Real hypersurfaces in complex two-plane Grassmannians and its applications to the Ricci tensor

Young Jin Suh
Department of Mathematics
College of Natural Sciences
Kyungpook National University
Taegu 702-701, Korea
yjsuh@mail.knu.ac.kr

January 25, 2007

1 Introduction

In the geometry of real hypersurfaces in complex space forms or in quaternionic space forms it can be easily checked that there do not exist any real hypersurfaces with parallel shape operator A by virture of the equation of Codazzi.

But if we consider a real hypersurface with parallel Ricci tensor S in such space forms, the proof of its non-existence is not so easy. In a class of Hopf hypersurfaces Kimura [7] has asserted that there do not exist any real hypersurfaces in a complex projective space $\mathbf{C}P^m$ with parallel Ricci tensor, that is $\nabla S = 0$.

Moreover, he has given a classification of Hopf hypersurfaces in $\mathbf{C}P^m$ with *commuting Ricci tensor*, that is $S\phi = \phi S$(see in [8]) and showed that M is locally congruent to one of real hypersurfaces of type A_1, A_2, B, C, D and E, that is, respectively, a tube of certain radius r over a totally geodesic $\mathbf{C}P^k$, a complex quadric $\mathbf{Q}^{m-1}$, $\mathbf{C}P^1 \times \mathbf{C}P^{\frac{n-1}{2}}$, a complex two-plane Grassmannian $G_2(\mathbf{C}^5)$ and an Hermitian symmetric space $SO(10)/U(5)$.

On the other hand, in a complex hyperbolic space $\mathbf{C}H^m$ Ki and the author [6] have given a complete classification of Hopf hypersurfaces in $\mathbf{C}H^m$ with *commuting Ricci tensor* and has proved that M is locally congruent to a horosphere, a geodesic hypersphere , a tube over a tally geodesic $\mathbf{C}H^k$ in $\mathbf{C}H^m$.

In a quaternionic projective space $\mathbf{Q}P^m$ Pérez [9] has considered the notion of $S\phi_i = \phi_i S$, $i = 1, 2, 3$, for real hypersurfaces in $\mathbf{Q}P^m$ and classified that M is locally congruent to of A_1, A_2-type, that is, a tube over $\mathbf{Q}P^k$ with radius $0 < r < \frac{\pi}{4}$.

Moreover, in also [9] he has classified real hypersurfaces in $\mathbf{Q}P^m$ satisfying *parallel Ricci tensor* is an open subset of a *geodesic hypersphere* whose radius r satisfies $\cot^2 r = \frac{1}{2m}$.

Let $G_2(\mathbf{C}^{m+2})$ be a complex two-plane Grassmannian consisting of complex 2-dimensional subspaces in $\mathbf{C}^{m+2}$, and a Kaehler and quaternionic Kaehler manifold but not hyper Kaehler manifold.

In [3] we consider the two natural geometrical conditions as follows:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbf{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $D^\perp$ are invariant under the shape operator of M if and only if*

(A) M is an open part of a tube around a totally geodesic $G_2(\mathbf{C}^{m+1})$ in $G_2(\mathbf{C}^{m+2})$, or

(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbf{Q}P^n$ in $G_2(\mathbf{C}^{m+2})$.

If the structure vector field ξ of M in $G_2(\mathbf{C}^{m+2})$ is invariant by the shape operator, M is said to be *Hopf real hypersurface*.

In such a case the integral curves of the structure vector field ξ are geodesic(See Berndt and Suh [4]). Moreover, the flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2(\mathbf{C}^{m+2})$ is said to be *geodesic Reeb flow*.

Moreover, we say that the Reeb vector field is *Killing*, that is, $L_\xi g = 0$ for the Lie derivative along the direction of the structure vector field ξ, where g denotes the Riemannian metric induced from $G_2(\mathbf{C}^{m+2})$.

Then this is equivalent to the fact that the structure tensor ϕ commutes with the shape operator A of M in $G_2(\mathbf{C}^{m+2})$.

In [3] due to Berndt and the present author we have given a characterization of real hypersurfaces of type (A) in Theorem A when the shape operator A of M in $G_2(\mathbf{C}^{m+2})$ *commutes with the structure tensor* ϕ.

This is equivalent to the condition that the Reeb flow on M is isometric, that is $L_\xi g = 0$, where L(resp. g) denotes the Lie derivative(resp. the induced Riemannian metric) of M in the direction of the Reeb vector field ξ as follows:

Theorem B. *Let M be a connected orientable real hypersurface in $G_2(\mathbf{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic $G_2(\mathbf{C}^{m+1})$ in $G_2(\mathbf{C}^{m+2})$.*

In the proof of Theorem A we have proved that the one-dimensional distribution $[\xi]$ is contained in either the 3-dimensional distribution $D^\perp$ or in the orthogonal complement D such that $T_xM = D \oplus D^\perp$.

The case (A) in Theorem A is just the result of Theorem B that the one dimensional distribution $[\xi]$ is contained in $D^\perp$. Of course they satisfy that the Reeb vector ξ is *Killing*, that is, *the structure tensor commutes with the shape operator*. Naturally, it satisfies that *the Ricci tensor commutes with the structure tensor*. But the Ricci tensor is not parallel.

When the Ricci tensor S of M in $G_2(\mathbf{C}^{m+2})$ commutes with the structure tensor ϕ, we say that M has *commuting Ricci tensor*.

Moreover, M is said to have *parallel Ricci tensor* if the Ricci tensor S of M in $G_2(\mathbf{C}^{m+2})$ has the property $\nabla S = 0$ for the induced covariant derivative ∇ of M.

In this talk, we introduce a new notion for the Ricci tensor S of M in $G_2(\mathbf{C}^{m+2})$ and will give three kinds of talks as follows:

1) We consider a non-existence theorem for M in $G_2(\mathbf{C}^{m+2})$ with *parallel and commuting Ricci tensor*, and give some conjectures related to such tensors (see [12], [13] and [15]).

2) We give a characterization of real hypersurfaces of type A in $G_2(\mathbf{C}^{m+2})$, that is, a tube over a totally geodesic $G_2(\mathbf{C}^{m+1})$ in $G_2(\mathbf{C}^{m+2})$ in terms of integral formulas related to the Ricci curvature $\mathrm{Ric}(\xi,\xi)$ along the direction of the structure vector field ξ for real hypersurfaces in $G_2(\mathbf{C}^{m+2})$ (see [4] and [16]).

3) We introduce a complete classification of *pseudo-Einstein* real hypersurfaces in $G_2(\mathbf{C}^{m+2})$ and also prove that there do not exist any *Enistein* real hypersurfaces in $G_2(\mathbf{C}^{m+2})$ (see [3], [11] and [14]).

References

1. D.V. Alekseevskii, *Compact quaternion spaces* , Func. Anal. Appl., **2**, 106–114.
2. J. Berndt, *Riemannian geometry of complex two-plane Grassmannians*, Rend. Sem. Mat. Univ. Politec. Torino, **55**(1997), 19–83.
3. J. Berndt and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians*, Monatshefte für Math. **127**(1999), 1–14.
4. J. Berndt and Y.J. Suh, *Isometric flows on real hypersurfaces in complex two-plane Grassmannians*, Monatshefte für Math. **137**(2002), 87-98.
5. T.E. Cecil and P.J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **69**(1982), 481-499.
6. U-H. Ki and Y.J. Suh, *On real hypersurfaces of a complex space form*, Math. J. Okayama Univ. **32**(1990), 207–221.
7. M.Kimura, *Real hypersurfaces of a complex projective space*, Bull. Austral Math. Soc., **33**(1986), 383–387.
8. M. Kimura, *Correction to "Some real hypersurfaces in complex projective space"*, Saitama Math. J., **10**(1992), 33-34.
9. J.D. Pérez, *On the Ricci tensor of a real hypersurfaces of quaternionic projective space* Int. J. Math. Soc., **19**(1996), 193–197.
10. J.D. Pérez and Y.J. Suh, *Real hypersurfaces of quaternionic projective space satisfying* $\nabla_{U_i}R=0$, Diff. Geom. and Its Appl., **7**(1997), 211–217.
11. Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with parallel shape operator II* , J. of Korean Math. Soc., **41**(2004), 535–565.
12. Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with vanishing Lie derivatives* , Canadian Math. Bull., **49**(2006), 134–143.
13. Y.J. Suh, *Real hypersurfaces of type B in complex two-plane Grassmannians* , Monatshefte für Math., **141**(2006), 337–355.
14. Y. J. Suh, *Pseudo-Einstein real hypersurfaces in complex two-plane Grassmannians*, Bull. of Austral Math. Soc., **73**(2006), 183–200.

15. Y. J. Suh and J.D. Pérez, *The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians*, J. of Korean Math. Soc., **44**(2007), 211–235.
16. Y.J.Suh and Y. Watanabe, *Real hypersurfaces in complex two-plane Grassmannians related to the Ricci curvature*, Math. J. of Toyama Univ., **28**(2005), 11–25.