

Harmonic Morphisms and Stable Minimal Submanifolds

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1 Introduction

The theory of harmonic morphisms is one of particularly interesting subclasses of harmonic maps. A harmonic map $\varphi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds is a critical point of the energy functional defined on each compact domain of M . A harmonic morphism between Riemannian manifolds is a map preserving harmonic structure. In other words, a map $\varphi : (M^n, g) \rightarrow (N^m, h)$ is called a harmonic morphism if for any harmonic function f defined on an open subset $V \subset N$ such that $\varphi^{-1}(V) \neq \emptyset$, the composition $f \circ \varphi : \varphi^{-1}(V) \rightarrow \mathbb{R}$ is also harmonic. Harmonic morphisms are characterized as harmonic maps which are horizontally (weakly) conformal.

2 Preliminaries

Let $\varphi : (M^n, g) \rightarrow (N^m, h)$ be a smooth map between Riemannian manifolds (M, g) and (N, h) . For a point $x \in M$, we set $\mathcal{V}_x = \ker(d\varphi_x)$. The space \mathcal{V}_x is called the vertical space at x . Let \mathcal{H}_x denote the orthogonal complement of \mathcal{V}_x in the tangent space $T_x M$. For a tangent vector $X \in T_x M$, we denote $X^\mathcal{V}$ and $X^\mathcal{H}$, respectively, the vertical component and the horizontal component of X . Let \mathcal{V} and \mathcal{H} denote the corresponding vertical and horizontal distributions in the tangent bundle TM . We say that φ is horizontally (weakly) conformal if, for each point $x \in M$ at which $d\varphi_x \neq 0$, the restriction $d\varphi_x|_{\mathcal{H}_x} : \mathcal{H}_x \rightarrow T_{\varphi(x)}N$ is conformal and surjective. Thus there exists a non-negative function λ on M such that

$$h(d\varphi(X), d\varphi(Y)) = \lambda^2 g(X, Y)$$

for horizontal vectors X, Y . The function λ is called the *dilation* of φ . Note that λ^2 is smooth and is equal to $|d\varphi|^2/m$, where $m = \dim(N)$.

Let $\varphi : M^n \rightarrow N^m$ be a horizontally (weakly) conformal map between Riemannian manifolds (M, g) and (N, h) . Denote the set of critical points of φ by $C_\varphi = \{x \in M : d\varphi_x = 0\}$ and let $M^* = M - C_\varphi$. We define two tensors T and A over M^* by

$$T_E F = (\bar{\nabla}_{E^\mathcal{V}} F^\mathcal{V})^\mathcal{H} + (\bar{\nabla}_{E^\mathcal{V}} F^\mathcal{H})^\mathcal{V}$$

and

$$A_E F = (\bar{\nabla}_{E^\mathcal{H}} F^\mathcal{H})^\mathcal{V} + (\bar{\nabla}_{E^\mathcal{H}} F^\mathcal{V})^\mathcal{H}$$

for vector fields E and F on M . Here $\bar{\nabla}$ denotes the Levi-Civita connection on M .

Let M^n be an n -dimensional complete Riemannian manifold and let P be a k -dimensional immersed submanifold of M . Then the tangent space of M can be decomposed into

$$TM = TP \oplus TP^\perp.$$

Define, for two tangent vectors X, Y on P , i.e., sections of TP , the symmetric 2-tensor $B(X, Y)$ by

$$B(X, Y) = (\bar{\nabla}_X Y)^\perp = (\bar{\nabla}_X Y)^\mathcal{H} = T_X Y,$$

where $\bar{\nabla}$ is the Levi-Civita connection on M and \perp denotes the normal component. We say P is minimal if the mean curvature trace(B) = 0.

Let E be a normal vector field on P with compact support. Then the second derivative of the volume functional \mathcal{A} in the direction E ([8]) is given by

$$(1) \quad \mathcal{A}''(0) = \int_P \langle -\Delta E + \bar{\mathcal{R}}(E) - \mathcal{B}(E), E \rangle.$$

Introducing a local orthonormal basis $\{e_1, \dots, e_k, \xi_{k+1}, \dots, \xi_n\}$ on TM such that $\{\xi_{k+1}, \dots, \xi_n\}$ is a local orthonormal frame on TP^\perp , the equation (1) becomes

$$(2) \quad \mathcal{A}''(0) = \int_P |\nabla^\perp E|^2 - \sum_{i=1}^k \langle \bar{\mathcal{R}}(e_i, E)E, e_i \rangle - \sum_{i,j=1}^k \langle B(e_i, e_j), E \rangle^2$$

It is well-known that P is totally geodesic if and only if $B = 0$ and hence $\mathcal{B} = 0$.

We say a minimally immersed submanifold P of M is *stable* (or volume-stable) if, for any normal variation E with compact support, the second derivative of the volume functional in the direction E is non-negative, i. e.,

$$\mathcal{A}''(0) \geq 0.$$

3 Stability of Minimal Fibers

Let $\varphi : M^n \rightarrow N^m$ be a horizontally (weakly) conformal map between Riemannian manifolds (M, g) and (N, h) . Suppose for a point $z \in N$, the fiber $P := \varphi^{-1}(z)$ is a k -dimensional minimal submanifold of M . Then the tangent vectors to P correspond vertical vectors of φ and normal vectors to P correspond to horizontal vectors of φ .

We have

$$(3) \quad \mathcal{A}''(0) = \int_P \sum_{i=1}^k \left(|(\bar{\nabla}_{e_i} E)^\mathcal{H}|^2 - |(\bar{\nabla}_{E} e_i)^\mathcal{H}|^2 \right) + \int_P \sum_{i=1}^k \left\langle (\bar{\nabla}_{E} T)_{e_i} e_i, E \right\rangle \\ + \int_P \left\{ \frac{|E|^2}{2} \sum_{i=1}^k \langle \nabla \log \lambda^2, e_i \rangle^2 - \frac{|E|^2}{2} \sum_{i=1}^k \left\langle \bar{\nabla}_{e_i} (\nabla \log \lambda^2)^\nu, e_i \right\rangle \right\}$$

Lemma 3.1 (Key Lemma 1) *Let $\varphi : (M^n, g) \rightarrow (N^m, h)$ be a horizontally conformal submersion with dilation λ . Assume $P := \varphi^{-1}(z), z \in N$ is a submanifold of M . Then*

$$\lambda^2 \Delta_P \left(\frac{1}{\lambda^2} \right) = \left| (\nabla \log \lambda^2)^\nu \right|^2 - \operatorname{div}_P (\nabla \log \lambda^2)^\nu,$$

where Δ_P and div_P denote the Laplacian and divergence on P , respectively, and ∇ denotes the gradient on M .

Theorem 3.2 (Codimension 1) *Let $\varphi : (M^n, g) \rightarrow (N^1, h)$ be a horizontally conformal submersion with dilation λ and suppose $P = \varphi^{-1}(t), t \in N$ is a minimal hypersurface of M . If T is parallel, then P is volume-stable.*

Sketch. Using Lemma 3.1 and other equations and formulae, We obtain

$$(4) \quad \mathcal{A}''_E(0) = \int_P \left| (\nabla f)^\nu \right|^2 - \frac{1}{4} \int_P f^2 \left| (\nabla \log \lambda^2)^\nu \right|^2 + \frac{1}{2} \int_P f^2 \lambda^2 \Delta \left(\frac{1}{\lambda^2} \right).$$

Applying the integration by parts, we have

$$\mathcal{A}''_E(0) = \int_P \left| (\nabla f)^\nu \right|^2 + \frac{1}{4} \int_P f^2 \left| (\nabla \log \lambda^2)^\nu \right|^2 + \int_P f \left\langle (\nabla f)^\nu, (\nabla \log \lambda^2)^\nu \right\rangle$$

Using the arithmetic-geometric inequality $2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2$ for $\epsilon > 0$, we obtain (with $\epsilon = 2$)

$$\left| f \left\langle (\nabla f)^\nu, (\nabla \log \lambda^2)^\nu \right\rangle \right| \leq \left| (\nabla f)^\nu \right|^2 + \frac{1}{4} f^2 \left| (\nabla \log \lambda^2)^\nu \right|^2.$$

Hence one can conclude that

$$\mathcal{A}''(0) \geq 0. \quad \blacksquare$$

Corollary 3.3 *Let $\varphi : (M^n, g) \rightarrow (N^1, h)$ be a horizontally conformal submersion whose fibers are totally geodesics. Then every fiber P is volume-stable.*

Theorem 3.4 (Codimension 2) *Let $\varphi : (M^n, g) \rightarrow (N^2, h)$ be a horizontally conformal submersion with dilation λ , where N is a 2-dimensional Riemannian manifold. Suppose $P = \varphi^{-1}(z), z \in N$ is a minimal submanifold of M . If T is parallel and the horizontal distribution \mathcal{H} is integrable, then P is volume-stable.*

Corollary 3.5 *Let $\varphi : (M^n, g) \rightarrow (N^2, h)$ be a submersive harmonic morphism from an n -dimensional Riemannian manifold M^n to a 2-dimensional Riemannian manifold N^2 . If T is parallel and the horizontal distribution \mathcal{H} is integrable, then any fiber is volume-stable.*

Corollary 3.6 *Let $\varphi : (M^n, g) \rightarrow (N^2, h)$ be a submersive harmonic morphism with totally geodesic fibers. If the horizontal distribution \mathcal{H} is integrable, then any fiber is volume-stable.*

Remark 3.7 In Theorem 3.4, Corollary 3.5 or Corollary 3.6, the condition that the horizontal distribution \mathcal{H} is integrable is indispensable. For instance, the Hopf map $\varphi : S^3 \rightarrow S^2$ is a submersive harmonic morphism with totally geodesic fibers, but the fibers are not volume-stable.

In [10], Montaldo proved if a submersive harmonic morphism $\varphi : (M^n, g) \rightarrow (N^2, h)$ from a compact Riemannian manifold to a surface has volume-stable minimal fibers, then φ is energy-stable, that is, the second derivative of the energy functional is non-negative. Thus Corollary 3.5 and Corollary 3.6 imply the following corollaries.

Corollary 3.8 *Let $\varphi : (M^n, g) \rightarrow (N^2, h)$ be a submersive harmonic morphism from a compact n -dimensional Riemannian manifold M^n to a 2-dimensional Riemannian manifold N^2 . If T is parallel and the horizontal distribution \mathcal{H} is integrable, then φ is energy-stable harmonic map.*

Corollary 3.9 *Let $\varphi : (M^n, g) \rightarrow (N^2, h)$ be a submersive harmonic morphism with totally geodesic fibers and M is compact. If the horizontal distribution \mathcal{H} is integrable, then φ is energy-stable.*

The converse for Corollary 3.8 or Corollary 3.9 is not true anymore. In fact, it is known ([10]) that the quotient map $\bar{\varphi} : \mathbb{R}P^3 \rightarrow S^2$ of the Hopf map $\varphi : S^3 \rightarrow S^2$ is energy-stable, but the horizontal distribution of $\bar{\varphi}$ is not integrable.

For higher codimensional case, we have the following properties.

Theorem 3.10 *Let $\varphi : (M^n, g) \rightarrow (N^m, h)$ be a horizontally conformal submersion from an n -dimensional Riemannian manifold M^n to an m -dimensional Riemannian manifold N^m ($n \geq m \geq 3$). Suppose a fiber of φ , $P = \varphi^{-1}(z), z \in N$ is a minimal submanifold of M . If the horizontal distribution \mathcal{H} is integrable and the tensor T is parallel, then P is volume-stable.*

Corollary 3.11 *Let $\varphi : (M^n, g) \rightarrow (N^m, h)$ be a submersive harmonic morphism with totally geodesic fibers. If the horizontal distribution \mathcal{H} is integrable, then every fiber is volume-stable.*

4 Horizontal Homotheticity

Definition 4.1 *A non-constant map $\varphi : (M, g) \rightarrow (N, h)$ is said to be horizontally homothetic if it is horizontally conformal and $(\nabla \log \lambda^2)^{\mathcal{H}} = 0$*

Theorem 4.2 ([2]) *Let $n > m \geq 2$, $\varphi : (M^n, g) \rightarrow (N^m, h)$ be a horizontally homothetic harmonic morphism and P be a submanifold of N . Then the followings are equivalent.*

- (1) P is minimal in N
- (2) $\varphi^{-1}(P)$ is minimal in M .

Let $\varphi : (M^n, g) \rightarrow (N^m, h)$ be a submersion with the horizontal distribution \mathcal{H} and vertical distribution \mathcal{V} , respectively. Let $P^l \subset N$ be an l -dimensional submanifold of N and define

$$L = \varphi^{-1}(P).$$

For each $x \in L = \varphi^{-1}(P)$, we define

$$\mathcal{W}_x = T_x L, \quad \mathcal{H}'_x = \mathcal{W}_x \cap \mathcal{H}, \quad \mathcal{H}''_x = \mathcal{W}_x^\perp$$

so that we have the following orthogonal decompositions

$$\mathcal{W} = \mathcal{V} \oplus \mathcal{H}', \quad \mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'', \quad TM = \mathcal{V} \oplus \mathcal{H} = \mathcal{W} \oplus \mathcal{H}''.$$

The second fundamental form B of $L = \varphi^{-1}(P)$ in M is defined by

$$(5) \quad B : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{H}'', \quad B(X, Y) = (\overline{\nabla}_X Y)^{\mathcal{H}''},$$

Note that

$$\dim L = \dim \varphi^{-1}(P) = n - m + l$$

Lemma 4.3 (Key Lemma 3) *Let $\varphi : (M^n, g) \rightarrow (N^m, h)$ be a horizontally conformal submersion with dilation λ . Let P^l be an l -dimensional submanifold of N and let $L = \varphi^{-1}(P)$. Assume $l \leq m - 1$. Then*

$$\lambda^2 \Delta_L \left(\frac{1}{\lambda^2} \right) = |(\nabla \log \lambda^2)^\top|^2 - \operatorname{div}_L (\nabla \log \lambda^2)^\top,$$

where Δ_L and div_L denote the Laplacian and divergence on L , respectively, ∇ denotes the gradient on M and \top denotes the tangential component of L , i.e., \mathcal{W} -component.

Theorem 4.4 *Let $\varphi : (M^n, g) \rightarrow (N^m, h)$ be a horizontally homothetic harmonic morphism with dilation λ to a Riemannian manifold N of non-positive sectional curvature. Let P be a totally geodesic submanifold of N and let $L = \varphi^{-1}(P)$. If T is parallel and the horizontal distribution of φ is integrable, then L is a stable minimal submanifold of M .*

As a direct application, we can obtain the following result.

Corollary 4.5 *Let $\varphi : (M^n, g) \rightarrow (N^m, h)$ be a horizontally homothetic harmonic morphism with totally geodesic fibers to a Riemannian manifold N of non-positive sectional curvature. Let P be a totally geodesic submanifold of N and let $L = \varphi^{-1}(P)$. If the horizontal distribution of φ is integrable, then L is a stable minimal submanifold of M .*

In case of hypersurfaces, the assumption on the sectional curvature condition can be weakened to non-positive Ricci curvature.

Corollary 4.6 *Let $\varphi : (M^n, g) \rightarrow (N^m, h)$ be a horizontally homothetic harmonic morphism to a Riemannian manifold N of non-positive Ricci curvature. Let P be a totally geodesic hypersurface of N and let $L = \varphi^{-1}(P)$. If T is parallel and the horizontal distribution of φ is integrable, then L is a stable minimal hypersurface of M .*

References

- [1] P. Baird and J. Eells, A conservation law for harmonic maps, Lecture Notes in Mathematics, 894, Springer(1981), pp. 1-25.
- [2] P. Baird and S. Gudmundsson, p -harmonic maps and minimal submanifolds, Math. Ann., 294 (1992), 611-624.
- [3] B. Fuglede, Harmonic morphisms between Riemannian manifolds, Ann. Inst. Fourier, Grenoble 28 (1978), 107-144.
- [4] S. Gudmundsson, The Geometry of Harmonic Morphisms, Ph. D Thesis, 1992.
- [5] T. Ishihara, A mappings of Riemannian manifolds which preserves harmonic functions, J. Math. Kyoto Univ. 19 (1979), 215-229.
- [6] H. Jin and X. Mo, On submersive p -harmonic morphisms and their stability, Contemporary Mathematics, 308 (2002), 205-209.
- [7] A. Kasue and T. Washio, Growth of equivariant harmonic maps and harmonic morphisms, Osaka Jour. of Math., 27 (1990), 899-928.
- [8] H. B. Lawson, Lectures on minimal submanifolds, Mathematics Lecture Series, 9, Publish or Perish, 1980.
- [9] E. Loubeau, On p -harmonic morphisms, Preprint.
- [10] S. Montaldo, Stability of harmonic morphisms to a surface, Intern. J. Math., 9 (1998), 865-875.
- [11] B. O'Neill, The fundamental equations of a submersion, Mich. Math. J., 13 (1966), 459-469.

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