Harmonic Morphisms and Stable Minimal Submanifolds

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1 Introduction

The theory of harmonic morphisms is one of particularly interesting subclasses of harmonic maps. A harmonic map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is a critical point of the energy functional defined on each compact domain of M. A harmonic morphism between Riemannian manifolds is a map preserving harmonic structure. In other words, a map $\varphi : (M^n, g) \to (N^m, h)$ is called a harmonic morphism if for any harmonic function f defined on an open subset $V \subset N$ such that $\varphi^{-1}(V) \neq \emptyset$, the composition $f \circ \varphi : \varphi^{-1}(V) \to \mathbb{R}$ is also harmonic. Harmonic morphisms are characterized as harmonic maps which are horizontally (weakly) conformal.

2 Preliminaries

Let $\varphi : (M^n, g) \to (N^m, h)$ be a smooth map between Riemannian manifolds (M, g) and (N, h). For a point $x \in M$, we set $\mathcal{V}_x = \ker(d\varphi_x)$. The space $\mathcal{V}_x$ is called the vertical space at x. Let $\mathcal{H}_x$ denote the orthogonal complement of $\mathcal{V}_x$ in the tangent space T_xM. For a tangent vector $X \in T_xM$, we denote $X^{\mathcal{V}}$ and $X^{\mathcal{H}}$, respectively, the vertical component and the horizontal component of X. Let $\mathcal{V}$ and $\mathcal{H}$ denote the corresponding vertical and horizontal distributions in the tangent bundle TM. We say that φ is horizontally (weakly) conformal if, for each point $x \in M$ at which $d\varphi_x \neq 0$, the restriction $d\varphi_x|_{\mathcal{H}_x} : \mathcal{H}_x \to T_{\varphi(x)}N$ is conformal and surjective. Thus there exists a non-negative function λ on M such that

$$h(d\varphi(X), d\varphi(Y)) = \lambda^2 g(X, Y)$$

for horizontal vectors X, Y. The function λ is called the *dilation* of φ. Note that λ^2 is smooth and is equal to $|d\varphi|^2/m$, where $m = \dim(N)$.

Let $\varphi : M^n \to N^m$ be a horizontally (weakly) conformal map between Riemannian manifolds (M, g) and (N, h). Denote the set of critical points of φ by $C_\varphi = \{x \in M : d\varphi_x = 0\}$ and let $M^* = M - C_\varphi$. We define two tensors T and A over M^* by

$$T_E F = (\overline{\nabla}_{E^{\mathcal{V}}} F^{\mathcal{V}})^{\mathcal{H}} + (\overline{\nabla}_{E^{\mathcal{V}}} F^{\mathcal{H}})^{\mathcal{V}}$$

and

$$A_E F = (\overline{\nabla}_{E^{\mathcal{H}}} F^{\mathcal{H}})^{\mathcal{V}} + (\overline{\nabla}_{E^{\mathcal{H}}} F^{\mathcal{V}})^{\mathcal{H}}$$

for vector fields E and F on M. Here $\overline{\nabla}$ denotes the Levi-Civita connection on M.

Let M^n be an n-dimensional complete Riemannian manifold and let P be a k-dimensional immersed submanifold of M. Then the tangent space of M can be decomposed into

$$TM = TP \oplus TP^{\perp}.$$

Define, for two tangent vectors X, Y on P, i.e., sections of TP, the symmetric 2-tensor $B(X,Y)$ by

$$B(X,Y) = (\overline{\nabla}_X Y)^{\perp} = (\overline{\nabla}_X Y)^{\mathcal{H}} = T_X Y,$$

where $\overline{\nabla}$ is the Levi-Civita connection on M and $\perp$ denotes the normal component. We say P is minimal if the mean curvature trace$(B) = 0$.

Let E be a normal vector field on P with compact support. Then the second derivative of the volume functional $\mathcal{A}$ in the direction E ([8]) is given by

$$\mathcal{A}''(0) = \int_P \langle -\Delta E + \overline{\mathcal{R}}(E) - \mathcal{B}(E), E\rangle. \tag{1}$$

Introducing a local orthonormal basis $\{e_1, \cdots, e_k, \xi_{k+1}, \cdots, \xi_n\}$ on TM such that $\{\xi_{k+1}, \cdots, \xi_n\}$ is a local orthonormal frame on $TP^{\perp}$, the equation (1) becomes

$$\mathcal{A}''(0) \;=\; \int_P |\nabla^{\perp} E|^2 - \sum_{i=1}^{k} \langle \overline{R}(e_i, E)E, e_i\rangle - \sum_{i,j=1}^{k} \langle B(e_i, e_j), E\rangle^2 \tag{2}$$

It is well-known that P is totally geodesic if and only if $B = 0$ and hence $\mathcal{B} = 0$.

We say a minimally immersed submanifold P of M is *stable* (or volume-stable) if, for any normal variation E with compact support, the second derivative of the volume functional in the direction E is non-negative, i. e.,

$$\mathcal{A}''(0) \geq 0.$$

3 Stability of Minimal Fibers

Let $\varphi : M^n \to N^m$ be a horizontally (weakly) conformal map between Riemannian manifolds (M, g) and (N, h). Suppose for a point $z \in N$, the fiber $P := \varphi^{-1}(z)$ is a k-dimensional minimal submanifold of M. Then the tangent vectors to P correspond vertical vectors of φ and normal vectors to P correspond to horizontal vectors of φ.

We have

$$\begin{aligned} \mathcal{A}''(0) \;=\; & \int_P \sum_{i=1}^{k} \left(\left| (\overline{\nabla}_{e_i} E)^{\mathcal{H}} \right|^2 - \left| (\overline{\nabla}_E e_i)^{\mathcal{H}} \right|^2 \right) + \int_P \sum_{i=1}^{k} \left\langle (\overline{\nabla}_E T)_{e_i} e_i, E \right\rangle \\ & + \int_P \left\{ \frac{|E|^2}{2} \sum_{i=1}^{k} \langle \nabla \log \lambda^2, e_i\rangle^2 - \frac{|E|^2}{2} \sum_{i=1}^{k} \left\langle \overline{\nabla}_{e_i} \left(\nabla \log \lambda^2\right)^{\mathcal{V}}, e_i \right\rangle \right\} \end{aligned} \tag{3}$$

Lemma 3.1 (Key Lemma 1) *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion with dilation λ. Assume $P := \varphi^{-1}(z), z \in N$ is a submanifold of M. Then*

$$\lambda^2 \Delta_P \left(\frac{1}{\lambda^2}\right) = \left|(\nabla \log \lambda^2)^{\mathcal{V}}\right|^2 - \mathrm{div}_P(\nabla \log \lambda^2)^{\mathcal{V}},$$

where Δ_P and div_P denote the Laplacian and divergence on P, respectively, and ∇ denotes the gradient on M.

Theorem 3.2 (Codimension 1) *Let $\varphi : (M^n, g) \to (N^1, h)$ be a horizontally conformal submersion with dilation λ and suppose $P = \varphi^{-1}(t), t \in N$ is a minimal hypersurface of M. If T is parallel, then P is volume-stable.*

Sketch. Using Lemma 3.1 and other equations and formulae, We obtain

$$\mathcal{A}''_E(0) = \int_P \left|(\nabla f)^{\mathcal{V}}\right|^2 - \frac{1}{4}\int_P f^2 \left|(\nabla \log \lambda^2))^{\mathcal{V}}\right|^2 + \frac{1}{2}\int_P f^2\lambda^2\Delta\left(\frac{1}{\lambda^2}\right). \tag{4}$$

Applying the integration by parts, we have

$$\mathcal{A}''_E(0) = \int_P \left|(\nabla f)^{\mathcal{V}}\right|^2 + \frac{1}{4}\int_P f^2 \left|(\nabla \log \lambda^2))^{\mathcal{V}}\right|^2 + \int_P f\left\langle(\nabla f)^{\mathcal{V}}, (\nabla \log \lambda^2)^{\mathcal{V}}\right\rangle$$

Using the arithmetic-geometric inequality $2ab \le \epsilon a^2 + \frac{1}{\epsilon} b^2$ for $\epsilon > 0$, we obtain (with $\epsilon = 2$)

$$\left|f\left\langle(\nabla f)^{\mathcal{V}}, (\nabla \log \lambda^2)^{\mathcal{V}}\right\rangle\right| \le \left|(\nabla f)^{\mathcal{V}}\right|^2 + \frac{1}{4} f^2 \left|(\nabla \log \lambda^2))^{\mathcal{V}}\right|^2.$$

Hence one can conclude that

$$\mathcal{A}''(0) \ge 0.$$

∎

Corollary 3.3 *Let $\varphi : (M^n, g) \to (N^1, h)$ be a horizontally conformal submersion whose fibers are totally geodesics. Then every fiber P is volume-stable.*

Theorem 3.4 (Codimension 2) *Let $\varphi : (M^n, g) \to (N^2, h)$ be a horizontally conformal submersion with dilation λ, where N is a 2-dimensional Riemannian manifold. Suppose $P = \varphi^{-1}(z), z \in N$ is a minimal submanifold of M. If T is parallel and the horizontal distribution $\mathcal{H}$ is integrable, then P is volume-stable.*

Corollary 3.5 *Let $\varphi : (M^n, g) \to (N^2, h)$ be a submersive harmonic morphism from an n-dimensional Riemannian manifold M^n to a 2-dimensional Riemannian manifold N^2. If T is parallel and the horizontal distribution $\mathcal{H}$ is integrable, then any fiber is volume-stable.*

Corollary 3.6 *Let $\varphi : (M^n, g) \to (N^2, h)$ be a submersive harmonic morphism with totally geodesic fibers. If the horizontal distribution $\mathcal{H}$ is integrable, then any fiber is volume-stable.*

Remark 3.7 In Theorem 3.4, Corollary 3.5 or Corollary 3.6, the condition that the horizontal distribution $\mathcal{H}$ is integrable is indispensable. For instance, the Hopf map $\varphi : S^3 \to S^2$ is a submersive harmonic morphism with totally geodesic fibers, but the fibers are not volume-stable.

In [10], Montaldo proved if a submersive harmonic morphism $\varphi : (M^n, g) \to (N^2, h)$ from a compact Riemannian manifold to a surface has volume-stable minimal fibers, then φ is energy-stable, that is, the second derivative of the energy functional is non-negative. Thus Corollary 3.5 and Corollary 3.6 imply the following corollaries.

Corollary 3.8 *Let $\varphi : (M^n, g) \to (N^2, h)$ be a submersive harmonic morphism from a compact n-dimensional Riemannian manifold M^n to a 2-dimensional Riemannian manifold N^2. If T is parallel and the horizontal distribution $\mathcal{H}$ is integrable, then φ is energy-stable harmonic map.*

Corollary 3.9 *Let $\varphi : (M^n, g) \to (N^2, h)$ be a submersive harmonic morphism with totally geodesic fibers and M is compact. If the horizontal distribution $\mathcal{H}$ is integrable, then φ is energy-stable.*

The converse for Corollary 3.8 or Corollary 3.9 is not true anymore. In fact, it is known ([10]) that the quotient map $\overline{\varphi} : \mathbb{R}P^3 \to S^2$ of the Hopf map $\varphi : S^3 \to S^2$ is energy-stable, but the horizontal distribution of $\overline{\varphi}$ is not integrable.

For higher codimensional case, we have the following properties.

Theorem 3.10 *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion from an n-dimensional Riemannian manifold M^n to an m-dimensional Riemannian manifold N^m $(n \geq m \geq 3)$. Suppose a fiber of φ, $P = \varphi^{-1}(z), z \in N$ is a minimal submanifold of M. If the horizontal distribution $\mathcal{H}$ is integrable and the tensor T is parallel, then P is volume-stable.*

Corollary 3.11 *Let $\varphi : (M^n, g) \to (N^m, h)$ be a submersive harmonic morphism with totally geodesic fibers. If the horizontal distribution $\mathcal{H}$ is integrable, then every fiber is volume-stable.*

4 Horizontal Homotheticity

Definition 4.1 *A non-constant map $\varphi : (M, g) \to (N, h)$ is said to be horizontally homothetic if it is horizontally conformal and $\left(\nabla \log \lambda^2\right)^{\mathcal{H}} = 0$*

Theorem 4.2 ([2]) *Let $n > m \geq 2$, $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally homothetic harmonic morphism and P be a submanifold of N. Then the followings are equivalent.*

(1) *P is minimal in N*

(2) *$\varphi^{-1}(P)$ is minimal in M.*

Let $\varphi : (M^n, g) \to (N^m, h)$ be a submersion with the horizontal distribution $\mathcal{H}$ and vertical distribution $\mathcal{V}$, respectively. Let $P^l \subset N$ be an l-dimensional submanifold of N and define

$$L = \varphi^{-1}(P).$$

For each $x \in L = \varphi^{-1}(P)$, we define

$$\mathcal{W}_x = T_x L, \quad \mathcal{H}'_x = W_x \cap \mathcal{H}, \quad \mathcal{H}''_x = W_x^{\perp}$$

so that we have the following orthogonal decompositions

$$\mathcal{W} = \mathcal{V} \oplus \mathcal{H}', \quad \mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'', \quad TM = \mathcal{V} \oplus \mathcal{H} = \mathcal{W} \oplus \mathcal{H}''.$$

The second fundamental form B of $L = \varphi^{-1}(P)$ in M is defined by

$$B : \mathcal{W} \times \mathcal{W} \to \mathcal{H}'', \qquad B(X, Y) = \left(\overline{\nabla}_X Y\right)^{\mathcal{H}''}, \tag{5}$$

Note that

$$\dim L = \dim \varphi^{-1}(P) = n - m + l$$

Lemma 4.3 (Key Lemma 3) *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion with dilation λ. Let P^l be an l-dimensional submanifold of N and let $L = \varphi^{-1}(P)$. Assume $l \leq m - 1$. Then*

$$\lambda^2 \Delta_L \left(\frac{1}{\lambda^2}\right) = \left|(\nabla \log \lambda^2)^{\intercal}\right|^2 - \mathrm{div}_L (\nabla \log \lambda^2)^{\intercal},$$

where Δ_L and div_L denote the Laplacian and divergence on L, respectively, ∇ denotes the gradient on M and $\intercal$ denotes the tangential component of L, i.e., $\mathcal{W}$-component.

Theorem 4.4 *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally homothetic harmonic morphism with dilation λ to a Riemannian manifold N of non-positive sectional curvature. Let P be a totally geodesic submanifold of N and let $L = \varphi^{-1}(P)$. If T is parallel and the horizontal distribution of φ is integrable, then L is a stable minimal submanifold of M.*

As a direct application, we can obtain the following result.

Corollary 4.5 *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally homothetic harmonic morphism with totally geodesic fibers to a Riemannian manifold N of non-positive sectional curvature. Let P be a totally geodesic submanifold of N and let $L = \varphi^{-1}(P)$. If the horizontal distribution of φ is integrable, then L is a stable minimal submanifold of M.*

In case of hypersurfaces, the assumption on the sectional curvature condition can be weakened to non-positive Ricci curvature.

Corollary 4.6 *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally homothetic harmonic morphism to a Riemannian manifold N of non-positive Ricci curvature. Let P be a totally geodesic hypersurface of N and let $L = \varphi^{-1}(P)$. If T is parallel and the horizontal distribution of φ is integrable, then L is a stable minimal hypersurface of M.*

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