varphi^z : \mathbb{C}^n = \mathbb{C}^{2n_1} \times \mathbb{C}^{n-2n_1} \to \mathbb{C}^n$ to be the unique complex linear extension of the map

$$T^z \times \mathrm{id}_{\mathbb{R}^{n-2n_1}} : \mathbb{R}^n = \mathbb{R}^{2n_1} \times \mathbb{R}^{n-2n_1} \to \mathbb{R}^n.$$

(The conditions (16) and $n_1 = \min_i n_i$ guarantee that this makes sense.) The map (67) is a Hamiltonian $S^1$-action on $\mathbb{C}^n$. (It is generated by the function $H : \mathbb{C}^n \to \mathbb{R}$ defined by $H(q+ip) := \sum_{i=1}^{n_1} q_i p_{n_1+i} - q_{n_1+i} p_i$.)

**Claim 1** *The set*

$$X := \bigcup_{z \in S^1} \varphi^z(X_0)$$

*satisfies the conditions of statement (ii).*

*Proof (of Claim 1)* Since the Stiefel manifolds are compact, the set $X_0$ and hence $X$ **is compact**. To see that $X$ **is contained in** $\overline{B}^{2n}$, note that $|\Psi(\Theta)| \leq |\Theta| \leq 1$, for every $\Theta = (\Theta_1, \dots, \Theta_\ell) \in \times_i V(k_i, n_i, a)$. It follows that $X_0 \subseteq \overline{B}^{2n}$. Since $\varphi^z$ is orthonormal, for every $z \in S^1$, this implies that $X \subseteq \overline{B}^{2n}$. To see that $X$ **has Hausdorff dimension at most** $d+1$, observe that

$$\dim\big(\times_{i=1}^\ell V(k_i, n_i, a)\big) = \sum_{i=1}^\ell k_i(2n_i - k_i) \leq d, \tag{68}$$

where in the second step we used the second inequality in (15). Note that $X$ is the image of $S^1 \times \times_{i=1}^\ell V(k_i, n_i, a)$ under the smooth map $(z, \Theta_1, \dots, \Theta_\ell) \mapsto \varphi^z \circ \Psi(\Theta_1, \dots, \Theta_\ell)$. Combining this with (68), a standard result (cf. [Fed, p. 176]) implies that $X$ has Hausdorff dimension at most $d+1$.

Recalling that $a = \pi/\mathrm{k}(n,d)$, **inequality (18)** is a consequence of definition (11) and the following claim.

**Claim 2** *We have*

$$\|\varphi^{-i}|_X\|_X^{\mathbb{R}^{2n},\omega_0} \geq a. \tag{69}$$

*Proof (of Claim 2)* We define $(M,\omega) := (\mathbb{C}^n, \omega_0)$, $z_0 := i \in S^1 \subseteq \mathbb{C}$ and the map $[0,1] \times \mathbb{C}^n \ni (t,x) \mapsto \psi^t(x) \in \mathbb{C}^n$ by

$$(\psi^t(x))^i := \begin{cases} (1+t)x^i, & \text{if } \mathrm{L}(i) = 1, \\ x^i, & \text{otherwise.} \end{cases} \tag{70}$$

**Claim 3** *The hypotheses of Lemma 15 are satisfied.*

*Proof (of Claim 3)* The **condition (46)** clearly holds. We define

$$f : \mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n-n_1} \to \mathbb{R}, \quad f(y, y') := |y|^2 - \frac{1}{\mathrm{k}(n,d)}. \tag{71}$$

That the pair $(X_0, df)$ **is rigidifying**, is a consequence of Lemma 17 and the following claim.

**Claim 4** *The hypothesis of Lemma 17 is satisfied.*

*Proof (of Claim 4)* We define

$$(\widetilde{M}, \widetilde{\omega}) := \big(\times_{i=1}^\ell \mathbb{C}^{k_i \times n_i}, \omega_0\big),\ (M', \omega') := (\ker \Psi, \omega_0|_{M'}),\ G := \times_{i=1}^\ell \mathrm{U}(k_i),$$

where $\Psi$ is defined as in (65) and $\mathrm{U}(k) \subseteq \mathbb{C}^{k \times k}$ denotes the unitary group. Furthermore, we define an action of $G$ on $\widetilde{M}$ by

$$(U_1, \dots, U_\ell) \cdot (\Theta_1, \dots, \Theta_\ell) := \big(U_1\Theta_1, \dots, U_\ell\Theta_\ell\big). \tag{72}$$

Moreover, we identify the Lie algebra $\mathfrak{g}$ of $G$ with its dual via the standard inner product, and we define the map $\mu : \times_{i=1}^{\ell} \mathbb{C}^{k_i \times n_i} \to \mathfrak{g}^* \cong \mathfrak{g}$ by

$$\mu(\Theta_1, \ldots, \Theta_\ell) := \frac{i}{2} \left( \frac{\mathbf{1}_{k_1}}{\mathrm{k}(n,d)} - \Theta_1 \Theta_1^*, \ldots, \frac{\mathbf{1}_{k_\ell}}{\mathrm{k}(n,d)} - \Theta_\ell \Theta_\ell^* \right).$$

Finally, we define $\Psi' : \widetilde{M} \to M' \subseteq \widetilde{M}$ to be the projection along $M'^{\omega_0}$ (the symplectic complement of the linear symplectic subspace $M'$ of $\widetilde{M}$), and

$$\widetilde{\Psi} := (\Psi, \Psi') : \widetilde{M} \to M \times M'. \tag{73}$$

We show that the conditions of Lemma 17 are satisfied with $\psi := \widetilde{\Psi}$: The action (72) is Hamiltonian, and $\mu$ is a moment map. We denote by $\mathrm{pr} : M \times M' \to M$ the canonical projection. Condition (52) follows from the facts $\mu^{-1}(0) = \times_i V(k_i, n_i, a)$ and $\mathrm{pr} \circ \widetilde{\Psi} = \Psi$, and from (66).

We define $g : \mathfrak{g} \to \mathbb{R}$ by $g(\xi_1, \ldots, \xi_\ell) := 2i(\xi_1)_1^1$. Using (71,63,65) and the fact $\mathrm{pr} \circ \widetilde{\Psi} = \Psi$, it follows that $g \circ \mu = f \circ \mathrm{pr} \circ \widetilde{\Psi}$. This proves that $f \circ \mathrm{pr} \circ \widetilde{\Psi}$ factors through $\mu$ and completes the proof of Claim 4.

**Claim 5** *The* **inclusions (47)** *hold.*

*Proof (of Claim 5)* Let $x := (y, y') \in X_0 \subseteq \mathbb{C}^{n_1} \times \mathbb{C}^{n-n_1}$. By (66) this means that there exists a tuple $\Theta := (\Theta_1, \ldots, \Theta_\ell) \in \times_i V(k_i, n_i, a)$ such that $\Psi(\Theta) = (y, y')$. To see the first inclusion in (47), observe that by (63,65), $y$ is the first row of $\Theta_1$. Using (71), it follows that $f(y, y') = 0$. This proves the first inclusion.

To see that the second inclusion holds, we denote

$$y' =: (y'', y''') \in \mathbb{C}^{n_1} \times \mathbb{C}^{n-2n_1}.$$

Using (67,71), we have

$$f \circ \varphi^i(y, y'', y''') = f(-y'', y, y''') = |y''|^2 - \frac{1}{\mathrm{k}(n,d)}. \tag{74}$$

It follows from (64,65) that $|y''|$ is bounded above by the norm of the first row of $\Theta_2$, i.e., $1/\sqrt{\mathrm{k}(n,d)}$. Combining this with (74), we have $f \circ \varphi^{-i}(x) \leq 0$. The second inclusion in (47) follows. This proves Claim 5.

We check the last hypothesis of Lemma 15:

**Claim 6** *For every $x \in X_0 \subseteq \mathbb{C}^n$ the* **inequalities (48)** *hold.*

*Proof (of Claim 6)* To see that the **first inequality** holds, we denote $x =: (y, y') \in \mathbb{C}^{n_1} \times \mathbb{C}^{n-n_1}$. By (66) there exists a tuple $\Theta := (\Theta_1, \ldots, \Theta_\ell) \in \times_i V(k_i, n_i, a)$ such that $\Psi(\Theta) = x$. It follows from (63,70,71) that

$$\left.\frac{d}{dt}\right|_{t=0} (f \circ \psi^t(x)) = 2|y|^2. \tag{75}$$

Furthermore, (63,65) imply that $y$ is the first row of $\Theta_1$, and therefore has norm $1/\sqrt{\mathrm{k}(n,d)}$. Combining this with (75), it follows that the first inequality in (48) holds.

We show that the **second inequality** holds: Using (64,70), we have $\big(\varphi^i \circ \psi^t(x)\big)^j = -x^{n_1+j}$, for every $j \in \{1,\dots,n_1\}$, and therefore,

$$\frac{d}{dt}\bigg|_{t=0} \big(f \circ \varphi^i \circ \psi^t(x)\big) = 0.$$

Hence the second inequality in (48) is satisfied. This proves Claim 6.

Hence all hypotheses of Lemma 15 are satisfied. This completes the proof of Claim 3.

By Claim 3, we may apply Lemma 15, to conclude that inequality (49) holds. Let $t \in [0,1]$. We define

$$N^t := V\big(k_1, n_1, (1+t)^2 a\big) \times \times_{i=2}^{\ell} V(k_i, n_i, a).$$

We denote by $\mathrm{pr} : \mathbb{C}^n \times \ker \Psi \to \mathbb{C}^n$ the canonical projection. Recall the definition (73) of $\widetilde{\Psi}$. Using (63,65,66,70), we have $\mathrm{pr} \circ \widetilde{\Psi}(N^t) = \Psi(N^t) = \psi^t(X_0)$. Hence it follows from Remark 26 below that

$$e\big(\psi^t(X_0), \mathbb{C}^n, \omega_0\big) \geq e\left(\widetilde{\Psi}(N^t), \mathbb{C}^n \times \ker \Psi, \omega_0\right). \tag{76}$$

Furthermore, by an elementary argument, the map $\widetilde{\Psi}$ is a (linear) symplectomorphism, and therefore,

$$e\left(\widetilde{\Psi}(N^t), \mathbb{C}^n \times \ker \Psi, \omega_0\right) = e\big(N^t, \times_i \mathbb{C}^{k_i \times n_i}, \omega_0\big). \tag{77}$$

Note that $N^t$ is a closed regular coisotropic submanifold of $\times_i \mathbb{C}^{k_i \times n_i}$. Hence applying Theorem 14, it follows that

$$e\big(N^t, \times_i \mathbb{C}^{k_i \times n_i}, \omega_0\big) \geq A_\times\big(\times_i \mathbb{C}^{k_i \times n_i}, \omega_0, N^t\big). \tag{78}$$

It follows from Proposition 25 below that

$$A_\times\big(\times_i \mathbb{C}^{k_i \times n_i}, \omega_0, N^t\big) \geq \min\big\{(1+t)^2 a, a\big\} = a.$$

Combining this with (in-)equalities (49,76,77,78), inequality (69) follows. This proves Claim 2 and hence Claim 1, and completes the proof of statement (ii) and therefore of Theorem 6.

## A Auxiliary results

### A.1 (Pre-)symplectic geometry

The following lemma was used in the proof of Proposition 12.

**Lemma 23** *Let $(M,\omega)$ be a symplectic manifold, $N \subseteq M$ a symplectic submanifold, and $r : M \to N$ a smooth retraction such that $\ker dr(x) = T_xN^\omega$ (as defined in (32)), for every $x \in N$. Let $H \in C^\infty(N,\mathbb{R})$. Then we have, for every $x \in N$,*

$$X^\omega_{H\circ r}(x) = X_H^{\omega|_N}(x). \tag{79}$$

For the proof of Lemma 23 we need the following. Let $(V,\omega)$ be a symplectic vector space and $W \subseteq V$ a linear subspace. Assume that $W$ is a symplectic subspace. We denote by $\mathrm{pr}^W$ the linear projection from $V$ onto $W$, along $W^\omega$.

**Remark 24** *Let $v \in V$ and $w \in W$ be vectors such that $\omega(v,\cdot) = \omega|_W(w,\mathrm{pr}^W\cdot)$. Then we have $v = w$. This follows from a straight-forward argument.*

*Proof (of Lemma 23)* Let $x \in N$. We have

$$\omega\big(X^\omega_{H\circ r}(x),\cdot\big) = d(H\circ r)(x) = dH(x)dr(x) = \omega|_N\big(X_H^{\omega|_N}(x), dr(x)\cdot\big).$$

Since $r$ is a retraction onto $N$, the map $dr(x) : T_xM \to T_xM$ is a projection onto $T_xN$. By hypothesis its kernel is $T_xN^\omega$. Hence equality (79) follows from Remark 24. This proves Lemma 23.

The next result was used in the proof of Theorem 6. For $k,n \in \mathbb{N}$ satisfying $k \le n$ we denote

$$V(k,n) := \big\{\Theta \in \mathbb{C}^{k\times n}\,\big|\,\Theta\Theta^* = \mathbf{1}_k\big\}.$$

**Proposition 25** *The Stiefel manifold $V(k,n)$ has minimal area*

$$A\big(\mathbb{R}^{2kn},\omega_0,V(k,n)\big) = \pi.$$

*Proof* For a proof we refer to [Zi1, Proposition 1.3].

We used the next remark in the proof of Theorem 6(ii). Recall the definition (42) of the displacement energy.

**Remark 26** *Let $(M,\omega)$ and $(M',\omega')$ be symplectic manifolds and $X \subseteq M$ a subset. Then we have*

$$e\big(X\times M', M\times M', \omega\oplus\omega'\big) \le e(X,M,\omega).$$

*This follows from a straight-forward argument.*

The next lemma was used in the proof of Proposition 2(ii). For a proof see [SZ1, Lemma 35]. We denote by $\mathrm{Ham}_c(M,\omega)$ the group of Hamiltonian diffeomorphisms of $M$ generated by a compactly supported function, and by $\|\cdot\|_c^{M,\omega}$ the compactly supported Hofer norm on this group.

**Lemma 27 ([SZ1])** *Let $(M,\omega)$ be a symplectic manifold, $K \subseteq M$ a compact subset, $\varphi \in \mathrm{Ham}(M,\omega)$, and $\varepsilon > 0$. Then there exists $\psi \in \mathrm{Ham}_c(M,\omega)$ such that*

$$\psi|_K = \varphi|_K, \quad \|\psi\|_c^{M,\omega} \le \|\varphi\|^{M,\omega} + \varepsilon. \tag{80}$$

*(Here our convention is that $\infty + \varepsilon := \infty$.)*

## A.2 Topology and manifolds

In Section 2.1 we used the following remark. Let $M$ be a $C^\infty$-manifold and $V$ a time-dependent vector field on $M$, i.e., a smooth map $[0,1]\times M \in (t,x) \mapsto V^t(x)\in TM$ such that $\pi\circ V^t = \mathrm{id}_M$, where $\pi : TM \to M$ denotes the canonical projection. We denote by $\mathcal{D}_V \subseteq [0,1]\times M$ the domain of the flow of $V$, and by $\varphi_V$ the flow of $V$.

**Remark 28** *The set $\mathcal{D}_V$ is open, and for every $t\in[0,1]$, the map*

$$\varphi_V^t : \mathcal{D}_V^t := \big\{x\in M \,\big|\, (t,x)\in\mathcal{D}_V\big\} \to M$$

*is injective and a smooth immersion. (This follows for example from [Le, Theorem 17.15, p. 451, and Problem 17-15, p. 463].)*

The following lemma was used in the proof of Proposition 12.

**Lemma 29** *Let $M$ be a smooth manifold and $A_i \subseteq M$ a closed subset, for $i = 0,1$. If $A_0\cap A_1=\emptyset$ then there exists a function $f\in C^\infty(M,[0,1])$ such that $f|_{A_i}\equiv i$, for $i=0,1$.*

*Proof (of Lemma 29)* This follows from a $C^\infty$-version of Urysohn's Lemma for $\mathbb{R}^n$ (see for example Theorem 1.1.3, p. 4 in [KP]) and a partition of unit argument.

In the proof of Proposition 12 we used the following result. Let $M$ be a smooth manifold, $N\subseteq M$ a submanifold, and $E\subseteq TM|_N$ a sub-bundle such that $TM|_N$ is the direct sum of $TN$ and $E$. For $x\in N$ we denote by $E_x$ the fiber of $E$ over $x$.

**Proposition 30** *Assume that $N$ is closed as a subset of $M$. Then there exists an embedding $\psi : E\to M$ such that, identifying $N$ with the zero section of $E$, we have*

$$\psi|_N = \mathrm{id}_N, \tag{81}$$

$$d\psi(x)v = v, \quad \forall v\in T_x(E_x)=E_x,\, x\in N. \tag{82}$$

*Proof (of Proposition 30)* This follows from a standard argument, along the lines of the proof of Theorem 5.2 in the book [Hi].

The following lemma was used in the proof of Lemma 15.

**Lemma 31** *Let $M$ be a $C^1$-manifold, $X_0\subseteq M$ a compact subset, $\varphi\in C^1(M,M)$ and $[0,1]\times M\ni(t,x)\mapsto\psi^t(x)\in M$ a $C^1$-map. Assume that $\psi^0=\mathrm{id}$ and there exists a function $f\in C^1(M,\mathbb{R})$ such that*

$$f(X_0)\subseteq(-\infty,0], \quad f\circ\varphi(X_0)\subseteq[0,\infty), \tag{83}$$

$$\frac{d}{dt}\Big|_{t=0}\big(f\circ\psi^t(x)\big)\le 0, \quad \frac{d}{dt}\Big|_{t=0}\big(f\circ\varphi\circ\psi^t(x)\big)>0,\ \forall x\in X_0. \tag{84}$$

*Then there exists $t_0>0$ such that for every $t\in(0,t_0]$, we have*

$$\varphi\circ\psi^t(X_0)\cap\psi^t(X_0)=\emptyset.$$

In the proof of this result, we will use the following remark:

**Remark 32** *Let $X$ be a compact topological space and $f:[0,1]\times X\ni(t,x)\to\mathbb{R}$ a function. Assume that the partial derivative $\partial_t f:[0,1]\times X\to\mathbb{R}$ exists and is continuous. Then we have*

$$\lim_{t\searrow 0}\frac{1}{t}\max_{x\in X}\big|f(t,x)-f(0,x)-t\partial_t f(0,x)\big| = 0.$$

*This follows from an elementary argument.*

*Proof (of Lemma 31)* Using that $\psi^0 = \mathrm{id}$ and that $X_0$ is compact, it follows from Remark 32 that

$$\lim_{t\searrow 0}\frac{1}{t}\max_{x\in X_0}\left|f\circ\psi^t(x)-f(x)-t\left.\frac{d}{dt}\right|_{t=0}(f\circ\psi^t(x))\right|=0. \tag{85}$$

Furthermore, defining the function $g:[0,1]\times M\to\mathbb{R}$ by

$$g(t,x):=f\circ\varphi\circ\psi^t(x)-f\circ\varphi(x)-t\left.\frac{d}{dt}\right|_{t=0}\big(f\circ\varphi\circ\psi^t(x)\big),$$

the same remark implies that

$$\lim_{t\searrow 0}\frac{1}{t}\max_{x\in X_0}|g(t,x)|=0. \tag{86}$$

We denote

$$c:=\min_{x,y\in X_0}\left(\left.\frac{d}{dt}\right|_{t=0}\big(f\circ\varphi\circ\psi^t\big)(y)-\left.\frac{d}{dt}\right|_{t=0}(f\circ\psi^t)(x)\right),$$
$$d(t):=\tfrac{1}{t}\min_{x,y\in X_0}\big(f\circ\varphi\circ\psi^t(y)-f\circ\psi^t(x)\big).$$

Compactness of $X_0$ and (84) imply that $c>0$. Furthermore, it follows from (83,85,86) that

$$\liminf_{t\searrow 0} d(t)\geq c.$$

Hence there exists $t_0\in(0,1]$ such that for $t\in(0,t_0]$ we have $d(t)\geq\frac{c}{2}$. It follows that for every $t\in(0,t_0]$ and $x,y\in X_0$, we have $f\circ\psi^t(x)\neq f\circ\varphi\circ\psi^t(y)$, and therefore, $\psi^t(x)\neq\varphi\circ\psi^t(y)$. Hence $\psi^t(X_0)\cap\varphi\circ\psi^t(X_0)=\emptyset$, for every $t\in(0,t_0]$. This proves Lemma 31.

The next lemma implies that the semi-norm $\|H\|_X$ given by (3) is well-defined.

**Lemma 33** *Let $X$ be a topological space and $f:[0,1]\times X\to\mathbb{R}$ be a continuous function. Assume that there exists a sequence of compact subsets $K_\nu\subseteq X$, $\nu\in\mathbb{N}$ such that $\bigcup_\nu K_\nu = X$. Then the map*

$$[0,1]\ni t\mapsto\sup_{x\in X}f(t,x)$$

*is Borel measurable.*

*Proof* This follows from an elementary argument.


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