

# Sharp Isoperimetric Inequalities for Stationary Varifolds and Area Minimizing Flat Chains mod $k$

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It is well known [4, 5, 8] that a smooth minimal surface  $\Sigma$  spanning a rectifiable Jordan curve  $C$  satisfies the isoperimetric inequality

$$4\pi \text{Area}(\Sigma) \leq \text{Length}(C)^2, \quad (1)$$

where equality holds if and only if  $C$  is a circle and  $\Sigma$  is a disk. Some smooth minimal surfaces in  $\mathbf{R}^3$  can be physically realized as soap films. However, the soap films formed by dipping some wire frames in soap solution contain interior singular curves. Here arises the main question of this paper: Does (1) still hold for this soap-film-like surface with singularities? In 1986 one of Almgren's results [2] answered this question affirmatively for area minimizing flat chains mod  $k$ . In this paper we extend his two-dimensional result and show that (1) holds also for two-dimensional stationary varifolds with connected boundary of multiplicity  $\geq 1$  (Theorem 2). Moreover, if the bounding curve  $C$  consists of  $k$  curves having the same end points, we obtain a new type of sharp isoperimetric inequality for area minimizing flat chains mod  $k$  spanned by  $C$ . Here, unlike (1), equality holds only for the union of  $k$  flat half disks with a common diameter (Theorem 3).

## 1 Arcs and sectors

In this section we derive sharp isoperimetric inequalities for domains in the plane where only a specific part of the boundary counts toward the length of the boundary.

**Lemma 1** *Let  $l_1$  and  $l_2$  be the rays emanating from a point  $O$  with an angle of  $\theta \leq \pi$ . Let  $C$  be a curve from a point of  $l_1$  to a point of  $l_2$ .*

(a) *Suppose that  $C$  lies in the smaller sector of the two formed by the rays ( $C$  may lie in either sector if  $\theta = \pi$ ). Define  $D$  as the domain bounded by  $l_1, l_2$ , and  $C$ . Then*

$$2\theta \text{Area}(D) \leq \text{Length}(C)^2,$$

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and equality holds if and only if  $C$  is a circular arc perpendicular to the rays.

(b) If  $C$  lies in the larger sector, then

$$2\pi \text{Area}(D) \leq \text{Length}(C)^2,$$

where equality holds if and only if  $C$  is a semicircle perpendicular to one of the two rays.

*Proof.* (a) For the existence of an optimal curve of fixed length enclosing the largest area, we refer to [10, pp 441-444]: the set  $S$  of convex curves of fixed length lying in the smaller sector, being equipped with the topology corresponding to the Hausdorff metric, is compact; the two functions defined on  $S$ , Length and Area, are continuous. Let  $C$  be an optimal curve which is convex and lies in the smaller sector, and let  $p_i \in l_i, i = 1, 2$ , be the end points of  $C$ . We claim that the triangle  $Op_1p_2$  is an isosceles triangle. If it is not, choose  $p'_i \in l_i$  such that  $Op'_1p'_2$  is an isosceles triangle with  $\text{Length}(\overline{p'_1p'_2}) = \text{Length}(\overline{p_1p_2})$ . Let  $C'$  be the curve from  $p'_1$  to  $p'_2$  which is congruent to  $C$ . Then the domain  $D'$  enclosed by  $C'$  and  $l_1, l_2$  has larger area than  $D$ . If  $C'$  lies in the smaller sector, the existence of  $C'$  contradicts the optimality of  $C$ , and so our claim follows. Suppose a part  $C''$  of  $C'$  leaves the smaller sector. Let  $p' \in l_1 \cup l_2$  be the end point of  $C''$  which is not an end point of  $C'$ . Rotate  $C''$  around  $p'$  by  $180^\circ$  and obtain a curve  $\bar{C}$  in the smaller sector. Then  $(C' \sim C'') \cup \bar{C}$  is a nonconvex curve of the same length as  $C$  in the smaller sector which encloses the same area as  $C$ . Hence one could find a convex curve of smaller length enclosing larger area than  $C$ , which is again a contradiction. Therefore  $Op_1p_2$  must be an isosceles triangle.

Next we claim that  $C$  is a circular arc.  $C$  lies on one side of the straight line  $l$  containing  $p_1$  and  $p_2$ . Find a circle  $\gamma$  through  $p_1$  and  $p_2$  such that an arc  $\alpha$  of  $\gamma$  from  $p_1$  to  $p_2$  has the same length as  $C$  and lies on the same side of  $l$  as  $C$ . Then the closed curve  $(\gamma \sim \alpha) \cup C$  gives a solution to the classical isoperimetric problem. The uniqueness of the solution implies  $C = \alpha$ .

So we have infinitely many candidates for the optimal curve: circular arcs of the same length that are perpendicular to the ray  $l_3$  which bisects the smaller sector formed by  $l_1$  and  $l_2$ . We claim that the arc we want should be perpendicular to the rays  $l_1$  and  $l_2$  too. For any circular arc  $A$  perpendicular to  $l_3$ ,  $l_3$  divides  $A$  into two subarcs,  $A_1$  and  $A_2$ . Move  $A_1$  and  $A_2$  separately to the opposite side of  $l_3$  by the rotations about  $O$  by the angles of  $\theta/2$  and  $-\theta/2$ . Then we obtain a curve  $A'$  in the sector which is of the same length as  $A$  and encloses the same area as  $A$  does. If  $A$  is not perpendicular to  $l_1$  and  $l_2$ , then  $A'$  is not smooth at  $A' \cap l_3$  and hence  $A'$  cannot enclose maximal area. Thus the circular arc perpendicular to  $l_1$  and  $l_2$  and only this arc maximizes the area of the domain bounded by  $l_1$  and  $l_2$ . Since this arc satisfies  $2\theta \text{Area}(D) = \text{Length}(C)^2$ , the desired inequality follows.

(b) Given any curve  $C$  joining  $l_1$  to  $l_2$  in the larger sector, one can increase the enclosed area of the domain bounded by  $l_1, l_2$  and  $C$  by moving (rotating and translating)  $C$  into a curve

$C'$  in such a way that both the end points of  $C'$  lie on  $l_1$ . Therefore part (a) completes the proof.

## 2 Cones with vertex on the boundary

Some two-dimensional cones in  $\mathbf{R}^n$  satisfy the classical isoperimetric inequality (see [5, Theorem 1]). This follows from the fact that two-dimensional cones, being flat, can be flattened (i.e., developed) to become a planar domain provided its density at the vertex is not smaller than 1. However, if the vertex lies on the boundary of the cone, the density hypothesis can be dropped [5, Corollary 1].

- Definition 1** i) A *compound Jordan curve* is a one-dimensional rectifiable connected set in  $\mathbf{R}^n$  which is the union of finitely many Jordan curves (= homeomorphic images of a circle).
- ii) For  $p \in \mathbf{R}^n$ ,  $p \ast C$  is the *cone* from  $p$  over  $C$ , the set of all line segments from  $p$  to the points of  $C$ .

**Lemma 2** *If  $C$  is a compound Jordan curve in  $\mathbf{R}^n$  and  $p$  a point of  $C$ , then*

$$4\pi \text{Area}(p \ast C) \leq \text{Length}(C)^2.$$

*Equality holds if and only if  $p \ast C$  can be developed, by cutting and inserting, one-to-one onto a disk.*

*Proof.* Choose a point  $q$  from  $C$  such that its distance from  $p$  is the maximum among all points of  $C$ . One can easily find a pair of curves  $\gamma', \gamma'' \subset C$  joining  $p$  to  $q$  and intersecting each other at finitely many points. Let  $\gamma_1 = \gamma' \cup \gamma''$ . Take  $\gamma_2, \gamma_3, \dots, \gamma_k \subset C$  to be pairwise disjoint curves such that

$$C = \bigcup_{i=1}^k \gamma_i \quad \text{and} \quad \text{Length}(C) = \sum_{i=1}^k \text{Length}(\gamma_i). \quad (2)$$

Now let us define a canonical parameter  $\theta$  for each  $\gamma_i$ . Let  $l_i(\theta) = \text{dist}(p, \gamma_i(\theta))$ . The desired canonical parameter  $\theta$  is assumed to satisfy the condition that

$$\text{Area}(p \ast \gamma_i([a, b])) = \frac{1}{2} \int_a^b l_i^2(\theta) d\theta \quad \text{for any } \gamma_i(a), \gamma_i(b) \in \gamma_i.$$

From this condition one can easily understand that  $\theta$  is the angle parameter viewed from  $p$ . Assume that each curve  $\gamma_i(\theta)$  is defined on an interval  $[0, a_i]$ , and assume  $\gamma_1(0) = \gamma_1(a_1) = p$ . But  $\gamma_i(0), i \geq 2$ , may or may not coincide with  $\gamma_i(a_i)$ . With each  $\gamma_i$  can one associate a

curve  $C_i$  on a plane  $\Pi$  which is the polar graph of  $l_i(\theta)$ : more precisely,  $C_i$  is the set of all points  $(l_i(\theta), \theta)$  in polar coordinates on  $\Pi$ . Then

$$\text{Length}(\gamma_i) = \text{Length}(C_i), \quad \text{Area}(p \ast \gamma_i) = \text{Area}(O \ast C_i), \quad (3)$$

where  $O$  is the origin of  $\Pi$ . Geometrically, one can obtain  $O \ast C_i$  by developing  $p \ast \gamma_i$  on  $\Pi$ , cutting  $p \ast \gamma_i$  along the line segment  $\overline{p\gamma_i(0)}$  if  $\gamma_i(0) = \gamma_i(a_i)$ .

Without loss of generality one can assume that  $a_i \leq \pi$  for all  $i \geq 2$ , for otherwise one can divide  $\gamma_i$  into smaller parametrized curves with the desired property. By Lemma 1(a) there exists a circular arc  $A_i, i \geq 2$ , on the sector formed by  $O \ast C_i$  such that

$$\text{Area}(O \ast C_i) = \text{Area}(O \ast A_i), \quad \text{Length}(C_i) \geq \text{Length}(A_i). \quad (4)$$

Note that since  $q$  is a maximum distance away from  $p$ ,

$$\text{dist}(O, A_i) \in \{\text{dist}(O, m) : m \in C_i\} = \{\text{dist}(p, s) : s \in \gamma_i\} \subset \{\text{dist}(p, r) : r \in \gamma_1\}.$$

Hence there exists  $b_i \in (0, a_1), i \geq 2$ , such that  $l_1(b_i) = \text{dist}(O, A_i)$ . Renumbering  $\gamma_i$  if necessary, we can assume  $b_2 \leq b_3 \leq \dots \leq b_k$ . Define a new function  $l : [0, \sum_{i=1}^k a_i] \rightarrow \mathbf{R}$  by

$$\begin{aligned} l(\theta) &= l_1(\theta), & \text{if } 0 \leq \theta \leq b_2 \\ l(\theta) &= l_1(b_2), & \text{if } b_2 \leq \theta \leq b_2 + a_2 \\ l(\theta) &= l_1(\theta - a_2), & \text{if } b_2 + a_2 \leq \theta \leq b_3 + a_2 \\ l(\theta) &= l_1(b_3), & \text{if } b_3 + a_2 \leq \theta \leq b_3 + a_2 + a_3 \\ &\dots & \cdot \dots \dots \dots \\ l(\theta) &= l_1\left(\theta - \sum_{i=2}^{j-1} a_i\right), & \text{if } b_{j-1} + \sum_{i=2}^{j-1} a_i \leq \theta \leq b_j + \sum_{i=2}^{j-1} a_i \\ l(\theta) &= l_1(b_j), & \text{if } b_j + \sum_{i=2}^{j-1} a_i \leq \theta \leq b_j + \sum_{i=2}^j a_i \\ &\dots & \cdot \dots \dots \dots \\ l(\theta) &= l_1\left(\theta - \sum_{i=2}^k a_i\right), & \text{if } \max b_j + \sum_{i=2}^k a_i \leq \theta \leq \sum_{i=1}^k a_i. \end{aligned}$$

Geometrically, the cone  $O \ast \bar{C}$ ,  $\bar{C}$  being the polar graph of  $l$  on  $\Pi$ , can be obtained from  $O \ast C_1$  by cutting out the line segments  $\overline{OC_1(b_i)}$ , widening these gaps by the angle of  $a_i$ , and inserting  $O \ast A_i$  in each gap. (See [5, Theorem 1] and [6, Lemma 4] for similar arguments.)

In the construction of  $l$  above, we tacitly assumed that  $b_i$  are all distinct. In case  $b_i = b_{i+1}$ , one can attach  $A_i$  to  $A_{i+1}$  to obtain a new arc  $A_i \cup A_{i+1}$  and then insert  $O \ast (A_i \cup A_{i+1})$  into the widened gap of  $\overline{OC_1(b_i)}$ .

Since the cutting and inserting process preserves the length of the curve and the area of the cone,  $\bar{C}$  satisfies

$$\begin{aligned} \text{Length}(\bar{C}) &= \text{Length}(C_1) + \sum_{i=2}^k \text{Length}(A_i), \\ \text{Area}(O \ast \bar{C}) &= \text{Area}(O \ast C_1) + \sum_{i=2}^k \text{Area}(O \ast A_i). \end{aligned} \quad (5)$$

Here the curve  $\bar{C}$  may have self intersections and the cone  $O \ast \bar{C}$  may have overlapping parts. In this case  $\text{Area}(O \ast \bar{C})$  should count the multiplicity of the overlapping.

By (2),(3),(4),(5), we have

$$\text{Area}(p \ast C) = \text{Area}(O \ast \bar{C}), \quad \text{Length}(C) \geq \text{Length}(\bar{C}).$$

Since  $l(0) = l(\sum_{i=1}^k a_i) = 0$ ,  $\bar{C}$  is a closed curve containing  $O$ . Therefore the conclusion follows from [5, Corollary 1].

### 3 Stationary varifolds

Our purpose in this section is to prove the isoperimetric inequality for a stationary 2-dimensional varifold in  $\mathbf{R}^n$ . It is known that the restriction of a stationary varifold  $V$  to the set of points of positive density is rectifiable [3, 1]. And Allard proved that if the density of a stationary varifold is essentially bounded away from zero, then an open dense subset of the support of  $V$  is a continuously differentiable submanifold of  $\mathbf{R}^n$  [1]. Following [1] and [9], we briefly introduce varifolds in  $\mathbf{R}^n$ , define stationary varifolds and their generalized boundary, and derive an area estimate of a stationary varifold from the first variation formula.

$m$ -dimensional varifolds in  $\mathbf{R}^n$  are simply Radon measures on  $G_m(\mathbf{R}^n) = \mathbf{R}^n \times G(n, m)$ , where  $G(n, m)$  is the space of  $m$ -dimensional subspaces of  $\mathbf{R}^n$ . Given such an  $m$ -varifold  $V$  in  $\mathbf{R}^n$ , there corresponds a Radon measure  $\mu_V$  on  $\mathbf{R}^n$  defined by

$$\mu_V(A) = V(\pi^{-1}(A)), \quad A \subset \mathbf{R}^n,$$

where  $\pi$  is the projection  $(x, S) \mapsto x$  of  $G_m(\mathbf{R}^n)$  onto  $\mathbf{R}^n$ . The *mass*  $\mathbf{M}(V)$  of  $V$  is defined by

$$\mathbf{M}(V) = \mu_V(\mathbf{R}^n) = V(G_m(\mathbf{R}^n)).$$

If  $M = \text{spt} \mu_V$  is rectifiable, then  $\mu_V = \mathcal{H}^m \llcorner \theta$ , where  $\mathcal{H}^m$  is the  $m$ -dimensional Hausdorff measure,  $\theta$  vanishes on  $\mathbf{R}^n \setminus M$  and is a positive locally  $\mathcal{H}^m$ -integrable function on  $M$ .  $\theta$  is called the *multiplicity function* of  $\mu_V$ . The support  $M$  and multiplicity  $\theta$  of  $\mu_V$  completely determine  $V$  when  $V$  is rectifiable. So the varifold  $V$  is also denoted by  $\underline{v}(M, \theta)$ . When  $\theta$  is

integer valued almost everywhere,  $V$  is called an *integral* varifold.

Let  $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be differentiable. Then we define the image varifold  $f_{\#}V$  of  $V$  by

$$f_{\#}V(A) = \int_{F^{-1}(A)} J_S f(x) dV(x, S), \quad A \subset G_m(\mathbf{R}^n),$$

where  $F : G_m(\mathbf{R}^n) \rightarrow G_m(\mathbf{R}^n)$  is defined by  $F(x, S) = (f(x), df_x(S))$  and where

$$J_S f(x) = \sqrt{\det((df_x|S)^* \circ (df_x|S))}, \quad (x, S) \in G_m(\mathbf{R}^n),$$

$(df_x|S)^*$  being the adjoint of  $df_x|S$ .

The *first variation*  $\delta V$  of  $V$  is a linear functional on the set  $\mathcal{X}(\mathbf{R}^n)$  of continuous vector fields with compact support on  $\mathbf{R}^n$ , defined by

$$\delta V(Y) = \frac{d}{dt} \mathbf{M}(\phi_{t\#}V)|_{t=0},$$

where  $\{\phi_t\}_{-1 < t < 1}$  is any 1-parameter family of diffeomorphisms in  $\mathbf{R}^n$  with  $Y$  as the initial velocity vector field. Differentiation under the integral gives

$$\delta V(Y) = \int_{G_m(\mathbf{R}^n)} \operatorname{div}_S Y(x) dV(x, S),$$

where

$$\operatorname{div}_S Y = \sum_{i=1}^m \langle \tau_i, D_{\tau_i} Y \rangle,$$

$\tau_1, \dots, \tau_m$  being an orthonormal basis for  $S$ , and  $D$  the Euclidean connection.

$V$  is said to be *stationary* in  $U$  if  $\delta V(Y) = 0$  for any  $Y \in \mathcal{X}(\mathbf{R}^n)$  with  $\operatorname{spt}|Y| \subset U$ .

Now we want to define  $\|\delta V\|$ , the *total variation measure* of  $\delta V$ . Assume that  $V$  has *locally bounded first variation* in  $\mathbf{R}^n$ , that is, for each  $W \subset \subset \mathbf{R}^n$  there is a constant  $c < \infty$  such that  $|\delta V(Y)| \leq c \sup|Y|$  for any  $Y \in \mathcal{X}(\mathbf{R}^n)$  with  $\operatorname{spt}|Y| \subset W$ . Then the Riesz representation theorem says that there exist a Radon measure  $\|\delta V\|$  on  $\mathbf{R}^n$  and a  $\|\delta V\|$ -measurable vector field  $\nu$  on  $\mathbf{R}^n$  such that  $|\nu| = 1$   $\|\delta V\|$ -a.e. and

$$\delta V(Y) = \int_{\mathbf{R}^n} \nu \cdot Y d\|\delta V\|,$$

where  $\|\delta V\|$  is characterized by

$$\|\delta V\|(W) = \sup\{\delta V(Y) : Y \in \mathcal{X}(\mathbf{R}^n), |Y| \leq 1, \text{ and } \operatorname{spt}|Y| \subset W\}$$

for any open  $W \subset \subset \mathbf{R}^n$ . Differentiating  $\|\delta V\|$  with respect to  $\mu_V$ , we see that

$$\frac{d\|\delta V\|}{d\mu_V}(x) = \lim_{\rho \rightarrow 0} \frac{\|\delta V\|(B_\rho(x))}{\mu_V(B_\rho(x))}$$

exists  $\mu_V$ -a.e. and that

$$\int_{\mathbf{R}^n} \nu \cdot Y d\|\delta V\| = - \int_{\mathbf{R}^n} \vec{H} \cdot Y d\mu_V + \int_{\mathbf{R}^n} \nu \cdot Y d\sigma,$$

where

$$\vec{H}(x) = - \frac{d\|\delta V\|}{d\mu_V}(x) \nu(x), \quad \sigma = \|\delta V\| \llcorner Z,$$

$$Z = \{x \in \mathbf{R}^n : \frac{d\|\delta V\|}{d\mu_V}(x) = \infty\}, \quad \text{and } \mu_V(Z) = 0.$$

Thus for  $Y \in \mathcal{X}(\mathbf{R}^n)$  we can write

$$\delta V(Y) = - \int_{\mathbf{R}^n} \vec{H} \cdot Y d\mu_V + \int_Z \nu \cdot Y d\sigma. \quad (6)$$

By analogy with the classical first variation formula for a smooth submanifold of  $\mathbf{R}^n$ , we call  $\vec{H}$  the *generalized mean curvature* of  $V$ ,  $Z$  the *generalized boundary* of  $V$ ,  $\sigma$  the *generalized boundary measure* of  $V$ , and  $\nu \llcorner Z$  the *generalized unit conormal* of  $V$ . We can easily see that  $V$  is stationary in  $U$  if and only if  $\vec{H} \llcorner U = 0$  and  $Z \cap U = \emptyset$ .

**Definition 2** (a) Let  $V$  be an  $m$ -dimensional varifold of locally bounded first variation in  $\mathbf{R}^n$  and  $Z$  the generalized boundary of  $V$  with the generalized boundary measure  $\sigma$ . Assume  $Z$  is  $(m-1)$ -rectifiable. Let

$$\psi(x) = \lim_{\rho \rightarrow 0} \frac{\sigma(B_\rho(x))}{\mathcal{H}^{m-1}(Z \cap B_\rho(x))}, \quad x \in Z.$$

Then define the *varifold boundary*  $\partial V$  of  $V$  to be the varifold  $\underline{\underline{v}}(Z, \psi)$ . In other words,  $\partial V$  is the  $(m-1)$ -dimensional rectifiable varifold with support  $Z$  and multiplicity  $\psi$ . Clearly  $\mu_{\partial V} = \sigma$ .

(b) For an  $m$ -varifold  $V = \underline{\underline{v}}(M, \theta)$ , the *varifold cone*  $p \ast V$  from  $p$  over  $V$  is the  $(m+1)$ -varifold  $\underline{\underline{v}}(p \ast M, \bar{\theta})$ , where  $\bar{\theta}(y) = \theta(x)$  whenever  $y$  lies on the line segment from  $p$  to  $x \in M$ .

**Example** Given a cube  $I^3$  of volume 1 in  $\mathbf{R}^3$ , let  $F$  be the union of the faces of  $I^3$ ,  $E$  the union of the edges of  $I^3$ . Define  $V$  to be the 2-dimensional varifold with support  $F$  and multiplicity 1 everywhere, i.e.,  $V = \underline{\underline{v}}(F, 1)$ . Then one can see that i) the generalized mean curvature  $\vec{H}$  of  $V$  vanishes on  $F \sim E$ , ii)  $E$  is the generalized boundary of  $V$ , iii)  $\sigma = (\mathcal{H}^1 \llcorner E) \llcorner \sqrt{2}$  is the generalized boundary measure of  $V$ , and iv) the generalized unit conormal  $\nu$  of  $V$  makes an angle of 45 degrees with the outward unit normals to  $F$  along  $E$ . It follows that  $V$  is stationary in  $\mathbf{R}^3 \sim E$ , the multiplicity of  $\partial V$  is  $\sqrt{2}$ , i.e.,  $\partial V = \underline{\underline{v}}(E, \sqrt{2})$ , and  $p \ast \partial V = \underline{\underline{v}}(p \ast E, \sqrt{2})$ ,  $p \in \mathbf{R}^3$ . Moreover  $\mathbf{M}(V) = 6$ ,  $\mathbf{M}(\partial V) = 12\sqrt{2}$ , and if  $p_0, p_1$  are the center of gravity and a vertex of  $I^3$  respectively,  $\mathbf{M}(p_0 \ast \partial V) = 6$ ,  $\mathbf{M}(p_1 \ast \partial V) = 3+3\sqrt{2}$ .

In [5, Proposition 1] we have proved a volume estimate for minimal submanifolds in  $\mathbf{R}^n$ . We extend this estimate to stationary varifolds in  $\mathbf{R}^n$  as follows.

**Theorem 1** *Let  $V$  be an  $m$ -varifold of locally bounded first variation in  $\mathbf{R}^n$ . If the generalized boundary  $Z$  of  $V$  is rectifiable and  $V$  is stationary in  $\mathbf{R}^n \sim Z$ , then for any  $p \in \mathbf{R}^n$*

$$\mathbf{M}(V) \leq \mathbf{M}(p \ast \partial V).$$

*Proof.* We see from (6) that

$$\delta V(Y) = \int_Z \nu \cdot Y d\sigma, \quad Y \in \mathcal{X}(\mathbf{R}^n).$$

Take  $Y$  to be the radial vector field defined by  $Y(x) = x - p$ . Then  $Y$  is the initial velocity vector field of the 1-parameter family of homothetic expansions  $\{\phi_t\}$  given by  $\phi_t(x) = (1+t)(x-p) + p$ . Hence

$$\mathbf{M}(\phi_{t\#}V) = (1+t)^m \mathbf{M}(V),$$

and so

$$\delta V(Y) = \frac{d}{dt}(1+t)^m \mathbf{M}(V)|_{t=0} = m \mathbf{M}(V).$$

On the other hand, since  $Z$  is rectifiable,  $Z$  has tangent spaces almost everywhere and  $\nu$  is normal to  $Z$ . Let  $\eta(x)$  be a unit vector which is perpendicular to  $Z$  at  $x \in Z$  and lies in the subspace of  $\mathbf{R}^n$  spanned by  $Y(x) = x - p$  and the tangent space to  $Z$  at  $x$ . Taking the negative of  $\eta$  if necessary, one can assume  $\eta \cdot Y \geq 0$ . It is not difficult to see that

$$\nu \cdot Y \leq \eta \cdot Y.$$

Let  $r(x) = |Y(x)|$ . Then  $dr$  is the 1-form dual to the unit radial vector field  $Y/|Y|$ . Hence

$$m \mathbf{M}(V) = \int_Z \nu \cdot Y d\sigma \leq \int_Z \eta \cdot Y d\sigma = \int_Z Y \cdot (dr \wedge d\sigma) = m \mathbf{M}(p \ast \partial V).$$

**Remark** In Theorem 1 we can see that  $\mathbf{M}(V) = \mathbf{M}(p \ast \partial V)$  if and only if  $\nu = \eta$  everywhere on  $Z$ , or equivalently,  $\nu(x)$  lies in the subspace spanned by  $Y(x)$  and the tangent space to  $Z$  at every  $x \in Z$ . This was the case in the example preceding Theorem 1 when  $p =$  the center of gravity of  $I^3$ . More generally, for a stationary integral varifold  $V$  arising from the boundary (=faces) of a regular polyhedron  $U$ ,  $\mathbf{M}(V) = \mathbf{M}(p \ast \partial V)$ , provided  $p$  is the center of gravity of  $U$ .

**Lemma 3** *Let  $W = \underline{v}(Z, \psi)$  be a rectifiable 1-varifold in  $\mathbf{R}^n$  with  $\psi \geq 1$  and let  $p$  be a point in  $Z$ . If  $Z$  is a compound Jordan curve, then*

$$4\pi \mathbf{M}(p \ast W) \leq \mathbf{M}(W)^2.$$

*Proof.* Lemma 2 is a special case of this lemma with  $Z = C$  and  $\psi = 1$ . So this lemma will be proved by Lemma 2 if we can construct a compound Jordan curve  $C$  in  $\mathbf{R}^n$  with  $p \in C$  such that

$$\text{Area}(p \ast C) = \mathbf{M}(p \ast W), \quad \text{Length}(C) \leq \mathbf{M}(W). \quad (7)$$

Let  $\gamma_1, \dots, \gamma_k \subset Z$  be the curves as defined in the proof of Lemma 2, and let  $J$  be the union of end points of  $\gamma_2, \dots, \gamma_k$ . Then  $Z \sim J$  is a union of curves none of which is a loop. Still let  $\gamma_1, \dots, \gamma_m$  denote the components of  $Z \sim J$ . And let  $p_i^1, p_i^2$  be the end points of  $\gamma_i$ . Define  $W_i$  to be the part of  $W$  corresponding to  $\gamma_i$ , i.e.,  $W_i = \underline{v}(\gamma_i, \psi|_{\gamma_i})$ . Parametrize  $\gamma_i$  by  $\theta, 0 \leq \theta \leq a_i$ . Let  $l(\theta) = \text{dist}(p, \gamma_i(\theta))$  and  $\psi(\theta) = (\psi|_{\gamma_i})(\theta)$ . Assume, as in the proof of Lemma 2, that  $\theta$  is the angle parameter so that  $l(\theta)$  satisfies

$$\mathbf{M}(p \ast W_i) = \frac{1}{2} \int_0^{a_i} l(\theta)^2 \psi(\theta) d\theta, \quad \mathbf{M}(W_i) = \int_0^{a_i} \sqrt{l^2 + \left(\frac{\partial l}{\partial \theta}\right)^2} \psi(\theta) d\theta.$$

Define a new parameter  $\phi$  of  $\gamma_i, 0 \leq \phi \leq b_i$ , such that  $\phi = f(\theta) = \int_0^\theta \psi(t) dt, f(a_i) = b_i$ .  $f$  is monotone increasing and its inverse function exists. Then  $l(\theta) = l(f^{-1}(\phi))$  and hence

$$\begin{aligned} \mathbf{M}(p \ast W_i) &= \frac{1}{2} \int_0^{b_i} l(f^{-1}(\phi))^2 d\phi, \\ \mathbf{M}(W_i) &= \int_0^{b_i} \sqrt{l^2 + \left(\frac{\partial l}{\partial \phi}\right)^2 \left(\frac{\partial \phi}{\partial \theta}\right)^2} d\phi \geq \int_0^{b_i} \sqrt{l^2 + \left(\frac{\partial l}{\partial \phi}\right)^2} d\phi, \end{aligned} \quad (8)$$

where we used the fact that  $\frac{\partial \phi}{\partial \theta} = \psi \geq 1$ . In view of the similarity of (8) to the formulas for the area and arc length in polar coordinates, one can easily construct a curve  $\bar{\gamma}_i$  from  $p_i^1$  to  $p_i^2$  parametrized by the angle parameter  $\phi$  such that  $l(f^{-1}(\phi)) = \text{dist}(p, \bar{\gamma}_i(\phi))$  and

$$\text{Area}(p \ast \bar{\gamma}_i) = \mathbf{M}(p \ast W_i), \quad \text{Length}(\bar{\gamma}_i) \leq \mathbf{M}(W_i). \quad (9)$$

One can further impose on  $\bar{\gamma}_i$  the condition that  $\gamma_i \cap J \subset \bar{\gamma}_i \cap J$ . Define  $C = \bigcup_{i=1}^m \bar{\gamma}_i$ . Clearly  $C$  is a compound Jordan curve, and  $p \in C$  since  $l = 0$  at  $p$ . Summing up (9) for  $i$ , we get (7) as desired.

**Theorem 2** *Suppose that  $V$  is a 2-varifold of locally bounded first variation in  $\mathbf{R}^n$ , the generalized boundary  $Z$  of  $V$  is rectifiable, and  $V$  is stationary in  $\mathbf{R}^n \sim Z$ . If the multiplicity of  $\partial V$  is  $\geq 1$  and  $Z$  is a compound Jordan curve, then*

$$4\pi \mathbf{M}(V) \leq \mathbf{M}(\partial V)^2.$$

*Proof.* Apply Theorem 1 and Lemma 3.

We conjecture that the theorem above can be extended in three ways: i) The theorem should hold without the hypothesis on the multiplicity of  $\partial V$  if the multiplicity of  $V$  is assumed to be 1 a.e.; ii) The optimal case (equality) should occur only when  $\text{spt}V$  is a disk. iii) The theorem should be true even when the generalized boundary  $Z$  of  $V$  is a disjoint union of compound Jordan curves. The difficulty lies in obtaining the angle estimate: The angle of  $Z$  viewed from a point of  $\text{spt}V \sim Z$  should be not smaller than  $2\pi$ . If the singular set of  $\text{spt}V$  is the union of differentiable curves (like J. Taylor's soap-film-like surfaces [11]) and if  $\text{spt}V$  is regular up to  $Z$ , then the angle estimate easily follows (see [5, Remarks]). Once we have the angle estimate, we can apply the arguments of [5, Theorem 1] and prove the isoperimetric inequality for  $V$  if  $Z$  is radially connected from a point of  $\text{spt}V \sim Z$ .

In case the multiplicity of  $\partial V$  is less than 1, one can modify the theorem as follows.

**Corollary 1** *Let  $V$  be a 2-varifold of locally bounded first variation in  $\mathbf{R}^n$  such that  $V$  is stationary outside the rectifiable generalized boundary  $Z$ . Write  $\partial V = \underline{v}(Z, \theta)$  and define  $\bar{\partial}V = \underline{v}(Z, \bar{\theta})$ ,  $\bar{\theta} = \max\{\theta, 1\}$ . If  $Z$  is a compound Jordan curve, then*

$$4\pi\mathbf{M}(V) \leq \mathbf{M}(\bar{\partial}V)^2.$$

## 4 Area minimizing flat chains mod $k$

In this section we derive a different type of sharp isoperimetric inequalities for certain area minimizing flat chains mod  $k$ . Roughly speaking, flat chains, or currents, are obtained by assigning an orientation to the tangent space of varifolds. First let us briefly define currents and related terminology.

Let  $\mathcal{D}^m$  be the space of smooth differential  $m$ -forms with compact support in  $\mathbf{R}^n$ . An  $m$ -dimensional *current* in  $\mathbf{R}^n$  is a continuous linear functional on  $\mathcal{D}^m$ . The set of such  $m$ -currents will be denoted  $\mathcal{D}_m$ . Any oriented  $m$ -dimensional rectifiable set  $M$  may be viewed as a current  $T_M$  in the following way. Let  $\vec{S}(x)$  denote the unit  $m$ -vector associated with the oriented tangent space to  $M$  at  $x$ . Then for any differential  $m$ -form  $\omega$ , define

$$T_M(\omega) = \int_M \langle \vec{S}(x), \omega \rangle d\mathcal{H}^m.$$

Furthermore, we will allow  $T_M$  to carry a positive integer multiplicity  $\theta(x)$ , and define

$$T_{M,\theta}(\omega) = \int_M \langle \vec{S}(x), \omega \rangle \theta(x) d\mathcal{H}^m. \quad (10)$$

Motivated by the classical Stokes' theorem, we are led to define the *boundary*  $\partial T \in \mathcal{D}_{m-1}$  of an  $m$ -current  $T$  by

$$\partial T(\omega) = T(d\omega), \quad \omega \in \mathcal{D}^{m-1}.$$

Again motivated by the above example,  $T_M$ , we define the *mass* of  $T$ ,  $\mathbf{M}(T)$ , for  $T \in \mathcal{D}_m$  by

$$\mathbf{M}(T) = \sup\{T(\omega) : |\omega| \leq 1, \omega \in \mathcal{D}^m\},$$

where  $|\omega| = \sup_{x \in \mathbf{R}^n} \langle \omega(x), \omega(x) \rangle^{1/2}$ . The *support* of a current  $T$ ,  $\text{spt}T$ , is the complement in  $\mathbf{R}^n$  of the largest open set on which  $T = 0$ .  $T$  is called a *rectifiable* current if  $\text{spt}T$  is rectifiable. The mass of a rectifiable current is just the Hausdorff measure of the associated rectifiable support (counting multiplicities). The integer multiplicity rectifiable currents  $T_{M,\theta}$  as defined in (10) are characterized by the property that they agree, to within a set of arbitrarily small  $\mathcal{H}^m$  measure, with  $m$ -dimensional  $C^1$  singular chains with integer coefficients. Notice that one can associate  $T_{M,\theta}$  with the integer multiplicity varifold  $V = \underline{v}(M, \theta)$  in  $\mathbf{R}^n$ .

$\mathcal{R}_m$  denotes the set of integer multiplicity rectifiable  $m$ -currents in  $\mathbf{R}^n$ . And  $\mathcal{I}_m^k$  denotes the space of  $m$ -dimensional rectifiable *flat chains modulo  $k$*  whose boundaries are also rectifiable flat chains modulo  $k$ , that is,

$$\mathcal{I}_m^k = \{T : T \in \mathcal{R}_m/k\mathcal{R}_m, \partial T \in \mathcal{R}_{m-1}/k\mathcal{R}_{m-1}\}$$

(see [7, 4.2.26]). We write the same notations  $\text{spt}, \partial, \mathbf{M}$  for flat chains mod  $k$  as we do for currents. One says  $T \in \mathcal{I}_m^k$  is *area minimizing* if

$$\mathbf{M}(T) \leq \mathbf{M}(S) \text{ for every } S \in \mathcal{I}_m^k \text{ with } \partial S = \partial T.$$

**Definition 3** Let  $Y^k \subset \mathbf{R}^3$  be the union of  $k$  great semicircles on a sphere meeting at the north and south poles at equal angles of  $2\pi/k$ . Define  $\mathcal{Y}_2^k \subset \mathcal{I}_2^k$  to be the set of 2-dimensional flat chains  $T \bmod k$  in  $\mathbf{R}^n$  with multiplicity 1 almost everywhere such that  $\text{spt}\partial T$  is homeomorphic to  $Y^k$  and the associated varifold  $V = \underline{v}(\text{spt}T, \theta)$  is locally of bounded first variation in  $\mathbf{R}^n$ .

**Theorem 3** Suppose that  $T$  is a 2-dimensional area minimizing flat chain mod  $k$  in  $\mathcal{Y}_2^k$ . If  $C_1, C_2, \dots, C_k$  are the curves that constitute  $\text{spt}\partial T$  and have common end points  $p, p'$ , then

$$2\pi\mathbf{M}(T) \leq \sum_{i=1}^k \text{Length}(C_i)^2.$$

And equality holds if and only if  $\text{spt}T$  is the union of  $k$  flat half disks meeting each other along the common diameter.

**Lemma 4** Let  $T \in \mathcal{Y}_2^k$  be a 2-dimensional area minimizing flat chain mod  $k$  and  $V$  the varifold associated with  $T$ . Then  $\text{spt}\partial V \subset \text{spt}\partial T$  and the multiplicity  $\psi$  of  $\partial V$  is less than or equal to 1 almost everywhere on  $\text{spt}\partial T$ .

*Proof of Lemma 4.* Since  $T$  is area minimizing,  $V$  is stationary in  $\mathbf{R}^3 \sim \text{spt}\partial T$ . Hence  $\text{spt}\partial V \subset \text{spt}\partial T$ . Now let us write  $\partial V = \underline{\nu}(\text{spt}\partial T, \psi)$  and suppose  $\psi > 1$  on a set  $U \subset \text{spt}\partial T$  with  $\mathcal{H}^1(U) > 0$ . As stated in the previous section, the generalized unit conormal  $\nu$  of  $V$  is  $\|\delta V\|$ -measurable. Hence we can approximate  $-\nu|_U$  by a smooth vector field  $Y$  with compact support with respect to  $L^1$  norm on  $U$ . Extend  $Y$  to a smooth vector field  $\bar{Y}$  on  $\mathbf{R}^n$ . Then by (6)  $\delta V(\bar{Y})$  can be made sufficiently close to  $-\sigma(U)$ . Note that

$$\sigma(U) = \int_U \psi d\mathcal{H}^1 > \mathcal{H}^1(U). \quad (11)$$

Let  $\{\phi_t\}_{-1 < t < 1}$  be a 1-parameter family of diffeomorphisms in  $\mathbf{R}^n$  with  $\bar{Y}$  as the initial velocity vector field and with  $\{x \in \text{spt}\partial T : \phi_t(x) \neq x\} \subset U$ . Then

$$\mathbf{M}(\phi_{t\#}V) = \mathbf{M}(V) - t[\sigma(U) + \epsilon] + O(t^2) \quad (12)$$

for some sufficiently small  $|\epsilon|$ . Now define a 2-dimensional rectifiable set  $D_t, t > 0$ , to be the union of the line segments from  $\phi_0(x)$  to  $\phi_t(x)$  for all  $x \in U$ . Clearly

$$\text{Area}(D_t) = t[\mathcal{H}^1(U) + \bar{\epsilon}] + O(t^2), \quad (13)$$

with  $|\bar{\epsilon}|$  sufficiently small. By giving a suitable orientation to  $D_t$ , one can make  $D_t$  into a current, still denoted  $D_t$ , such that  $S_t = \phi_{t\#}T + D_t$  is a flat chain mod  $k$  with  $\partial S_t = \partial T$ . Moreover we have from (12),(13)

$$\mathbf{M}(S_t) = \mathbf{M}(\phi_{t\#}V) + \mathbf{M}(D_t) = \mathbf{M}(T) + t[-\sigma(U) + \mathcal{H}^1(U) - \epsilon + \bar{\epsilon}] + O(t^2).$$

Therefore we can deduce from (11) that for some  $t$ ,  $\mathbf{M}(S_t) < \mathbf{M}(T)$ , which is a contradiction.

*Proof of Theorem 3.* By Theorem 1 and Lemma 4 we have

$$\mathbf{M}(T) \leq \mathbf{M}(p \times \partial V) \leq \sum_{i=1}^k \text{Area}(p \times C_i).$$

Since each  $p \times C_i$  is flat, one can develop it on a plane and apply Lemma 1(a) to get

$$2\pi \text{Area}(p \times C_i) \leq \text{Length}(C_i)^2.$$

Hence

$$2\pi \mathbf{M}(T) \leq \sum_{i=1}^k \text{Length}(C_i)^2.$$

Here equality holds if and only if  $\mathbf{M}(T) = \sum_{i=1}^k \text{Area}(p \times C_i)$  and if each  $p \times C_i$  can be developed onto a half disk. So if equality holds, then the flat chain mod  $k, T_M$ , associated with  $M = \bigcup_{i=1}^k p \times C_i$  is area minimizing. Let  $r(x) = |x - p|$ . Then the slice of  $T_M$  by

$r$  at  $\rho$ ,  $\langle T_M, r, \rho \rangle$ , is a 1-dimensional flat chain mod  $k$  in  $\partial B_\rho(p)$  for almost all  $\rho > 0$ . Furthermore  $\langle T_M, r, \rho \rangle$  is locally length minimizing, i.e.,  $\langle T_M, r, \rho \rangle \llcorner D$  is length minimizing for any sufficiently small domain  $D$  in  $\partial B_\rho(p)$ . Hence  $\text{spt} \langle T_M, r, \rho \rangle$  is the union of part of great circles in  $\partial B_\rho(p)$ . Therefore  $M$  is a polyhedral set and each  $C_i$  is a planar curve. Now let  $\bar{C}_i$  be the semicircle centered at  $p$ , ending at  $p'$ , and of radius  $|p' - p|$  such that the half disk  $\bar{D}_i$  determined by  $\bar{C}_i$  (i.e.,  $\partial \bar{D}_i \supset \bar{C}_i$ ) contains the half disk  $D_i$  determined by  $C_i$ . Define a new flat chain mod  $k$   $S$  such that  $\text{spt} S = \text{spt} T \cup (\cup_{i=1}^k (\bar{D}_i \sim D_i))$  and  $\text{spt} \partial S = \cup_{i=1}^k \bar{C}_i$ . Then we have  $\mathbf{M}(S) = 4\mathbf{M}(T)$  since  $\mathbf{M}(T) = \sum_{i=1}^k \text{Area}(D_i)$ . Note that  $\bar{C}_i$  is similar to  $C_i$  and that  $\text{Length}(\bar{C}_i) = 2\text{Length}(C_i)$ . Hence one can easily see that  $S$  also is area minimizing. Let  $p''$  be the other end point of  $\bar{C}_i$ . Along the open line segment of  $\overline{pp''}$ , the density of  $S$  is  $k/2$ . Then by the upper semicontinuity of the density, the density of  $S$  at  $p$  is at least  $k/2$ . And the monotonicity of the mass ratio (a flat chain version of [7, 5.4.3]) says that

$$\Theta^2(S, p, r) = \mathbf{M}(S \llcorner B(p, r)) / \pi r^2$$

is a monotonically nondecreasing function of  $r$ . But for  $d = \text{dist}(p, p')$ ,  $\Theta^2(S, p, d) = k/2$ , and  $\lim_{r \rightarrow 0} \Theta^2(S, p, r) \geq k/2$ . Hence  $\Theta^2(S, p, r)$  is a constant function and  $\text{spt} S$  is a cone. Therefore  $\text{spt} T = \cup_{i=1}^k p \ast C_i$  whenever equality holds in the isoperimetric inequality.

Let  $Y$  be a union of three half disks meeting each other along their common diameter at equal angles of 120 degrees. Let  $T$  be the intersection with the unit ball  $B_1(O)$  of an infinite cone from  $O$  through the 1-skeleton of a regular tetrahedron with its center of mass at  $O$ . In [11] J. Taylor proved that the disk,  $Y$ , and  $T$  are the only three cones that are area minimizing under Lipschitz maps leaving the boundary fixed. In view of this fact we raise the following problem as an analogue of Theorem 3.

**Open Problem:** Suppose that  $V$  is a 2-varifold with multiplicity 1 almost everywhere and is locally of bounded first variation in  $\mathbf{R}^n$  such that  $V$  is stationary outside the rectifiable boundary  $\text{spt} \partial V$ . Suppose also that  $\text{spt} V$  is homeomorphic to  $T$ . Let  $C_1, C_2, \dots, C_6 \subset \text{spt} \partial V$  be the curves that constitute  $\text{spt} \partial V$  and lie between 4 junctions of  $\text{spt} \partial V$ . Show that

$$[2 \cos^{-1}(-\frac{1}{3})] \mathbf{M}(V) \leq \sum_{i=1}^6 \text{Length}(C_i)^2,$$

where equality holds if and only if  $\text{spt} V$  is a homothetic expansion (or contraction) of  $T$ .

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