SYMPLECTIC VORTICES WITH FIXED HOLONOMY AT INFINITY

E. GONZALEZ, A. OTT, C. WOODWARD, AND F. ZILTENER

ABSTRACT. Let Σ be a Riemann surface with cylindrical ends, G a compact, connected Lie group and let X denote a compact symplectic manifold with a Hamiltonian G-action. Given a conjugacy class associated to each end, we define a moduli space of symplectic vortices which generalizes the moduli spaces of parabolic bundles introduced by Mehta and Seshadri. Using the moduli spaces we construct gauged Gromov-Witten invariants for convex target.

CONTENTS

Preliminary version.

1. INTRODUCTION

In this paper we study moduli spaces of symplectic vortices on surfaces with cylindrical ends with fixed holonomies around the punctures. These moduli spaces have the same relation to the moduli spaces of symplectic vortices constructed by Mundet [17] and Cieliebak, Gaio, and Salamon [5] as the moduli spaces of bundles with fixed holonomy introduced by Mehta-Seshadri [14] have to the moduli spaces of flat bundles studied by Narasimhan Seshadri [18]. These moduli spaces are expected to play a role in various factorization theorems, in particular, the generalization of orbifold quantum cohomology

Date: July 8, 2008.

to continuous Lie groups. In the case of the circle group, part of this project is carried out by Mundet-Tian [11].

We now describe the results more precisely. Let G be a compact group, X a compact, connected Hamiltonian G-manifold with moment map $\Phi : X \to \mathfrak{g}^*$ equipped with a compatible, G-invariant almost complex structure, Σ a compact complex surface equipped with an area form ω_Σ, and $P \to \Sigma$ a principal G-bundle. The space $\mathcal{A}(P, X)$ of pairs (A, u) consisting of a connection A on P and a pseudoholomorphic section u of the associated bundle with fiber X has a natural action of the group $\mathcal{G}(P)$ of gauge transformations. The formal symplectic quotient

$$M(P, X) = \mathcal{A}(P, X) /\!\!/ \mathcal{G}(P)$$

is the moduli space of solutions to the *symplectic vortex equations*

$$\overline{\partial}_A u = 0, \qquad F_A + \omega_\Sigma u^* P(\Phi) = 0. \tag{1}$$

A solution is called a *symplectic vortex*. These are natural equivariant versions of pseudoholomorphic maps, which arise naturally in a number of settings. On the other hand, these moduli spaces can be viewed as generalizations of moduli spaces of flat connections. Symplectic vortices on a closed curve were introduced by Mundet [17] and Cieliebak, Gaio, and Salamon [5], see also Cieliebak, Gaio, Mundet, and Salamon [8], [4].

We first construct a compactification $\overline{M}(P, X, \underline{\mu})$ of *polystable vortices with fixed holonomies* by allowing sphere bubbles in the fibers of P, and twisted holomorphic cylinders along the cylindrical ends, where $\underline{\mu}$ are the holonomy parameters around the punctures. We then construct evaluation maps at infinity for each cylindrical end. Becaue the energy of the section must be finite, the limit of each section is forced to lie in the fixed point set for the limiting holonomy of the connection. The resulting evaluation map (ignoring problems with finite covers that we explain later) maps

$$\mathrm{ev} : \overline{M}(P, X, \underline{\mu}) \to \prod_{i=1}^{n} (X^{\mu_i})_{G_{\mu_i}}$$

where X^{μ_i} is the fixed point set of the element $\exp(\mu_i)$, G_{μ_i} is the centralizer of $\exp(\mu_i)$ and $(X^{\mu_i})_{G_{\mu_i}}$ is the classifying space for the action of G_{μ_i} on X^{μ_i}. In certain fortuitous cases, namely X aspherical or convex and every underlying connection irreducible, we show how to achieve transversality and define *gauged Gromov-Witten invariants*

$$\prod_{i=1}^{n} H_{G_{\mu_i}}(X^{\mu_i}, \mathbb{Q}) \to \mathbb{Q}$$

by pairing with the pseudocycle defined by the evaluation map above.

One would like to use the invariants to construct a “gauged version” of small equivariant quantum cohomology, which should be a deformation of the non-abelian equivariant cohomology of orbit-type strata of X in the sense of [1]. Roughly speaking, equivariant quantum cohomology of X should count pseudoholomorphic curves in $X_G = X \times_G EG$. Givental has introduced an analog of equivariant quantum cohomology, which is a module over $H(BG)$, and counts curves in X. Thus, in Givental's version, only the X-direction is quantized. Since $H(BG)$ is free, it does not admit any interesting deformation, so in some sense $H_G(X)$ cannot be quantized further. But this is taking quantization too literally: a curve in BG is a G-bundle, and thus one expects the ring structure on $QH_G(pt)$ to count bundles. There are a number of theories which do this, topological Yang-Mills in two dimension being the simplest, and various version of twisted K-theory being the most sophisticated to date. For the former, the corresponding Frobenius algebra is the convolution algebra of invariant distributions on G (which contains the center of the universal enveloping algebra, and hence $H(BG)$, as a sub-algebra.) One might hope to construct a similar theory for more general target whose Frobenius algebra is some version of equivariant cohomology of the inertia space of X using the gauged Gromov-Witten invariants we describe above. It is not yet clear to us whether one should expect this product to be strictly associative.

We thank Constantin Teleman for encouragement.

2. Vortices with fixed holonomy

2.1. **Bundles with fixed holonomy.** We briefly review the Mehta-Seshadri theory of parabolic bundles [14]. Let G be a compact, 1-connected Lie group. We denote by T a maximal torus and W the Weyl group. Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{t}_+$ a choice of positive chamber, so that $\alpha_0 \in \mathfrak{t}^*$ is the highest root. The *Weyl alcove* $\mathfrak{A}$ is the subset of $\mathfrak{t}_+$ defined by

$$\mathfrak{A} := \{\xi \in \mathfrak{t}_+, \quad \alpha_0(\xi) \leq 1\}.$$

The sequence of maps $\mathfrak{A} \to T \to G$ given by exponential and inclusion respectively induce isomorphisms of quotient spaces

$$\mathfrak{A} \cong T/W \cong G/\operatorname{Ad}(G).$$

For any $\mu \in \mathfrak{A}$ we denote by $\mathcal{C}_\mu$ the corresponding conjugacy class.

Let Σ be a compact, oriented surface with $n \geq 0$ boundary components, $P \to \Sigma$ be a principal G-bundle. The space $\mathcal{A}(P)$ of connections on P is an affine space modelled on $\Omega^1(\Sigma, P(\mathfrak{g}))$ the space of one-forms with values in the adjoint bundle $P(\mathfrak{g})$. The choice of an invariant inner product $\langle \cdot, \cdot \rangle_\mathfrak{g}$ on $\mathfrak{g}$ induces a map

$$\Omega^1(\Sigma, P(\mathfrak{g}))^2 \to \Omega^2(\Sigma), \quad (a_1, a_2) \mapsto \langle a_1 \wedge a_2 \rangle_\mathfrak{g}$$

by combining the wedge product and metric. This induces on $\mathcal{A}(P)$ the structure of an infinite-dimensional symplectic manifold with symplectic form given by

$$(a_1, a_2) \mapsto \int_\Sigma \langle a_1 \wedge a_2 \rangle_\mathfrak{g}. \tag{2}$$

The group of gauge transformations

$$\mathcal{G}(P) = \{a : P \to P, a(pg) = a(p)g \forall g \in G\}$$

acts on $\mathcal{A}(P)$ via pull-back. The infinitesimal action is given by

$$\Omega^0(\Sigma, P(\mathfrak{g})) \to \Omega^1(\Sigma, P(\mathfrak{g})), \quad \xi \mapsto -\mathrm{d}_A\xi.$$

The action preserves the symplectic form (13) and has moment map given by the curvature

$$\mathcal{A}(\Sigma, G) \to \Omega^2(\Sigma, \mathfrak{g}), \quad A \mapsto F_A.$$

In the case that Σ has boundary, the gauge group fits into an exact sequence

$$1 \to \mathcal{G}_\partial(P) \to \mathcal{G}(P) \to \mathcal{G}(\partial P) \to 1$$

where $\mathcal{G}_\partial(\Sigma)$ is the group of gauge transformations that equal to the identity on the boundary. The symplectic quotient

$$M(P, \partial P) = \mathcal{A}_\flat(P)/\mathcal{G}_\partial(P)$$

may be identified with the moduli space of flat connections with framing on the boundary. The residual group $\mathcal{G}(\partial P)$ acts on $M(P, \partial P)$ by changing the framing on the boundary, with moment map given by restriction to the boundary

$$\Phi : M(P, \partial P) \to \mathcal{A}(\partial P).$$

The $\mathcal{G}(\partial P)$-orbits on $\mathcal{A}(\partial P)$ are parametrized by conjugacy classes, via the map given by measuring the holonomy around each boundary component. For any $\underline{\mu} \in \mathfrak{A}^n$, where n is the number of boundary components, we denote by $\mathcal{O}_{\underline{\mu}} \subset \mathcal{A}(\partial P)$ the orbit labelled by $\mu \in \mathfrak{A}^n$. The symplectic quotient

$$M(P, \underline{\mu}) = \Phi^{-1}(\mathcal{O}_{\underline{\mu}})/\mathcal{G}(\partial P),$$

is the *moduli space of flat connections with fixed holonomies* $\exp(\underline{\mu})$ around the boundary. Let

$$M(\Sigma, \underline{\mu}) = \bigcup_P M(P, \underline{\mu})$$

denote the union over topological types of bundles $P \to \Sigma$, the *moduli space of flat bundles with fixed holonomies.*

Lemma 2.1.1. *There exists a homeomorphism*

$$M(\Sigma, \underline{\mu}) \to \{\varphi \in \mathrm{Hom}(\pi_1(\Sigma), G), \quad \varphi(\gamma_i) \in \mathcal{C}_{\mu_i}, i = 1, \dots, n\}/G.$$

For generic $\underline{\mu}$, the space $M(\Sigma, \underline{\mu})$ is a compact orbifold.

Proof. Any flat bundle is determined by its holonomies up to conjugacy; conversely, it is straightforward to construct from a representation a bundle with that holonomy. For the second claim, see [15]. □

The moduli space $M(\Sigma, \underline{\mu})$ came to play a fundamental role in the theory of moduli of flat bundles through various *factorization theorems* introduced by physicists. In the first of these, Witten showed how by cutting a surface along a circle one obtains the *Migdal formula* for the symplectic volumes of the moduli spaces $M(\Sigma, \underline{\mu})$ [22]. Another example is the *Verlinde formula* (see e.g. [2]) which expresses the dimension of the space of non-abelian theta functions on the closed surface as a sum over theta functions for the cut surface, after a sum over markings μ satisfying a certain integrality condition.

The moduli spaces $M(\Sigma, \underline{\mu})$ have a holomorphic description due to Mehta-Seshadri [14], see also [3], [20], as follows. Let $P_{\mathbb{C}} \to \Sigma$ be a holomorphic principal $G_{\mathbb{C}}$-bundle.

Definition 2.1.2. A *quasiparabolic structure* at a point $z \in \Sigma$ is a reduction of structure group of E_z to a parabolic subgroup $P \subset G$, that is, a point in the quotient $\sigma_z \in E_z/P$. A *parabolic structure* is a quasiparabolic structure $\sigma_z \in E_z/P$ together with conjugacy class $\mathcal{C} \subset G$ of the type specified by the parabolic subgroup. That is, the Levi subgroup L of the parabolic P is isomorphic to the centralizer Z_g of any point g in the conjugacy class $\mathcal{C}$. A parabolic bundle is *semistable* if a certain inequality is satisfied for each reduction of E to a maximal parabolic subgroup (at least, if none of the markings μ_j are contained in the opposite wall of the Weyl alcove, see [20]).

On the set of semistable parabolic bundles one defines a *grade equivalence* relation, which equates parabolic bundles if their associated graded bundles are isomorphic. Let $M_{\mathbb{C}}(\Sigma, \underline{\mu})$ denote the moduli space of grade-equivalence classes of semistable parabolic bundles. Narasimhan-Mehta-Seshadri [14] and extensions [7], [3], [20] show that

Theorem 2.1.3. *$M_{\mathbb{C}}(\Sigma, \underline{\mu})$ is a normal projective variety homeomorphic to $M(\Sigma, \underline{\mu})$.*

In the case that each conjugacy class μ_j has finite order, one can describe $M_{\mathbb{C}}(\Sigma, \underline{\mu})$ in terms of equivariant bundles for a finite group action. Let $\tilde{\Sigma} \to \Sigma$ be a totally ramified $\mathbb{Z}_N$-covering, such that the order of each μ_j divides N. Suppose we are given a $\mathbb{Z}_N$-equivariant holomorphic principal $G_{\mathbb{C}}$-bundle $\tilde{E} \to \tilde{\Sigma}$, with the following property: consider a local trivialization near s_j in which the generator of $\mathbb{Z}_N$ acts by g_j, and suppose that $\exp(\mu_j) = g_j$. At each puncture glue in the trivial bundle $D \times G_{\mathbb{C}}$ over the disk via the transition map

$$(z, g) \mapsto (z, \exp(N \ln(z)\mu_j/2\pi i)g). \tag{3}$$

The group $\mathbb{Z}_N$ acts freely on the resulting completed bundle and the quotient is a holomorphic principal $G_{\mathbb{C}}$-bundle E over $\overline{\Sigma}$. The bundle E has a parabolic reduction at the fiber s_j, given as the image of $(0, P_j)$ in the local trivialization, where P_j is the parabolic corresponding to ξ_j; this is independent of the local trivializations used above. Mapping $\tilde{E}$ to E defines a correspondence between equivariant bundles on $\tilde{\Sigma}$ and parabolic bundles on Σ, see [14], [20].

2.2. **Gauged holomorphic maps.**

Let Σ be a Riemann surface with n cylindrical ends. Let $P \to \Sigma$ be a principal G-bundle.

Definition 2.2.1. A *framing* on the i-th end of P is a principal G-bundle $P_i \to S^1$ and an isomorphism of P with $\pi^* P_i$ along the i-th end, where $\pi : S^1 \times \mathbb{R} \to S^1$ is the projection.

Suppose that P is equipped with a framing on each end. We denote by $\mathcal{A}(P)$ the space of smooth connections on P with asymptotic limits at infinity along each end, that is, there exists a connection $\lim_i(A)$ on P_i such that

$$\lim_i(A)(\theta_i) = \lim_{r_i \to \infty} A(r_i, \theta_i), \quad i = 1, \ldots, n. \tag{4}$$

Let X be a compact, connected Hamiltonian G-manifold with moment map $\Phi : X \to \mathfrak{g}^*$. Consider the associated bundle

$$\pi : P(X) = (P \times X)/G \to \Sigma.$$

Let $\Gamma(\Sigma, P(X))$ denote the space of asymptotically constant sections of $P(X)$, that is, sections u with a constant limit

$$\lim_i(u) := \lim_{r_i \to \infty} u_i(r_i, \theta_i)$$

along each end. I we denote by z_i the point at infinity in each cylindrical end, then let $\overline{u}$ be the extension of u so that $\overline{u}(z_i) = \lim_i(u)$. Let

$$\mathcal{A}(P) \times \mathcal{J}(X)^G \to \mathcal{J}(P(X)), \quad (A, J) \mapsto J_A$$

denote the map which assigns to any connection and invariant almost complex structure on X the almost complex structure on $P(X)$ determined by the splitting induced by A. Later on we will need an explicit formula for this map and an extensions of this constructions to certain Sobolev completions. We will discuss this in Section 3.

2.3. **Energy and equivariant symplectic area.**

We define a suitable notion of energy of a pair (A, u) as follows.

Equip Σ with the metric with cylindrical form on the ends. We wish to consider vortices for area forms with suitable exponential decay on the cylindrical ends.

Definition 2.3.1. An area form $\omega_\Sigma \in \Omega^2(\Sigma)$ is *admissible* if and only if on each cylindrical end, with coordinates s, t, there exists constants $c, C > 0$ such that the following estimates hold: if $\omega_\Sigma = \lambda^2(s,t) ds \wedge dt$ then

$$|\lambda(s,t)| \leq Ce^{-cs}, \qquad 9|d\lambda|^2 - \Delta(\lambda^2) \leq c$$

check ....

In particular, the form $e^{-cs} ds \wedge dt$ is admissible for $c > 0$.

Let $X, \omega, G, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \Phi$ be as above, Σ be a real surface (not necessarily orientable), P a principal G-bundle over Σ, $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_\Sigma$ be Riemannian metrics on X and Σ, and $(A, u) \in \mathcal{A}(P) \times \Gamma(P(X))$. We define the *energy density* of (A, u) to be the function

$$e_{(A,u)} := e_{(A,u)}^{\langle \cdot, \cdot \rangle_X, \langle \cdot, \cdot \rangle_\Sigma} := \frac{1}{2}\big(|\mathrm{d}_A u|^2 + |F_A|^2 + |u^* P(\Phi)|^2\big) : \Sigma \to \mathbb{R},$$

where the norms are w.r.t. the metrics $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_\Sigma$. Furthermore, we define the *energy* of (A,u) to be the integral

$$E(A,u) := \int_\Sigma e_{(A,u)} d\mu_{\langle\cdot,\cdot\rangle_\Sigma},$$

where $\mu_{\langle\cdot,\cdot\rangle_\Sigma}$ is the measure on Σ induced by $\langle\cdot,\cdot\rangle_\Sigma$. As in the case of ordinary pseudo-holomorphic maps, there is a relation of the energy to a suitable notion of *equivariant symplectic area*. Recall that the Cartan construction gives rise to a map

$$\mathcal{A}(P) \times \Omega_G(X) \to \Omega(P(X))$$

given by

$$(A,\beta) \mapsto \pi_{\mathrm{basic}}((p_2^*\beta) \wedge (p_1^* A))$$

where p_1, p_2 are the projections of $P \times X$ on the factors and π_{basic} is projection on the subspace of basic forms. Applying this to the equivariant symplectic form $\omega_G \in \Omega_G^2(X)$, for each A we obtain a closed two-form

$$\omega_{P(X),A} \in \Omega^2(\Sigma, P(X)), \qquad \pi^*\omega_{P(X),A} = p_2^*\omega_G + \mathrm{d}(p_1^*A, p_2^*\Phi).$$

Definition 2.3.2. The *equivariant symplectic area* of $(A,u) \in \mathcal{A}(P) \times \Gamma(P(X))$ is

$$D(A,u) := \int_\Sigma u^*(\omega_{P(X),A})$$

if finite.

Definition 2.3.3. Given $J_X \in \mathcal{J}(X)^G$, for each section $u \in \Omega^0(\Sigma; P(X))$ define the operator

$$\overline{\partial}_A(u) := \frac{1}{2}(\mathrm{d}u + J_A(u) \circ \mathrm{d}u \circ j) \in \Omega^{0,1}(\Sigma; u^* T^{\mathrm{vert}} P(X)).$$

Lemma 2.3.4. *The energy and equivariant symplectic area of a pair (A,u) are related by*

$$E(A,u) = D(A,u) + \int_\Sigma \left(|\overline{\partial}_A u|^2 + \frac{1}{2}|F_A + u^*P(\Phi)\omega_\Sigma|^2 \right) \omega_\Sigma. \tag{5}$$

Proof. As in [4, Proposition 3.1]. □

Definition 2.3.5. A *gauged holomorphic map* is a pair $(A,u) \in \mathcal{A}(P) \times \Gamma(P(X))$ with $\overline{\partial}_A u = 0$. Let $\mathcal{A}(P,X)$ denote the space of gauged holomorphic maps of finite energy.

Later it will be useful to consider $u : \Sigma \to P(X)$ as a pseudoholomorphic map to $P(X)$. The latter can be given a symplectic form as follows. Let $\omega_\Sigma \in \Omega^2(\Sigma)$ be an area form on Σ. For any $c > 0$ let

$$\omega_{A,c} = \omega_A + c\pi^*\omega_\Sigma \in \Omega^2(P(X)). \tag{6}$$

Lemma 2.3.6. *Let Σ, X be compact. For any $c_1 > 0$, there exists a $c_2 > 0$ such that if $\sup |A|_{C_1} < c_1$ and $c > c_2$ then $\omega_{A,c} \in \Omega^2(P(X))$ is symplectic.*

Proof. It suffices to add on a sufficiently large multiple of $\pi^*\omega_\Sigma$ so that $\omega_{A,c}$ is positive on the horizontal subspace. Since the norm of ω_A on the horizontal subspace depends linearly on the C^1-norm of the connection and the moment map, the claim follows. □

The almost complex structure J_A determined by $J \in \mathcal{J}(X)^G$ and the connection is automatically compatible with $\omega_{A,c}$. We denote by $g_{A,c}$ the metric determined by $J_A, \omega_{A,c}$ on $P(X)$, and $\| \cdot \|_{A,c}$ the corresponding norm. For sections u of $P(X)$ define

$$E_{A,c}(u) = \int_\Sigma g_{A,c}(\mathrm{d}u, \mathrm{d}u), \qquad D_{A,c}(u) = \int_\Sigma u^*\omega_{A,c}$$

denote the corresponding energy and equivariant symplectic area, so that the energy-action relation for pseudoholomorphic u holds

$$E_{A,c}(u) = D_{A,c}(u).$$

2.4. **Vortices with fixed holonomy.** Let $\underline{\mu} \in \mathfrak{A}^n$.

Definition 2.4.1. A *gauged holomorphic map with fixed holonomies* $\underline{\mu}$ is a pair $(A, u) \in \mathcal{A}(P, X)$ with $\lim_i A = \mu_i \mathrm{d}\theta_i$ for each cylindrical end $i = 1, \ldots, n$ as in (4). Denote by $\mathcal{A}(P, X, \underline{\mu})$ the space of gauged holomorphic maps with fixed holonomies $\underline{\mu}$.

One can define a formal (possibly degenerate) closed two-form on $\mathcal{A}(P, X, \underline{\mu})$ as follows. Let $\omega_\Sigma \in \Omega^2(\Sigma)$ be the metric with exponential decay constants C_Σ, κ_Σ as above, so that on the ends we have

$$\omega_\Sigma(r_i, \theta_i) = C_i e^{-\kappa_i r_i} \mathrm{d}r_i \mathrm{d}\theta_i \tag{7}$$

for some $\kappa_i > 0$ and locally constant function C_i, possibly zero. The formula

$$(\xi_1, \xi_2) \mapsto \int_\Sigma u^*\omega_{P(X)}(\xi_1, \xi_2)\omega_\Sigma \tag{8}$$

defines a formal two-form on $\mathcal{A}(P, X)$.

Let $\mathcal{G}(P)$ denote the group of gauge transformations of P with limits at infinity, that is, there exists a gauge transformation $\lim_i g \in \mathcal{G}(P_i)$ with

$$\lim_i g(\theta_i) = \lim_{r_i \to \infty} g(\theta_i, r_i).$$

$\mathcal{G}(P)$ naturally acts on $\mathcal{A}(P, X)$ preserving the two-form (17). It has generating vector fields given by

$$\xi_{\Gamma(\Sigma, P(X))}(s) = (\xi(s))_X(u(s)).$$

Let ∂P denote the bundle at infinity

$$\partial P = \bigcup_{i=1}^n P_i.$$

The action of $\mathcal{G}(P)$ on $\mathcal{A}(P, X)$ is formally Hamiltonian with moment map given by

$$\mathcal{A}(P, X) \to \Omega^2(\Sigma, P(\mathfrak{g})) + \mathcal{A}(\partial P), \quad (A, u) \mapsto (F_A + \omega_\Sigma u^* P(\Phi), A|_{\partial P}).$$

These formal considerations motivate the following definition. Let $\underline{\mu} \in \mathfrak{A}^n$ be a n-tuple of elements of the Weyl alcove, parametrizing a gauge orbit in $\mathcal{A}(\partial \overline{P})$. Let $\mathcal{G}(P, \underline{\mu})$ denote the subgroup of $\mathcal{G}(P)$ fixing the connections $\mu_i \mathrm{d}\theta_i$ at infinity.

Definition 2.4.2. A *vortex with fixed holonomy* is a pair $(A, u) \in \mathcal{A}(P, X, \underline{\mu})$ with $F_A + \omega_\Sigma u^* P(\Phi) = 0$. An isomorphism of vortices $(A_j, u_j), j = 0, 1$ is a gauge transformation $g \in \mathcal{G}(P, \underline{\mu})$ with $g(A_0, u_1) = (A_1, u_1)$. Denote by $\mathcal{A}_0(P, X, \underline{\mu})$ the set of symplectic vortices with fixed holonomies $\underline{\mu}$. A vortex is *stable* if it has finite automorphism group. Let $M(P, X, \underline{\mu})$ denote the moduli space of isomorphism classes vortices with fixed holonomy,

$$M(P, X, \underline{\mu}) := \mathcal{A}_0(P, X, \underline{\mu}) /\!\!/ \mathcal{G}(P, \underline{\mu}). \tag{9}$$

2.5. **Framed vortices and evaluation maps.**

Definition 2.5.1. A *framing* of P at infinity at the i-th end is a trivialization $\phi_i : \overline{P}_i \to G$. A *framed vortex* is a vortex (A, u) with holonomies at infinite $\underline{\mu} = (\mu_1, \ldots, \mu_n)$ is a vortex (A, u) equipped with framings at infinity for each cylindrical end such that

$$\phi_i : \overline{P}_i^{\mathrm{Hol}_i(A)} \to G_{\mu_i}.$$

An *isomorphism* of framed vortices is an isomorphism of vortices intertwining the framings.

Let $\mathcal{A}_0^{\mathrm{fr}}(P, X, \underline{\mu})$ resp. $M^{\mathrm{fr}}(P, X, \underline{\mu})$ denote the space of framed vortices with fixed holonomy at infinity resp. moduli space of isomorphism classes of framed vortices with fixed holonomy. The evaluation maps at infinity

$$\mathcal{A}_0^{\mathrm{fr}}(P, X, \underline{\mu}) \to X^n, \qquad [A, u] \mapsto (\overline{u}(z_i))$$

are gauge-invariant and descend to maps

$$\mathrm{ev}^{\mathrm{fr}} : M^{\mathrm{fr}}(P, X, \underline{\mu}) \to X^n.$$

We will see in Theorem **??**, that the finite energy condition forces the evaluation map ev to take values in the fixed point set of the limit of the connection. That is, if $\lim_i A = \mu_i \mathrm{d}\theta_i$ then

$$\mathrm{ev}_i^{\mathrm{fr}}(A, u) \in X^{\mu_i}, \quad i = 1, \ldots, n.$$

We have a canonical map

$$\pi : M^{\mathrm{fr}}(P, X, \underline{\mu}) \to M(P, X, \underline{\mu})$$

given by forgetting the framings. Suppose that every vortex is stable with trivial stabilizer, and suppose that there exists a classifying map

$$\psi : M^{\mathrm{fr}}(P, X, \underline{\mu}) \to EG_{\underline{\mu}} := \prod_{i=1}^{n} EG_{\mu_1} \times \ldots \times EG_{\mu_n}$$

for the bundle π. Combining the classifying map with the evaluation maps above gives a map

$$\mathrm{ev} : M(P, X, \underline{\mu}) \to X^{\underline{\mu}}_{G_{\underline{\mu}}} := \prod_{i=1}^{n} X^{\mu_i} \times_{G_{\mu_i}} EG_{\mu_i}.$$

Later we explain how to construct, in good cases, a classifying map up to problems with finite covers.

2.6. **Correspondence with parabolic vortices.**

In this section we describe the holomorphic objects corresponding to vortices. Suppose that X is a $G_{\mathbb{C}}$-variety, and Σ is a compact Riemann surface.

Definition 2.6.1. A *holomorphic gauged map* from Σ to X is a pair $(P_{\mathbb{C}}, u)$ consisting of a holomorphic principal $G_{\mathbb{C}}$-bundle $P_{\mathbb{C}} \to \Sigma$ together with a holomorphic section $u : \Sigma \to E_{\mathbb{C}}(X)$.

In the language of stacks, this is a holomorphic map from Σ to the quotient stack $X/G_{\mathbb{C}}$. The correspondence between connections on a given principal G-bundle $E \to \Sigma$ and holomorphic structures on its complexification extends to a correspondence between vortices and holomorphic gauged maps with underlying bundle E.

Suppose now that Σ has cylindrical ends. Let $\overline{\Sigma}$ denote the associated closed surface, obtained by adding points at infinity $z_1, \ldots, z_n$.

Definition 2.6.2. A *parabolic holomorphic map* from Σ to X consists of a parabolic bundle $\overline{P}_{\mathbb{C}} \to \overline{\Sigma}$ (see Section 2.1) with markings at $z_1, \ldots, z_N$ together with a holomorphic section $u : \overline{\Sigma} \to \overline{P}_{\mathbb{C}}(X)$.

Theorem 2.6.3. *There exists a one-to-one correspondence between pseudoholomorphic gauged maps (E, A, u) with holonomies μ_j and limits along the j-th cylindrical end in $X^{\mu_j}_k$ and parabolic holomorphic maps $(\overline{E}_{\mathbb{C}}, \overline{u})$ with $\overline{u}(z_j)$ in $P_{\mu_j} X^{\mu_j}_k$.*

Proof. Given a connection on a principal G-bundle $E \to \Sigma$ with fixed holonomies $\mu_1, \ldots, \mu_n$ around the ends, one obtains a parabolic $G_{\mathbb{C}}$ bundle $E_{\mathbb{C}} \to \Sigma$ by assigning to each fiber $(E_{\mathbb{C}})_{z_i}$ at infinity the parabolic reduction determined by μ_i, see e.g. [20], by gluing in trivial bundles using the twistings (3). By removal of singularities, a holomorphic section $u : \Sigma \to E_{\mathbb{C}}(X)$ of finite energy extends automatically to a section $\overline{u} : \overline{\Sigma} \to \overline{E}_{\mathbb{C}}(X)$. The value of $\overline{u}$ at the points $z_1, \ldots, z_n$ is described as follows. Let $B \subset G_{\mathbb{C}}$ denote the Borel subgroup whose Lie algebra contains the positive root spaces. Let P_{μ_j} be the parabolic determined by μ_j,

$$P_{\mu_j} = \{g \in G_{\mathbb{C}} | \lim_{t \to \infty} \mathrm{Ad}(\exp(t\mu_j))g \in B\}$$

whose Lie algebra is that of B plus the sum of negative root spaces vanishing on μ_j. The map $x \mapsto \exp(t\mu_j)x$ restricts on each P_{μ_j}-orbit on X to a retraction of $P_{\mu_j}x$ to

$P_{\mu_j} x \cap X^{\mu_j}$; in particular, each P_{μ_j} orbit contains an element of X^{μ_j}. We write X^{μ_j} in terms of components.

$$X^{\mu_j} = \bigcup_k X_k^{\mu_j}, \qquad X = \bigcup_k P_{\mu_j} X_k^{\mu_j}$$

where $X_k^{\mu_j}$ are the connected components of X^{μ_j}. In the trivializations at the punctures, the section $\overline{u}$ is given by

$$\exp(N \ln(z)\mu_j / 2\pi i) u(z), z \neq 0.$$

Thus $u(0)$ lies in $X_k^{\mu_j}$ if and only if $\overline{u}(0)$ takes values in $P_{\mu_j} X_k^{\mu_j}$, that is, the limit of $\overline{u}(0)$ under the flow defined by μ_j is $u(0)$. This shows □

Recall that S-equivalence is the equivalence relation defined by orbit-closure.

Definition 2.6.4. A parabolic holomorphic map is *polystable* if it is complex gauge equivalent to vortex, *stable* if it is complex gauge equivalent to a stable vortex, and *semistable* if it is S-equivalent to a stable vortex.

The definition of semistability depends on the choice of volume form ω_Σ. In particular, if we choose ω_Σ identically zero, then the stable parabolic holomorphic maps are those whose underlying parabolic bundles are parabolic semistable, by the Mehta-Seshadri theorem [14]. It would be interesting to investigate the Hilbert-Mumford criterion for stability more generally in this context. Mundet [16] has described the answer in the case without markings.

3. Fredholm theory

To show that the moduli spaces above are finite dimensional, we introduce appopriate Sobolev completions. Using the implicit function theorem we show that the moduli space of regular solutions of a given Sobolev class is a smooth finite dimensional manifold, with tangent space the kernel of the Fredholm linearized operator. Elliptic regularity then implies that any solution of a given Sobolev class is gauge equivalent to a smooth solution.

3.1. **Weighted Sobolev spaces.**

Let $\overline{\Sigma}$ be a compact, connected surface, and $z_1, \ldots, z_n \in \overline{\Sigma}$ distinct points. Let Σ denote the punctured surface $\overline{\Sigma} \backslash \{z_1, \ldots, z_n\}$.

Definition 3.1.1. A *cylindrical end* at z_i is a holomorphic, proper embedding $\rho_i : (0, \infty) \times S^1 \to \Sigma$ such that $\lim_{r \to \infty} \rho_i(r, \theta) = z_i$ uniformly in $\theta \in S^1$.

We will denote the coordinates on the i-th end by (r_i, θ_i).

Let E be a Euclidean rank r vector bundle over Σ, equipped with a trivialization on the cylindrical ends. Weighted Sobolev spaces of sections of E are defined as follows.

Definition 3.1.2. A connection

$$D : \Omega^0(\Sigma, E) \to \Omega^1(\Sigma, E), \quad D(f\sigma) = \mathrm{d}f\sigma + fD(\sigma)$$

on E is *asymptotically trivial* if it is equal to the connection on the i-th cylindrical end $r_i > R_i$ given by the trivialization on the i-th end.

Let $r_\Sigma \in C^\infty(\Sigma)$ be a function equal to r_i and supported on each cylindrical end. Let $m > 0$, and let δ be a real number, and $\Omega^0_c(\Sigma, E)$ the space of compactly-supported sections.

Definition 3.1.3. For an asymptotically trivial connection D on E, the *δ-weighted (m,p)-Sobolev norm* on $\Omega^0_c(\Sigma; E)$ with respect to D is

$$\|\xi\|^p_{m,p,\delta} := \int_\Sigma e^{\delta r_\Sigma}\Big(\sum_{|\alpha|\le m} |D^\alpha(\xi^i)|^p\Big)$$

Let $W^p_{m,\delta}(\Sigma, E)$ be the completion of $\Omega^0_c(\Sigma; E)$ with respect to the weighted Sobolev norm $\|\cdot\|_{m,p,\delta}$.

These spaces satisfy the following multiplication and embedding theorems, see e.g. [12, Lemma 7.2], [6, Proposition 3.23].

Proposition 3.1.4. (a) *If $m-2/p \ge m'-2/q$ and $\delta' < \delta$ then the identity on smooth sections induces a compact embedding $W^p_{m,\delta}(E) \to W^q_{m',\delta'}(E)$.*

(b) *If E, E' are vector bundles over Σ then tensor product on smooth sections induces an embedding*

$$W^p_{m,\delta}(E) \times W^p_{m',\delta'}(E') \to W^p_{m'',\delta''}(E \otimes E') \tag{10}$$

for $\delta'' \ge \delta + \delta'$ and $m + m' > m" + 2/p$.

(c) *The identity induces a compact embedding $W^p_{m,\delta}(E) \to C^0(E)$ if $2/p < m$ and $\delta > 0$.*

(d) *A metric on E and integration over Σ induces a perfect pairing*

$$W^p_{m,\delta}(E) \times W^q_{m,-\delta}(E) \to \mathbb{R}$$

where $1/p + 1/q = 1$.

Applying these constructions to the vector bundle $\Lambda^k T^*\Sigma \otimes E$ we define the weighted Sobolev space

$$\Omega^k(\Sigma, E)_{m,p,\delta} = W_{m,p,\delta}(\Lambda^k T^*\Sigma \otimes E)$$

of k-forms with values in E. If $F : \Omega^{k+l}(\Sigma, E)_{m,2,\delta} \to \Omega^k(\Sigma, E)_{m,2,\delta}$ is a differentiable operator, the formal *weighted adjoint* in the space $W_{m,2,\delta}$ is

$$F^* = e^{-\delta r_\Sigma} F^\star e^{\delta r_\Sigma} \tag{11}$$

where $F^\star$ is the usual adjoint. This follows from the identity

$$\int_\Sigma e^{\delta r_\Sigma}\langle F\phi, \psi\rangle = \int_\Sigma e^{\delta r_\Sigma}\langle \phi, e^{-\delta r_\Sigma} F^\star e^{\delta r_\Sigma}\psi\rangle.$$

A similar analysis gives weighted adjoints for $p > 2$. In this paper we will always use formal weighted adjoints whenever we are using differential operators on weighted Sobolev spaces.

3.2. **Connections.** Let P be a G-principal bundle over Σ, equipped with a trivialization over the cylindrical ends. Let A be a connection on P. Denote by $\rho_k^* A$ the restriction of A at the k-th cylindrical end.

Definition 3.2.1. A connection A is *asymptotically constant* if $\rho_k^* A = \xi_k d\theta$ for some $\xi_k \in \mathrm{Map}((0,\infty)\times S^1, \mathfrak{g})$ with ξ_k is constant on $(R,\infty)\times S^1$ for some $R > 0$. The *asymptotic holonomy* $\mathrm{Hol}_k(A) \in G$ of an asymptotically constant connection A is

$$\mathrm{Hol}_k(A) = \lim_{r\to\infty} \exp(\xi_k(r,\theta)).$$

A gauge transformation $g \in \mathcal{G}(P)$ is *asymptotically constant* if for each k, there exists an r_0 such that $\rho_k^* g(r,\theta)$ is constant for $r > r_0$. We say that the limit $\lim_{r\to\infty} \rho_k^* g(r,\theta)$ is the *asymptotic value* of g.

Given a collection $\underline{\mu} \in \mathfrak{A}^n$ let A_0 be an asymptotically constant connection with asymptotic holonomies $\underline{\mu}$. Consider the space of connections

$$\mathcal{A}(\underline{\mu})_{m,p,\delta} := A_0 + \Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta}.$$

Using Sobolev multiplication (10), for $\delta > 0$ the curvature defines a smooth map

$$F : \mathcal{A}(\underline{\mu})_{m,p,\delta} \to \Omega^2(\Sigma, P(\mathfrak{g}))_{m,p,\delta}, \qquad A \mapsto F_A.$$

Let $\mathcal{G}(P,\underline{\mu})$ denote the subgroup of $\mathcal{G}(P)$ consisting of asymptotically constant gauge transformations whose asymptotic value on the k-th end centralizes $\exp(\mu_k)$.

Definition 3.2.2. Let $\mathcal{G}(P,\underline{\mu})_{m+1,p,\delta}$ denote the set of products of the form $g\exp(\xi)$ with $g \in \mathcal{G}(P,\underline{\mu})$ and $\xi \in \Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta}$. Let $\mathcal{G}_\partial(P)_{m+1,p,\delta}$ denote the subgroup of maps with asymptotic limit the identity at infinity.

Lemma 3.2.3. *$\mathcal{G}(P,\underline{\mu})_{m+1,p,\delta}$ is a Banach Lie group.*

Proof. One can construct the metric for $\mathcal{G}(P,\underline{\mu})_{m+1,p,\delta}$ using a representation embedding $G \to V$ to a linear metric space and the induced norm on $\Omega^0(\Sigma, P(\mathrm{End}(V))_{m+1,p,\delta}$. Charts near the identity are defined using the exponential map

$$\Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta} \oplus \mathfrak{g}^n \to \mathcal{G}(\underline{\mu})_{m+1,p,\delta}, \quad (\xi,\psi) \mapsto g\exp(\xi + \sum_k \varphi_k \psi_k). \tag{12}$$

And extedned to the whole group by left multiplication. Compatibility of charts follows from the Sobolev multiplication theorems. □

We denote by $G_{\underline{\mu}}$ the product of centralizers

$$G_{\underline{\mu}} = \prod_{i=1}^n G_{\mu_i}.$$

Proposition 3.2.4. *The map $\mathcal{G}(P,\underline{\mu})_{m+1,p,\delta} \to G_{\underline{\mu}}$ defined by asymptotic value $g \mapsto \lim_i g$ is a smooth homomorphism of Lie groups with kernel $\mathcal{G}_\partial(P)_{m+1,p,\delta}$.*

Proof. In the charts (12), the asymptotic value is simply the map $(g, \underline{\xi}) \mapsto \exp(\underline{\xi})$. $\mathcal{G}_\partial(P)_{m+1,p,\delta}$ is the kernel is immediate from the definition. □

Lemma 3.2.5. *There is an exact sequence*

$$1 \to \mathcal{G}_\partial(P)_{m+1,p,\delta} \to \mathcal{G}(P, \underline{\mu})_{m+1,p,\delta} \to G_{\underline{\mu}}.$$

Proof. One only has to show that the last map is surjective. Given an element $g \in G_{\underline{\mu}}$, choose ξ so that $g = \exp(\xi)$ (which exists since G is connected and compact) and extend ξ to an infinitesimal gauge transformation on Σ using a cutoff function. Then the exponential of ξ is the required gauge transformation. □

The formula

$$(a_1, a_2) \mapsto \int_\Sigma \langle a_1 \wedge a_2 \rangle_{\mathfrak{g}}. \tag{13}$$

defines a two-form on the Banach manifold $\mathcal{A}(\underline{\mu})_{m,p,\delta}$. This form is weakly non-degenerate in the sense that it defines an injection of the tangent bundle into the cotangent bundle. The group $\mathcal{G}(\underline{\mu})_{m+1,p,\delta}$ acts smoothly on $\mathcal{A}(\underline{\mu})_{m,p,\delta}$, with moment map given by the curvature. Define

$$M(P, \underline{\mu})_{m,p,\delta} = \mathcal{A}_\flat(\underline{\mu})_{m,p,\delta} / \mathcal{G}(\underline{\mu})_{m+1,p,\delta}.$$

Standard gauge-theoretic slice arguments show that the inclusion of smooth connections induces a homeomorphism

$$M(P, \underline{\mu}) \to M(P, \underline{\mu})_{m,p,\delta}.$$

That is, the moduli space is independent of the choice of Sobolev norm.

3.3. **Gauged sections.** Let X be a compact, connected Hamiltonian G-manifold with moment map $\Phi : X \to \mathfrak{g}^*$. For each principal G-bundle P on Σ consider its associated bundle

$$\pi : P(X) = (P \times X)/G \to \Sigma.$$

Each connection A on P determines a symplectic connection on $TP(X)$:

$$TP(X) = TP(X)^{\mathrm{hor}} \oplus TP(X)^{\mathrm{vert}} \cong \pi^* T\Sigma \oplus TP(X)^{\mathrm{vert}}.$$

The moment map $\Phi : X \to \mathfrak{g}^*$ induces a map

$$P(\Phi) : P(X) \to P(\mathfrak{g}^*).$$

Let $\Gamma(\Sigma, P(X))_\infty$ denote the space of asymptotically constant sections of $P(X)$, that is, sections independent of both coordinates r, θ for r sufficiently large on each cylindrical end. The tangent space to $\Gamma(\Sigma, P(X))_\infty$ at u is the space $\Omega^0(\Sigma, u^* T^{\mathrm{vert}} P(X))_\infty$ of asymptotically constant sections of the vertical tangent bundle $\Omega^0(\Sigma; u^* T^{\mathrm{vert}} P(X))$.

Definition 3.3.1. Let $\Gamma(\Sigma, P(X))_{m,p,\delta}$ denote the space of sections of $P(X)$ of the form $\exp_u(\xi)$ where $u \in \Gamma(\Sigma, P(X))_\infty$ is asymptotically constant and $\xi \in \Omega^0(\Sigma, u^* T^{\mathrm{vert}} P(X))_{m,p,\delta}$.

Proposition 3.3.2. *For $m > 2/p$ the space $\Gamma(\Sigma, P(X))_{m,p,\delta}$ has the structure of a Banach manifold. The action of the Banach Lie group $\mathcal{G}_{m+1,p,\delta}(P)$ given by $u \mapsto g \cdot u$ is smooth.*

Proof. A metric on $\Gamma(\Sigma, P(X))_{m,p,\delta}$ is induced from a metric on X. Define

$$T_u\Gamma(\Sigma, P(X))_{m,p,\delta} := \Omega^0(\Sigma, u^*T^{\text{vert}}P(X))_{m,p,\delta} \oplus \bigoplus_{i=1}^{n} \overline{u}(z_i)^*T^{\text{vert}}P(X), \tag{14}$$

where $z_i \in \overline{\Sigma}$ is the point at infinity on the i-th cylindrical end. Define a map

$$\iota : T_u\Gamma(\Sigma, P(X))_{m,p,\delta} \to T_u\Gamma(\Sigma, P(X))), \quad (\xi_0, \xi_1, \ldots, \xi_n) \mapsto \xi_0 + \sum_{i=1}^{n} \varphi_i \xi_i \tag{15}$$

where φ_i is function equal to 1 in a neighborhood of infinity on the i-th cylindrical end, with support on the cylindrical end. Charts are defined by

$$\varphi_u : T_u\Gamma(\Sigma, P(X))_{m,p,\delta} \to \Gamma(\Sigma, P(X)), \quad \xi \mapsto \exp_u(\iota(\xi)).$$

It follows immediately from the definitions that φ_u is a homeomorphism onto an open neighborhood of u. Compatibility of charts follows from Sobolev mutliplication again. The claim on the action of the gauge group is left to the reader. □

By definition, any section u in the space $\Gamma(\Sigma, P(X))_{m,p,\delta}$ has a limit at infinity on the cylindrical ends and thus extends to a continuous section $\overline{\Sigma} \to \overline{P}(X)$. That is, we have a map

$$\text{ev} : \Gamma(\Sigma, P(X))_{m,p,\delta} \to X^n$$

given by the asymptotic values at infinity.

Definition 3.3.3. A two-form $\omega_\Sigma \in \Omega^2(\Sigma)$ has *exponential decay* on the cylindrical ends if it is given by an expression of the form

$$\omega_\Sigma(r, \theta) = C_\Sigma e^{-\kappa r} \mathrm{d}r\mathrm{d}\theta \tag{16}$$

on the cylindrical ends, for some $\kappa > 0$ and locally constant function C_Σ, possibly zero.

For the most part, we assume that C_Σ is identically equal to 1, to simplify the notation. The formula

$$(\xi_1, \xi_2) \mapsto \int_\Sigma u^*\omega_{P(X)}(\xi_1, \xi_2)\omega_\Sigma \tag{17}$$

defines a symplectic form on $\Gamma(\Sigma, P(X))_\infty$. This form extends naturally to the completion $\Gamma(\Sigma, P(X))_{m,p,\delta}$ for $m, \delta \geq 0$, by Sobolev multiplication. The action of $\mathcal{G}_{m+1,p,\delta}(P)$ on $\Gamma(\Sigma, P(X))_{m,p,\delta}$ preserves the two-form (17). It has generating vector fields given by

$$\xi_{\Gamma(\Sigma,P(X))}(s) = (\xi(s))_X(u(s)).$$

Definition 3.3.4. A *gauged pseudoholomorphic map* of Sobolev class m, p, δ is a pair $(A, u) \in \mathcal{A}(P)_{m,p,\delta} \times \Gamma(P(X))_{m,p,\delta}$.

Definition 3.3.5. The *energy* of (A, u) is

$$E(A, u) = \frac{1}{2} \int_\Sigma \left(C_\Sigma e^{\kappa r_\Sigma} |\mathrm{d}_A u|^2 + C_\Sigma^2 e^{2\kappa r_\Sigma} |F_A|^2 + |u^*P(\Phi)|^2 \right) \omega_\Sigma.$$

Remark 3.3.6. On the cylindrical ends, the energy is given by the integral of the energy density function

$$\chi(r,\theta) = \frac{1}{2}(|\mathrm{d}_A u|^2 + C_\Sigma^{-1} e^{\kappa r}|F_A|^2 + C_\Sigma e^{-\kappa r}|u^* P(\Phi)|^2) \tag{18}$$

with respect to the standard metric on the cylindrical ends. Note in particular that for finite energy, the term $u^*P(\Phi)$ is not required to go to zero at infinity.

Finally we define the equivariant symplectic area of a gauged pseudoholomorphic map. The basic 2-form on the product $P \times X$ given by $p_2^*\omega_X + \mathrm{d}(p_1^*A, p_2^*\Phi)$ descends to a closed 2-form on $P(X)$ denoted $\omega_{P(X),A}$ or just by $\omega_{P(X)}$ assuming that the dependency on the connection is understood.

Definition 3.3.7. The *equivariant symplectic area* of (A,u) is

$$D(A,u) := \int_\Sigma u^*(\omega_{P(X),A}).$$

Lemma 3.3.8. *The equivariant symplectic area of a pair (A,u) is a topological invariant in the following sense: If u_t is a smooth family of maps then $D(A,u_t)$ is constant in t.*

Proof. Let $X_t = \frac{d}{dt}u_t$. Since the maps u_t have exponential decay, so does X_t, and

$$\int_0^1 \frac{d}{dt} u_t^* \omega_{P(X),A} = \int_0^1 \mathrm{d}u_t^* \omega_{P(X),A}(X_t, \cdot) = 0$$

by Stokes theorem. □

3.4. **Almost complex structures.** We now introduce a suitable class of almost complex structures on $P(X)$. Let j denote the complex structure on Σ.

Definition 3.4.1. An almost complex structure I on $P(X)$ is $\omega_{P(X)}, j$*-compatible* if

(a) the projection $\pi : P(X) \to \Sigma$ is (I,j)-holomorphic,
(b) $I|_{\pi^{-1}(z)}$ is $\omega_{P(X)}|_{\pi^{-1}(z)}$-compatible on each fibre $\pi^{-1}(z)$,
(c) I is asymptotically constant on each cylindrical end.

Let $\mathcal{J}(P(X))_\infty$ be the space of $\omega_{P(X)}, j$-compatible almost complex structures. Define

$$\mathcal{A}(P,\underline{\mu}) \times \mathcal{J}(X)^G \to \mathcal{J}(P(X)), \quad (A, J_X) \mapsto J_A \tag{19}$$

given by defining J_A on the horizontal space by j and on the vertical space by J_X, using the induced splitting by A.

The map (19) extends to suitable Sobolev completions as follows. Let $\mathcal{J}(P(X))_{m,p,loc}$ denote the space of $\omega_{P(X)}, j$-compatible almost complex structures locally of Sobolev class $1,p$, and asymptotically constant in a neighborhood of infinity.

Lemma 3.4.2. *Given $J_0 \in \mathcal{J}(P(X))_\infty$, the space of compatible almost complex structures near J_0 is a smooth manifold with tangent space $T_{J_0}\mathcal{J}(P(X))$ the set of $Y \in \operatorname{End}(TX)$ such that*

$$YJ_0 + J_0 Y = 0, \quad \pi \circ Y = 0, \qquad \omega(Y\cdot, J_0\cdot) = -\omega(J_0\cdot, (\Delta J)\cdot).$$

Proof. [13, p. 47]. □

To define a suitable Banach space of almost complex structures, we choose a local diffeomorphism

$$\exp_{J_0} : B_\epsilon(0, T_{J_0}\mathcal{J}(P(X))) \to \mathcal{J}(P(X)).$$

Definition 3.4.3. Let $\mathcal{J}(P(X), J_0)_{m,p,\delta}$ be the subspace of $\mathcal{J}(P(X))_{m,p,\mathrm{loc}}$ such that $J = \exp_{J_0}(Y)$ for some section Y of Sobolev class (m,p,δ) near infinity.

Lemma 3.4.4. *The map $(A+a, J) \mapsto J_{A+a}$ of (19) extends naturally to a map*

$$\mathcal{A}(\underline{\mu})_{m,p,\delta} \times \mathcal{J}(X)^G \to \mathcal{J}(P(X), J_A)_{m,p,\delta}.$$

Proof. Consider the splitting induced by a connection $A \in \mathcal{A}(P)$

$$T_{[p,x]}P(X) \to T_{\pi(p)}\Sigma \oplus T_x X, \quad [w,v] \mapsto (D\pi(w - A(w)), A(w)_X(x) + v).$$

The complex structure J_A is given by

$$J_A[w,v] = [D\pi^{-1}j(D\pi)w - A(D\pi^{-1}j(D\pi)w), J(A(w)_X(x) + v)]. \tag{20}$$

The connection enters as a multiplication operator, and the claimed regularity follows from the Sobolev multiplication theorem. Given a base connection A_0 with limiting holonomy μ, and an element $A \in \mathcal{A}(\underline{\mu})_{m,p,\delta}$ sufficiently close to A_0, we have $J_A \in \mathcal{J}(P(X), J_{A_0})_{m,p,\delta}$, again using the formula (20). □

Given $J \in \mathcal{J}(X)^G$, for each section $u \in \Omega^0(\Sigma; P(X))_{m,p,\delta}$ define the Cauchy-Riemann operator

Definition 3.4.5.

$$\overline{\partial}_A(u) := \frac{1}{2}(\mathrm{d}u + J_A(u) \circ \mathrm{d}u \circ j) \in \Omega^{0,1}(\Sigma; u^* T^{\mathrm{vert}} P(X))_{m-1,\delta}.$$

Lemma 3.4.6. *The energy and equivariant symplectic area of a pair (A,u) are related by*

$$E(A,u) = D(A,u) + \int_\Sigma |\overline{\partial}_{J_A} u|^2 + \frac{1}{2}\int |F_A + u^* P(\Phi)\omega_\Sigma|^2. \tag{21}$$

Proof. As in [4, Proposition 3.1]. □

Corollary 3.4.7. *If (A,u) is a symplectic vortex, then $E(A,u) \leq E(A',u')$ for any pair (A',u') homotopic to (A,u).*

Proof. By the symplectic vortex equations (1) and the energy-action relation (21). □

3.5. **Moduli space of vortices.** The moduli space of (finite energy) symplectic vortices can be defined by a symplectic quotient construction as follows. Define

$$\mathcal{A}(P,X,\underline{\mu})_{m,p,\delta} := \{(A,u) \in \mathcal{A}(\underline{\mu})_{m,p,\delta} \times \Gamma(P(X))_{m,p,\delta}, \quad \overline{\partial}_A(u) = 0\}.$$

The group of gauge transformations $\mathcal{G}(\underline{\mu})_{m+1,p,\delta}$ acts on $\mathcal{A}(P,X,\underline{\mu})_{m,p,\delta}$. The action is formally Hamiltonian with moment map given by the sum of the curvature and pull-back of moment map. These formal considerations motivate the following definition.

Definition 3.5.1. A *vortex with fixed holonomy* of Sobolev class m,p,δ is a pair $(A,u) \in \mathcal{A}(P,X,\underline{\mu})_{m,p,\delta}$ with $F_A + \omega_\Sigma u^*P(\Phi) = 0$ and $E(A,u) < \infty$. An isomorphism of vortices $(A_j,u_j), j = 0,1$ is a gauge transformation $g \in \mathcal{G}(P,\underline{\mu})_{m,p,\delta}$ with $g(A_0,u_1) = (A_1,u_1)$. A vortex is *stable* if it has finite automorphism group.

The moduli space of symplectic vortices with fixed holonomy is therefore the symplectic quotient

$$M(P,X,\underline{\mu})_{m,p,\delta} := \mathcal{A}(P,X,\underline{\mu})_{m,p,\delta} /\!\!/ \mathcal{G}(P,\underline{\mu})_{m+1,p,\delta}. \tag{22}$$

3.6. **Slices for the gauge group action.** The construction of slices for the gauge action is given by Gaio-Salamon [8], in the case without cylindrical ends. For simplicity consider $p = 2$, the general case follows similarly. We first show that the action of $\mathcal{G}(P,\underline{\mu})$ on the space $\mathcal{A}(\underline{\mu})_{m,p,\delta} \times \Gamma(\Sigma, P(X))_{m,p,\delta}$ admits slices. Recall that the generating vector fields for the gauge group action on $\mathcal{A}(\underline{\mu})_{m,p,\delta} \times \Gamma(\Sigma, P(X))_{m,p,\delta}$ are given by

$$\eta_{\mathcal{A}(P,X,\underline{\mu})} = (-d_A(\eta), (\eta_X)(u)) \text{ for } \eta \in \Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta}.$$

A pair (a, xi) is orthogonal to the generating vector fields if

$$\langle -d_A\eta, a\rangle + \langle \eta_X(u), \xi\rangle = 0, \text{ for all } \eta \in \Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta}.$$

Equivalently, using the formal weighted adjoints as in (11) we have

$$\langle \eta, -d_A^* a\rangle + Z(\xi)\rangle = 0$$

where d_A^* is the weighted adjoint of d_A and

$$Z : \Omega^0(\Sigma, u^*TX)_{m,p,\delta} \to \Omega^0(\Sigma, P(\mathfrak{g}))_{m,p,\delta}$$

is the one corresponding to $\eta \mapsto \eta_X(u)$.

Definition 3.6.1. Let $S_{A,u}$ denote the kernel of the operator $(a,\xi) \mapsto -\mathrm{d}_A^* a + Z(\xi)$.

Theorem 3.6.2. *The susbset $A + a, \exp_u(\xi)$ for $(a,\xi) \in S_{A,u}$ is a slice for the action of $\mathcal{G}(P,\underline{\mu})_{m+1,p,\delta}$ on $\mathcal{A}(P,X,\underline{\mu})_{m,p,\delta}$.*

Proof. Using the definition of the moment map and the compatibility condition of the metric

$$\langle \xi, \eta_X(u)\rangle = \omega(J\xi, \eta_X(u)) = d(u^*\Phi\eta, J\xi)$$

so that the formal adjoint of $\eta \mapsto \eta_X(u)$ is the Lie derivative $L_{J\xi}(u^*\Phi)$ which in turn gives the weighted adjoint

$$Z(\xi) = e^{-\delta r} L_{J e^{\delta r}\xi}(u^*\Phi).$$

We have a splitting of the tangent space

$$T_{(A,u)}\mathcal{A}(P,X,\underline{\mu})_{m,p,\delta} = T_{(A,u)}(\mathcal{G}(\underline{\mu})_{m+1,p,\delta}\cdot(A,u)) \oplus S_{A,u}.$$

We claim that there exists a neighborhood U of the origin in

$$\Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta} \times \Omega^0(\Sigma, u^*T^{\mathrm{vert}}PX)_{m,p,\delta} \tag{23}$$

and a map $G : U \to \Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta} \times K$, such that

$$(a,\xi) \mapsto (\mathrm{d}_A^* a + Z(\xi), G(a,\xi)) \tag{24}$$

is a diffeomorphism onto a neighborhood of the origin. Consider the map

$I : \Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta} \times \Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta} \times \Omega^0(\Sigma, u^*T^{\mathrm{vert}}P(X))_{m,p,\delta} \to \Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta}$

$$I(\eta,(a,\xi)) = d_A^*(e^\eta\cdot(A+a)-A) + Z(\log_u(e^\eta\cdot\exp_u\xi)). \tag{25}$$

We claim that I is smooth with and its partial derivative respect the first term is

$$D_1 I : \eta \mapsto \mathrm{d}_A^*\mathrm{d}_A\eta + Z(\eta_X(u)).$$

Indeed, the derivative of I can be computed by expanding each term in Equation (25) to obtain

$$I(\eta,(a,\xi)) = \mathrm{d}_A^*\mathrm{d}_A\eta + \mathrm{d}_A^* a + Z(\xi) + Z(\eta_X(u)) + \text{ quadratic terms.}$$

Its partial derivative respect to η at $(a,\xi) = (0,0)$ is $d_A^* d_A\eta + Z(\eta_X(u))$. Moreover,

$$D_1 I : \eta \mapsto \mathrm{d}_A^*\mathrm{d}_A\eta + Z(\eta_X(u))$$

is invertible. Indeed, the second term $Z(\eta_X(u))$ has order zero and the operator $\mathrm{d}_A^*\mathrm{d}_A$ is elliptic, and DI has trivial kernel, since

$$\langle \eta, \mathrm{d}_A^*\mathrm{d}_A\eta + Z(\eta_X(u))\rangle = \|\mathrm{d}_A\eta\|^2 + \|\eta_X(u)\|^2.$$

The Theorem now follows from the implicit function theorem applied to the map (25). □

3.7. **Hamiltonian perturbations.** We introduce Hamiltonian perturbations of the vortex equations as follows. We remark that though this section we fix an almost complex structure $J \in \mathcal{J}(X)^G$.

Let $\mathrm{Map}_G(P, C^\infty(X))$ denote vector bundle whose fiber at $z \in \Sigma$ is the space of G-equivariant maps $\mathrm{Map}_G(P_z, C^\infty(X))$.

Definition 3.7.1. An *admissible Hamiltonian perturbation* is a $\mathcal{G}(P,\underline{\mu})$-equivariant map

$$H : \mathcal{A}(P,\underline{\mu}) \to \Omega^1_c(\Sigma, \mathrm{Map}_G(P, C^\infty(X))$$

that is, a 1-form with compact support on Σ with values in $\mathrm{Map}_G(P, C^\infty(X))$, depending equivariantly on a connection $A \in \mathcal{A}(P,\underline{\mu})$. Let $\mathcal{H}(P,X)$ denote the space of admissible Hamiltonian perturbations.

Mapping each Hamiltonian function to its Hamitonian vector field induces a map

$$\mathcal{H}(P,X) \to \mathrm{Map}_{\mathcal{G}(P,\underline{\mu})}(\mathcal{A}(P,\underline{\mu}), \Omega^1_c(\Sigma, T^{\mathrm{vert}}(P(X)))), \quad H \mapsto X_H.$$

Definition 3.7.2. An *H-vortex* is a solution to the perturbed vortex equations

$$F_A + \omega_\Sigma u^* P(\Phi) = 0, \quad \overline{\partial}_{J(A)} u - X_{H(A)}(u)^{0,1} = 0. \tag{26}$$

An H-vortex (A, u) is *stable* if it has finite automorphism group.

Lemma 3.7.3. *The gauge group $\mathcal{G}(P, \underline{\mu})$ acts on space of H-vortices.*

Proof. Suppose that (A, u) is an H-vortex and $g \in \mathcal{G}(P, \underline{\mu})$. We have

$$F_{gA} + \omega_\Sigma (gu)^* P(\Phi) = \mathrm{Ad}(g)(F_A + \omega_\Sigma u^* P(\Phi)) = 0$$

and

$$\begin{aligned} \overline{\partial}_{J(gA)} gu - X_{H(gA)}(gu)^{0,1} &= g\overline{\partial}_{J(A)} g^{-1} gu - X_{g^* H(A)}(gu)^{0,1} \\ &= g\overline{\partial}_{J(A)} u - g X_{H(A)} g^{-1}(gu)^{0,1} \\ &= g\overline{\partial}_{J(A)} u - g X_{H(A)}(u)^{0,1} \\ &= 0. \end{aligned}$$

□

Definition 3.7.4. An *isomorphism* of H-vortices $(A_j, u_j), j = 0, 1$ is a gauge transformation $g \in \mathcal{G}(P, \underline{\mu})$ with $g(A_0, u_0) = (A_1, u_1)$. Let $M(P, X, \underline{\mu}, H)$ denote the coarse moduli space of isomorphism classes of H-vortices.

For any pair (A, u) of a connection and a section of the bundle $P(X)$ we consider the non-linear map

$$\begin{aligned} \mathcal{F}_{(A,u,H)} : \Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta} \times \Omega^0(\Sigma, u^* T^{\mathrm{vert}} P(X))_{m,p,\delta} \to \\ \Omega^0 \oplus \Omega^2(\Sigma, P(\mathfrak{g})) \oplus \Omega^{0,1}(\Sigma, u^* T^{\mathrm{vert}} P(X))_{m-1,\delta} \\ \xi \mapsto (F_A + \omega_\Sigma u^* P(\Phi), -\mathrm{d}_A^* a + \omega_\Sigma L_{J\xi} u^* P(\Phi), \Psi_u(\xi)^{-1} \overline{\partial}_{J,H(A)}(\exp_u(\xi))). \end{aligned}$$

Here Ψ_u and $\exp_u$ are the parallel transport and exponential map, both along vertical geodesics on fibres using the adapted vertical connection

$$\tilde{\nabla}_v := \nabla_v - \frac{1}{2} J_{H(A)}(\nabla_v J_{H(A)}),$$

obtained from the vertical Levi-Civita connection ∇. In this way $\tilde{\nabla}$ does preserves $J_{H(A)}$ as well as the metric.

Definition 3.7.5. The *linearized operator* for an H-perturbed vortex (A, u) is the combination

$$\begin{aligned} \tilde{D}_{A,u} : \Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta} \oplus \Omega^0(\Sigma, u^* T^{\mathrm{vert}} P(X))_{m,p,\delta} \to \\ (\Omega^0 \oplus \Omega^2)(\Sigma, P(\mathfrak{g}))_{m-1,p,\delta} \oplus \Omega^{0,1}(\Sigma, u^* T^{\mathrm{vert}}(P(X)))_{m,p,\delta}, \\ (a, \xi) \mapsto (*d_A a + L_\xi u^*(\Phi), -d_A^* a + L_{J\xi}(u^* \Phi), D_{u,J(A),H(A)} \xi + L_u^{0,1}(a)), \end{aligned} \tag{27}$$

where $D_{u,J(A),H(A)}$ is the standard linearization of the perturbed Cauchy-Riemann operator. We say that (A, u) is *regular* if $\tilde{D}_{A,u}$ is surjective.

Lemma 3.7.6. *The operator $\tilde{D}_{A,u}$ is equal to the linearization $\tilde{D}_{A,u} = d\mathcal{F}_{(A,u,H)}(0)$, and is Fredholm.*

Proof. The first claim is left to the reader. To show $\tilde{D}_{A,u}$ is Fredholm, note that the terms involving the Lie derivatives are of lower order, and thus they can be ignored. We can assume that $D_{u,J(A),H(A)}$ is the operator associated to the base connection A_0 since the difference $D_{u,J(A),H(A)} - D_{u,J(A_0),H(A_0)}$ is a compact operator. Also, $L_u^{0,1}(a)$ is compact thus $D_{u,J_A,H_A}\xi + L_u^{0,1}(a)$ is a lower order perturbation of the operator $D_{u,J(A_0),H(A_0)}$. Thus, it is enough to show that the map

$$(a,\xi) \to (d_A \oplus d_A^* a,\ D_{u,J(A_0),H(A_0)}(\xi))$$

is Fredholm. To see this, take the weight δ_0 smaller than any element in the spectrum of the limiting operators. If $\delta < \delta_0$ then $d_{A_0} \oplus d_{A_0}^*$ and $D_{u,J(A_0),H(A_0)}$ are both Fredholm, since the second factor in (14) is finite dimensional. Thus $\tilde{D}_{(A,u)}$ is Fredholm. □

Definition 3.7.7. Let $M^{\mathrm{reg}}(P, X, H, \underline{\mu})_{m,p,\delta}$ denote the locus of regular, stable vortices of Sobolev class (m, p, δ).

Theorem 3.7.8. *$M^{\mathrm{reg}}(P, X, H, \underline{\mu})_{m,p,\delta}$ has the structure of a smooth, finite dimensional manifold, with tangent space given by the kernel of $\tilde{D}_{A,u}$.*

Proof. This is an application of the implicit function theorem for Banach spaces and will be omitted. □

Definition 3.7.9. Let $M^{\mathrm{irr}}(P, X, \underline{\mu})_{m,p,\delta}$ denote the subset of $M(P, X, \underline{\mu})_{m,p,\delta}$ consisting of vortices (A, u) such that A is irreducible, that is, A has automorphism $\mathrm{Aut}(A) = Z$.

Theorem 3.7.10. *Suppose that Z acts trivially on X. Then there exists a subset $\mathcal{H}^{\mathrm{reg}}(P, X) \subset \mathcal{H}(P, X)$ of Baire second category such that every element of $M^{\mathrm{irr}}(P, X, \underline{\mu})_{m,p,\delta}$ is regular.*

Proof. To apply the implicit function theorem, we need to construct a Banach space of Hamiltonian perturbations, for that we follow Floer []. Let $\phi : \Sigma \to \mathbb{R}$ be a positive function approaching infinite on each cylindrical end, and let C^k_ϕ be the subspace of $f \in C^k_{\mathrm{loc}}$ such that the norm

$$\|f\|^2_{\phi,k} = \sum_{j=0}^{k} \sup_{s\in\Sigma} \|\phi(s)(D_j f)(s)\|^2$$

is finite. Let $\mathcal{H}(P, X)_k$ denote the space of $\mathcal{G}(\underline{\mu})$-equivariant maps from $\mathcal{A}(P, \underline{\mu})$ to $\Omega^1(\Sigma, \mathrm{Hom}_G(P, C^k(X)))$ that are class C^k_ϕ. Given an element $A \in \mathcal{A}(P, \underline{\mu})_{m,p,\delta}$, let S_A be a slice for the action of $\mathcal{G}(\underline{\mu})_{m+1,p,\delta}$ at A. Local charts for $\mathcal{H}(P, X)_k$ are given by maps $S_A \to \Omega^1(\Sigma, \mathrm{Map}_G(P, C^k(X)))$ of class C^k_ϕ.

Consider the universal moduli space

$$\begin{aligned}\mathcal{A}^{\mathrm{univ,irr}}(P,X,\underline{\mu}) =& \big\{(A,u,J,H) | \bar\partial_{J_{A,H}} u = 0, \quad F_A + \omega_\Sigma u^* P(\Phi) = 0, \quad \#\operatorname{Aut}(A) < \infty \big\} \\ & \subset \mathcal{A}(P,\underline{\mu})_{m,p,\delta} \times \Gamma(\Sigma, P(X))_{m,p,\delta} \times \mathcal{J}(P,X)_{m,p,\delta} \times \mathcal{H}(P,X)_m.\end{aligned}$$

Since the perturbations H vanish at infinity on the cylindrical ends, zero is not contained in the spectrum of the limiting operators on the cylindrical ends and so for sufficiently small Sobolev weights on the cylindrical ends the universal linearized operator

$$\begin{aligned}D^{\mathrm{univ}} = D^{\mathrm{univ}}_{A,u,J,H} : \Omega^0(\Sigma, u^* T^{\mathrm{vert}} P(X))_{m,p,\delta} \times \Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta} \times T_H \mathcal{H}(P,X)_m \to \\ \Omega^{0,1}(\Sigma, u^* T^{\mathrm{v}} P(X))_{m,p,\delta} \\ (a,\xi,h) \mapsto \tilde{D}_{(A,u)}(a,\xi) + \Delta_H D(h)\end{aligned}$$

Here we denote by $h \mapsto \Delta_H D(h)$ the partial operator at H in the direction of h, and note that it is (j,J)-anti-linear.

We claim that D^{univ} is onto. Suppose otherwise, so that some η is orthogonal to the image of D^{univ}. Denote by $(D^{\mathrm{univ}})^*$ the dual operator defined by integration and inner product

$$\int_\Sigma \langle D^{\mathrm{univ}} \xi, \eta \rangle \omega_\Sigma = \int_\Sigma \langle \xi, (D^{\mathrm{univ}})^* \eta \rangle \omega_\Sigma.$$

By integration by parts we get $(D^{\mathrm{univ}})^* \eta = 0$ and also

$$\langle \eta, Y \circ D^{\mathrm{univ}} \circ j \rangle = 0, \qquad \text{for all } Y \in T_H \mathcal{H}(P,X).$$

Suppose that there is a point $z \in \Sigma$ where $\eta_z \neq 0$. Let $x = u(z)$ and take a (j,J)-anti-linear map $Z : T_z\Sigma \to T_x^{\mathrm{vert}} P(X)$, such that $\langle \eta_x, Z \circ j \rangle \neq 0$. Since $\mathcal{A}(P,\underline{\mu})$ is reducible-free, there exists an infinitesimal Hamiltonian perturbation Y such that $\langle \eta_z, (Y \circ D \circ j)_z \rangle \neq 0$. Indeed, choose Y for A, extend it to a slice for the gauge group action, and then to the flow-out by equivariance. Since by assumption Z acts trivially on X, Z acts trivially on the slice and the flow-out is well-defined. After multiplying Y by a bump function supported near $x = u(z)$, we may assume that Y lies in the space $T_I \mathcal{J}(J)$, and that $\langle \eta, (Y \cdot Du \cdot j) \rangle \neq 0$ which is a contradiction. Therefore $\eta = 0$ and D^{u} is onto. By applying the implicit function theorem to the map I we obtain a slice, that is the desired neighborhood U and map G similar to those of (23),(24). A similar construction shows that the universal space $\mathcal{A}^{\mathrm{univ,irr}}(P,X,\underline{\mu})$ is Banach and admits slices for the $\mathcal{G}$ action. The quotient

$$M^{\mathrm{univ,irr}}(P,X,\underline{\mu}) = \mathcal{A}^{\mathrm{univ,irr}}(P,X,\underline{\mu}) / \mathcal{G}(P,\underline{\mu})$$

is a smooth Banach manifold. The Sard-Smale theorem applied to the projection

$$\operatorname{proj} : M^{\mathrm{univ}}(\underline{\mu}) \to \mathcal{H}(P,X)_k.$$

shows that the space $\mathcal{H}(P,X)_k^{\mathrm{reg}}$ is of second category in $\mathcal{H}(P,X)_k$. □

In Section we generalize the discussion to non-constant almost complex structures, depending on the connection.

3.8. **Framed vortices and evaluation maps.** Let $M^{\mathrm{fr,reg}}(P,X,\underline{\mu})_{m,p,\delta}$ denote the moduli space of framed vortices whose underlying vortex is regular and stable.

Theorem 3.8.1. *$M^{\mathrm{fr,reg}}(P,X,\underline{\mu})_{m,\mathfrak{p},\delta}$ has the structure of a smooth principal G^n orbifold bundle over $M^{\mathrm{reg}}(P,X,\underline{\mu})_{m,p,\delta}$. Moreover, the framed evaluation maps*

$$\mathrm{ev}^{\mathrm{fr}} : M^{\mathrm{fr,reg}}(P,X,\underline{\mu})_{m,p,\delta} \to X^{\underline{\mu}}$$

are smooth.

Proof. The first part is similar to Theorem 3.7.8, and will be omitted. Smoothness is a consequence of the description of charts (15). □

Lemma 3.8.2. *If G^n acts freely on $M^{\mathrm{fr,reg}}(P,X,\underline{\mu})_{m,p,\delta}$ then there exists a classifying map for the principal bundle*

$$\pi : M^{\mathrm{fr,reg}}(P,X,\underline{\mu})_{m,p,\delta} \to M^{\mathrm{reg}}(P,X,\underline{\mu})_{m,p,\delta}.$$

More generally, if G^n acts locally freely, π has the structure of a smooth principal orbifold bundle, and the classifing map exists after passing to the classifying space for the orbifold.

Proof. Immediate from the existence of slices in Theorem 3.6.2. □

In the case that G^n acts only locally freely, the lemma gives a well defined map

$$\psi : BM^{\mathrm{fr,reg}}(P,X,\underline{\mu})_{m,p,\delta} \to EG^n.$$

Together with the framed evaluation maps, this gives a map

$$\mathrm{ev} : BM^{\mathrm{reg}}(P,X,\underline{\mu})_{m,p,\delta} \to \prod_{i=1}^{n} X^{\mu_i}_{G_{\mu_i}}$$

and a pull-back map in rational cohomology

$$\mathrm{ev}^* : \prod H_{G_{\mu_i}}(X^{\mu_i},\mathbb{Q}) \to H(M^{\mathrm{reg}}(P,X,\underline{\mu})_{m,p,\delta},\mathbb{Q}).$$

3.9. **Regularity.**

Proposition 3.9.1. *Let δ_0 be the constant of Theorem* **??***, and consider $m \geq 1$ and $p > 2$ and $0 < \delta < \delta_0$. Then any symplectic vortex (A,u) of class m,p,δ is gauge equivalent to a smooth vortex (A',u').*

Proof. The case of closed curves is Proposition [4, Theorem 3.1]; the proof for the case of cylindrical ends is similar. □

The converse of this statement, that any smooth vortex is of class m,p,δ for sufficiently small δ depending on $\underline{\mu}$, will be proved in the following section.

4. Exponential decay

In this section we prove various versions of exponential decay used for compactness in the next section. We prove the results for unperturbed symplectic vortices only; the perturbed case can be reduced to this case by the graph construction in [8, Appendix A].

4.1. **Invariant symplectic action.** Let (X,ω) be a symplectic manifold, G be a compact connected Lie group acting on X in a Hamiltonian way, with equivariant moment map $\Phi : X \to \mathfrak{g}^*$, $\langle\cdot,\cdot\rangle_X$ be a G-invariant Riemannian metric on X, and $\pi : Q \to S^1 \simeq \mathbb{R}/\mathbb{Z}$ be a (necessarily trivial) principal G-bundle. Consider a pair (x, a_0), where $x : S^1 \to Q(X) := (Q \times X)/G$ is a smooth section and $a_0 \in \mathcal{A}(Q)$ is a connection one-form. We denote by

$$\ell(x,a_0) := \int_{S^1} |d_{a_0}x|\, dt$$

its (twisted) length w.r.t.the metric $\langle\cdot,\cdot\rangle_X$.

Definition 4.1.1. An *a_0-compatible extension* of x is a smooth section $u : [0,1]\times S^1 \to [0,1]\times Q(X)$ such that

$$u(1,\cdot) = x, \quad d_{a_0}(u(0,\cdot)) = 0. \tag{28}$$

We denote by $\mathrm{Ext}_{a_0}(x)$ the set of such extensions and call (x,a_0) *admissible* iff there exists $u \in \mathrm{Ext}_{a_0}(x)$ such that the following holds. If $\hat{u} \in \mathrm{Ext}_{a_0}(x)$ is another extension satisfying

$$\max_{s\in[0,1]} \ell\big(\hat{u}(s,\cdot),a_0\big) \le \max_{s\in[0,1]} \ell\big(u(s,\cdot),a_0\big) \tag{29}$$

then

$$\int_{[0,1]\times S^1} \omega\big(\partial_s \hat{u}, d_{a_0}(\hat{u}(s,\cdot))\cdot\big)\wedge ds = \int_{[0,1]\times S^1} \omega\big(\partial_s u, d_{a_0}(u(s,\cdot))\cdot\big)\wedge ds. \tag{30}$$

Definition 4.1.2. The invariant symplectic action is the map

$$\mathcal{A} : \big\{(x,a,a_0)\,\big|\,(x,a_0) \text{ admissible, } a\in\mathcal{A}(Q)\big\} \to \mathbb{R}$$

defined by

$$\mathcal{A}(x,a,a_0) := \int_{[0,1]\times S^1} \omega\big(\partial_s u, d_{a_0}(u(s,\cdot))\cdot\big)\wedge ds + \int_{S^1}\big\langle Q(\Phi)\circ x, a-a_0\big\rangle, \tag{31}$$

where u is as in the definition of admissibility, and we view $a-a_0$ as a one-form on S^1 with values in the adjoint bundle $Q(\mathfrak{g})$.

Remark 4.1.3. Sections $x : S^1 \to Q(X)$ and $u : [0,1]\times S^1 \to Q(X)$ of the bundles are in natural bijection with equivariant maps $Q \to X$ and $[0,1]\times Q \to X$ respectively. We will sometimes use this identification.

Remark 4.1.4. The following gives a more intrinsic but less explicit definition of the action. Let $A \in \mathcal{A}([0,1] \times Q)$ be a connection one-form on the bundle $[0,1] \times Q$. The form

$$\tilde{\omega}_A = \pi_3^*\omega + \mathrm{d}\big\langle \pi_3^*\Phi, \pi_2^* A\big\rangle \in \Omega^2([0,1] \times Q \times X)$$

where π_2, π_3 are the projections on Q, X, descends to a form $\omega_A \in \Omega^2([0,1] \times Q(X))$.

Now let $A \in \mathcal{A}([0,1] \times Q)$ be a connection one-form with $A|_{s=0} = a_0$ and $A|_{s=1} = a$, and let A_0 denote the pull-back of a_0 to $[0,1] \times Q$ under projection on the second factor. Let (x, a_0) be an admissible pair and $u \in \mathrm{Ext}_{a_0}(x)$ be an extension as in the definition of admissibility. We have

$$\begin{aligned}\mathcal{A}(x,a,a_0) &= \int_{[0,1]\times S^1} u^*\big(\omega_{A_0} + \mathrm{d}\big\langle([0,1]\times Q)(\Phi), A - A_0\big\rangle\big) \\ &= \int_{[0,1]\times S^1} u^*\omega_A.\end{aligned}$$

The following lemma proves in particular that the invariant symplectic action is invariant under gauge transformation. We prove a stronger statement which includes bundle morphisms not necessarily covering the identity. We view such a morphism as a section $\psi : S^1 \to Q(Q) = (Q \times Q)/G$. If $x : S^1 \to Q(X)$ is a section we define the pull-back section $\psi^* x : S^1 \to Q(X)$ by $\psi^* x(t) := \big[q, \tilde{x} \circ \tilde{\psi}(q)\big]$, for every $t \in S^1$, where $q \in Q$ is an arbitrary point over t. Here $\tilde{x} : Q \to X$ and $\tilde{\psi} : Q \to Q$ are the equivariant maps corresponding to x and ψ.

Lemma 4.1.5. *Let X, ω, G, Q and $\langle\cdot,\cdot\rangle_X$ be as above and $\Psi : [0,1] \times Q \to Q$ a smooth equivariant map such that for every $s \in [0,1]$ the following holds. The map $f_s : S^1 \simeq \mathbb{R}/\mathbb{Z} \to S^1$ defined by the equation $f_s \circ \pi = \pi \circ \Psi(s,\cdot)$ satisfies $\frac{d}{dt} f_s(t) \geq 0$, for every $t \in S^1$, or $\frac{d}{dt} f_s(t) \leq 0$, for every $t \in S^1$. Then for every smooth section $x : S^1 \to Q(X)$ and every $a_0 \in \mathcal{A}(Q)$, the pair $\big(x \circ \Psi(1,\cdot), \Psi(0,\cdot)^* a_0\big)$ is admissible if and only if (x, a_0) is, and*

$$\mathcal{A}\big(\Psi(1,\cdot)^* x, \Psi(0,\cdot)^* a_0, \Psi(1,\cdot)^* a\big) = \deg(f_0)\mathcal{A}(x,a,a_0).$$

Proof of Lemma 4.1.5. We denote $d := \deg(f)$.

...

It suffices to construct maps

$$\mathrm{Ext}_{a_0}(x) \to \mathrm{Ext}(\phi^*(x,a_0)), \quad \mathrm{Ext}(\phi^*(x,a_0)) \to \mathrm{Ext}(x,a_0)$$

preserving the quantities in (30), (29). We choose smooth functions $\psi, \chi : [0,1] \to [0,1]$ satisfying

(32) $$\psi(0) = 0,\ \psi(s) = 1,\ \forall s \in [1/2,1], \quad \chi(s) = 0,\ \forall s \in [0,1/2],\ \chi(1) = 1.$$

Given an extension u' of $\phi^* x$ we define

$$u(s,t) := \begin{cases} (\zeta^*_{\psi(s)} x)(t) & \text{if } 0 \le s \le \frac{1}{2}, \\ ((\chi \times 1)^* u')(s,t) & \text{if } \frac{1}{2} \le s \le 1. \end{cases} \tag{33}$$

That is, on the first region u is equal to the section given by pull-back of x by the map $(s,q) \mapsto \zeta_{\psi(s)}(q)$ and on the second region u is the pull-back of u' by the map $(s,q) \mapsto (\chi(s), q)$. The section u is smooth since $u'(0,\cdot) = \phi^* x = \zeta_1^* x$. If A' is an extension of $a' = \phi^* a$ with limit a_0 as in Remark 4.1.4, define $\rho(s,q) = \zeta_{\psi(s)}(q)$ and

$$A(s,t) := \begin{cases} (\rho^* a_0)(t) & \text{if } 0 \le s \le \frac{1}{2}, \\ ((\chi \times 1)^* A')(s,t) & \text{if } \frac{1}{2} \le s \le 1. \end{cases} \tag{34}$$

We have

$$\max_s \ell(u(s,\cdot), a_0) \le \max_s \ell\big(u'(s,\cdot), a_0\big) \tag{35}$$

since $\frac{d}{dt}\pi(\zeta(s,t)) \ge 0$. On the other hand,

$$\int_\Sigma u^* \omega_A = \int_{[0,1/2]\times S^1} (\rho^* x)^* \omega_{\rho^* A} + \int_{[1/2,1]\times S^1} (\chi \times 1)^* \omega_{A'}.$$

To see that the first integral vanishes, note that $(\rho^* x)\omega_{\rho^* A} = \zeta^* x^* \omega_{a_0}$ where $\zeta(s,t) = \pi(\rho(s, \pi^{-1}(t)))$ is the map covered by ρ. Since ω_{A_0} is degree two, $x^* \omega_{a_0} = 0$. To see that the second integral vanishes, consider the map $\eta(\lambda, s, t) = ((1-\lambda)s + \chi(s), t)$. By Stokes' theorem

$$\begin{aligned} 0 &= \int_{[0,1]\times\Sigma} (\eta^* u')^* \mathrm{d}\omega_{\eta^* A} \\ &= \int_{(\{0,1\}\times\Sigma)\cup([0,1]\times\partial\Sigma)} (\eta^* u')^* \omega_{\eta^* A} \\ &= \int_\Sigma (u')^* \omega_{A'} - \int_\Sigma u^* \omega_A \end{aligned} \tag{36}$$

where to obtain the third equality we have used that $\eta([0,1] \times \partial\Sigma) \subseteq \partial\Sigma$ and so the pull-back of $(\rho^* u')\omega_{\rho^* A}$ to $[0,1] \times \partial\Sigma$ vanishes. Hence

$$\int_\Sigma u^* \omega_A = \int_{[0,1]\times S^1} (u')^* \omega_{A'}.$$

Conversely, given an extension $u \in \mathrm{Ext}(x, a_0)$ define

$$u'(s,t) := \begin{cases} (\zeta^*_{\psi(1-s)} x)(t) & \text{if } 0 \le s \le \frac{1}{2}, \\ ((\chi \times 1)^* u)(s,t) & \text{if } \frac{1}{2} \le s \le 1 \end{cases} \tag{37}$$

and with $\rho(s,q) = \zeta_{\psi(1-s)}(q)$

$$A'(s,t) := \begin{cases} (\rho^* a_0)(t) & \text{if } 0 \le s \le \frac{1}{2}, \\ ((\chi \times 1)^* A)(s,t) & \text{if } \frac{1}{2} \le s \le 1. \end{cases} \tag{38}$$

Similar arguments to those before show that the maximal length of u' and the integral of $(u')^* \omega_{A'}$ are the same as those for A, u. This proves Lemma 4.1.5. □

Proposition 4.1.6 (Equivariant isoperimetric inequality)**.** *Let $X, \omega, G, \langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and $\Phi, \langle\cdot,\cdot\rangle_X$ be as above, and let $\mathcal{C} \subseteq G$ be a conjugacy class. Assume that X is closed. Then there exist constants $\delta, C > 0$ such that for every principal G-bundle Q over S^1, every connection $a_0 \in \mathcal{A}(Q)$ with holonomy in $\mathcal{C}$ and every section $x : S^1 \to (Q \times X)/G$ satisfying*

$$\ell(x, a_0) \leq \delta \tag{39}$$

the following holds. The pair (x, a_0) is admissible and for every connection $a \in \mathcal{A}(Q)$ and every $\epsilon > 0$ and $1 \leq p \leq \infty$ the following inequality holds:

$$|\mathcal{A}(x, a, a_0)| \leq C\|d_a x\|_p^2 + \left(\frac{1}{4\epsilon} + C\right)\|a - a_0\|_p^2 + \epsilon\|x^*Q(\Phi)\|^2_{\frac{p}{p-1}}. \tag{40}$$

Proposition 4.1.7 (Energy action identity)**.** *Let $X, \omega, G, \langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and Φ be as above, J be a G-invariant ω-compatible almost complex structure, and let $\mathcal{C} \subseteq G$ be a conjugacy class. Assume that X is closed. Then there exists a constant $\delta > 0$ such that the following holds. Let $s_- \leq s_+$ be numbers, Q be a principal G-bundle over S^1, a_0 be a connection on Q, ω_Σ be an area form on $\Sigma := [s_-, s_+] \times S^1$, and (u, A) be a solution of the vortex equations (1) on $[s_-, s_+] \times Q$. Assume that*

$$\ell\big(u(s,\cdot), a_0\big) \leq \delta, \ \forall s \in [s_-, s_+]. \tag{41}$$

Then the pairs $\big(u(s_\pm,\cdot), a_0\big)$ are admissible and

$$E(u, A) = -\mathcal{A}\big((u, A)(s_+,\cdot), a_0\big) + \mathcal{A}\big((u, A)(s_-,\cdot), a_0\big). \tag{42}$$

To prove Propositions 4.1.6 and 4.1.7, we use the symplectic action for paths with Lagrangian boundary conditions. More precisely, let (X, ω) be a symplectic manifold, $L_0, L_1 \subseteq X$ be Lagrangian submanifolds and let $\langle\cdot,\cdot\rangle_X$ be a Riemannian metric on X. Let $x : [0, 1] \to X$ be a smooth path such that $x(i) \in L_i$, for $i = 0, 1$. We denote by $\ell(x) := \int_0^1 |\dot{x}|\, dt$ its length w.r.t.$\langle\cdot,\cdot\rangle_X$.

Definition 4.1.8. An (L_0, L_1)-*compatible extension* of x is a smooth map $u : \Sigma := [0, 1] \times [0, 1] \to X$ such that

$$u(0, t) = u(0, 0), \ \forall t, \quad u(1,\cdot) = x, \quad u(s, i) \in L_i, \forall s \in [0, 1], \ i = 0, 1. \tag{43}$$

We denote by $\mathrm{Ext}_{L_0,L_1}(x)$ the set of all such extensions, and we call the path x *admissible* iff there exists $u \in \mathrm{Ext}_{L_0,L_1}(x)$ such that the following holds. If $\hat{u} \in \mathrm{Ext}_{L_0,L_1}(x)$ is another extension satisfying

$$\max_s \ell\big(\hat{u}(s,\cdot)\big) \leq \max_s \ell\big(u(s,\cdot)\big), \quad \forall s \in [0, 1], \tag{44}$$

then

$$\int_\Sigma \hat{u}^*\omega = \int_\Sigma u^*\omega. \tag{45}$$

We define the *relative action* to be the map

$$\mathcal{A}_{L_0,L_1} : \big\{x \in C^\infty([0, 1], X) \,\big|\, x \text{ admissible}\big\} \to \mathbb{R},$$

$$\mathcal{A}_{L_0,L_1}(x) := -\int_\Sigma u^*\omega,$$

where $u : \Sigma \to X$ is a smooth map with the above properties. Admissibility and the relative symplectic action are invariant under changes of coordinates. This is the content of the following.

Lemma 4.1.9. *Let X, ω, L_0, L_1 and $\langle\cdot,\cdot\rangle_X$ be as above, $x : [0,1] \to X$ a path, and $\phi : [0,1] \to [0,1]$ be a smooth map fixing 0 and 1 such that $\phi'(t) \geq 0$, for every $t \in [0,1]$. Then $x \circ \phi$ is admissible if and only if x is, and*

$$\mathcal{A}_{L_0,L_1}(x \circ \phi) = \mathcal{A}_{L_0,L_1}(x)$$

Proof of Lemma 4.1.9. The proof is similar to that of Lemma 4.1.5. It suffices to construct maps

$$\mathrm{Ext}_{L_0,L_1}(x) \to \mathrm{Ext}_{L_0,L_1}(x \circ \phi), \quad \mathrm{Ext}_{L_0,L_1}(x \circ \phi) \to \mathrm{Ext}_{L_0,L_1}(x)$$

preserving the symplectic area and maximal length in (45), (44). Let ψ, χ be as in (32). Given an extension $u' \in \mathrm{Ext}_{L_0,L_1}(x \circ \phi)$, define $u \in \mathrm{Ext}_{L_0,L_1}(x)$ by

$$u(s,t) := \begin{cases} x((1-\psi(s))t + \psi(s)\phi(t)) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ u(\chi(s),t) & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} .$$

One sees easily that u has the same area and maximal length as u', The reverse construction is similar.

□

Remark 4.1.10. Their are alternative ways of defining admissibility and the relative symplectic action, for example, one could define it as follows. If $x, y \in X$ are points, we denote their distance by $d(x,y)$, if $x : [0,1] \to X$ is a path, we denote its length by $\ell(x)$, and if $S \subseteq X$ is a subset, we denote its diameter by

$$\mathrm{diam}(S) := \sup_{x,y \in S} d(x,y).$$

(All these quantities are with respect to $\langle\cdot,\cdot\rangle_X$.) We denote by $\Sigma := [0,1] \times [0,1]$ the unit square. We call a path $x \in W^{1,\infty}([0,1], X)$ *admissible* iff $x(i) \in L_i$, for $i = 0,1$, and there exists a map $u \in W^{1,\infty}(\Sigma, X)$ such that

$$u(0,t) = u(0,0),\ \forall t, \quad u(1,\cdot) = x, \quad u(s,i) \in L_i, \forall s \in [0,1],\ i = 0,1 \tag{46}$$

and the following holds. If $\hat{u} \in W^{1,\infty}(\Sigma, X)$ is another map satisfying (46) and the condition $\mathrm{diam}(\hat{u}(\Sigma)) \leq \mathrm{diam}(u(\Sigma))$ then

$$\int_\Sigma \hat{u}^*\omega = \int_\Sigma u^*\omega.$$

Proposition 4.1.11 (Relative isoperimetric inequality)**.** *Let X, ω, L_0, L_1 and $\langle\cdot,\cdot\rangle_X$ be as above. Assume that L_0 and L_1 are closed. Then there exists constants $\delta, C > 0$ such that the following holds. If $x : [0,1] \to X$ is a path satisfying $x(i) \in L_i$, for $i = 0,1$ and $\ell(x) < \delta$ then x is admissible and*

$$|\mathcal{A}_{L_0,L_1}(x)| \leq C\|\dot{x}\|_2^2. \tag{47}$$

Proof of Proposition 4.1.11. This follows from [19, Lemma 3.4.5] .... □

Lemma 4.1.12. *Let X be a manifold and $\phi : X \to X$ be a diffeomorphism such that X admits a ϕ-invariant metric. Let $Y := \big\{(x,x)\,\big|\, x \in X\big\} \subset X^2$ denote the diagonal and $Z := \big\{(x,\phi(x))\,\big|\, x \in X\big\} \subset X^2$ the graph of ϕ. Then the intersection $Y \cap Z$ is clean.*

Proof. The tangent spaces to Y, Z are

$$TY = \{((x,v),(x,v)), \quad x \in X, v \in T_xX\}$$

$$TZ = \{((x,v),(\phi(x), D_x\phi(v))), \quad x \in X, v \in T_xX\}.$$

Hence

$$Y \cap Z = \{(x,x) \in X^2,\ \phi(x) = x\}$$

$$TY \cap TZ = \{((x,v),(x,v)) \in TX^2,\ (\phi(x), D_x\phi(v)) = (x,v)\}.$$

Projection onto the first factor gives identifications

$$Y \cap Z \to X^\phi, \qquad TY \cap TZ \to (TX)^{D\phi}$$

where X^ϕ resp. $(TX)^\phi$ denotes the fixed point set of the action of ϕ on X resp. TX. Thus the intersection is clean if and only if X^ϕ is a submanifold and

$$T(X^\phi) = (TX)^{D\phi}. \tag{48}$$

Let $x \in Y \cap Z$. Let g be a ϕ-invariant metric on X and denote by $\exp_x : T_xX \to X$ the exponential map. It follows from the uniqueness of geodesics with given initial conditions that

$$\exp_x \circ D_x\phi = \phi \circ \exp_x. \tag{49}$$

We choose a number $r > 0$ such that $\exp_x : B_r \to X$ is a diffeomorphism onto its image, where $B_r \subset T_xX$ denotes the ball of radius r around 0 w.r.t. the metric on T_xX. By equation (49) the fixed point set $B_r \cap T_xX^{D_x\phi}$ is mapped bijectively onto the fixed point set $\exp(B_r) \cap X^\phi$. Because $T_xX^{D_x\phi}$ is a subspace, X^ϕ is a manifold near x. Taking tangent spaces we obtain

$$T_x(X^\phi) = D\exp_x(0)T_0(T_xX)^{D_x\phi} = (T_xX)^{D_x\phi}.$$

Hence (48) which proves the Lemma. □

We now relate the invariant and relative symplectic actions. Let $\mathcal{C}$ be as in Proposition 4.1.6. We choose a representative $g \in \mathcal{C}$. Consider the manifold $\tilde{X} := X \times X$ with the symplectic structure $\tilde{\omega} := (-\omega) \oplus \omega$ and the Riemannian metric $\langle\cdot,\cdot\rangle_{\tilde{X}}$ induced by $\langle\cdot,\cdot\rangle_X$, and the Lagrangian submanifolds

$$L_0 := \big\{(x,x)\,|\, x \in X\big\}, \quad L_1 := \big\{(x, g^{-1}x)\,\big|\, x \in X,\ g \in G\big\}.$$

Let $a_0 \in \mathcal{A}(Q)$ be a connection with holonomy in $\mathcal{C}$ and $x : S^1 \to Q(X)$ be a section satisfying $\ell(x, a_0) \le \delta$. We define $\rho : [0,1] \to S^1 \simeq \mathbb{R}/\mathbb{Z}$ by $\rho(t) := t + \mathbb{Z}$. Since a_0 has holonomy conjugate to g, there exists a map $f : [0,1] \to Q$ satisfying the conditions

$$\pi \circ f = \rho, \quad a_0(\partial_t f) = 0, \quad f(1) = f(0)g. \tag{50}$$

We define

$$\tilde{x} : [0,1] \to \tilde{X}, \quad \tilde{x}(t) := \big(x \circ f((1-t)/2), x \circ f((1+t)/2)\big). \tag{51}$$

Then, denoting by $\tilde{\ell}(\tilde{x})$ the length of $\tilde{x}$ w.r.t.$\langle \cdot, \cdot \rangle_{\tilde{X}}$, we have

$$\tilde{x}(i) \in L_i, \text{ for } i = 0, 1, \quad \tilde{\ell}(\tilde{x}) = \ell(x, a_0).$$

Furthermore, by Lemma 4.1.12 with $\phi(x) := gx$, the Lagrangians L_0 and L_1 intersect cleanly.

Lemma 4.1.13. *Let $L_0, L_1, x, \tilde{x}, a_0$ be above. The path $\tilde{x}$ is admissible if and only if (x, a_0) is, and*

$$\mathcal{A}(x, a, a_0) = \mathcal{A}_{L_0,L_1}(x) + \int_{S^1} \big\langle x^* Q(\Phi), a - a_0 \big\rangle.$$

Proof. Let $\phi : [0,1] \to [0,1]$ be a smooth map with

$$\frac{d}{dt}\phi(t) \geq 0, \forall t \in [0,1], \quad \phi(t) = 0, t \leq 1/4, \quad \phi(t) = 1, t \geq 3/4. \tag{52}$$

By Lemma 4.1.9, x is admissible if and only if $x \circ \phi$ is, and the two paths have the same action. Similarly, let $\psi : S^1 \to S^1$ denote the corresponding smooth map, obtained from the identification $S^1 = [0,1]/(0 \sim 1)$. Let $\rho : Q \to Q$ be a lift of ψ preserving a_0; given a single value $\rho(0)$, the remaining values $\rho(t)$ are defined by parallel transport of $\rho(0) = \rho(\psi(0))$ along a path from 0 to $\psi(t)$. By Lemma 4.1.5, $\rho^* x$ is admissible if and only if x is, and $\mathcal{A}(\rho^* x, a, a_0) = \mathcal{A}(x, a, a_0)$.

This reduces to the case that $\tilde{x}$ is constant on $[0, 1/4] \cup [3/4, 1]$, and x is covariant constant on $[-1/4, 1/4] \in S^1$. We claim that $\tilde{x}$ is admissible if and only if x is. To prove this, it suffices to define maps

$$\mathrm{Ext}_{a_0}(x) \to \mathrm{Ext}_{L_0,L_1}(\tilde{x}), \quad \mathrm{Ext}_{L_0,L_1}(\tilde{x}) \to \mathrm{Ext}_{a_0}(x)$$

preserving the maximal length and integrals in the definitions of admissibility, where $\mathrm{Ext}_{a_0}(x)$ resp. $\mathrm{Ext}_{L_0,L_1}(\tilde{x})$ is the set of extensio of x resp. $\tilde{x}$, compatible with a_0 and (L_0, L_1) respectively. Given any extension $u \in \mathrm{Ext}_{a_0}(x)$ define $\tilde{u} \in \mathrm{Ext}_{L_0,L_1}(\tilde{x})$ by pullback under f, with the same maximal length and integral. Conversely, given any extension $\tilde{u} \in \mathrm{Ext}_{L_0,L_1}(\tilde{x})$ define an extension $\tilde{u}_1$ of $\tilde{x}$ by pull-back under the map $1 \times \phi$, where ϕ is the map of (52). Since $\tilde{u}_1$ is constant near $0, 1$, it descends to a smooth map

$$u_1 : [0,1] \times S^1 \to Q, \quad u_1(1, \cdot) = x, \quad \ell(u_1(s, \cdot), a_0) = \ell(\tilde{u}(s, \cdot)).$$

One sees easily that u_1 has the same maximal length and

$$\mathcal{A}(x, a, a_0) = \int_{[0,1] \times S^1} u_1^* \omega_{A_0} = \int_{[0,1] \times [0,1]} \tilde{u}^* \omega.$$

Hence x is admissible if and only if $\tilde{x}$ is and

$$\mathcal{A}(x, a, a_0) = \mathcal{A}(x, a, a_0) + \int_{S^1} \big\langle x^* Q(\Phi), a - a_0 \big\rangle = \mathcal{A}_{L_0,L_1}(\tilde{x}) + \int_{S^1} \big\langle x^* Q(\Phi), a - a_0 \big\rangle$$

which proves Lemma 4.1.13. □

Proof of Proposition 4.1.6. Let $\tilde{X}, \tilde{\omega}, L_0, L_1, \langle\cdot,\cdot\rangle_{\tilde{X}}$ and $\tilde{\ell}$ be as above. We choose constants $\delta > 0$ and $C_1 := C$ as in Lemma **??**, with $X, \omega, \langle\cdot,\cdot\rangle_X$ and ℓ replaced by $\tilde{X}, \tilde{\omega}, \langle\cdot,\cdot\rangle_{\tilde{X}}$ and $\tilde{\ell}$. We define

$$C_2 := \max\left\{|\xi_X(x)| \,\middle|\, x \in X,\, \xi \in \mathfrak{g} : |\xi| = 1\right\}, \quad C := \max\left\{2C_1, 2C_2^2\right\}.$$

Let x, a, a_0 be as in the Proposition, $f : [0,1] \to Q$ be a map satisfying (50) and $\tilde{x} : [0,1] \to X$ be the path defined by (51). By the statement of Lemma **??** the path $\tilde{x}$ is admissible. Hence by Lemma 4.1.13 the pair (x, a_0) is admissible. Furthermore, for every $\epsilon > 0$,

$$\begin{aligned} \left|\mathcal{A}(x,a,a_0)\right| &\leq \left|\mathcal{A}_{L_0,L_1}(\tilde{x})\right| + \int_{S^1} \left|\left\langle Q(\Phi)\circ x, a - a_0\right\rangle\right| \\ &\leq C_1 \ell(\tilde{x})^2 + \|Q(\Phi)\circ x\|_2 \|a - a_0\|_2 \\ &\leq C_1 \|d_{a_0} x\|_2^2 + \epsilon \|Q(\Phi)\circ x\|_2^2 + \frac{1}{4\epsilon}\|a - a_0\|_2^2. \end{aligned} \tag{53}$$

Here in the first inequality we used Lemma 4.1.13, in the second inequality we used **??** and Hölder's estimate, and in the last inequality we used that $\ell(\tilde{x}) = \|d_{a_0}x\|_1 \leq \|d_{a_0}x\|_2$. Furthermore,

$$|d_{a_0}x|^2 = \left|d_a x + (a - a_0)_X(x)\right|^2 \leq 2|d_a x|^2 + 2|(a-a_0)_X(x)|^2. \tag{54}$$

Since $|(a-a_0)_X(x)| \leq C_2 |a - a_0|$, combining (53) with (54) implies

$$\left|\mathcal{A}(x,a,a_0)\right| \leq C\|d_{a_0}x\|_2^2 + \left(C + \frac{1}{4\epsilon}\right)\|a-a_0\|_2^2 + \epsilon\|Q(\Phi)\circ x\|_2^2.$$

This proves Proposition **??**. □

Proof of Proposition 4.1.7. ...

This proves Proposition 4.1.7. □

...

The proof depends on the a priori estimate for symplectic vortices proved by Gaio-Salamon:

Theorem 4.1.14 (Mean Value Inequality for Vortices)**.** *There exists a constant c_1 depending only on the supremum of $\omega_{\Sigma(T_1,T_2)}$ such that if (A,u) is a vortex on $\Sigma(T_1,T_2)$ with $E_{A,u}(B_{\frac{1}{2}}(r,\theta)) < c_1$ then*

$$e_{A,u} \leq \frac{32}{\pi} E_{A,u}(B_{\frac{1}{2}}(r,\theta)).$$

Proof. The energy density satisfies $\Delta e_{A,u} \geq -C e_{A,u}^2$ for some constant C depending on an upper bound for the norm of $\omega_{\Sigma(T_1,T_2)}$, see [23, Lemma 11], which in turn uses results of Gaio-Salamon [8]. Note that the case needed here is substantially easier than that of [23], which considers the case of unbounded area form. □

Proof of Proposition **??**. Let $e_{A,u}$ denote the energy density of A, u and $E_{A,u}(B_{\frac{1}{2}}(r,\theta))$ the energy on a ball of radius $\frac{1}{2}$ around r, θ. By Proposition 4.1.14, $\mathrm{d}_A u$ satisfies the pointwise estimate

$$\|\mathrm{d}_A u(r,\theta)\| \leq e_{A,u}(r,\theta)\| \leq E_{A,u}(B_{\frac{1}{2}}(r,\theta)). \tag{55}$$

Integrating gives the first condition in the definition of admissibility. By assumption, A is C^0-close to a_0. Hence (A_T, u_T) is admissible. By (55), the diameter of the image of u is bounded by the total energy. It follows that the image of u is contained in $P(B_{r_X}(x))$, if c_2 is sufficiently small. This proves Theorem **??**. □

4.2. **Decay on cylinders.** I'M WORKING ON THIS SUBSECTION (Fabian).

Let (X, ω) be a symplectic manifold, G be a compact connected Lie group acting on X in a Hamiltonian way, with moment map $\Phi : X \to \mathfrak{g}^*$, $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ be an invariant inner product on $\mathfrak{g}$, and J be a ω-compatible G-invariant almost complex structure on X. We denote by ι^G the injectivity radius of G and by d^G the distance function on G, both w.r.t. the metric on G induced by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. Furthermore, if $\mathcal{C}, \mathcal{C}' \subseteq G$ are conjugacy classes then we define

$$\bar{d}^G(\mathcal{C}, \mathcal{C}') := \min \left\{ d(g, g') \,\middle|\, g \in \mathcal{C},\, g' \in \mathcal{C}' \right\}$$

Then $\bar{d}$ is a distance function that induces the quotient topology on the set of conjugacy classes, see for example Lemma A.10 in [**?**].

Theorem 4.2.1. *Assume that X is closed. Then for every conjugacy class $\mathcal{C} \subseteq G$ there exists a constant $\epsilon_0 > 0$ such that for every constant $0 < c < \epsilon_0$ and every constant $K \in \mathbb{R}$ there exists a constant C with the following property. Let $s_0, s_1 \in \mathbb{R}$ be such that $s_1 \geq s_0 + 2$, $P \to \Sigma := [s_0, s_1] \times S^1$ be a principal G-bundle, $\lambda \in C^\infty\big([s_0, s_1] \times S^1, (0, \infty)\big)$, $p > 2$ and (u, A) be a $W^{1,p}$-solution of the vortex equations on P with area form $\omega_\Sigma := \lambda^2 ds \wedge dt$. Assume that*

$$\lambda \leq K e^{-c \min\{s - s_0, s_1 - s\}}, \tag{56}$$

$$K^2 \geq \max_{x \in X} |\mu(x)|^2 \left(\sup_{[s_0, s_1] \times S^1} \frac{9 |d\lambda|^2}{\Delta(\lambda^2)} - 1 \right), \tag{57}$$

$$E(u, A) \leq \epsilon_0, \tag{58}$$

$$\bar{d}^G(\mathcal{C}, \mathcal{C}_s) \leq K e^{-2c \min\{s - s_0, s_1 - s\}},\ \forall s \in [s_0, s_1], \tag{59}$$

where $\mathcal{C}_s \subseteq G$ denotes the conjugacy class of the holonomy of A around the circle $\{s\} \times S^1$. Then

$$|d_A u(s,t)| \leq C e^{-c \min\{s - s_0, s_1 - s\}}, \tag{60}$$

for every $(s,t) \in [s_0 + 1, s_1 - 1] \times S^1$.

Let $X, \omega, G, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \Phi$ and J be as above, Σ be a real surface equipped with two Riemannian metrics $\langle \cdot, \cdot \rangle_\Sigma$ and $\langle \cdot, \cdot \rangle_\Sigma^0$ and a pair $(u, A) \in \Gamma(P(X)) \times \mathcal{A}(P)$ we define the *energy density of* (u,A) *w.r.t.* $\big(\langle\cdot,\cdot\rangle_\Sigma, \langle\cdot,\cdot\rangle^0_\Sigma\big)$ to be

$$e^{\langle\cdot,\cdot\rangle_\Sigma,\langle\cdot,\cdot\rangle^0_\Sigma}_{(u,A)} := \frac{1}{2}\Big(|d_Au|_0^2 + \lambda^2|\mu\circ u|^2 + \lambda^{-2}|F_A|_0^2\Big),$$

where $\lambda:\Sigma\to(0,\infty)$ is the function defined by the equation $\langle\cdot,\cdot\rangle_\Sigma = \lambda^2\langle\cdot,\cdot\rangle^0_\Sigma$, and the point-wise norms $|\cdot|_0$ and $|\cdot|$ are taken w.r.t. $\langle\cdot,\cdot\rangle^0_\Sigma$ and $\langle\cdot,\cdot\rangle_\Sigma$ respectively. In the case $\Sigma\subseteq\mathbb{C}$ or $\Sigma\subseteq\mathbb{R}\times S^1$ and $\langle\cdot,\cdot\rangle^0_\Sigma = ds^2+dt^2$, we abbreviate

$$e^\lambda_{(u,A)} := e^{\langle\cdot,\cdot\rangle_\Sigma, ds^2+dt^2}_{(u,A)}.$$

Lemma 4.2.2 (Point-wise bound on energy density)**.** *Let* $X,\omega,G,\langle\cdot,\cdot\rangle_{\mathfrak{g}},\Phi$ *and* J *be as above. Assume that* X *is closed. Then there exists a constant* $\epsilon>0$ *with the following property. Let* $K\in\mathbb{R}$*,* $r>0$*,* $\lambda\in C^\infty\big(B_r,(0,\infty)\big)$*,* $P\to B_r$ *be a principal* G*-bundle,* $p>2$ *and* (u,A) *be a* $W^{1,p}$*-solution the vortex equations on* P *with area form* $\lambda^2 ds\wedge dt$*. Assume that the inequalities (57) and*

$$E^\lambda(A,u) + K^2\int_{B_r}\lambda^2 \mathrm{d}s\,\mathrm{d}t \le K\epsilon \tag{61}$$

are satisfied. Then

$$\big(e^\lambda_{(A,u)} + K^2\lambda^2\big)(0) \le \frac{8}{\pi r^2}\left(E^\lambda(A,u) + K^2\int_{B_r}\lambda^2\mathrm{d}s\,\mathrm{d}t\right). \tag{62}$$

For the proof of Lemma 4.2.2 we need the following.

Lemma 4.2.3 (Mean value inequality)**.**

Proof of Lemma 4.2.2. Let $X,\omega,G,\langle\cdot,\cdot\rangle_{\mathfrak{g}},\Phi$ and J be as above, and assume that X is compact. By the calculation in the proof of the pointwise bound on the energy density [**?**, Lemma ?] there exists a constant $C_0\ge 0$ such that for every $r>0$, $\lambda\in C^\infty\big(B_r,(0,\infty)\big)$, every principal G-bundle P over B_r and every λ-vortex (A,u) on P we have

$$\Delta e^\lambda_{(A,u)} \ge \Big(-9|d\lambda|^2 + \Delta(\lambda^2) - C_0\lambda^4\Big)|\mu\circ u|^2 - C_0\big(e^\lambda_{(A,u)}\big)^2. \tag{63}$$

Let now $K\in\mathbb{R}$. We choose constants $C\in\mathbb{R}$ and $E>0$ as in Lemma 4.2.3, corresponding to $n=2$ and with K replaced by

$$K' := \max\big\{K + C_0K^2, C_0\big\}.$$

Let $r>0$, $\lambda\in C^\infty\big(B_r,(0,\infty)\big)$, $P\to B_r$ be a principal G-bundle, $p>2$ and (u,A) be a $W^{1,p}$-solution the vortex equations on P with area form $\lambda^2 ds\wedge dt$. Assume that the conditions (**??**,**??**,61) are satisfied. By elliptic regularity and invariance of energy density under gauge transformation **??** the density $e^\lambda_{(A,u)}$ is smooth. Furthermore, the first inequality in (**??**) holds by the definition of $e^\lambda_{(A,u)}$, and the second inequality with K replaced by K' follows from (63,**??**,**??**) and the fact $\lambda^2|\mu\circ u|^2\le e^\lambda_{(A,u)}$. Finally, the third inequality from (61). Therefore, inequality (62) follows from the statement of 4.2.3. This proves Lemma 4.2.2. □

Lemma 4.2.4. *Let G be a compact Lie group and $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ be an invariant inner product on $\mathfrak{g} := \mathrm{Lie}(G)$, Q be a principal G-bundle over $S^1 \simeq \mathbb{R}/\mathbb{Z}$, X be a manifold (possibly with boundary), $a_0 \in \mathcal{A}(Q)$ and a be a smooth section of the real vector bundle*

$$\big\{(x,q,\phi)\,\big|\,(x,q) \in X \times Q,\ \phi : T_qQ \to \mathfrak{g} \text{ equivariant},\ \phi(q\xi) = \xi,\ \forall \xi \in \mathfrak{g}\big\} \to X \times Q.$$

Denoting by $\mathcal{C}$ and $\mathcal{C}_x$ the conjugacy classes of the holonomy of a_0 and $a(x,\cdot)$ around S^1, we assume that

$$\bar{d}^G(\mathcal{C},\mathcal{C}_x) < \iota^G.$$

Then there exists an equivariant diffeomorphism $g : X \times Q \to Q$ that induces the canonical projection $X \times S^1 \to S^1$ and satisfies

$$\big|\big(a(x,\cdot,\cdot)\mathrm{d}(g(x,\cdot)) - a_0\big)(q)\big| = \bar{d}^G(\mathcal{C},\mathcal{C}_x), \tag{64}$$

for every $x \in X$, $q \in Q$.

Remark 4.2.5. Philosophically, in this lemma a is a family of connections on Q and g is a family of gauge transformations on Q, both depending smoothly on $x \in X$. In the case in which X is a point the lemma says that given two connections on $Q \to S^1$ we may gauge transform one of them in such a way that it lies at distance from the other connection given by the distance of the conjugacy classes. For a general manifold X, the lemma can be seen as a parametrized version of this.

Proof of Lemma 4.2.4.

Claim 4.2.6. There exists smooth map $\widetilde{g} : X \times [0,1] \to G$ such that

(65)

Proof of Claim 4.2.6. ...

This proves Claim 4.2.6. □

...

This proves Lemma 4.2.4. □

Proof of Theorem 4.2.1. For $\xi \in \mathfrak{g}$ and $x \in X$ we denote by $\xi_X(x) \in T_xX$ the infinitesimal action of ξ at x, and we define

$$C_1 := \max\big\{|\xi_X(x)|\,\big|\,x \in X,\ \xi \in \mathfrak{g} :\ |\xi| = 1\big\}.$$

Let $\mathcal{C} \subseteq G$ be a conjugacy class. We fix constants $\delta > 0$ and $C_2 := C$ as in Proposition 4.1.6. Shrinking δ if necessary, we may assume that it satisfies the condition of Proposition 4.1.7. We define

$$c_0 := \frac{1}{C_2}, \quad \epsilon := \min\left\{\frac{\iota^G}{2}, \frac{\delta}{2C_1}\right\}. \tag{66}$$

Let $K \geq 0$. We choose constants $C_3 := C$ and $\tilde{E} := E > 0$ as in Lemma 4.2.2, with K replaced by $\tilde{K} := \min\{K, \sqrt{\epsilon}\}$. We define

$$E := \min\left\{\frac{\tilde{E}}{4}, \frac{\delta^2}{16C_3}\right\}. \tag{67}$$

Let $0 < c < c_0$, $s_0, s_1 \in \mathbb{R}$ be such that $s_1 \geq s_0 + 2$, $P \to \Sigma := [s_0, s_1] \times S^1$ be a principal G-bundle, $\lambda \in C^\infty\big([s_0, s_1] \times S^1, (0,\infty)\big)$, $p > 2$ and (u, A) be a $W^{1,p}$-solution of the vortex equations on P with area form $\omega_\Sigma := \lambda^2 ds \wedge dt$. Assume that conditions (**??**,58,**??**) are satisfied. Let $(s,t) \in [s_0 + 1, s_1 - 1] \times S^1$. Consider the map $\phi : B_{\frac{1}{2}} \to [s_0, s_1] \times S^1$ given by $\phi(s', t') := (s + s', t + t')$. We define $\tilde{\lambda} := \lambda \circ \phi : B_{1/2} \to \mathbb{R}$. By the assumptions (**??**) the conditions (**??**) and (**??**) hold with $r := 1/2$ and λ, K replaced by $\tilde{\lambda}, \tilde{K}$. Moreover, by (58) and (67) the condition (61) is satisfied with E, P, A, u replaced by $\tilde{E}$ and $(\tilde{P}, \tilde{A}, \tilde{u}) := \phi^*(P, A, u)$. It follows that the inequality (62) holds with C replaced by C_3, and hence

$$\begin{aligned} |d_A u|(s,t)^2 &\leq e^{\lambda}_{(u,A)}(s,t) \\ &= e^{\tilde{\lambda}}_{(\tilde{A},\tilde{u})}(0,0) \\ &\leq 4C_3 E^{\tilde{\lambda}}(\tilde{A}, \tilde{u}) \\ &\leq 4C_3 E^{\lambda}(A, u). \end{aligned} \tag{68}$$

We define

$$C_4 :=$$

and the function $f : \big[s_0, (s_0 + s_1)/2\big]$ by

$$f(s) := E\big(u, A, \big[s, s_1 + s_0 - s\big] \times S^1\big) + C_4\big(e^{-2c(s - s_0)} + e^{-2c(s - s_1)}\big). \tag{69}$$

Claim 4.2.7. We have $f'(s) \leq -2cf(s)$, for every $s \in \big[1, (s_1 - s_0)/2 - 1\big]$.

Proof of Claim 4.2.7. We may assume without loss of generality that there exists a principal G-bundle Q over S^1 such that $P = [s_0, s_1] \times Q$. We fix a connection a_0 on Q whose holonomy lies in $\mathcal{C}$.

Claim 4.2.8. There exists a gauge transformation $g : P \to P$ such that

$$\ell\big(gu(s, \cdot), a_0\big) \leq \delta, \tag{70}$$

for every $s \in [s_0, s_1]$.

Proof of Claim 4.2.8. Denoting by $\mathcal{C}_s$ the conjugacy class of the holonomy of $A|_{\{s\}\times Q}$ inequalities (**??**) and (66) imply that $\bar{d}^G(\mathcal{C}, \mathcal{C}_s) \leq \iota^G/2$, for every $s \in [s_0, s_1]$. Hence by Lemma 4.2.4 with $X := [s_0, s_1]$ and the section a defined by $av := A(0, v)$ for $v \in T_q Q$ and $q \in Q$, there exists an equivariant diffeomorphism $g : [s_0, s_1] \times Q \to Q$ such that condition (64) is satisfied. We define the gauge transformation $\tilde{g} : P \to P$ by $\tilde{g}(s, q) := g(s)$, for every $s \in [s_0, s_1]$ and $q \in Q$, and we define $(A', u') := \tilde{g}^*(A, u)$. We fix $s \in [s_0+1, s_1-1]$. We have

$$\ell\big((u'(s,\cdot), a_0\big) \leq \ell\big((A',u')(s,\cdot)\big) + \int_{S^1} \Big|\big(A'(s,\cdot) - a_0\big)_X \big(u'(s,\cdot)\big)\Big|(t)dt \tag{71}$$

$$\leq \ell\big((A,u)(s,\cdot)\big) + C_1 \int_{S^1} \big|A'(s,\cdot) - a_0\big|dt. \tag{72}$$

By inequality (65) we may estimate

$$\begin{aligned} \ell\big((A,u)(s,\cdot)\big) &\leq \int_{S^1} |d_A u|(s,t)dt \\ &\leq \int_{S^1} \sqrt{e^\lambda_{(A,u)}(s,t)}dt \\ &\leq 2\sqrt{C_3 E^\lambda(A,u)} \leq \frac{\delta}{2}. \end{aligned} \tag{73}$$

Here in the third step we used (68). On the other hand, by (64), assumption (**??**) and (66) we have

$$\big|A'|_{\{s\times Q\}} - a_0\big|(t) \leq \bar{d}^G(\mathcal{C}, \mathcal{C}_s) \leq \frac{\delta}{2C_1} e^{-2c\min\{s-s_0, s_1-s\}}, \tag{74}$$

for every $t \in S^1$. Substituting this into (72) and combining with (73), we get

$$\ell\big((u'(s,\cdot), a_0\big) \leq \delta.$$

This proves Claim 4.2.8. □

We choose a gauge transformation $g : P \to P$ as in Claim 4.2.8 and define $(A', u') := g(A, u)$. Then by (70) the condition (41) of Proposition 4.1.7 is satisfied with (A, u) replaced by (A', u'). It follows that the pairs $\big(u'(s_0,\cdot), a_0\big)$ and $\big(u'(s_1,\cdot), a_0\big)$ are admissible, and equality (42) holds with (A, u) replaced by (A', u'). We fix now $s \in \big[1, (s_1-s_0)/2-1\big]$. Using again (70) the condition (39) of Proposition 4.1.6 is satisfied with $x := u'(s,\cdot)$ and $a := A'|_{\{s\}\times Q}$. So applying this proposition with $p := 2$ and ϵ replaced by e^{-2cs}, we have

$$\begin{aligned} \big|\mathcal{A}\big((A',u')(s,\cdot), a_0\big)\big| &\leq C_2 \big\|\mathrm{d}_{A'}(u'(s,\cdot))\big\|_2^2 + \left(\frac{e^{2cs}}{4} + C_2\right) \big\|A'|_{\{s\}\times Q} - a_0\big\|_2^2 \\ &\quad + e^{-2cs}\big\|u'^* P(\Phi)(s,\cdot)\big\|_2^2 \\ &\leq C_2 \int_{S^1} \big|\mathrm{d}_{A'}(u'(s,\cdot))\big|^2 dt + \end{aligned} \tag{75}$$

(68) (74)

We have

$$
\begin{aligned}
f'(s) &= \frac{d}{ds}\left(\int_{[s_0+s,s_1-s]}\left(\int_0^1 e_{(u,A)}^{e^{-2\kappa s}}dt + C_3\big(e^{-2\kappa(s_0+s)} - e^{-2\kappa(s_1-s)}\big)\right)ds\right)\\
&= -\int_0^1 \big(e_{(u,A)}^{e^{-2\kappa s}}(s_1,t) + e_{(u,A)}^{e^{-2\kappa s}}(s_0,t)\big)dt - 2C_3\kappa\big(e^{-2\kappa(s_0+s)} + e^{-2\kappa(s_1-s)}\big)\\
&= -\|d_Au|_{\{s_1\}\times TS^1}\|_2^2 - e^{-2\kappa s_1}\|u^*P(\Phi)(s_1,\cdot)\|_2^2\\
&\quad -\|d_Au|_{\{s_0\}\times TS^1}\|_2^2 - e^{-2\kappa s_0}\|u^*P(\Phi)(s_0,\cdot)\|_2^2\\
&\quad -2C_3\kappa\big(e^{-2\kappa(s_0+s)} + e^{-2\kappa(s_1-s)}\big)\\
&\geq \ldots.. - \|d_Au|_{\{s_1\}\times TS^1}\|_2^2 - \|d_Au|_{\{s_0\}\times TS^1}\|_2^2 +\\
&\quad \ldots\ldots\ldots.\|A|_{s_0\times Q} - a_0\|_2^2 + \|A|_{s_1\times Q} - a_0\|_2^2\\
&\quad +\ldots\ldots\ldots\ldots..\|u(s_1,\cdot)^*Q(\Phi)\|_2^2 + \|u(s_1,\cdot)^*Q(\Phi)\|_2^2\\
&\geq \ldots\ldots\ldots.\mathcal{A}\big((u,A)(s_1,\cdot),a_0\big) - \mathcal{A}\big((u,A)(s_0,\cdot),a_0\big)\\
&\leq -\ldots\ldots\ldots.f(s)
\end{aligned}
$$

...

This proves Claim 4.2.7. □

Claim 4.2.7 implies that on $\big[0,(s_1-s_0)/2\big]$,

$$\frac{d}{ds}\big(fe^{2cs}\big) = f'e^{2cs} + f2ce^{2cs} \leq 0,$$

and hence

$$f(s) \leq f(0)e^{-2cs} \leq E(u,A)e^{-2cs}. \tag{76}$$

Combining this with (**??**) and (69), we get, for every $\hat{s} \in \big[s_0+1,(s_0+s_1)/2\big]$ and $\hat{t} \in S^1$,

$$|d_Au(\hat{s},\hat{t})|^2 \leq \frac{32}{\pi}f(\hat{s}-s_0-1/2) \leq \frac{32e^c}{\pi}E(u,A)e^{-2c(\hat{s}-s_0)}.$$

Similarly, we get for every $\hat{s} \in \big[(s_0+s_1)/2, s_1-1\big]$ and $\hat{t} \in S^1$,

$$|d_Au(\hat{s},\hat{t})|^2 \leq \frac{32}{\pi}f(s_1-\hat{s}-1/2) \leq \frac{32e^c}{\pi}E(u,A)e^{-2c(s_1-\hat{s})}.$$

Inequality (60) follows from this and (**??**). This proves Theorem 4.2.1. □

Corollary 4.2.9 (Decay for vortices on semi-infinite cylinders).

Proof of Corollary 4.2.9. ...

This proves Corollary 4.2.9. □

...

We show that for any vortex (A, u) on a finite cylinder

$$\Sigma = \Sigma(T_0, T_1) = [T_0, T_1] \times S^1$$

with sufficiently small vortex energy, the energy of u considered as a pseudoholomorphic map to $P(X)$ satisfies a convexity estimate. This is used in the compactness Theorem 5.2.6 to show bubbles connect. We remark that we do not know how to prove directly that the vortex energy density on small cylinders has exponential decay; rather, we show that the energy density of u, considered as a J_A-holomorphic map to $P(X)$, has exponential decay. In some sense, this is more natural we already know that A has exponential decay, after gauge transform, because the second vortex equation implies exponential decay of the curvature.

To prove that u satisfies an exponential decay estimate, let

$$\omega_\Sigma \in \Omega^2(\Sigma), \quad \omega_\Sigma = e^{-\kappa t} \mathrm{d}t\mathrm{d}\theta.$$

Recall that if A is a sufficiently small connection then the form

$$\omega_A + \pi^* \omega_\Sigma \in \Omega^2(P(X))$$

is symplectic. Let

$$\| \cdot \|_A = (\omega_A + \pi^* \omega_\Sigma)(\cdot, J \cdot)$$

denote the norm for the corresponding metric on $P(X)$. We denote by

$$e_A(u) = \|\mathrm{d}u\|_A^2, \quad E_A(u) = \int_\Sigma e_A(u)$$

the resulting energy density and total energy. If A_t, u_t denotes the restriction of (A, u) to $\{t\} \times S^1$ we denote by

$$\begin{aligned} D_{A_t}(u_t) &= D(A_t, u_t) + \int_{S^1 \times [t, \infty)} e^{-\kappa r} \mathrm{d}r\mathrm{d}\theta \\ &= D(A_t, u_t) + e^{-\kappa t}/\kappa. \end{aligned}$$

In the case that a_0 is trivial, this is the usual action of u_t considered as a loop in in the extension of $P(X)$ over the one-point compactification. We denote by $E_A(u; T)$ the energy on the sub-cylinder $[T_0 + T, T_1 - T] \times S^1$.

Theorem 4.2.10. *For any $\kappa > 0$ and $g_0 \in G$, there exist constants $\kappa_1, c_0, c_1, c_2, c_3, c_4, c_5 > 0$ such that for all $T_0 > 0$, all vortices (A, u) on $\Sigma(-T_0, T_0)$, the following holds: If*

$$\sup \|A|_{\{t\} \times S^1} - a_0\| \leq c_3 e^{-\kappa t}$$

$$T_0 > c_0, \quad T_1 > c_0 + 1, \quad E_A(u) < c_2$$

then

$$E_A(u; T) < c_4 e^{-\kappa_1 T} E_A(u; 0).$$

Furthermore, the distance between $u(z)$ and $u(z')$ for $z, z' \in \Sigma(-T_0+T, T_0-T)$ measured using the metric $g_{A,c}$ satisfies the exponential decay estimate

$$\sup_{z,z'} \operatorname{dist}(u(z), u(z')) \leq c_5 e^{-\kappa_1 T} E_A(u;0)^{1/2}$$

Proof. After gauge transformation, we may assume that we are in the setting of Corollary **??**. Let A_t, u_t denote the pull-back of (A, u) to $\{t\} \times S^1$, and A_∞ the pull-back of a_0 to P. By Proposition **??** we have that for any $T > 0$,

$$E_A(u;T) = D_{A_T}(u_T) - D_{A_{-T}}(u_{-T}).$$

By the isoperimetric inequality Proposition **??**

$$\begin{aligned} D_{A_T}(u_T) &= D(A_T, u_T) + e^{-\kappa T}/\kappa \\ &\leq C \int_{S^1} \|\mathrm{d}_{A_T} u_T\|^2 + \|A_T - A_\infty\| \|\Phi\| + e^{-\kappa T}/\kappa \\ &\leq C \int_{S^1} \|\mathrm{d}_{A_T} u_T\|^2 + (1 + c_3) e^{-\kappa T}/\kappa \\ &\leq C' \int_{S^1} \|\mathrm{d} u_T\|_A^2 \end{aligned}$$

where $C' = \max(1 + c_3, C)$. Hence

$$\begin{aligned} E_A(u;T) &= D_{A_T}(u_T) - D_{A_{-T}}(u_{-T}) \\ &\leq C' \int_{S^1 \times \{T, -T\}} \|\mathrm{d} u_{\pm T}\|_A^2 \\ &= -C' \frac{\mathrm{d}}{\mathrm{d}T} E_A(u;T). \end{aligned}$$

Thus

$$\frac{d}{dt} E_A(u;T) \leq -E_A(u;T)/C'$$

which implies

$$E_{A,c}(u;T) \leq C \exp(-\kappa_1 T) E_A(u;0)$$

for some constant C and $\kappa_1 = 1/C'$ which proves the third claim. Applying the a priori estimate in Theorem 4.1.14 completes the proof of the second claim. □

4.3. **Removal of singularities.** We use the exponential decay results of the previous sections to prove a removal of singularities theorem for vortices (A, u) on the punctured disk. The result is not a full removal of singularities theorem: we assume that the connection has a C^0 extension. If we knew that the connection had a smooth extension, then removal of singularities would be a straightforward consequence of removal of singularities for pseudoholomorphic maps, viewing u as a pseudoholomorphic map to $P(X)$. However, Uhlenbeck compactness only provides a C^0 limit to the connection, so we have no control over the higher derivatives.

Proposition 4.3.1. *Suppose that (A, u) is a smooth finite energy vortex on the punctured disk $B - \{0\}$, and that A admits a C^0 extension over B. Then u admits a C^0 extension over B, and the pair (A, u) is gauge equivalent to a smooth vortex on B.*

Sketch of proof. Removal of singularities for pseudoholomorphic maps is [13, Lemma 4.5.1]. The proof for vortices is similar: one first shows that u admits a $W^{1,p}$ extension for $p > 2$ with $2-p$ sufficiently small. Elliptic regularity then implies that the pair (A, u) is gauge equivalent to a smooth vortex.

Suppose that the mean value inequality of Proposition 4.1.14 holds for all balls of energy at most C. Choose r_0 so that $E(A, u; B_{2r_0}) < C$. For $0 < r \leq r_0$ let

$$\epsilon(r) = E(A, u; B_r)$$

denote the energy of the restriction of (A, u) to B_r. Then ϵ extends to a continuous function on $[0, r_0]$ with $\epsilon(0) = 0$. By the mean value inequality 4.1.14

$$|\nabla_A u(re^{i\theta})|^2 \leq C\epsilon(2r).$$

This implies that the length of $\gamma_r(\theta) = u(re^{i\theta})$ satisfies

$$\ell(\gamma_r) \leq \sqrt{C\epsilon(2r)}$$

which goes to zero as $r \to 0$. From the isoperimetric inequality (**??**) (for trivial twisting $g_0 = 1$) we obtain

$$E(A, u; B_r) \leq c\ell(\gamma_r)^2.$$

Hence

$$\begin{aligned}\epsilon(r) &\leq c\ell(\gamma_r)^2 \\ &\leq \pi c r^2 \int_{S^1} |\nabla_A u(re^{i\theta})|^2 \\ &= 2\pi c r \frac{d}{dr}\epsilon(r).\end{aligned}$$

Integrating from r to r_1 gives for $\mu = 1/2\pi c$

$$\epsilon(r) \leq c_1 r^\mu.$$

So

$$|\mathrm{d}u(\rho e^{i\theta})|^2 \leq C\rho^{-2}\epsilon(2\rho) \leq C\rho^{2\mu-2}.$$

Hence for p with $2 < p < 2/(1-\mu)$

$$\int_{B_r} |\nabla_A u|^p \leq C \int_0^r \rho^{1-p(1-\mu)} \mathrm{d}\rho < \infty.$$

It follows as in [13] that u lies in $W^{1,p}(B_r)$. Now elliptic regularity Proposition 3.9.1 implies that (A, u) is gauge equivalent to a smooth solution. □

5. Compactness

5.1. **Vortices with bounded first derivative.** The following extends compactness for bounded first derivative in Cieliebak-Gaio-Mundet-Salamon [4] to the case with cylindrical ends:

Theorem 5.1.1. *Let (A_α, u_α) be a sequence of vortices with bounded energy $E(A_\alpha, u_\alpha)$. If $\mathrm{d}u_\alpha$ is bounded in C^0 on compact sets, then there exists a smooth vortex (A_∞, u_∞) such that after gauge transformation and passing to a subsequence (A_α, u_α) converges to (A_∞, u_∞) uniformly in all derivatives on compact sets.*

Proof. Let (A_α, u_α) be a sequence as above. The vortex equation and bound on Φ give a pointwise bound on the curvature,

$$\|F_A\| < Ce^{\kappa_\Sigma r_\Sigma}.$$

This implies an L^2 bound for the curvature, see Proposition 4.2.10. Uhlenbeck compactness implies that there is a subsequence (still denoted) A_α and a sequence of gauge transformations $g_\alpha \in \mathcal{G}(P)$ such that $g_\alpha \cdot A_\alpha$ converges weakly to a connection A_∞ in the local Sobolev topology $W_{m,p,\mathrm{loc}}$ and strongly in the C^0-topology (see Wehrheim [21, Theorem A']). By hypothesis the sequence $g_\alpha \cdot u_\alpha$ is bounded in $W_{m,p,loc}$ the theorems Alaoglu and Rellich, after passing to a subsequence, $g_\alpha \cdot u_\alpha$ converges weakly in $W_{m,p,loc}$ and strongly in C^0 to a section u_∞. Since F_{A_α} converges to F_{A_∞} and $\overline{\partial}_{J_{A_\alpha}} u_\alpha$ converges to $\overline{\partial}_{J_{A_\infty}} u_\infty$ both weakly in $W_{0,p,\delta}$, the pair (A_∞, u_∞) is a weak solution to the vortex equations in $W_{m,p,loc}$. By Proposition **??**, after gauge transformations, we can assume that (A_α, u_α) converges to (A_∞, u_∞) in $W_{m,p,\delta}$, and by Proposition 3.9.1 we may gauge transform so that (A_∞, u_∞) is actually smooth.

To show convergence in all derivatives on compact sets, we use the bootstrapping method of the proof of 3.9.1, as follows. By Coulomb gauge, there is a sequence of gauge transformations $g_\alpha \in \mathcal{G}_{m+1,p,loc}$ such that $d^*_{A_\infty}(g_\alpha A_\alpha - A_\infty) = 0$. The sequence $g_\alpha A_\alpha$ has also L^2-bounded curvature and thus a subsequence, still denoted by $g_\alpha A_\alpha$, converges to A_∞ weakly in $W_{m,p,loc}$ and strongly in C^0. g_α is uniformly bounded in $W_{m+1,p,loc}$ and converges to an element $g \in \mathcal{G}_{m+1,p,loc}$ strongly in $W_{m,p,loc}$ and weakly in $W_{m+1,p,loc}$. By Proposition **??** a subsequence $g_\alpha \cdot u_\infty$ converges to $g \cdot u_\infty$ strongly in C^0 and weakly in $W_{m,p,\delta}$, and $g \cdot A_\infty = A_\infty$. Using equation (20), one sees that the convergence of A_α gives the convergence of almost complex structures J_{A_α} in $W_{m,p,loc}$. In particular, for every small open set U of compact support, this sequence is bounded uniformly, that is there is a constant c_0 such that, $\|J_{A_\alpha}\| \le c_0$, for all α. Now, by [4, Lemma 3.3] we have a constant c, depending on c_0 and U so that, the sequence u_α of J_{A_α}-holomorphic sections is also bounded in $W_{m+1,p,loc}$, and by Proposition **??** we can assume it is also bounded in $W_{m+1,p,\delta}$. The sequences $a_\alpha := A_\alpha - A_\infty, u_\alpha$ are bounded in $W_{m,p,\delta}$, and they satisfy

$$d^*_{A_\infty}(a_\alpha) = 0, F_{A_\infty} + d_{A_\infty} a_\alpha + [a_\alpha, a_\alpha] + \omega_\Sigma (u_\alpha)^* P(\Phi) = 0.$$

Therefore, $d_{A_\infty}(a_\alpha)$ is bounded in $W_{m,p,\delta}$ as well as $d^*_{A_\infty}(a_\alpha)$, elliptic regularity shows that a_α is bounded in $W_{m+1,p,\delta}$. By passing to a subsequence if necessary, we can now assume that (A_α, u_α) converges in $W_{m+1,p,\delta}$. Continuing this process we get the convergence on compact sets in all derivatives up to gauge transformation. □

The case when the sequence $\mathrm{d}u_\alpha$ is not bounded on compact sets in the C^0 norm yields the existence of bubbles. More formally, we introduce the space of stable vortices which will be the proper compactification for the moduli space of vortices.

5.2. **Nodal Vortices.** Let Σ be a connected, oriented surface with n cylindrical ends.

Definition 5.2.1. A *combinatorial type* consists of a rooted tree Γ together with a

(a) partition of the vertices $\mathrm{Vert}(\Gamma) = \{0\} \cup \mathrm{Vert}(\Gamma)_C \cup \mathrm{Vert}(\Gamma)_s$ where vertex 0 is the *root vertex*. The vertices $\mathrm{Vert}(\Gamma)_C$ resp $\mathrm{Vert}(\Gamma)_s$ are *cylindrical* resp. *spherical*;
(b) a labelling of the cylindrical vertices by $\{1, \ldots, n\}$;
(c) a bijection from the set of semiinfinite edges of Γ to $\{1, \ldots, n\}$;

satisfying the condition that for each $i \in \{1, \ldots, n\}$, the vertices labelled i should form a linear string connected adjacent to the root vertex to the edge for the i-th marking.

In Figure 1, the root resp. cylindrical resp. spherical vertices are black resp. grey resp. white. The three seminfinite edges are labeled $1, 2, 3$ and the single cylindrical vertex connects the root vertex with the third semininfinite edge.

Figure 1. Example of combinatorial type

The set of spherical vertices $\mathrm{Vert}(\Gamma)_s$ admits a partition

$$\mathrm{Vert}(\Gamma)_s = \mathrm{Vert}(\Gamma)_{\Sigma,0} \cup \mathrm{Vert}(\Gamma)_{\Sigma,\infty}$$

depending on whether they are connected to the root vertex by a path of spherical vertices or not. We call the second set the *spherical vertices at infinity*.

Definition 5.2.2. Let Γ be a combinatorial type. A *marked nodal curve with cylindrical ends of combinatorial type* Γ *with principal component* Σ is a nodal curve $\underline{\Sigma}$ consisting of components $\Sigma_0, \ldots, \Sigma_k$ and nodes $\{\{w_1^-, w_1^+\}, \ldots, \{w_m^-, w_m^+\}\}$ such that graph obtained by replacing components with vertices and nodes/markings with edges is the graph underlying Γ. A component is a *cylindrical* resp. *spherical* if the corresponding vertex is cylindrical resp. spherical.

Each cylindrical component Σ_i has two distinguished special points, given by the connecting points in the chain of cylinders connecting the root vertex with the marking, and hence a distinguished isomorphism $\Sigma_i \to \mathbb{P}^1$ mapping the two special points to $0, \infty$.

FIGURE 2. Example of a nodal curve with cylindrical ends

Definition 5.2.3. Let Γ be a combinatorial type, $P \to \Sigma$ a principal G-bundle, $J \in \mathcal{J}(P, X)$ and $\mathcal{H}(P, X)$ compatible almost complex structures and Hamiltonian perturbations respectively. A *nodal* (J, H)*-vortex* of combinatorial type Γ is a datum $(\underline{\Sigma}, \underline{A}, \underline{u}, \underline{z})$ consisting of

(a) a nodal curve $\underline{\Sigma}$ of combinatorial type Γ;
(b) a (J, H)-perturbed vortex on the principal component $\Sigma \cong \Sigma_0$.
(c) For each $i \in \mathrm{Vert}(\Gamma)_C$, a flat J-holomorphic vortex (A_i, u_i) on the cylinder $\Sigma_i - \{0, \infty\} \cong S^1 \times \mathbb{R}$ with holonomy given by the limiting holonomy of A_0 on the l-th end, where l is the label of i.
(d) For each $j \in \mathrm{Vert}(\Gamma)_s$ a $J_{\hat{z}_j}$-holomorphic sphere $v_j; \Sigma_j \to P(X)_{\hat{z}_j}$ in some fiber of $P(X)$;

satisfing matching conditions $\underline{u}(w_j^-) = \underline{u}(w_j^+)$ at the nodes $w_j^{\pm}, j = 1, \ldots, m$. We call the pair (Σ_j, v_j) a *spherical bubble* and the triple (Σ_i, A_i, u_i) a *cylinder bubble*. A bubble is *trivial* if the map is covariant constant on the bubble. A nodal vortex is *polystable* if each trivial bubble has at least three special points; that is, each cylinder bubble on which u is trivial is attached to at least one sphere bubble, and each sphere bubble has at least three special points. A nodal vortex is *stable* if it has finite automorphism group.

Definition 5.2.4. An *isomorphism* of polystable vortices $(\underline{A}_\alpha, \underline{u}_\alpha), \alpha = 0, 1$ consists of

(a) a gauge transformation over the principal component
(b) automorphisms of the domains of the bubbles preserving the special points

such that the action of the gauge transformation on the principal component and on the bundles over the fibers by evaluation at the special points, together with the automorphisms of domains, transforms $(\underline{A}_0, \underline{u}_0)$ to $(\underline{A}_1, \underline{u}_1)$.

Let $M_\Gamma(P, X, \underline{\mu})$ denote the moduli space of isomorphism classes of polystable vortices of combinatorial type Γ, and $M_\Gamma^{\rm fr}(P, X, \underline{\mu})$ the moduli space of isomorphism classes of framed polystable vortices. As in the case with smooth domain, $M_\Gamma^{\rm fr}(P, X, \underline{\mu})$ admits an evaluation map

$$\mathrm{ev}^{\rm fr} : M_\Gamma^{\rm fr}(P, X, \underline{\mu}) \to X^{n,\underline{\mu}} := \prod_{i=1}^n X^{\mu_i}.$$

If $M_\Gamma^{\rm fr}(P, X, \underline{\mu}) \to M_\Gamma(P, X, \underline{\mu})$ is a principal $G_{\underline{\mu}}^n$ bundle with a classifying map then combining this with the evaluation map defines

$$\mathrm{ev} : M_\Gamma(P, X, \underline{\mu}) \to X_{G_{\underline{\mu}}}^{n,\underline{\mu}}.$$

Let $\overline{M}(P, X, \underline{\mu})$ denote the union over combinatorial types

$$\overline{M}(P, X, \underline{\mu}) = \bigcup_\Gamma M_\Gamma(P, X, \underline{\mu}).$$

For any nodal curve $\underline{\Sigma}$, be denote by $Z_i \subset \Sigma_i$ the set of nodal points in Σ_i, that is, points at which other components of $\underline{\Sigma}$ are attached.

Definition 5.2.5. Suppose that (A_α, u_α) is a sequence of vortices on Σ and $(\underline{A}, \underline{u})$ is a polystable vortex. We say that (A_α, u_α) *Gromov converges* to $(\underline{A}, \underline{u})$ if there exist a sequence g_α of gauge transformations such that

(a) $g_\alpha A_\alpha$ converges uniformly to A on compact subsets of Σ_0;
(b) $u_{0,\alpha}$ converges to u_∞ uniformly on compact subsets of the complement of $Z_0 \subset \Sigma_0$ of $\underline{u}$;
(c) for every bubble component Σ_i of $\underline{\Sigma}$, there exists a sequence $\epsilon_{i,\alpha} \to \infty$ and maps $\phi_{i,\alpha} : \Sigma_i - B_{\epsilon_{i,\alpha}(w_i)} \to \Sigma$ such that $u_\alpha \circ \phi_{i,\alpha}$ converges uniformly on compact subsets of the complement of $Z_i \subset \Sigma_i$ to $u_{i,\infty}$.
(d) for any $w_j \in Z_j$, the energy lost

$$m(w_j) := \lim_{\epsilon \to 0} \lim_{\alpha \to \infty} E(u_\alpha \circ \phi_{i,\alpha}; B_\epsilon(z_j))$$

is equal to the sum of the energies on the components of $\underline{u}$ attached to w_j.
(e) for any $w_j \in Z_j$, $\phi_{\alpha,i}^{-1} \circ \phi_{\alpha,j}$ converges to w_j uniformly on compact sets in a neighborhood of w_i.
(f) if z_i is contained in Σ_j, then $z_i = \lim_{\alpha \to \infty} \phi_{j,\alpha}^{-1}(z_{i,\alpha})$.

A sequence $(A_\alpha, \underline{u}_\alpha)$ of polystable vortices Gromov-converges to a polystable vortex $(A, \underline{u})$ if there exists a sequence of contractions on the trees $\Gamma_\alpha \to \Gamma$, and properties similar to those above. The definition is similar to that in [13, Section 5.5] and omitted.

A subset C of $\overline{M}(P,X,\underline{\mu})$ is *Gromov closed* if any Gromov convergent sequence in C has limit point in C, and *Gromov open* if its complement is closed. This induces a topology in $\overline{M}(P,X,\underline{\mu})$.

Theorem 5.2.6. *$\overline{M}(P,X,\underline{\mu})$ is compact and Hausdorff. Furthermore, any convergent sequence is Gromov convergent.*

The proof takes up the rest of the section.

5.3. **Energy quantization.** The following is an energy quantization result for bubbles in the fibers of $P(X)$:

Lemma 5.3.1. *There exists a constant $\hbar > 0$ such that any non-constant holomorphic map $u : \mathbb{P}^1 \to P(X)$ having values in the fiber $P(X)$ has energy at least $\hbar$.*

Proof. By the standard result [13], since each fiber is equivariantly isomorphic (non-canonically) to X. □

Next we prove energy quantization of cylindrical bubbles with flat connections.

Proposition 5.3.2. *For any conjugacy class $\mathcal{C} \subset G$, there is a constant $\epsilon = \epsilon(\Sigma, \mathcal{C})$ such that any zero-area vortex (A,u) on the cylinder with holonomy in $\mathcal{C}$ and energy $E(A,u) < \epsilon(\Sigma,\mathcal{C})$ is trivial in the sense that u is covariant constant.*

We reduce this to energy quantization for holomorphic strips with Lagrangian boundary conditions.

Proposition 5.3.3. *Let (X,ω) be a compact symplectic manifold and $L_+, L_- \subset X$ compact Lagrangian submanifolds intersecting cleanly and $J \in \mathcal{J}(X)$ a tame almost complex structure. There exists a constant $\epsilon = \epsilon(L_+,L_-,J)$ such that if $u : \mathbb{R}\times[-1,1] \to X$ is a J-holomorphic map satisfying the boundary conditions $u(\mathbb{R}\times\{\pm 1\}) \subset L_\pm$ and the energy estimate $E(u) < \epsilon$. then u is trivial.*

For the proof of Proposition 5.3.3 we need the following result. If X is a manifold and $Y \subset X$ is a submanifold, we denote by $N_X Y$ the normal bundle of Y in X.

Lemma 5.3.4. *Let X be a manifold, $i_\pm : Y_\pm \to X$ submanifolds intersecting cleanly, and $\omega \in \Omega(X)$ a closed differential form with $i_\pm^*\omega = 0$. Then there exists a neighborhood U of $Y_+ \cap Y_-$ and a form $\alpha \in \Omega(U)$ such that $\mathrm{d}\alpha = \omega|_U$ and $i_\pm^*\alpha = 0$.*

Proof. By the local model theorem for clean intersections [10, Proposition C.3.1], there exists a neighborhood U of $Y_+\cap Y_-$ in X and a diffeomorphism φ of U to a neighborhood $\varphi(U)$ of the zero section in $N_x(Y_+ \cap Y_-)$ such that

$$\varphi(U \cap Y_\pm) = \varphi(U) \cap N_{Y_\pm}(Y_+ \cap Y_-).$$

In particular, $\varphi(U \cap Y_\pm)$ is a sub-bundle of $N_X(Y_+ \cap Y_-)$. Scalar multiplication on the fibers of $N_X(Y_+ \cap Y_-)$ defines a strong deformation retraction to $Y_+ \cap Y_-$,

$$\psi : [0,1] \times U \to U, \quad (\lambda, x) \mapsto \varphi^{-1}(\lambda\varphi(x)).$$

Let $V_t \in \mathrm{Vect}(U)$ be the time-dependent vector field generating ψ,

$$V_t = \frac{d}{dt}\psi(x,t).$$

The Poincaré formula

$$\alpha = \int_0^1 \psi_t^* \iota(V_t)\omega \mathrm{d}t$$

produces the required primitive since

$$\begin{aligned} \mathrm{d}\alpha &= \int_0^1 \psi_t^* L_{V_t}\omega \mathrm{d}t \\ &= \int_0^1 \frac{d}{dt}\psi_t^*\omega \mathrm{d}t \\ &= \psi_1^*\omega - \psi_0^*\omega \\ &= \omega. \end{aligned}$$

The restriction of α to $Y_\pm$ is

$$\begin{aligned} i_\pm^*\alpha &= \int_0^1 \psi_t^* i_\pm^* \iota(V_t)\omega \mathrm{d}t \\ &= \int_0^1 \psi_t^* \iota(V_t)\iota_\pm^*\omega \mathrm{d}t \\ &= 0 \end{aligned}$$

since V_t is tangent to $Y_\pm$. The Lemma follows. □

Proof of Proposition 5.3.3. We set $Y_\pm := L_\pm$ and denote by $i_\pm : Y_\pm \to X$ the inclusions. Since $L_\pm$ is Lagrangian the hypotheses of Lemma 5.3.4 are satisfied. So let U and α be as in the conclusion of that lemma. Since X, L_+, L_- are compact, there exists a constant $\delta > 0$ such that if $x \in X$ then

$$\max(\mathrm{dist}(x, L_+), \mathrm{dist}(x, L_-)) < \delta \implies x \in U. \tag{77}$$

By the apriori lemma for holomorphic maps with Lagrangian boundary conditions [13, Lemma 4.3.1] there exists a constant $\epsilon_0 > 0$ such that if $u : \mathbb{R} \times [-1,1] \to X$ is a holomorphic map satisfying $u(\mathbb{R} \times \{\pm 1\}) \subseteq L_\pm$ and $E(u) \le \epsilon_0$ then

$$|du(z)|^2 \le \frac{8}{\pi}\int_{B_2(s,\pm 1)\cap\mathbb{R}\times[-1,1]} |du|^2, \tag{78}$$

for every $s \in \mathbb{R}$ and $z \in \bar{B}_1(s, \pm 1) \cap \mathbb{R} \times [-1,1]$. Let u be such a map. It follows that

$$\|\partial_t u(s,\cdot)\|_{L^1([-1,1])} \le 2\sqrt{\frac{8}{\pi}E\big(u, [s-2, s+2] \times [-1,1]\big)}, \tag{79}$$

for every $s \in \mathbb{R}$. We define

$$\epsilon := \min \big\{\epsilon_0, \frac{\pi}{32}\delta^2\big\}. \tag{80}$$

Assume that $E(u) \leq \epsilon$. Fix $(s,t) \in \mathbb{R} \times [-1,1]$. Then

$$\begin{aligned} d\big(u(s,-1), u(s,t)\big) &\leq \int_{-1}^{t} |\partial_t u(s,t')|\, dt' \\ &\leq 2\sqrt{\frac{8}{\pi}E(u)} < \delta. \end{aligned} \tag{81}$$

Here in the second inequality we used (79) and in the last inequality we used (80). It follows that $\mathrm{dist}(u(s,t), L_-) < \delta$. Similarly, we have $\mathrm{dist}(u(s,t), L_+) < \delta$. By (77) we have

$$u(s,t) \in U. \tag{82}$$

Hence, denoting $E(s) := E\Big(u, \big((-\infty,-s) \cup (s,\infty)\big) \times [-1,1]\Big)$, we have for every $s \geq 2$

$$\begin{aligned} E(u) &= \int_{[-s,s]\times[-1,1]} u^*\omega + E(s) \\ &= \int_{\{s\}\times[-1,1]} u^*\alpha - \int_{\{-s\}\times[-1,1]} u^*\alpha + 0 + E(s) \\ &\leq \Big(\|\partial_t u(s,\cdot)\|_{L^1([-1,1])} + \|\partial_t u(-s,\cdot)\|_{L^1([-1,1])}\Big)\|\alpha\|_{L^\infty(X)} + E(s). \end{aligned}$$

Here in the second step we used the fact $d\alpha = \omega|_U$, Stokes' theorem and the fact $i_\pm^*\alpha = 0$. By (79) the last expression tends to 0, as $s \to \infty$. It follows that $E(u) = 0$. This proves Proposition 5.3.3. □

Proof of Proposition 5.3.2. We choose a representative $g \in \mathcal{C}$ and define $\tilde{X} := X \times X$ with the symplectic form $\tilde{\omega} := -\omega \oplus \omega$ and the Lagrangian submanifolds

$$L_- := \big\{(x,x) \,\big|\, x \in X\big\}, \quad L_+ := \big\{(x, g^{-1}x) \,\big|\, x \in X\big\}.$$

Let (A,u) be a zero-area vortex on P; we view u as an equivariant map from P to X and A as a $\mathfrak{g}$-valued one-form on P. The pair (A,u) defines a J-holomorphic strip $\tilde{u}$ with boundary in $L_\pm$ as follows. Let $\pi : P \to \mathbb{R} \times S^1$ denote the projection and $\rho : \mathbb{R} \times [-1,1] \to \mathbb{R} \times S^1$ the be map $\rho(s,t) := (s, e^{\pi i t})$. Since A is flat and the conjugacy class of the holonomy around the circle $\{0\} \times S^1$ equals $\mathcal{C}$, there exists a map $f : \mathbb{R} \times [-1,1] \to P$ solving the equations

$$A(\partial_s f) = 0, \quad A(\partial_t f) = 0, \quad f(s,1) = f(s,-1)g, \quad \pi \circ f = \rho$$

for every $s \in \mathbb{R}$. Let $\tilde{X} = X \times X$ denote the Cartesian product equipped with the almost complex structure $\tilde{J} = -J \times J$. Let $\tilde{u} : \mathbb{R} \times [-1,1] \to X \times X$ denote the $\tilde{J}$-holomorphic map

$$\tilde{u}(s,t) = ((u \circ f)(s/2, -(t+1)/2), (u \circ f)(s/2, (t+1)/2)).$$

Then $\tilde{u}$ satisfies the boundary conditions $\tilde{u}(s, \pm 1) \in L_\pm$ and its energy is $E(\tilde{u}) = E(u, A)$. Since G is compact, X admits a G-invariant metric. Therefore, the hypothesis of Lemma 4.1.12 are satisfied and hence the intersection $L_- \cap L_+$ is clean. Therefore it follows by Proposition 5.3.3 that if $E(\tilde{u})$ has energy less than $\epsilon(L_-, L_+, \tilde{J})$ then $\tilde{u}$ is trivial, and hence u is covariant constant. This proves Proposition 5.3.2. □

5.4. **Proof of compactness.** Suppose that (A_α, u_α) is a sequence of vortices.

Definition 5.4.1. $z \in \Sigma$ is a *bubble point* for the sequence (A_α, u_α) if there exists a convergent sequence $z_\alpha \to z$ such that $\mathrm{d}_{A_\alpha} u_\alpha(z_\alpha) \to \infty$.

Proposition 5.4.2. *If (A_α, u_α) is a sequence of vortices whose energy is bounded by C, then there is a finite set of bubbling points $Z \in \Sigma$, and a vortex (A_∞, u_∞) on Σ such that a subsequence still denoted (A_α, u_α), converges after gauge transformations to (A_∞, u_∞) on compact sets of $\Sigma \backslash Z$ in all derivatives.*

Proof. To show this, for a bubbling point s, $\lim_{z_\alpha \to s} |\mathrm{d}_{A_\alpha} u_\alpha(z_\alpha)| = \infty$. Let $\epsilon > 0$ small enough so that a neighborhood of s is consider as an open set in $\mathbb{C}$. By Hofer's lemma [13, 4.6.4] to the function $z \mapsto |d_{A_\alpha} u(z)|$ for $|z - s| < \epsilon$, the points z_α and the constants $\delta_\alpha := |d_{A_\alpha} u_\alpha(z_\alpha)|^{-1/2}$, there exist sequences $\zeta_\alpha \in \Sigma$, $\epsilon_\alpha > 0$ such that

$$\zeta_\alpha \to s; \quad \sup_{|z - \zeta_\alpha| < \epsilon_\alpha} |d_{A_\alpha} u_\alpha| \leq 2c_\alpha; \quad \epsilon_\alpha \to 0; \quad \epsilon_\alpha c_\alpha \to \infty,$$

where $c_\alpha := |d_{A_\alpha} u_\alpha(\zeta_\alpha)|$. Let ψ_α be the sequence defined on the ball $B_{\epsilon_\alpha c_\alpha}(0)$ given by $\psi_\alpha(z) := (\zeta_\alpha + z/c_\alpha)$. This sequence converges to s uniformly in compact sets. The rescaled sequence

$$v_\alpha(z) := u_\alpha(\psi_\alpha(z))$$

has uniformly bounded first derivative on compact sets, on arbitrarily large domains, since the sequence $\epsilon_\alpha c_\alpha \to \infty$. The maps v_α are pseudoholomorphic with respect to the almost complex structure determined by the re-scaled connections $c_\alpha^{-1} A_\alpha$, which have uniformly bounded curvature. Using Theorem 5.1.1 we get, after gauge transformations, that the sequence $(c_\alpha^{-1} A_\alpha, v_\alpha)$ converges uniformly to a pair (A_0, v) on compact sets, in all derivatives. Note that the limit A_0 is necessarily the trivial connection, since $c_\alpha \to \infty$, and that the map v necessarily lies on the fibre s,

$$v : \mathbb{C} \to P(X)_s$$

since standard removal of singularities for finite energy maps implies that v extends to a smooth J-holomorphic map $v : \mathbb{P}^1 \to P(X)_s$, where J is the given almost complex structure on X, and v is non constant.

By energy quantization, sphere bubbles can develop at most at finitely many points Z. On the complement $\Sigma - Z$, $d_\alpha u_\alpha$ is uniformly bounded in compact sets and (A_α, u_α) has bounded energy, then by Theorem 5.1.1 A_α converges to a connection A_∞ on compact subsets of Σ and u_α converges to a section u_∞ on compact subsets of $\Sigma - Z$ in all derivatives, so that (A_∞, u_∞) is a solution to the vortex equations on $\Sigma - Z$. Since u_∞ is a finite energy pseudoholomorphic map from Σ to $P(X)$, removal of singularities implies that it extends to all of Σ. (A_∞, u_∞) is the *principal component* of the limiting sequence. □

Let $s \in \Sigma$ be a bubbling point, and v_α the sequence obtained by re-scaling as in the proof of Proposition 5.4.2. Fix a trivialization of P in a neighborhood of s. Let

$$m(s) := \lim_{\epsilon \to 0} \lim_{\alpha} (u_\alpha; B_\epsilon(s))$$

be the energy of the sequence being captured at s. By choosing a subsequence, this limit exists since the energy is uniformly bounded. Since bubbling occurs near s, we can restrict the sequence u_α to the ball $B_R(s)$ for $R > 0$. Moreover, let $\zeta'_\alpha \in B_R(s)$ be the point where the function $d_{A_\alpha} u_\alpha$ attains its supremum on $B_R(s)$. The section $\tilde{u}_\alpha := u_\alpha(z + \zeta'_\alpha)$ is such that attain its sup at $z = 0$ and thus the sequence $\tilde{u}_\alpha$ has a bubbling point at $z = 0$, since the sequence $\zeta'_\alpha \to s$.

Definition 5.4.3. The sequence obtained by *soft rescaling* is the sequence $v_\alpha(z) = u_\alpha(\epsilon_\alpha z)$ where the rescaling constants ϵ_α are defined so that

$$E(\tilde{u}_\alpha; B_{\epsilon_\alpha}(0)) = m(s) - h/2, \tag{83}$$

where $h < \min\{\hbar/2, c_2/2\}$ where $\hbar$ is the energy quantization constant and c_2 is the constant guaranteeing exponential decay in Theorem 4.2.10.

For any $\epsilon > \epsilon_\alpha$ the energy of $\tilde{u}_\alpha$ on the annulus $B_\epsilon(0) \backslash B_{\epsilon_\alpha}(0)$ is at the most $\delta/2$ and then there is not enough energy to form another bubble in this annulus. Let $\psi_\alpha(z) := \zeta'_\alpha + \epsilon_\alpha z$ and let $(A'_\alpha(z), v_\alpha(z)) := (A_\alpha(\psi_\alpha), u_\alpha(\psi_\alpha))$ denote the rescaled sequence defined on the ball $B_{R/\epsilon_\alpha}(0)$.

Proposition 5.4.4 (Bubbles connect)**.** *Consider the sequence v_α constructed by soft rescaling above.*

(a) *There is a finite energy pseudoholomorphic map $v : \mathbb{C} \to P(X)_s$ and a finite set of points $Z_1 \subset \mathbb{C}$ so that the sequence v'_α converges uniformly in all derivatives on $\mathbb{C} \backslash Z_1$ to v. By removal of singularities v extends to $\mathbb{P}^1$.*
(b) *Let $m_j := \lim_{\epsilon \to 0} \lim_\alpha E(v_\alpha; B_\epsilon(s_j))$ is the energy being captured by the bubbling point $s_j \in Z_1$, then*
$$m_0 := E(v) + \sum_{s_j \in Z_1} m_j.$$
Therefore, there is no other possible bubbles forming at s, only at the points $s_j \in Z_1$.
(c) *We have $u(z) = v(\infty)$ in $P(X)_z$.*

Proof. This sequence by construction has uniformly bounded energy. By the previous discussion, there exists a finite set $Z_1 \subset \mathbb{C}$, and a vortex (A'_0, v) such that a subsequence still denoted (A'_α, v_α) converges uniformly on compact sets in all derivatives to (A'_0, v). Since $\epsilon_\alpha \to 0$, A' is necessarily the trivial connection. Also note that the function $|dv(z)|$ has its maximum at 0, thus $0 \in Z_1$. v has finite energy, and since $\delta_\alpha \to 0$, it is defined on arbitrarily big subsets of $\mathbb{C}$. Removal of singularities shows that it extends to a J-holomorphic map $v : \mathbb{P}^1 \to P(X)_s$, with J exactly the almost complex structure on X, since the connection A' is trivial. Now, recall that by Step 1, a subsequence of (A_α, u_α) converges in all derivatives on compact sets of $\Sigma \backslash Z$ to the principal component vortex (A_∞, u_∞). The pair (A_0, v) is the *first bubble* that appears attached to the principal component.

The rest of the proposition is as in the proof of [13, 4.7.1], using that the exponential decay for the energy on annuli near s for the sequence u_α of J_{A_α}-holomorphic curves proved in Proposition 4.2.10.

Let $m(s)$ be the energy lost at the point s. Note that we do not need to distinguish the Yang-Mills-Higgs energy with the twisted energy $E_{A_\alpha}(u_\alpha; B_\epsilon(s))$ here, since the energy of the connection on $B_\epsilon(s)$ approaches zero. That is, we also have

$$m(s) = \lim_{\epsilon \to 0} \lim_{\alpha \to \infty} E_{A_\alpha}(u_\alpha; B_\epsilon(s)).$$

Note that $\lim_{R \to \infty} \lim E_{A_\alpha}(u_\alpha; B_{R\epsilon_\alpha}) = m(s)$, since otherwise it would exist a subsequence still denoted by u_α and a constant $\rho > 0$ such that for $R \geq 1$,

$$\lim_\alpha E_{A_\alpha}(u_\alpha; B_{R\epsilon_\alpha}) \leq m(s) - \rho.$$

Thus for $R > 1$, the energy in the annuli satisfies $\lim_\alpha E_{A_\alpha}(u_\alpha; A(\epsilon_\alpha, R\epsilon_\alpha)) \leq c_2/2 - \rho$, which is a contradiction.

To finish the proof, note that $E_{A_\alpha}(u_\alpha; B_{\epsilon_\alpha}(s)) = E_{A_\alpha}(v_\alpha; B_1(s)) = m_0 - h/2 \geq m_0 - \hbar/2$, and both sequences u_α, v_α capture energy $m(s)$ at s. This shows that $Z_1 \subset B_1(s)$, that is all bubbling points of v_α are in the unit ball of s. Then, for all balls $B_\epsilon(s) \subset B_1(s)$ we have

$$\begin{aligned} m(s) &= \lim_{R \to \infty} \lim_\alpha E_{A_\alpha}(v_\alpha; B_R(s)) \\ &= \lim_{R \to \infty} \lim_\alpha E_{A_\alpha}(v_\alpha; B_R(s) \backslash B_\epsilon(s)) + \lim_\alpha E_{A_\alpha}(v_\alpha; B_\epsilon(s)) \\ &= \lim_{R \to \infty} E(v; B_R(s) \backslash B_\epsilon(s)) + \lim_\alpha E_{A_\alpha}(v_\alpha; B_\epsilon(s)) \\ &= E(v; \mathbb{C} \backslash B_\epsilon(s)) + \lim_{\rho \to 0} \lim_\alpha E_{A_\alpha}(v_\alpha; B_\epsilon(s) \backslash \cup_{j \in Z_1} B_\rho(s_j)) + \sum_{s_j \in Z_1} m(s_j) \\ &= E(v) + \sum_{s_j \in Z_1} m(s_j). \end{aligned}$$

By definition, there exists a sequence $\kappa_\alpha \to 0$ such that

$$\lim_{\alpha \to \infty} E_{A_\alpha}(u_\alpha; B_{\kappa_\alpha}(s)) \to m(s).$$

$[\log(\delta_\alpha), \log(\kappa_\alpha)] \times S^1$. By the uniform mean value inequality (4.1.14), on the subset $[\log(\delta_\alpha) + 1, \log(\kappa_\alpha) - 1] \times S^1$ the twisted derivatives $\mathrm{d}_{A_\alpha} u_\alpha$ are uniformly bounded. The exponential decay lemma 4.2.10 shows exponential decay of the energy on this region. Recall from [13, p. 103] that since there is not enough energy for bubbling, the energy on the outer region of the annulus must approach zero. Hence the energy density on the annulus is controlled by the energy on the inner region.

Since the connections A_α are already in Coulomb gauge with respect to the trivial connection, exponential decay of the distance. We have

$$u_\infty(z) = \lim_{\alpha\to\infty}(u_\alpha(\kappa_\alpha)), \quad v_\infty(\infty) = \lim_{\alpha\to\infty}(u_\alpha(\delta_\alpha))$$

since there is not enough energy on the annulus $A(\delta_\alpha, \kappa_\alpha)$ for further bubbling.

Hence

$$\mathrm{dist}(v_\infty(\infty), u_\infty(0)) \leq \lim_{\alpha\to\infty} \mathrm{dist}(u_\alpha(\kappa_\alpha), u_\alpha(\delta_\alpha)) = 0$$

by exponential decay. □

We can now inductively construct the spherical components of a *stable* holomorphic section on the fibre $P(X)_s$ by applying the previous step to all points $s_j \in Z_1$. All bubble maps are holomorphic with respect to the trivial connection, and thus one can just think of lines in X. This process stops since the energy m_0 is finite. The case when Z has more than one element is left to the reader.

It remains to construct the bubble trees attached to the cylindrical ends. If $\sup_\alpha |\mathrm{d}_{A_\alpha}(u_\alpha(z_\alpha))|$ is unbounded for some sequence $z_\alpha \to z_i$ going to infinity on the i-th cylindrical end, there are two possibilities that can happen. First, on the cylindrical ends there is also translational symmetry, which means that other vortices can form at infinity on the cylindrical ends. Second, spherical bubbles on these cylindrical bubbles can also form. For any time $s \geq 0$, let $\tau_s : S^1 \times (0, \infty) \to S^1 \times (0, \infty)$ denote translation by s.

Suppose that there is a sequence of points z_α on the cylindrical end such that the r-coordinate of z_α goes to infinity and $\mathrm{d}_{A_\alpha} u(z_\alpha)$ is bounded from above and below. Thus, there must be a sequence of numbers r_α,a vortex (B, v) on the trivial bundle over the cylinder $S^1 \times \mathbb{R}$, satisfying the flat limit of the vortex equations

$$F_B = 0, \quad \overline{\partial}_{J_B} v = 0, \tag{84}$$

and a finite set Z so that $\tau_{r_\alpha}(A_\alpha, u_\alpha)$ converges on compact sets on the complement $\Sigma \backslash Z$ to (B, v) in all derivatives. Note that after gauge transformation any connection on the cylinder is in temporal gauge, if flat it is then of the form $A = \mathrm{d} + a\mathrm{d}\theta$, for a constant element $a \in \mathfrak{g}$, and d the trivial connection. Then, the holonomy at infinity of the limit vortex agrees with the holonomy at infinity of the sequence. By energy quantization for g-twisted pseudoholomorphic cylinders Theorem **??**, only finitely many cylindrical bubbles can occur.

To capture the first bubble, let $m_0 := \lim_T \lim_\alpha (A_\alpha, u_\alpha; (T; \infty) \times S^1)$ be the energy of the sequence that dissipates at infinity. Let h be a constant smaller than $\min\{\hbar, k\}$, where k is the constant in Theorem 4.2.10. Choose r_α so that the energy $E(u_\alpha; (r_\alpha, \infty) \times S^1) = m_0 - h/2$, by gauge transforming the pairs (A_α, u_α) if necessary and choosing a subsequence, we can assume that r_α is bigger than the constant r_0 of Theorem 4.2.10, so that the covariant derivative $\nabla_A u$ and the energy $E(u)$ have exponential decay on the cylindrical end. Let $(B_\alpha, v_\alpha) := \tau_{r_\alpha}(A_\alpha, u_\alpha)$ be the rescaled sequence. This sequence has bounded energy vortices. By using Step 1, there is a finite set $Z \subset S^1 \times \mathbb{R}$ and a flat vortex (B, v) on the cylinder such that (B_α, v_α) converges in $\mathbb{R} \times S^1 \backslash Z$ in compact sets with all derivatives. Moreover, the choice of rescaling ensures that no further bubbling for the original sequence can happen at infinity in the i-th cylindrical end. Exponential decay of u at infinity, as well as exponential decay for flat vortices as stated in Proposition **??** ensures the existence of limits which by construction connect: $u(\infty) = v(-\infty)$.

We now apply this process inductively, after capturing the first bubble, the limiting sequence is as follows. For $j = 1, \ldots, M$ there are sequences $\{r^j_\alpha\}$ of positive numbers such that $r^j_\alpha \to \infty$ as $\alpha \to \infty$, flat vortices (A_j, v_j) on the cylinder $\mathbb{R} \times S^1$ and finite sets of points $Z_j \subset \mathbb{R} \times S^1$ so that $\tau_{r_\alpha}(A_\alpha, u_\alpha)$ converges to (B_j, v_j) on $\mathbb{R} \times S^1 \backslash Z_j$ on compact sets and there is spherical fibre bubbling occurring on points $s \in Z_j$. Now apply the arguments used in Case 1 to build the bubble tree on the fibres $P(X)_s, s \in Z_j$. The cylindrical bubbles also have limits at infinity, since the exponential decay for finite energy flat vortices Proposition **??** and by construction they connect: $v_j(\infty) = v_{j+1}(-\infty)$. This finishes the proof of the Theorem 5.2.6.

5.5. **Local distance functions.** To show that the topology is Hausdorff we need to describe Gromov convergence of stable vortices in terms of an auxiliary function called the distance function. Our function is the same as in [13, p. 134] with an added term including the connection.

Definition 5.5.1. The *local distance function* ρ_ϵ for $\epsilon > 0$ is

$$\rho_\epsilon((\underline{A}, \underline{u}), (\underline{A}', \underline{u}')) = \inf_{f : T \to T'} \inf_{g \in \mathcal{G}(P)} \inf_{\underline{\phi}} \rho_\epsilon((\underline{A}, \underline{u}), g(\underline{A}', \underline{u}'); f, \underline{\phi}),$$

where

$$\begin{aligned} \rho_\epsilon((\underline{A}, \underline{u}), (\underline{A}', \underline{u}'); f, \underline{\phi}) :=& \|A' - A\|_{L^2} \\ &+ \sup_j |E(A, \underline{u}; B_\epsilon(w_j)) - E(A', \underline{u}'; \underline{\phi}(B_\epsilon(w_j)))| \\ &+ \sup_{i \in \Gamma} \sup_{z \notin B_\epsilon(w_i)} d(u'_{f(i)} \phi_i, u_i) \\ &+ \sup_{i \neq j, f(i) = f(j)} \sup_{z \notin B_\epsilon(w_j)} d(\phi^{-1}_{i^+(j)} \phi_{i^-(j)}, w_j) \\ &+ \sup_{f(i) \neq f(j)} d(\phi_j^{-1}(z_{f(i)f(j)}), w_i) \\ &+ \sup_{i \in T, 1 \leq j \leq n} d(\phi_i^{-1}(z'_{f(i)j}), w_i) \end{aligned}$$

depends on the contraction $f : \Gamma \to \Gamma'$ such that map the nodes $i \to i'$ and

$$\underline{\phi} = (\phi_1, \ldots, \phi_n), \quad \phi_i : \Sigma_i - B_\epsilon(Z_i) \to \Sigma'_{f(i)}$$

is holomorphic isomorphism of $\Sigma_i - B_\epsilon(Z_i)$ onto its image. We set $\rho_\epsilon = \infty$ if there are no contractions f.

Lemma 5.5.2. *For ϵ sufficiently small, $(\underline{A}_\alpha, \underline{u}_\alpha)$ Gromov converges to $\underline{A}, \underline{u}$, if and only if $\rho_\epsilon((\underline{A}_\alpha, \underline{u}_\alpha), (\underline{A}, \underline{u}))$ converges to zero.*

Proof. This is the vortex version of [13, Lemma 5.5.8]. The forward direction is immediate from the definition of Gromov convergence. The reverse implication holds for any ϵ sufficiently small so that the energy of $(A, \underline{u})$ on B_ϵ is less than $\hbar/2$ where $\hbar$ is the energy quantization constant of the previous section. Since $\rho_\epsilon \to 0$, $(\underline{A}_\alpha, \underline{u}_\alpha)$ converges to $(A, \underline{u})$ uniformly on compact subsets of the complements of the balls $B_\epsilon(w_j)$. Since there is no additional bubbling, $(\underline{A}_\alpha, \underline{u}_\alpha)$ converges to some limit $(\underline{A}_\infty, \underline{u}_\infty)$ with the same combinatorial type as $(\underline{A}, \underline{u})$ and $(\underline{A}_\infty, \underline{u}_\infty) = (\underline{A}, \underline{u})$ on the complement of the balls $B_\epsilon(w_j)$. By unique continuation for pseudoholomorphic maps, $\underline{u}_\infty = \underline{u}$ everywhere on $\underline{\Sigma}$, which completes the proof. □

Proposition 5.5.3. *The Gromov open sets form a topology for which any convergent sequence is Gromov convergent. Furthermore, any convergent sequence has a unique limit.*

Proof. By [13, Lemma 5.6.5] it suffices to show that for all $(\underline{A}, \underline{u})$ there is an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, the function ρ_ϵ satisfies the following:

(a) $\rho_\epsilon((\underline{A}, \underline{u}), (\underline{A}', \underline{u}')) = 0$ if and only if $(\underline{A}, \underline{u}) = (\underline{A}', \underline{u}')$.
(b) $(\underline{A}_\alpha, \underline{u}_\alpha)$ converges to $(\underline{A}, \underline{u})$ if and only if $\rho_\epsilon((\underline{A}_\alpha, \underline{u}_\alpha), (\underline{A}, \underline{u}))$ converges to 0.
(c) Suppose that $(\underline{A}_\alpha, \underline{u}_\alpha)$ converges to $(\underline{A}, \underline{u})$. Then $\limsup_\alpha \rho_\epsilon((A_\alpha, \underline{u}_\alpha), (A', \underline{u}')) \leq \rho_\epsilon((\underline{A}, \underline{u}), (A', \underline{u}'))$.

(a) Suppose $\rho_\epsilon((\underline{A}, \underline{u}), (\underline{A}', \underline{u}')) = 0$. Then after gauge transformation and reparametrization $\underline{A} = \underline{A}'$, and $\underline{u} = \underline{u}'$. (b) and (c) follow from Lemma 5.5.2. □

6. Transversality

6.1. **Fredholm theory for nodal vortices.** The tangent space to $M_\Gamma(P, X, \underline{\mu})$ is formally the kernel of a *linearized operator* defined as follows. For each $w \in \mathrm{Vert}_s(\Gamma)$ we denote by $T_w^{\mathrm{vert}} P(X)$ the corresponding linearized fiber of P, and for each $w \in \mathrm{Vert}_c(\Gamma)$ we denote by $T_w X^\mu$ the corresponding tangent fiber of X^μ.

Definition 6.1.1. The *linearized operator* associated to a nodal vortex $\underline{A}, \underline{u}$ is the operator $\tilde{D}_{\underline{A},\underline{u}}$ from

$$\Omega^1(\Sigma, P(\mathfrak{g})) \oplus \Omega^0(\Sigma, u_0^* T^{\mathrm{vert}} P(X)) \oplus \bigoplus_\alpha (\Omega^1(S^1 \times \mathbb{R}, \mathfrak{g}) \oplus \Omega^0(S^1 \times \mathbb{R}, u_\alpha^* X)) \oplus \bigoplus_\beta \Omega^0(\mathbb{P}^1, v_\beta^* TX)$$

to

$$(\Omega^0 \oplus \Omega^2)(\Sigma, P(\mathfrak{g})) \oplus \Omega^{0,1}(\Sigma, u_0^* T^{\mathrm{vert}} P(X)) \oplus \bigoplus_\alpha ((\Omega^0 \oplus \Omega^2)(S^1 \times \mathbb{R}, \mathfrak{g}) \oplus \Omega^{0,1}(S^1 \times \mathbb{R}, u_\alpha^* X)) \oplus \bigoplus_\beta \Omega^{0,1}(\mathbb{P}^1, v_\beta^* TX) \oplus \bigoplus_{w \in \mathrm{Vert}_c(\Gamma)} T_w X^\mu \oplus \bigoplus_{w \in \mathrm{Vert}_s(\Gamma)} T_w^{\mathrm{vert}} P(X)$$

obtained by combining the various linearized operators on the principal component and bubbles, and taking the difference of the infinitesimal sections at the nodes. We say that (Γ) is *regular* if $\tilde{D}_{A,\underline{u}}$ is surjective.

6.2. **Transversality for fixed type.** We now prove transversality for the moduli space of vortices of a fixed combinatorial type.

Definition 6.2.1. A *connection dependent almost complex structure* is a map

$$J : \mathcal{A}(P, \underline{\mu}) \to \mathrm{Map}_G(P, \mathcal{J}(X))$$

equivariant with respect to the action of $\mathcal{G}(P, \underline{\mu})$, equal to a constant almost complex structure J_i in a neighborhood of infinity on the i-th end, for $i = 1, \ldots, n$. A *perturbation datum* consists of a connection-dependent almost complex structure J and an admissible Hamiltonian perturbation $H \in \mathcal{H}(P, X)$ from Section 3.7.

Let $\mathcal{J}(P, X)$ denote the space of connection-dependent almost complex structures, and $\mathcal{JH}(P, X) = \mathcal{J}(P, X) \times \mathcal{H}(P, X)$ the space of perturbation data. $\mathcal{JH}(P, X)$ admits a Sobolev completion by almost complex structures and perturbations of class $1, p, \delta$, as in Theorem **??**.

Definition 6.2.2. Given $(J, H) \in \mathcal{JH}(P, X)$, a *stable* J, H*-vortex* is a nodal vortex $(A, \underline{u})$ such that

(a) The principal component (A, u_0) is a $(J(A), H(A))$-vortex;
(b) Each cylindrical and spherical bubble u_i is $J_{\hat{z}_i}$-holomorphic, where $\hat{z}_i \in \Sigma_0$ is the attaching point onto the principal component.

Let $M_\Gamma(P, X, J, H, \underline{\mu})$ or $M_\Gamma(P, X, \underline{\mu})$ for short resp. $M_\Gamma^{\mathrm{fr}}(P, X, \underline{\mu})$ denote the moduli space of J, H-vortices with fixed holonomies $\underline{\mu}$, of combinatorial type Γ resp. vortices framed at infinity.

Definition 6.2.3. The *linearized operator* $D_{A,\underline{u}}$ associated to a (J, H)-holomorphic vortex $A, \underline{u}$ is the operator is defined as in (6.1.1), by adding the perturbation terms on the principal component in Section 3.7. The vortex is *regular* of the linearized operator is surjective.

In particular, regularity implies that the difference of evaluation maps from the sum of kernels of linearized operators on the components to the fiber at the node is surjective.

Let $M_\Gamma^{\text{reg}}(P, X, \underline{\mu}, J, H)$ denote the locus of regular, stable vortices, and $M^{\text{fr,reg}}(P, X, \underline{\mu}, J, H)$ its inverse image in $M^{\text{fr}}(P, X, \underline{\mu}, J, H)$.

Theorem 6.2.4. *$M_\Gamma^{\text{reg}}(P, X, \underline{\mu}, J, H)$ resp. $M_\Gamma^{\text{fr,reg}}(P, X, \underline{\mu}, J, H)$ resp. has the structure of a smooth orbifold with tangent space at $A, \underline{u}$ isomorphic to the quotient of the kernel of the operator $\tilde{D}_{A,\underline{u}}$ by* $\text{aut}(A, \underline{u})$. *Moreover, the evaluation map*

$$\text{ev}_\Gamma^{\text{fr}} : M_\Gamma^{\text{fr,reg}}(P, X, \underline{\mu}, J, H) \to X^{\underline{\mu}}$$

is smooth.

The proof is similar to the one of Theorem 3.7.8 and will be omitted.

Definition 6.2.5. We say that a pair $(J, H) \in \mathcal{JH}$ is *regular* if every simple pair $(A, \underline{u})$ is regular.

Theorem 6.2.6. *Suppose that for every vortex $(A, \underline{u})$, the underlying connection A has* $\text{Aut}(A) = Z$. *Then the space $\mathcal{JH}^{reg}(\underline{\mu}, \Gamma)$ of regular perturbations is of second category in $\mathcal{JH}$.*

Proof. The proof is an application of Sard-Smale again. Consider the universal moduli space

$$M^{\text{univ}}(P, X, \underline{\mu}) = \{(A, \underline{u}, J, H), [A, \underline{u}] \in M(P, X, J, H)^{\text{simple}}\}.$$

It is straightforward to check that the universal moduli space is smooth, as in Theorem 3.7.10. By Sard-Smale, the set of regular values of the projection

$$M^{\text{univ}}(P, X, \underline{\mu}) \to \mathcal{JH}$$

is Baire second-category. Any such regular value is regular in the sense of Definition 6.2.5. □

7. Invariants

In this section we construct invariants associated to compact Hamiltonian G-manifolds equipped with convex invariant almost complex structure $J \in \mathcal{J}(X)^G$.

7.1. **Classifying maps.** For later use we wish to note the following construction of the classifying map, which does not use smoothness of $M^{\text{reg,fr}}$. Unfortunately, the construction is very restrictive. Recall that the center $Z \subset G$ embeds in $\mathcal{G}(P)^n$ as constant loops.

Definition 7.1.1. Let $\overline{M}^*(P, X, \underline{\mu})$ resp. $\overline{M}_\Gamma^{*,\text{fr}}(P, X, \underline{\mu})$ denote the subset of $\overline{M}(P, X, \underline{\mu})$ resp. $\overline{M}^{\text{fr}}(P, X, \underline{\mu})$ of pairs $[A, \underline{u}]$ such that $\text{Aut}(A) = Z$.

Let G^n/Z denote the quotient of G^n by the diagonal action of Z. Note that G^n/Z has a natural group structure.

By the *genus* of a surface with cylindrical ends Σ we mean the genus of the closed surface $\overline{\Sigma} = \Sigma \cup \{z_1, \ldots, z_n\}$. The following is well-known:

Lemma 7.1.2. *Let $G = SU(r)$. There exists a dense, open subset $\mathfrak{A}^{\mathrm{reg},g,n} \subset \mathfrak{A}^n$ such that for every $\underline{\mu} \in \mathfrak{A}^{\mathrm{reg},g,n}$, every element of $M(P, \underline{\mu})$ has $Aut(A) = Z$.*

Proof. Existence of a one-parameter family of stabilizers implies that A admits a reduction to $S(U(j) \times U(n-j))$ for some j. But this implies an affine linear relation on holonomies, see [14] $\square$

Proposition 7.1.3. *Let $G = SU(r)$. Let Σ be a surface of genus g with n cylindrical ends. For every $\underline{\mu} \in \mathfrak{A}^{\mathrm{reg},g,n}$ of the previous lemma, there exists an $\epsilon_0 > 0$ such that if $\epsilon > \epsilon_0$ then every ϵ-vortex $(A, \underline{u})$ with fixed holonomies $\underline{\mu}$ has $\mathrm{Aut}(A) = Z$.*

Proof. Any sequence A_α of connections with vanishing curvature $F_{A_\alpha} \to 0$ converges (by Uhlenbeck compactness) to a flat connection A_∞ weakly in $W^{m,p}$. We may assume that A_α is in Coulomb gauge with respect to A_∞, that is, $\mathrm{d}^*_{A_\infty}(A_\alpha - A_\infty) = 0$. Let ξ_α be an infinitesimal gauge transformation fixing A_α,

$$\mathrm{d}_{A_\alpha} \xi_\alpha = 0$$

with $W^{m+1,p}$ norm equal to 1. Now $[A_\alpha, \xi_\alpha]$ has bounded $W^{3-p/m+1,p}$-norm by Sobolev multiplication, so by elliptic regularity ξ_α has bounded $W^{4-2/p,p}$ norm. Hence we may assume, after passing to a subsequence, that ξ_α converges in $W^{m+1,p}$ to some non-zero ξ_∞. Then

$$\mathrm{d}_{A_\infty} \xi_\infty = \lim_{\alpha \to \infty} \mathrm{d}_{A_\alpha} \xi_\alpha = 0$$

implies that A_∞ is reducible. This contradicts Lemma 7.1.2. $\square$

For any $C > 0$ let $\overline{M}(\Sigma, X, \underline{\mu}, C)$ denote the subset of energy bounded by C.

Proposition 7.1.4. *Let $G = SU(r)$ and $\underline{\mu} \in \mathfrak{A}^{\mathrm{reg},g,n}$ as in Lemma 7.1.2. Suppose that ϵ_0 is sufficiently large so that every ϵ vertex $(A, \underline{u})$ with fixed holonomies $\underline{\mu}$ and $\epsilon > \epsilon_0$ has $\mathrm{Aut}(A) = Z$, as in the previous lemma. For any $C > 0$ there exists a classifying map*

$$\psi : \overline{M}^{\mathrm{fr}}_n(\Sigma, X, \underline{\mu}, C) \to E(G^n/Z)$$

taking values in a finite-dimensional approximation to $E(G^n/Z)$, such that the restriction of ψ to the regular locus of any stratum $M^{\mathrm{fr,reg}}_{n,\Gamma}(\Sigma, X, \underline{\mu}, C)$ is smooth, and equal to the pull-back of the map $M^{\mathrm{fr,reg}}_{n,\Gamma}(\Sigma, X, \underline{\mu}, C) \to M^{\mathrm{fr,reg}}_n(\Sigma, X, \underline{\mu}, C)$ given by forgetting the bubbles.

Proof. Let $\mathcal{A}^{\mathrm{fr}}(P, \underline{\mu})_{m,p,\delta}$ denote the moduli space of framed connections on P, that is, connections together with framings at infinity of Sobolev class $1, p, \delta$ as described above. The locus $\mathcal{A}^{*,\mathrm{fr}}(P, \underline{\mu})_{m,p,\delta}$ of connections A with $\mathrm{Aut}(A) = Z$ has a free action of the group G^n/Z changing the framings. It follows that the map

$$\pi_{\mathcal{A}} : \mathcal{A}^{\mathrm{fr}}(P, \underline{\mu})_{m,p,\delta}/\mathcal{G}(P, \underline{\mu})_{m+1,p,\delta} \to \mathcal{A}(P, \underline{\mu})_{m,p,\delta}/\mathcal{G}(P, \underline{\mu})_{m+1,p,\delta}$$

has the structure of a smooth principal G^n/Z-bundle [9, Remark B.9]. Partitions of unity exist for Banach manifolds modelled on separable Hilbert spaces; from this the usual construction of classifying maps yields a map

$$\psi_{\mathcal{A}} : \mathcal{A}^{\mathrm{fr}}(P, \underline{\mu})/\mathcal{G}(P, \underline{\mu})_{m+1,p,\delta} \to E(G^n/Z).$$

Let $K(C)$ be a compact set containing the image of $M_n^{\mathrm{fr}}(P, \underline{\mu}, C)$. The classifying map $\psi_{\mathcal{A}}$ may be constructed so that $\psi_{\mathcal{A}}$ takes values in a finite dimensional approximation on $K(C)$ and is smooth as a map of Banach manifolds on $K(C)$ (by covering $K(C)$ with finitely many open sets). Pulling back $\psi_{\mathcal{A}}$ to $M_n^{\mathrm{fr}}(P, X, \underline{\mu})$ completes the proof. □

7.2. **Evaluation maps.** Recall that $\overline{M}^{\mathrm{fr}}(\Sigma, X, \underline{\mu})$ denotes the moduli space of vortices with framings at the points $\underline{z} = (z_1, \ldots, z_n)$. Define the *evaluation map*

$$\mathrm{ev} : \overline{M}^{\mathrm{fr}}(\Sigma, P, X, \underline{\mu}) \to X^{\underline{\mu}} := X^{\mu_1} \times \ldots \times X^{\mu_n}$$

by

$$\mathrm{ev}_i^{\mathrm{fr}}([A, u]) := \lim_{r \to \infty} u(\rho_i(r, \theta))$$

and

$$\mathrm{ev}([A, u]) = \prod_{i=1}^{n} \mathrm{ev}_i^{\mathrm{fr}}([A, u]).$$

Lemma 7.2.1. *The evaluation map* $\mathrm{ev} : \overline{M}^{\mathrm{fr}}(P, X, \underline{\mu})$ *is smooth on the regular locus of any stratum.*

Proof. In the local charts (14), the evaluation map is given by

$$\mathrm{ev}_i([A, u]) = \exp_{u(z_i)}(\xi_i)$$

and is therefore smooth. □

Combining the evaluation map with the classifying map gives an *evaluation map*

$$\mathrm{ev}_C : \overline{M}(P, X, \underline{\mu}, C) \to (X^{\underline{\mu}})_{G_{\underline{\mu}}/Z}.$$

7.3. **Invariants.**

Theorem 7.3.1. *Suppose that* ϵ_0, C *are as above, so that every vortex with* $\epsilon > \epsilon_0$ *and energy* $E(A, \underline{u}) < C$ *has automorphism group* $\mathrm{Aut}(A) = Z$. *Let* X *be a compact symplectic manifold such that* $Z \subset G$ *acts trivially on* X, *and* $J \in \mathcal{J}(X)^G$ *is an invariant convex almost complex structure. There exists a subset* $\mathcal{H}^{reg}(P, X, \underline{\mu}, C)$ *of Baire second category in* $\mathcal{H}(P, X, \underline{\mu})$ *such that for any* $H \in \mathcal{H}^{\mathrm{reg}}(\underline{\mu}, C)$, *then* ev_C *is a pseudo-cycle.*

Proof. Let $\mathcal{H}^{\mathrm{reg}}(\underline{\mu})$ be the intersection of the sets $\mathcal{H}^{\mathrm{reg}}(\underline{\mu}, \Gamma)$ of Theorem 6.2.6, for all combinatorial types and for all $\underline{\mu}'$ with $N\underline{\mu}' = \underline{\mu}$ for some integer $N > 0$. For any $H \in \mathcal{H}^{\mathrm{reg}}(\underline{\mu})$, the vortices of every combinatorial type are regular. It follows from monotonicity that the images of the multiply covered components are contained in smooth manifolds of codimension at least two lower than the expected dimension. Therefore ev_C is a pseudocycle. □

For an equivariant class $\alpha \in H_{G_{\underline{\mu}}/Z}(X^{\underline{\mu}})$, Let Y denote a pseudocycle representing the Poincaré dual $(\alpha_N)^\vee \in H(X^{\underline{\mu}}_{G_{\underline{\mu}}/Z})$. Define

$$Z(\alpha) := \lim_{C\to\infty} \mathrm{ev}_C \cdot Y, \tag{85}$$

the intersection number of pseudo-cycles in $X_{G_{\underline{\mu}}/Z}$. The intersection number $Z(\alpha)$ is independent of the choice of finite dimensional approximation to $E(G_{\underline{\mu}}/Z)$, by a diagram chase, see [8].

Consider the projection

$$\prod_{i=1}^n G_{\mu_i} \to \prod_{i=1}^n G_{\mu_i}/Z.$$

The induced map on universal bundles

$$E(\prod_{i=1}^n G_{\mu_i}) \to E(\prod_{i=1}^n G_{\mu_i}/Z)$$

induces a map

$$\pi_Z : (X^{\underline{\mu}})_{G_{\underline{\mu}}} \to (X^{\underline{\mu}})_{G_{\underline{\mu}}/Z}$$

with fibers BZ. Since BZ has torsion cohomology, π_Z induces an isomorphism

$$\pi_{Z,*} : H_{G_{\underline{\mu}}}(X^{\underline{\mu}}, \mathbb{Q}) \to H_{G_{\underline{\mu}}/Z}(X^{\underline{\mu}}, \mathbb{Q})$$

in cohomology with rational coefficients.

Definition 7.3.2. Let X, ϵ_0 be as above. For $\alpha \in H_{G_{\underline{\mu}}}(X^{\underline{\mu}}, \mathbb{Q})$, the *vortex invariant*

$$Z(\alpha) = Z(\pi_Z(\alpha)) = \lim_{C\to\infty} (\mathrm{ev}_C\, M(\Sigma, X, \underline{\mu}, C).\pi_{Z,*}(\alpha)).$$

References

[1] A. Alekseev and E. Meinrenken. The non-commutative Weil algebra. *Invent. Math.*, 139(1):135–172, 2000.

[2] A. Beauville. Conformal blocks, fusion rules and the Verlinde formula. In *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, number 9 in Israel Math. Conf. Proc., pages 75–96. Bar-Ilan Univ., Ramat Gan, 1996.

[3] U. Bhosle and A. Ramanathan. Moduli of parabolic G-bundles on curves. *Math. Z.*, 202(2):161–180, 1989.

[4] Kai Cieliebak, A. Rita Gaio, Ignasi Mundet i Riera, and Dietmar A. Salamon. The symplectic vortex equations and invariants of Hamiltonian group actions. *J. Symplectic Geom.*, 1(3):543–645, 2002.

[5] Kai Cieliebak, Ana Rita Gaio, and Dietmar A. Salamon. J-holomorphic curves, moment maps, and invariants of Hamiltonian group actions. *Internat. Math. Res. Notices*, (16):831–882, 2000.

[6] S. K. Donaldson. *Floer homology groups in Yang-Mills theory*, volume 147 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2002. With the assistance of M. Furuta and D. Kotschick.

[7] M. Furuta and B. Steer. Seifert fibred homology 3-spheres and the Yang-Mills equations on Riemann surfaces with marked points. *Adv. Math.*, 96(1):38–102, 1992.
[8] Ana Rita Pires Gaio and Dietmar A. Salamon. Gromov-Witten invariants of symplectic quotients and adiabatic limits. *J. Symplectic Geom.*, 3(1):55–159, 2005.
[9] V. Guillemin, V. Ginzburg, and Y. Karshon. *Moment maps, cobordisms, and Hamiltonian group actions*, volume 98 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. Appendix J by Maxim Braverman.
[10] L. Hörmander. *The analysis of linear partial differential operators. III.* Springer-Verlag, Berlin, 1994. Pseudo-differential operators, Corrected reprint of the 1985 original.
[11] Ignasi Mundet i Riera and Gang Tian. A compactification of the moduli space of twisted holomorphic maps. arXiv:math.SG/0404407.
[12] Robert B. Lockhart and Robert C. McOwen. Elliptic differential operators on noncompact manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):409–447, 1985.
[13] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.
[14] V. B. Mehta and C. S. Seshadri. Moduli of vector bundles on curves with parabolic structure. *Math. Ann.*, 248:205–239, 1980.
[15] E. Meinrenken and C. Woodward. Hamiltonian loop group actions and Verlinde factorization. *Journal of Differential Geometry*, 50:417–470, 1999.
[16] Ignasi Mundet i Riera. A Hitchin-Kobayashi correspondence for Kähler fibrations. *J. Reine Angew. Math.*, 528:41–80, 2000.
[17] Ignasi Mundet i Riera. Hamiltonian Gromov-Witten invariants. *Topology*, 42(3):525–553, 2003.
[18] M. S. Narasimhan and C. S. Seshadri. Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math. (2)*, 82:540–567, 1965.
[19] Marcin Poźniak. Floer homology, Novikov rings and clean intersections. In *Northern California Symplectic Geometry Seminar*, volume 196 of *Amer. Math. Soc. Transl. Ser. 2*, pages 119–181. Amer. Math. Soc., Providence, RI, 1999.
[20] C. Teleman and C. Woodward. Parabolic bundles, products of conjugacy classes and Gromov-Witten invariants. *Ann. Inst. Fourier (Grenoble)*, 53(3):713–748, 2003.
[21] Katrin Wehrheim. *Uhlenbeck compactness*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2004.
[22] E. Witten. On quantum gauge theories in two dimensions. *Comm. Math. Phys.*, 141:153–209, 1991.
[23] Fabian Ziltener. Optimal decay for symplectic vortices and a sharp equivariant isoperimetric inequality, 2006. arXiv.org:math/0611768.