THE ISOPERIMETRIC INEQUALITY OF A MINIMAL SURFACE

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Most histories of the isoperimetric problem begin with its legendary origins in the "Problem of Queen Dido" about 800 B.C. In Vergil's *Aeneid* there is a reference to the story in question. Dido, fleeing from the Phoenician city of Tyre ruled by King Pygmalion, her tyrannical brother, and arriving at the site that was to become Carthage, sought to purchase land from the natives. They asserted that they would sell only as much ground as she could surround with a bull's hide. She accepted the terms and made the most of them by cutting a bull's hide into narrow strips which she pieced together to form a single, very long strip. Then, by sheer intuition, she reasoned that the maximum area could be encompassed by shaping the strip into the circumference of a circle. Thus she was able to lead a comfortable life in the big land. But her peaceful life did not last long; King Pygmalion, ever wanting more power and land, invaded Carthage. Queen Dido was forced to flee again. This time she decided to move to the wonderland, where the natives inhabit a minimal surface. There she purchased land surrounded by the same strip that she had used in Carthage. And Queen Dido asked herself whether her land in the wonderland was bigger than that in Carthage......

A rigorous mathematical proof that Dido made the optimum choice in Carthage was not achieved until the nineteenth century [St]. But her question that was raised in the wonderland has not been completely settled yet. This survey note concerns Queen Dido's new problem and will summarize the results so far obtained.

1. Hélein's proof

Let us begin with a proof of the original problem of Dido. There are numerous proofs for the problem; among these we will introduce the most recent one given by Helein [H] in 1994. Queen Dido's characterization of the circle is most succinctly expressed in the isoperimetric inequality

$$4\pi A \ \leq \ L^2, \tag{1}$$

where A is the area enclosed by a curve C of length L, and where equality holds if and only if C is a circle. In fact, there are equivalent isoperimetric inequalities for curves on a sphere and

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a hyperbolic plane whose Gaussian curvatures are $+1$ and -1, respectively:

$$4\pi A \ \leq \ L^2 + KA^2 \tag{2}$$

for a curve C on a surface with constant Gaussian curvature K.

Helein's proof uses a calibration argument which works for all constant K in (2).

Proof of (2). Let D be the domain enclosed by a smooth curve C on a surface with constant Gaussian curvature K. (x, y) denotes a point in $D \times \partial D$. For fixed $y \in \partial D$, cover D with the set of all circular arcs emanating from y and perpendicular to ∂D at y. Let $V(x, y)$ be the unit tangent vector to the arc pointing away from y. Then $V(x, y)$ is a unit vector field on $D \times \partial D$. One can easily compute

$$\mathrm{div} V \ = \ \frac{1 + f'(r)}{f(r)} < \nu, \nabla r > \tag{3}$$

where ν is the unit inward normal to ∂D at y, $r = \mathrm{dist}(x, y)$, and $f(r) = r, \ \sin r, \ \sinh r$ if $K = 0, +1, -1$, respectively. Moreover one can show that if $r = \mathrm{dist}(x, z), x, z \in D$, then

$$\mathrm{div} \frac{1 + f'(r)}{f(r)} \nabla r \ = \ 4\pi\delta_x - K, \tag{4}$$

where δ_x is the Dirac function centered at x. Let ω, dl be the volume forms of $D, \ \partial D$, respectively. Then we have a two-form $\alpha = V \lrcorner\, \omega \wedge dl$ such that

$$d\alpha \ = \ \mathrm{div} V \omega \wedge dl \ = \ \frac{1 + f'(r)}{f(r)} < \nu, \nabla r > \omega \wedge dl.$$

Therefore

$$\begin{aligned}
\int_{D \times \partial D} d\alpha \ &= \ \int_D \left(\int_{\partial D} \frac{1 + f'(r)}{f(r)} < \nu, \nabla r > dl \right) \omega \ = \ \int_D \left(\int_D d\left(\frac{1 + f'(r)}{f(r)} \nabla r \lrcorner\, \omega \right) \right) \omega \\
&= \ \int_D \left(\int_D \mathrm{div} \left(\frac{1 + f'(r)}{f(r)} \nabla r \right) \omega \right) \omega \ = \ \int_D \left(\int_D (4\pi\delta_x - K)\omega \right) \omega \\
&= \ \int_D (4\pi - KA)\omega \ = \ 4\pi A - KA^2.
\end{aligned}$$

On the other hand

$$\int_{D \times \partial D} d\alpha \ = \ \int_{\partial D \times \partial D} \alpha \ \leq \ \int_{\partial D \times \partial D} dl \wedge dl \ = \ L^2.$$

Thus we get (2).

2. Li-Schoen-Yau's proof.

A minimal surface, being a locally least area surface, can be said to be a generalized plane. For this reason it is natural to conjecture that the isoperimetric inequality (2) should hold for domains in a minimal surface in a simply connected space form. The first partial proof for a minimal surface in $\mathbf{R}^n$ was obtained by Carleman in 1921 [Ca]. He proved the inequality for a simply connected minimal surface. Then in 1959 Reid [R] and Hsiung [Hs] proved it for a minimal surface with connected boundary, and in 1975 a proof was obtained by Osserman-Schiffer for a doubly connected minimal surface in $\mathbf{R}^3$, and in 1977 by Feinberg [F] for a doubly connected minimal surface in $\mathbf{R}^n$. Then in 1983 Li-Schoen-Yau proved (1) for a minimal surface with two boundary components in $\mathbf{R}^3$. Since Li-Schoen-Yau's theorem extends all the previous theorems except Feinberg's, let us briefly introduce their proof.

Fix a point $p \in \mathbf{R}^n$ and define $r(x) = \mathrm{dist}(p, x)$. Since rectangular coordinate functions $x_1, ..., x_n$ of $\mathbf{R}^n$ are harmonic on a minimal surface $\Sigma^2 \subset \mathbf{R}^n$ we have on Σ

$$\triangle r^2 \;=\; 4, \tag{5}$$

where we take p as the origin. Translating Σ suitably, we may assume $\int_{\partial\Sigma} x_i = 0$. Then

$$\begin{aligned}
4\mathrm{Area}(\Sigma) &= \int_\Sigma \triangle r^2 \;=\; \int_{\partial\Sigma} 2r\frac{\partial r}{\partial \nu} \qquad (\nu\text{: the outward unit conormal to } \partial\Sigma) \\
&\le 2\int_{\partial\Sigma} r \;\le\; 2\mathrm{Length}(\partial\Sigma)^{1/2}\left(\int_{\partial\Sigma}\sum x_i^2\right)^{1/2} \\
&\le \frac{1}{\pi}\mathrm{Length}(\partial\Sigma)^{3/2}\left[\int_{\partial\Sigma}\sum\left(\frac{dx_i}{ds}\right)^2\right]^{1/2} \qquad (\text{by the Poincaré inequality}) \\
&= \frac{1}{\pi}\mathrm{Length}(\partial\Sigma)^2, \qquad \left(\text{because } \sum\left(\frac{dx_i}{ds}\right)^2 = 1\right)
\end{aligned}$$

which proves (1) when $\partial\Sigma$ is connected. In case $\partial\Sigma$ is not connected, Li-Schoen-Yau assumed that for any $i = 1, ..., n$ the components of $\partial\Sigma$ can be assembled together to become a connected curve C_i after translations parallel to the hyperplane $\{x_i = 0\}$ of $\mathbf{R}^n$. They called such $\partial\Sigma$ *weakly connected*. Under this assumption one has

$$\begin{aligned}
\int_{\partial\Sigma} x_i^2 &= \int_{C_i} x_i^2 \;\le\; \frac{1}{2\pi}\mathrm{Length}(C_i)\int_{C_i}\left(\frac{dx_i}{ds}\right)^2 \qquad (\text{by the Poincaré inequality}) \\
&= \frac{1}{2\pi}\mathrm{Length}(\partial\Sigma)\int_{\partial\Sigma}\left(\frac{dx_i}{ds}\right)^2.
\end{aligned}$$

Adding up this inequality for all $i = 1, ..., n$, one gets (1) as above. Equivalently, $\partial\Sigma$ is said to be weakly connected if there exists a rectangular coordinate system $\{y_1, ..., y_n\}$ of $\mathbf{R}^n$ such that no hyperplane $\{y_i = const\}$ in $\mathbf{R}^n$ separates $\partial\Sigma$. And they proved that given a minimal surface Σ in $\mathbf{R}^3$ $\partial\Sigma$ is weakly connected if $\partial\Sigma$ has two components.

3. Proof by comparisons of area and angle

The author extended Li-Schoen-Yau's theorem by taking a more geometric point of view [C1]. In their proof they used the inequality $\frac{\partial r}{\partial \nu} \le 1$. But if η is the unit normal to $\partial\Sigma$ which makes the smallest angle with ∇r, then

$$\frac{\partial r}{\partial \nu} \;\le\; \frac{\partial r}{\partial \eta} \;\le\; 1.$$

In fact, η is the outward unit conormal to $\partial\Sigma$ on the cone $p \divideontimes \partial\Sigma$, the union of the line segments from p to the points of $\partial\Sigma$. Although $p \divideontimes \partial\Sigma$ is not minimal the identity $\triangle r^2 = 4$ holds there too because

$$\triangle r^2 \;=\; 4 + 2 < X, \triangle X > \;=\; 4 + 2 < X, \vec{H} > \;=\; 4,$$

where X is the coordinate vector and $\vec{H}$ is the mean curvature vector of $p \divideontimes \partial\Sigma$. Therefore

$$4\mathrm{Area}(\Sigma) \;=\; \int_\Sigma \triangle r^2 \;=\; \int_{\partial\Sigma} 2r\frac{\partial r}{\partial \nu} \;\le\; \int_{\partial\Sigma} 2r\frac{\partial r}{\partial \eta} \;=\; \int_{p \divideontimes \partial\Sigma} \triangle r^2 \;=\; 4\mathrm{Area}(p \divideontimes \partial\Sigma),$$

which gives an area comparison between Σ and $p \divideontimes \partial\Sigma$. A nice thing about the cone $p \divideontimes \partial\Sigma$ is that $p \divideontimes \partial\Sigma$ is flat and hence is locally developable. If $\partial\Sigma$ is connected, cut along a line segment l from p to a point in $\partial\Sigma$ and then one can develop $p \divideontimes \partial\Sigma$ into a cone $O \divideontimes C$ on a plane. $O \divideontimes C$ has the same area as $p \divideontimes \partial\Sigma$ and C has the same length as $\partial\Sigma$. C may not be a closed curve. But to make it closed one takes $p \in \Sigma$ and then on Σ one can show

$$\triangle \log r \ \geq \ 2\pi\delta_p.$$

Hence

$$2\pi \ \leq \ \int_\Sigma \triangle \log r \ = \ \int_{\partial\Sigma} \frac{1}{r}\frac{\partial r}{\partial \nu} \ \leq \ \int_{\partial\Sigma} \frac{1}{r}\frac{\partial r}{\partial \eta} \ = \ \mathrm{Angle}(\partial\Sigma, p),$$

where $\mathrm{Angle}(\partial\Sigma, p)$ is the angle of $\partial\Sigma$ viewed from p. This angle estimate implies that $\partial\Sigma$ rotates around p more than $360°$ and consequently C should intersect itself. Then cutting $O \divideontimes C$ into two pieces and assemblying them appropriately gives rise to a new domain $D \subset \mathbf{R}^2$ with

$$\mathrm{Area}(D) \ \geq \ \mathrm{Area}(O \divideontimes C) \ \geq \ \mathrm{Area}(\Sigma), \quad \mathrm{Length}(\partial D) \ = \ \mathrm{Length}(C) \ = \ \mathrm{Length}(\partial\Sigma).$$

Therefore the isoperimetric inequality for D gives (1) for Σ. So far $\partial\Sigma$ has been assumed to be connected. But if $\partial\Sigma$ is *radially connected* from p, that is, if $\{r : r = \mathrm{dist}(p,q),\ q \in \partial\Sigma\}$ is a connected interval, then we can apply the argument of "cutting and inserting" as well as "cutting and assemblying" to obtain the desired domain D and hence to prove (1) for such Σ. See [C1, Theorem 1] for more details. Although there is no relationship between radial connectivity and weak connectivity, we have a stronger corollary than Li-Schoen-Yau's: if Σ is in $\mathbf{R}^n$ such that $\partial\Sigma$ has two components then from a point in Σ which is a midpoint between the two components, $\partial\Sigma$ is radially connected and hence (1) holds for such Σ.

The author and Gulliver [CG1, 2] investigated the possibility of extending the geometric method above to minimal surfaces Σ in S^n and H^n. They showed that the area comparison $\mathrm{Area}(\Sigma) \leq \mathrm{Area}(p \divideontimes \partial\Sigma)$ does hold for $\Sigma \subset H^n$ but not for $\Sigma \subset S^n$ [CG1, Proposition 2, Remark 1] whereas the angle estimate $\mathrm{Angle}(\partial\Sigma, p) \geq 2\pi$ holds for Σ in S^n as well as in H^n [CG2, Proposition 2]. And they proved the isoperimetric inequality

$$4\pi A \ \leq \ L^2 - A^2$$

for a minimal surface $\Sigma \subset H^n$ whose boundary is radially connected from a point of Σ. They did not employ the aforementioned cutting and inserting argument; instead, they used Bol's isoperimetric inequality directly for $p \divideontimes \partial\Sigma$ [B].

Some minimal surfaces in $\mathbf{R}^3$, like compound soap films, contain singular curves. They are not smooth (although smooth almost everywhere) and in some literature they are called stationary varifolds or area minimizing currents. Here one can ask whether the isoperimetric inequality (1) holds also for these surfaces with singularities. In [C2] the author gave an affirmative answer; moreover he derived a new type of optimal isoperimetric inequality for certain types of soap films with singularities.

4. The ambient space of varying curvature

So far we have considered minimal surfaces in a Riemannian manifold $\bar{M}$ of constant curvature. In this section let us extend the results of the preceding section to minimal surfaces in a manifold M of varying curvature. The main obstacle to this extension is that one cannot prove the area comparison

$$\text{Area}(\Sigma) \ \leq \ \text{Area}(p \divideontimes \partial\Sigma)$$

for Σ and $p \divideontimes \partial\Sigma$ in M. But we will get around this difficulty by comparing the area of $\Sigma \subset M$ with that of a cone in $\bar{M}$ associated with $p \divideontimes \partial\Sigma$.

When we study a Riemannian manifold of varying curvature the comparison theorems are very useful tools. Among several comparison theorems the one that we need most for our purposes is the Hessian comparison theorem for the distances in M and in $\bar{M}$. From this comparison we can get the following lemmas on the Laplacian of some functions of distance. (See [C3] for their proofs.)

Lemma 1. *Let Σ^2 be a minimal surface in a simply connected Riemannian manifold M of sectional curvature bounded above by a constant K. Define $r(x) = \text{dist}(p, x)$ for fixed $p \in M$. If $K = 0$, we have on Σ*
(a) $\triangle r^2 \ \geq \ 4$;
(b) $\triangle r \ \geq \ \frac{1}{r}(2 - |\nabla r|^2)$;
(c) $\triangle \log r \ \geq \ 2\pi\delta_p \ \ if \ p \in \Sigma$.
If $K = -k^2 < 0$, then
(d) $\triangle r \ \geq \ k(2 - |\nabla r|^2)\coth kr$;
(e) $\triangle \log(1 + \cosh kr) \ \geq \ -K$;
(f) $\triangle \log \frac{\sinh kr}{1+\cosh kr} \ \geq \ 2\pi\delta_p \ \ if \ p \in \Sigma$;
(g) $\triangle \log \sinh kr \ \geq \ 2\pi\delta_p - K \ \ if \ p \in \Sigma$.
If $K = k^2 > 0$, then
(h) $\triangle r \ \geq \ k(2 - |\nabla r|^2)\cot kr$;
(i) $\triangle \log \sin kr \ \geq \ 2\pi\delta_p - K \ \ if \ p \in \Sigma \ and \ r \leq \frac{\pi}{2k}$;
(j) $\triangle \log \frac{\sin kr}{1+\cos kr} \ \geq \ 2\pi\delta_p \ \ if \ p \in \Sigma \ and \ r \leq \frac{\pi}{2k}$.

Lemma 2. *Let $\Gamma = \bar{p} \divideontimes C$ be the cone from $\bar{p}$ over a curve C in a Riemannian manifold $\bar{M}$ of nonpositive constant sectional curvature $K = -k^2$. Then on Γ*
(a) $\triangle \bar{r}^2 \ = \ 4 \ \ if \ K = 0$; $\qquad \triangle \log(1 + \cosh k\bar{r}) = -K \ \ if \ K < 0$;
(b) $\triangle \log \bar{r} \ = \ \alpha\delta_{\bar{p}} \ \ if \ K = 0$; $\qquad \triangle \log \frac{\sinh k\bar{r}}{1+\cosh k\bar{r}} \ = \ \alpha\delta_{\bar{p}} \ \ if \ K < 0$,
where $\alpha = \text{Angle}(C, \bar{p})$.

Now we have the main theorem as follows.

Theorem 1. *Let Σ be a minimal surface in a complete simply connected Riemannian manifold M with sectional curvature bounded above by a nonpositive constant K. If $\partial\Sigma$ is radially* *connected from a point of Σ, then Σ satisfies the isoperimetric inequality*

$$4\pi A \;\le\; L^2 + KA^2,$$

where equality holds if and only if Σ is a geodesic disk in a surface of constant Gaussian curvature K.

Sketchy Proof. First, suppose $K < 0$. Integrate Lemma 1(e) to get

$$-K\,\mathrm{Area}(\Sigma) \;\le\; \int_\Sigma \triangle \log(1+\cosh kr) \;=\; \int_{\partial\Sigma} \frac{k\sinh kr}{1+\cosh kr}\frac{\partial r}{\partial \nu} \;\le\; \int_{\partial\Sigma} \frac{k\sinh kr}{1+\cosh kr}\frac{\partial r}{\partial \eta} \quad (6)$$

$$= \int_{\partial\Sigma} \frac{k\sinh kr}{1+\cosh kr}\sqrt{1-<\nabla r, \tau>^2}, \quad (7)$$

where ν, η are as in the preceding section and τ is a unit tangent to $\partial\Sigma$. Now the key step in the extension to the variable curvature case is to carry the integral in (7) over to the simply connected space form $\bar{M}$ of curvature K. Let $C_1, ..., C_l$ be the components of $\partial\Sigma$. Fix $\bar{p} \in \bar{M}$, define $\bar{r}(y) = \mathrm{dist}(\bar{p}, y),\ y \in \bar{M}$, and choose $q_i \in C_i$ for each $i = 1, ..., l$. Then choose $\bar{q}_1, ..., \bar{q}_l \in \bar{M}$ in such a way that $r(q_i) = \bar{r}(\bar{q}_i)$. Suppose that each curve C_i is parametrized by $c_i(s)$ with arclength parameter s such that $q_i = c_i(0) = c_i(\lambda_i),\ \lambda_i = \mathrm{Length}(C_i)$. Then we construct a curve $\bar{C}_i$ in $\bar{M}$ starting from $\bar{q}_i$ and parametrized by $\bar{c}_i(s)$ with arclength parameter $s \in [0, \lambda_i]$ and $\bar{c}_i(0) = \bar{q}_i$ such that the unit tangent vector $\bar{c}_i'(s)$ makes an angle of $\cos^{-1} < \nabla r, c_i'(s) >$ with $\nabla\bar{r}$. Of course the curve $\bar{C}_i$ is not unique; but given a two-dimensional infinite cone $\bar{p}⋇C$ containing $\bar{q}_i$, one can uniquely determine a curve $\bar{C}_i$ on $\bar{p}⋇C$ with the prescribed properties. Since $\bar{p}⋇C$ is developable, one can also assume without loss of generality that $\bar{c}_i(0) = \bar{c}_i(\lambda_i)$, or equivalently, $\bar{C}_i$ is closed. Anyhow, r on C_i coincides with $\bar{r}$ on $\bar{C}_i$ in the sense that

$$r(c_i(s)) \;=\; \bar{r}(\bar{c}_i(s)) \quad \text{and} \quad <\nabla r, c_i'(s)> \;=\; <\nabla \bar{r}, \bar{c}_i'(s)>.$$

Hence

$$\begin{aligned} -K\,\mathrm{Area}(\Sigma) \;&\le\; \sum_{i=1}^{l} \int_{C_i} \frac{k\sinh kr}{1+\cosh kr}\sqrt{1-<\nabla r, c_i'(s)>^2} \\ &=\; \sum_{i=1}^{l} \int_{\bar{C}_i} \frac{k\sinh k\bar{r}}{1+\cosh k\bar{r}}\sqrt{1-<\nabla \bar{r}, \bar{c}_i'(s)>^2}. \end{aligned}$$

If $\bar{\eta}$ is the outward unit conormal to $\bar{C}_i$ on $\bar{p}⋇\bar{C}_i$, then

$$\begin{aligned} \mathrm{Area}(\Sigma) \;&\le\; -\frac{1}{K}\sum_{i=1}^{l} \int_{\bar{C}_i} \frac{k\sinh k\bar{r}}{1+\cosh k\bar{r}}\frac{\partial \bar{r}}{\partial \bar{\eta}} \;=\; \sum_{i=1}^{l} \int_{\bar{p}⋇\bar{C}_i} \frac{1}{-K}\triangle \log(1+\cosh k\bar{r}) \\ &=\; \sum_{i=1}^{l} \mathrm{Area}(\bar{p}⋇\bar{C}_i) \quad \text{(by Lemma 2(a))} \\ &=\; \mathrm{Area}(\bar{p}⋇\bar{C}), \quad \bar{C} = \bigcup_{i=1}^{l} \bar{C}_i. \end{aligned}$$

Also it follows from the definition of $\bar{C}_i$ that

$$\mathrm{Length}(\partial\Sigma) \;=\; \mathrm{Length}(\bar{C}).$$

Similarly, integrating Lemma 1(f) over Σ and using Lemma 2(b) as above, we get

$$2\pi \;\leq\; \mathrm{Angle}(\bar{C}, \bar{p}).$$

Moreover, since $r|_{\partial\Sigma}$ coincides with $\bar{r}|_{\bar{C}}$, $\bar{C}$ is also radially connected from $\bar{p}$. Hence by [CG1, Lemma 4] we get

$$4\pi \mathrm{Area}(\bar{p} \divideontimes \bar{C}) \;\leq\; \mathrm{Length}(\bar{C})^2 + K\mathrm{Area}(\bar{p} \divideontimes \bar{C})^2.$$

Therefore using the comparisons of area and angle obtained above and the monotonicity of the quadratic function $4\pi A - KA^2$ of $A > 0$, we obtain the desired isoperimetric inequality for Σ in case $K < 0$.

Second, suppose $K = 0$. Lemma 1(a) and Lemma 2(a) imply

$$\mathrm{Area}(\Sigma) \;\leq\; \mathrm{Area}(\bar{p} \divideontimes \bar{C}),$$

and Lemma 1(c) and Lemma 2(b) imply

$$2\pi \;\leq\; \mathrm{Angle}(\bar{C}, \bar{p}).$$

Thus the theorem follows from [C1].

5. Weak isoperimetric inequality

It would be beautiful if Hélein's argument could work out well for minimal surfaces as well. But various attempts made by the author with that wish have ended up with no results as yet. In this section, instead, we will exploit Simon's argument which resembles Hélein's (see [CG2, p.181]), and obtain an isoperimetric inequality, though not sharp, which holds for all minimal surfaces without any connectivity assumption.

Theorem 2. *Let Σ^2 be a minimal surface in a complete simply connected Riemannian manifold with sectional curvature bounded above by a constant K. If $K \leq 0$, then*

$$2\pi A \;\leq\; L^2 + KA^2. \tag{8}$$

In case $K > 0$, (8) holds under the additional assumption $\mathrm{diam}(\Sigma) \leq \frac{\pi}{2\sqrt{K}}$.

Proof. i) $K = -k^2 < 0$. Integrating Lemma 1(g) for fixed $p \in \Sigma$, we get

$$2\pi - KA \;\leq\; \int_\Sigma \triangle \log \sinh kr \;\leq\; \int_{\partial\Sigma} k \coth kr. \tag{9}$$

Since (9) holds for all $p \in \Sigma$ we can integrate it over Σ and apply Fubini's theorem to obtain

$$\begin{aligned} 2\pi A - KA^2 \;&\leq\; \int_\Sigma \int_{\partial\Sigma} k \coth kr \;=\; \int_{\partial\Sigma} \int_\Sigma k \coth kr \\ &\leq\; \int_{\partial\Sigma} \int_\Sigma \triangle r \quad \text{(by Lemma 1(d))} \\ &=\; \int_{\partial\Sigma} \int_{\partial\Sigma} \frac{\partial r}{\partial \nu} \;\leq\; L^2. \end{aligned}$$

ii) $K = 0$. Integrate Lemma 1(c) twice and apply Lemma 1(b) as in i).
iii) $K > 0$. Integrate Lemma 1(i) twice and apply Lemma 1(h).

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