SPACELIKE MEAN CURVATURE ONE SURFACES IN DE SITTER 3-SPACE

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Introduction

The results mentioned in this article are extracted from:

S. Fujimori, W. Rossman, M. Umehara, K. Yamada, and S.-D. Yang, *Spacelike mean curvature one surfaces in de Sitter 3-space*, preprint.

Fujimori studied spacelike constant mean curvature one (CMC-1) surfaces in de Sitter 3-space S^3_1 when the surfaces have no singularities except within some compact subset and are of finite total curvature on the complement of this compact subset. However, there are many CMC-1 surfaces whose singular sets are not compact. In fact, such examples have already appeared in the construction of trinoids given by Lee and Yang via hypergeometric functions.

So, we develop a fundamental framework that allows the singular set to be non-compact. The following two theorems are the main results.

Theorem A. *A complete end of a weakly complete CMC-1 face of finite topology in S^3_1 is never hyperbolic, so must be either elliptic or parabolic. Moreover, the total curvature over a neighborhood of such an end is finite.*

Theorem B. *Suppose a CMC-1 face $f\colon M^2 \to S^3_1$ is complete. Then there exist a compact Riemann surface $\overline{M}^2$ and a finite number of points $p_1, \dots, p_n \in \overline{M}^2$ such that M^2 is biholomorphic to $\overline{M}^2 \setminus \{p_1, \dots, p_n\}$, and $2\deg(G) \geq -\chi(\overline{M}^2) + 2n$, where G is the hyperbolic Gauss map of f and $\chi(\overline{M}^2)$ is the Euler characteristic of $\overline{M}^2$. Furthermore, the equality holds if and only if each end is regular and embedded.*

1. Preliminaries

The representation formula. We identify $X = (x_0, x_1, x_2, x_3) \in \boldsymbol{R}^4_1$ with

$$X = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in \mathrm{Herm}(2) := \{X \in M_2(\boldsymbol{C}) : X^* = X\}.$$

Hyperbolic 3-space and De Sitter 3-space are

$$\begin{aligned} H^3 &= \{(x_0, x_1, x_2, x_3) \in \boldsymbol{R}^4_1\,;\, -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1, x_0 > 0\} \\ &= \{X \in \mathrm{Herm}(2); \det X = 1, \mathrm{Tr}\, X > 0\} = \{FF^*\,;\, F \in \mathrm{SL}_2\,\boldsymbol{C}\}, \\ S^3_1 &= \{(x_0, x_1, x_2, x_3) \in \boldsymbol{R}^4_1\,;\, -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\} \\ &= \{X \in \mathrm{Herm}(2); \det X = -1\} = \{F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F^*\,;\, F \in \mathrm{SL}_2\,\boldsymbol{C}\}. \end{aligned}$$

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Due to the page limitation for this article, citations for references and proper credits to various results mentioned here have been omitted, which can be found in the original paper.

An immersion into S^3_1 is called *spacelike* if the induced metric on the immersed surface is positive definite. There is only one, up to congruency, complete spacelike CMC-1 immersion, which we call an S^3_1-horosphere.

A *CMC*-1 *face* is a C^∞-map f from an oriented 2-manifold M^2 into S^3_1 such that

1. for some open dense subset $W \subset M^2$, $f|_W$ is a spacelike CMC-1 immersion,
2. for any singular point $p \in M^2$ (i.e., where the induced metric degenerates), there is a C^1-differentiable function $\lambda : U \cap W \to (0,\infty)$, where $U \subset M^2$ is a neighborhood of p, so that $\lambda\, ds^2$ extends to a C^1-differentiable Riemannian metric on U, and
3. $df(p) \neq 0$ for any $p \in M^2$.

Theorem 1.1 (Fujimori)**.** *Let M^2 be a Riemann surface. Let g be a meromorphic function and ω a holomorphic 1-form on the universal cover $\widetilde{M}^2$ such that*

$$d\hat{s}^2 = (1+|g|^2)^2\omega\bar{\omega} \tag{1.1}$$

is a Riemannian metric on $\widetilde{M}^2$ and $|g|$ is not identically 1. Take a holomorphic immersion $F = (F_{jk}) : \widetilde{M}^2 \to \mathrm{SL}_2\boldsymbol{C}$ satisfying $F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\omega$. Then $f : \widetilde{M}^2 \to S^3_1$ defined by $f := F\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}F^*$ is a CMC-1 face which is conformal away from its singularities. The induced metric ds^2 on M^2, the second fundamental form h, and the Hopf differential Q of f are given as follows:

$$ds^2 = (1-|g|^2)^2\omega\bar{\omega}, \quad h = Q + \overline{Q} + ds^2, \quad Q = \omega\, dg. \tag{1.2}$$

The singularities of the CMC-1 face occur at points where $|g| = 1$.

The converse also holds.

Remark 1.2. g is called a *secondary* Gauss map of f. The pair (g,ω) is called *Weierstrass data* of f, and F is called a *holomorphic null lift* of f.

The holomorphic 2-differential Q as in (1.2) is called the *Hopf differential* of f. $G := \frac{dF_{11}}{dF_{21}}$ is called the *hyperbolic Gauss map.*

Remark 1.3. Let K_{ds^2} be the Gaussian curvature of ds^2 on the set of regular points of f. Then $d\sigma^2 := K_{ds^2}\, ds^2 = \frac{4\, dg\, d\bar{g}}{(1-|g|^2)^2}$ is a pseudometric of constant curvature -1, which degenerates at isolated umbilic points.

Completeness. We say a CMC-1 face $f\colon M^2 \to S^3_1$ is *complete* if there exists a symmetric 2-tensor field T which vanishes outside a compact subset $C \subset M^2$ such that the sum $T + ds^2$ is a complete Riemannian metric on M^2.

We say that f is *weakly complete* if it is congruent to a S^3_1-horosphere or if the lift metric (1.1) is a complete Riemannian metric on M^2.

f is of *finite type* if there exists a compact set C of M^2 such that the first fundamental form ds^2 is positive definite and has finite total (absolute) curvature on $M^2 \setminus C$.

Conjugacy classes of $SU_{1,1}$. For any real number t, we set

$$\Lambda_e(t) := \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix},\ \Lambda_p(t) := \begin{pmatrix} 1+it & -it \\ it & 1-it \end{pmatrix},\ \Lambda_h(t) := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

A matrix in $SU_{1,1} := \{S \in SL(2,\mathbf{C}); S\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}S^* = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\}$ is called *elliptic*, *parabolic* or *hyperbolic* if it is conjugate in $\mathrm{SU}_{1,1}$ to one of $\Lambda_e(t)$ for $-\pi < t \le \pi$, $\pm\Lambda_p(\pm 1)$ or $\pm\Lambda_h(t)$ for $t > 0$, respectively. Any matrix in $\mathrm{SU}_{1,1}$ is of one of these three types.

Monodromy of ends of CMC-1 faces. Let $f\colon M^2 \to S^3_1$ be a CMC-1 face of a Riemann surface M^2 and take a deck transformation $\tau \in \pi_1(M^2)$ in the universal cover $\widetilde{M}^2$. There exists a $\tilde\rho(\tau) \in \mathrm{SU}_{1,1}$ such that $F \circ \tau = F\tilde\rho(\tau)$. The representation $\tilde\rho\colon \pi_1(M^2) \to \mathrm{SU}_{1,1}$ is called the *monodromy representation*, which induces a $\mathrm{PSU}_{1,1}$-representation $\rho\colon \pi_1(M^2) \to \mathrm{PSU}_{1,1}$ satisfying $g \circ \tau^{-1} = \rho(\tau) \star g$.

Let $f\colon M^2 \to S^3_1$ be a weakly complete CMC-1 face of finite topology, where M^2 is diffeomorphic to a compact Riemann surface $\overline{M}^2$ with finitely many punctures $\{p_1, \dots, p_n\}$. Any puncture p_j, or occasionally a small neighborhood U_j of p_j, is called an *end* of f.

An end is called *elliptic*, *parabolic* or *hyperbolic* when the monodromy matrix $\tilde\rho(\tau) \in \mathrm{SU}_{1,1}$ is elliptic, parabolic or hyperbolic, respectively, where $\tau \in \pi_1(M^2)$ is the deck transformation corresponding the counterclockwise loop about p_j.

The Schwarzian derivative. The Schwarzian derivative of h is

$$S_z(h) := \left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2 \qquad \left(' = \frac{d}{dz}\right).$$

Let $f\colon M^2 \to S^3_1$ be a CMC-1 face with the hyperbolic Gauss map G, a secondary Gauss map g and the Hopf differential Q. Then $S(g) - S(G) = 2Q$.

2. Monodromy of punctured hyperbolic metrics

Lifts of $\mathrm{PSU}_{1,1}$-projective connections on a punctured disk. Let $P = p(z)dz^2$ be a holomorphic 2-differential on $\Delta^* = \{z \in \boldsymbol{C}\,;\, 0 < |z| < 1\}$. Then there exists a holomorphic developing map $g_P\colon \widetilde{\Delta}^* \to \boldsymbol{C} \cup \{\infty\}$ such that $S(g_P) = P$, where $\widetilde{\Delta}^*$ is the universal cover of Δ^*. For any other holomorphic function h such that $S(h) = P$, there exists an $A \in \mathrm{SL}_2\boldsymbol{C}$ so that $A \star g_P = h$. Thus there exists a matrix $T \in \mathrm{PSL}_2\boldsymbol{C}$ such that $g \circ \tau^{-1} = T \star g$, where τ is the generator of $\pi_1(\Delta^*)$ corresponding to a counterclockwise loop about the origin. We call T the monodromy matrix of g. If $T \in \mathrm{PSU}_{1,1}$, P is called a $\mathrm{PSU}_{1,1}$-projective connection on Δ^* and g_P is called a $\mathrm{PSU}_{1,1}$-lift of P. A $\mathrm{PSU}_{1,1}$-projective connection on Δ^* has a removable singularity, a pole or an essential singularity at 0, and is said to have a *regular singularity* at 0 if it has at most a pole of order 2 at 0. When $T \in \mathrm{PSU}_{1,1}$, it is conjugate to one of the matrices in (1). The $\mathrm{PSU}_{1,1}$-projective connection P is then called *elliptic*, *parabolic* or *hyperbolic* when T is *elliptic*, *parabolic* or *hyperbolic*, respectively. Note that a $\mathrm{PSU}_{1,1}$-lift g has the $\mathrm{PSU}_{1,1}$ ambiguity $g \mapsto A \star g$ for $A \in \mathrm{PSU}_{1,1}$. The property that $|g| > 1$ (resp. $|g| < 1$) is independent of this ambiguity.

Proposition 2.1. *Let $g : \widetilde{\Delta}^* \to \boldsymbol{C} \cup \{\infty\}$ be a $\mathrm{PSU}_{1,1}$-lift of a $\mathrm{PSU}_{1,1}$-projective connection P on Δ^*. Then the following assertions hold, where $R := \frac{1}{2}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}$:*

(1) Suppose that P is elliptic. Then, (i) *there exists an $A \in \mathrm{SU}_{1,1}$ and $\mu \in \boldsymbol{R}$ such that*

$$h(z) := z^{-\mu} A \star g(z)$$

is a single-valued meromorphic function on Δ^**. Moreover,* (ii) P *has a removable singularity or a regular singularity at* $z = 0$ *if and only if* $h(z)$ *has at most a pole at* $z = 0$*.*

(2) Suppose that P *is parabolic and that* $s \in \boldsymbol{R}_+$ *is arbitrary. Then* (i) *there exists an* $A \in \mathrm{SU}_{1,1}$ *such that*

$$h(z) := (RA) \star g(z) + \frac{t}{\pi i} \log z$$

is a single-valued meromorphic function on Δ^**, where* $t = s$ *or* $t = -s$ *as specified in the proof. The sign of* t *does not depend on the* $\mathrm{PSU}_{1,1}$*-ambiguity of* g*. Moreover,* (ii) $h(z)$ *has at most a pole at* $z = 0$ *if and only if* P *has a pole of order* 2 *at* $z = 0$*, and* (iii) $h(z)$ *is holomorphic at* $z = 0$ *if and only if* $P - \frac{1}{2z^2} dz^2$ *has at most a pole of order* 1*.* (iv) *When* h *is holomorphic,* $|g| > 1$ *(resp.* $|g| < 1$*) on a neighborhood of* 0 *if and only if* $t > 0$ *(resp.* $t < 0$*).*

(3) Suppose that P *is hyperbolic. Then* (i) *there exist an* $A \in \mathrm{SU}_{1,1}$ *and* $\mu \in \boldsymbol{R} \setminus \{0\}$ *such that*

$$h(z) := z^{-i\mu} (RA) \star g(z)$$

is a single-valued meromorphic function on Δ^**. Moreover,* (ii) $h(z)$ *has at most a pole at* $z = 0$ *if and only if* P *has a pole of order* 2 *at* $z = 0$*.*

In particular, P *has a pole of order exactly* 2 *at* $z = 0$ *if* P *is either parabolic or hyperbolic and* 0 *is a regular singularity of* P*.*

Monodromy of punctured hyperbolic metrics.

Definition 2.2. We say that a hyperbolic punctured metric $d\sigma^2$ has a *regular singularity* at the origin if $S(d\sigma^2)$ has at most a pole of order 2 at the origin.

Theorem 2.3. *Any conformal hyperbolic metric on* Δ^* *has a regular singularity at* $z = 0$*.*

3. Intrinsic behavior of regular ends

A puncture-type end p_j of f is called *regular* if the hyperbolic Gauss map G has at most a pole at p_j. On the other hand, a puncture-type end p_j is called *g-regular* if the Schwarzian derivative $S(g)$ of the secondary Gauss map g has at most a pole of order 2 at p_j.

Theorem 3.1. *All ends of a complete CMC-*1 *face are g-regular. In particular, all ends are of puncture-type.*

Theorem 3.2. *The Hopf differential of a weakly complete CMC-*1 *face of finite topology has at most a pole at any complete elliptic end.*

Definition 3.3. An elliptic end of a weakly complete CMC-1 face of finite topology is *integral* if the monodromy of the secondary Gauss map is the identity, and *non-integral* otherwise.

Lemma E1. *Let* $f\colon \Delta^* \to S^3_1$ *be a g-regular non-integral elliptic end. Then the singular set does not accumulate the end* 0*.*

Lemma E2. *Suppose* $f : \Delta^* \to S^3_1$ *is a g-regular integral elliptic end. If the singular set accumulates at the end, then there are an* $m \in \boldsymbol{N}$ *and a* $\delta \in \boldsymbol{R}$ *such that, for any* $\varepsilon > 0$*, there exists an* $r > 0$ *so that the singular set of* f *in* $\{z; 0 < |z| < r\}$ *lies in* $S(m, \varepsilon, \delta)$*.*

Definition 3.4. A parabolic end of a weakly complete CMC-1 face is of the *first kind* if

$$S(d\sigma^2) - \frac{dz^2}{2z^2} = S(g) - \frac{dz^2}{2z^2} = S(G) + 2Q - \frac{dz^2}{2z^2}$$

has at most a pole of order 1. Otherwise, it is of the *second kind.*

Lemma P. *Let $f : \Delta^* \to S^3_1$ be a g-regular parabolic end of a weakly complete CMC-1 face. If the end is of the first kind, the singular set does not accumulate at the end. If the end is of the second kind, then the singular set does accumulate at the end. In this case, there exist an $m \in \boldsymbol{N}$ and a $\delta \in \boldsymbol{R}$ such that, for all $\varepsilon > 0$, there exists an $r > 0$ such that the singular set of f in $\{z; 0 < |z| < r\}$ lies in $S(m, \varepsilon, \delta)$.*

Lemma H. *Let $f : \Delta^* \to S^3_1$ be a g-regular hyperbolic end of a weakly complete CMC-1 face. Then any ray in Δ^* emanating from the origin meets the singular set infinitely many times.*

Corollary 3.5. *The monodromy of a hyperbolic metric on Δ^* is either elliptic or parabolic. That is, hyperbolic monodromy never occurs.*

Corollary 3.6 (Characterization of hyperbolic ends)**.** *A g-regular end $f : \Delta^* \to S^3_1$ of a weakly complete CMC-1 face is hyperbolic if and only if any ray in Δ^* emanating from the origin meets the singular set infinitely many times.*

Theorem 3.7. *Any complete end of a weakly complete CMC-1 face is either g-regular elliptic or g-regular parabolic of the first kind.*

Theorem 3.8. *Let $f : M^2 \to S^3_1$ be a CMC-1 face. If f is complete, then*

1. *f is weakly complete,*
2. *the singular set of f is compact, and*
3. *M^2 has finite topology and each end is of puncture-type.*

Conversely, if f satisfies (1), (2) *and* (3), *and if the parabolic ends of f, if there is any, are regular, then f is complete.*

Theorem 3.9. *Any complete CMC-1 face is of finite type.*

4. The Osserman type inequality.

Lemma 4.1. *The Hopf differential of a CMC-1 face has a pole of order 2 at any complete regular parabolic end.*

Lemma 4.2. *Let $f : \Delta^* \to S^3_1$ be a complete regular end at $z = 0$ of a CMC-1 face with the Hopf differential Q and the hyperbolic Gauss map G. Then the metric $d\sigma^2_\# = \frac{4dGd\overline{G}}{(1+|G|^2)^2}$ on Δ^* satisfies*

$$\operatorname*{Ord}_{z=0}(d\sigma^2_\#) \geq \operatorname*{Ord}_{z=0}(Q) + 2\ .$$

Theorem 4.3. *A complete regular end of a CMC-1 face which is not an S^3_1-horosphere is properly embedded if and only if the degree of the hyperbolic Gauss map is 1.*