# Radial viewpoint on minimal surfaces

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**Abstract**

Some mysteries of minimal surface theory can be unravelled through geometric quantities and functions such as the area of cones, the cone angles, and the Laplacian of functions of distance. In this article I will present this radial point of view, telling a story behind some papers of mine.

## 1 Introduction

A minimal surface $\Sigma$ in $\mathbb{R}^3$ can be called a generalized plane. This is because $\Sigma$ shares some common properties with a plane: $\Sigma$ has zero mean curvature, $\Sigma$ is locally area minimizing, and Euclidean coordinates $x, y, z$ are harmonic on $\Sigma$.

I learned minimal surface theory in a course given by Rick Schoen at the University of California, Berkeley, in Spring quarter, 1983. And I learned geometric measure theory from him again in Fall and Winter quarters, 1984 at UCSD. It was at this time that Rick gave me the thesis problem: Given a compact 3-dimensional Riemannian manifold $M$, find a fundamental domain of $M$ with least boundary area in its universal cover [C1].

After my thesis the first problem I worked on was the isoperimetric inequality for minimal surfaces in Euclidean space. I learned about this problem in Rick's class at Berkeley. I succeeded in getting a partial answer to this isoperimetric problem, which became my second paper [C2]. But there was a 2-year gap between my first paper and the second. During this period I was under heavy pressure to write a new paper as a postdoctor, so I regretted that my thesis had no subsequent problem for a sequel.

But only after many years had passed did I realize that I had gained a new sight from my thesis to solve the isoperimetric problem. I had to prove the existence and regularity of a fundamental domain with least boundary in the thesis. For the regularity part I used Jean Taylor's result [T], which originated from Reifenberg's epiperimetric inequality [R1] and from his proof of the analyticity of minimal surfaces [R2]:

**Epiperimetric Inequality**. *Suppose $Y$ is an orientable polyhedral cone, with vertex 0, of dimension $m$ in $\mathbb{R}^n$, whose boundary lies on the unit sphere with center 0. Then, if $Y$ lies sufficiently near to a diametral plane, we can construct a new surface $Y^*$ with the same boundary such that*

$$\mathrm{Vol}(Y^*) \leq k\mathrm{Vol}(Y) + (1-k)\omega_m, \tag{1}$$

*where $\omega_m$ is the volume of the unit $m$-ball and $k$ is a constant depending only on $m$ and $n$.*

Here Reifenberg is comparing the area of an area minimizing surface $\Sigma$ with that of the cone $Y := 0 \divideontimes \partial\Sigma$. There was another occasion for me to encounter a similar inequality in Rick's class at Berkeley. He proved the monotonicity of the volume of minimal submanifolds using the coarea formula. But just before that he showed

$$\mathrm{Vol}(\Sigma) \leq \mathrm{Vol}(0 \divideontimes \partial\Sigma) \tag{2}$$

if $\Sigma$ is minimal in $\mathbb{R}^n$ and $\partial\Sigma$ lies on a sphere with center 0.

In retrospect I think these inequalities were the starting point of my journey after the thesis. In this survey article I will tell how these inequalities led me to a radial viewpoint on minimal surfaces. And I will derive those results on the isoperimetric inequality, index and embeddedness of a minimal surface in my papers [C1], [C2], [C3], [C4], [CG1], [CG2], [CG3].

## 2 Cone

The day Rick taught the monotonicity in the class he introduced the problem of the isoperimetric inequality for a minimal surface $\Sigma \subset \mathbb{R}^n$: Show that $\Sigma$ satisfies

$$4\pi A \leq L^2, \quad A = \mathrm{Area}(\Sigma), \ L = \mathrm{Length}(\partial\Sigma), \tag{3}$$

and that $4\pi A = L^2$ if and only if $\Sigma$ is a disk on $\mathbb{R}^2$. The first affirmative partial answer was obtained by Carleman [Ca] in 1921 for a simply connected $\Sigma$. But the general case is still open.

Rick himself, together with Li and Yau, had a partial result on this problem [LSY]. Their argument goes as follows. Harmonicity of the Euclidean coordinates $x_1, ..., x_n$ on $\Sigma$ gives

$$\Delta r^2 = 4, \quad r^2 = x_1^2 + \cdots + x_n^2.$$

Integrating this over $\Sigma$ yields

$$4\,\mathrm{Area}(\Sigma) = \int_{\partial\Sigma} 2r\frac{\partial r}{\partial \nu},$$

where $\nu$ is the outward unit conormal to $\partial\Sigma$ on $\Sigma$. Translating $\Sigma$ suitably, one may assume $\int_{\partial\Sigma} x_i = 0$. Then

$$\begin{aligned} 4\,\mathrm{Area}(\Sigma) &= \int_{\partial\Sigma} 2r\frac{\partial r}{\partial \nu} \leq 2\int_{\partial\Sigma} r \leq 2\,\mathrm{Length}(\partial\Sigma)^{1/2}\Big(\int_{\partial\Sigma}\sum x_i^2\Big)^{1/2} \\ &\leq \frac{1}{\pi}\,\mathrm{Length}(\partial\Sigma)^{3/2}\Big(\int_{\partial\Sigma}\sum\Big(\frac{dx_i}{ds}\Big)^2\Big)^{1/2} = \frac{1}{\pi}\,\mathrm{Length}(\partial\Sigma)^2 \end{aligned}$$

(the last inequality follows from the Poincaré inequality, and the last equality from $\sum(dx_i/ds)^2 = 1$). This gives $4\pi A \leq L^2$ when $\partial\Sigma$ is connected.

I extended Li-Schoen-Yau's theorem by taking a more geometric point of view [C2]. In their proof they used the inequality $\frac{\partial r}{\partial \nu} \leq 1$. But if $\eta$ is the unit normal to $\partial\Sigma$ which makes the smallest angle with $\nabla r$, then

$$\frac{\partial r}{\partial \nu} \leq \frac{\partial r}{\partial \eta} \leq 1. \tag{4}$$

In fact $\eta$ is the outward unit conormal to $\partial\Sigma$ on the cone $p \divideontimes \partial\Sigma$, the union of the line segments from $p$ to the points of $\partial\Sigma$. Since $p \divideontimes \partial\Sigma$ is flat, we actually have $\Delta r^2 = 4$ on $p \divideontimes \partial\Sigma$ as well. Therefore

$$4\,\mathrm{Area}(\Sigma) = \int_{\partial\Sigma} 2r\frac{\partial r}{\partial \nu} \leq \int_{\partial\Sigma} 2r\frac{\partial r}{\partial \eta} = \int_{p \divideontimes \partial\Sigma} \Delta r^2 = 4\,\mathrm{Area}(p \divideontimes \partial\Sigma),$$

which gives an *area comparison* between $\Sigma$ and $p \divideontimes \partial\Sigma$:

$$\mathrm{Area}(\Sigma) \leq \mathrm{Area}(p \divideontimes \partial\Sigma). \tag{5}$$

Note the similarity between (5) and (1), (2).

A nice thing about the cone $p \divideontimes \partial\Sigma$ is that $p \divideontimes \partial\Sigma$ is flat and hence is locally developable. Cut $p \divideontimes \partial\Sigma$ along a line segment from $p$ to a point of $\partial\Sigma$ and then one can develop $p \divideontimes \partial\Sigma$ into a cone $O \divideontimes C$ on $\mathbb{R}^2$. $O \divideontimes C$ has the same area as $p \divideontimes \partial\Sigma$ and $C$ has the same length as $\partial\Sigma$. $C$ may or may not have self-intersection depending on whether the cone angle of $p \divideontimes \partial\Sigma$ at $p$ is $\geq 2\pi$ or $< 2\pi$. So we need a *cone angle comparison* similar to the area comparison (5). Recall that (5) follows from $\Delta r^2 = 4$. From this one can also derive

$$\Delta \log r \geq 2\pi\, \delta_p \quad \text{on } \ \Sigma \ni p,$$

and

$$\Delta \log r = 2\pi\, \Theta_{p \divideontimes \partial\Sigma}(p)\, \delta_p \quad \text{on } \ p \divideontimes \partial\Sigma,$$

where $\delta_p$ is the Dirac delta function. Hence

$$2\pi \leq \int_{\Sigma} \Delta \log r = \int_{\partial\Sigma} \frac{1}{r}\frac{\partial r}{\partial \nu} \leq \int_{\partial\Sigma} \frac{1}{r}\frac{\partial r}{\partial \eta} = \int_{p \divideontimes \partial\Sigma} \Delta \log r = \mathrm{Angle}(\partial\Sigma, p), \tag{6}$$

where $\mathrm{Angle}(\partial\Sigma, p)$ is the *cone angle* of $\partial\Sigma$ viewed from $p$, i.e., $\mathrm{Angle}(\partial\Sigma, p) = 2\pi\Theta_{p \divideontimes \partial\Sigma}(p)$. This angle estimate implies that $\partial\Sigma$ rotates around $p$ by at least $360°$ and consequently $C$ should intersect itself. Then cutting $O \divideontimes C$ into two pieces and pasting them appropriately, one can get a domain $D \subset \mathbb{R}^2$ with

$$\mathrm{Area}(D) \geq \mathrm{Area}(O \divideontimes C) \geq \mathrm{Area}(\Sigma), \quad \mathrm{Length}(\partial D) = \mathrm{Length}(C) = \mathrm{Length}(\partial\Sigma).$$

(See [C2, Lemma 1] for the construction of $D$). Therefore the classical isoperimetric inequality for $D$ gives (3) for $\Sigma$.

So far $\partial\Sigma$ has been assumed to be connected. However, even if $\partial\Sigma$ is not connected, $C$ as defined above may become a connected curve. This motivates the following.

**Definition** A set $\Gamma \subset \mathbb{R}^n$ is said to be *radially connected from* $p \in \mathbb{R}^n$ if $\{r : r = \mathrm{dist}(p, q),\ q \in \Gamma\}$ is a connected interval.

If $\partial\Sigma$ is radially connected from $p$, then we can apply to $p \divideontimes \partial\Sigma$ the argument of *cutting and inserting* to obtain a cone $O \divideontimes C \subset \mathbb{R}^2$ with $C$ connected. Moreover, if $p$ is in $\Sigma$, then $C$ has a self-intersection and so we can obtain the domain $D \subset \mathbb{R}^2$ as above and hence the desired isoperimetric inequality for $S$. See [C2, Theorem 1] for more details. Thus we have outlined the proof of the following.

**Theorem 1** [C2] *If $S$ is a minimal surface whose boundary is radially connected from a point of the surface, then it satisfies $4\pi A \leq L^2$.*

In case $\partial\Sigma$ has two components, $\partial\Sigma$ is radially connected from a midpoint($\in \Sigma$) of the components. Hence we have the following.

**Corollary 1** *A minimal surface in $\mathbb{R}^n$ with one or two boundary components satisfies $4\pi A \leq L^2$.*

A salient difference between the two comparisons (5) and (6) is that equality can hold even for a nonflat minimal surface $\Sigma$ in (5), but in (6) equality holds only for a flat star-shaped surface. Therefore it tempted me to guess that a surface satisfying equality in (5) should have some peculiar property. Indeed there was something. It is the topic of the next section.

# 3 Horizon

Note that

$$\text{Area}(\Sigma) = \text{Area}(p ⋇ \partial\Sigma)$$

if and only if

$$\frac{\partial r}{\partial \nu} = \frac{\partial r}{\partial \eta} \ \text{ along } \ \partial\Sigma.$$

This happens when $\overrightarrow{px}$ is tangent to $\Sigma$ at every $x \in \partial\Sigma$. The vector field $X = \{\overrightarrow{px} : x \in \mathbb{R}^3\}$ is the variation field of the 1-parameter family of homothetic expansions $\psi_r, r > 0$ in $\mathbb{R}^3$ defined by $\psi_r(x) = r(x-p)+p$. If $\Sigma$ is minimal, so is $\psi_r(\Sigma)$. Therefore we can obtain a foliation of a tubular neighborhood of $\Sigma$ whose leaves are minimal surfaces $\psi_r(\Sigma)$. The normal variation field of $\Sigma$ arising from this foliation is in fact $\text{proj}_{\Sigma^\perp}(X)$, where $\text{proj}_{\Sigma^\perp}$ is the orthogonal projection onto the normal bundle of $\Sigma$. Moreover, $\text{proj}_{\Sigma^\perp}(X)$ is a Jacobi field on $\Sigma$, that is, it is in the kernel of the Jacobi operator $\Delta - 2K$. And it vanishes at $x \in \Sigma$ when $\overrightarrow{px}$ is tangent to $\Sigma$. Thus if $\text{Area}(\Sigma) = \text{Area}(p ⋇ \partial\Sigma)$, then $\partial\Sigma$ is a subset of the nodal set of the Jacobi field on $\Sigma$.

In [C3] I defined the *horizon* of $\Sigma$ with respect to $X$ to be the set $\{x \in \Sigma : \overrightarrow{px}$ is tangent to $\Sigma\}$, that is, the nodal set of $\text{proj}_{\Sigma^\perp}(X)$. By a well-known theory of eigenvalues and eigenfunctions, each nodal component of $\text{proj}_{\Sigma^\perp}(X)$ is stable, and by Courant's nodal domain theorem the number of the nodal components is a lower bound of the index of the Jacobi operator. This fact is a generalization of the Morse index theorem that the index on a geodesic $\gamma$ from $p$ to $q$ is equal to the number of conjugate points of $p$ on $\gamma$. Therefore the horizon of a minimal surface is a higher dimensional version of the conjugate point on a geodesic.

There is another natural way of getting Jacobi fields on $\Sigma$: Use Killing vector fields of $\mathbb{R}^3$. Suppose $V$ is a variation vector field associated with a family of parallel translations, or rotations about a line. Then $\text{proj}_{\Sigma^\perp}(V)$ is a Jacobi field. Similarly rotations on a 2-plane in $\mathbb{R}^{n+1}$ gives rise to a Killing vector field on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, which in turn generates a Jacobi field on a minimal submanifold of $\mathbb{S}^n$. In [C3] it is shown that the number of the nodal components of this natural Jacobi field on many well-known complete minimal surfaces of $\mathbb{R}^3$ and $\mathbb{S}^3$ is a lower bound of "1 + the index of the surface".

## 4 Non-Euclidean space

Given a domain $D$ in $\mathbb{S}^2$ or in $\mathbb{H}^2$, it is known that the area $A$ and perimeter $L$ of $D$ satisfy

$$4\pi A \le L^2 + A^2 \text{ on } D \subset \mathbb{S}^2, \quad 4\pi A \le L^2 - A^2 \text{ on } D \subset \mathbb{H}^2.$$

As in $\mathbb{R}^n$ it is conjectured that a minimal surface $\Sigma$ in $\mathbb{S}^n, \mathbb{H}^n$ should satisfy the isoperimetric inequality

$$4\pi A \le L^2 + A^2 \text{ on } \Sigma \subset \mathbb{S}^n, \quad 4\pi A \le L^2 - A^2 \text{ on } \Sigma \subset \mathbb{H}^n. \tag{7}$$

In [CG1] and [CG2] Gulliber and I extended the cone method of Section 2 to the minimal surfaces in $\mathbb{H}^n$. The cone method in $\mathbb{H}^n$ will be reviewed in this section.

The cone method was based on

$$\Delta r^2 = 4$$

for the minimal surfaces in $\mathbb{R}^n$. This originates from

$$\operatorname{Hess} r^2 = 2\bar{g},$$

where $\operatorname{Hess} r^2$ denotes the Hessian of $r^2$ and $\bar{g}$ the metric of $\mathbb{R}^n$. How can we compute the Hessian of a distance function in the non-Euclidean space? We can do this by considering a Jacobi field. Let $r(x) = \operatorname{dist}(p, x)$, $p, x \in \mathbb{S}^n, \mathbb{H}^n$ and let $\gamma$ be a geodesic from $p$. If $V(x)$ is a Jacobi field along $\gamma$ such that $V(p) = 0$ and $V \perp \gamma'$, then $|V(x)| = \sin r(x)$ or $|V(x)| = \sinh r(x)$ in $\mathbb{S}^n, \mathbb{H}^n$, respectively. One can then show that

$$\operatorname{Hess} \cos r = -(\cos r)\bar{g} \text{ on } \mathbb{S}^n, \quad \operatorname{Hess} \cosh r = (\cosh r)\bar{g} \text{ on } \mathbb{H}^n, \tag{8}$$

where $\bar{g}$ is the metric of $\mathbb{S}^n$, $\mathbb{H}^n$.

Let $\Sigma^k$ be a submanifold of $M^n$ with mean curvature vector $\vec{H}$ and let $e_{k+1}, ..., e_n$ be orthonormal vector fields perpendicular to $\Sigma$. Then

$$\Delta_\Sigma f = \Delta_M f + \vec{H} f - \sum_{\alpha=k+1}^{n} \operatorname{Hess} f(e_\alpha, e_\alpha) \tag{9}$$

for a smooth function $f$ defined on $M$. Then (8) and (9) yield on a minimal submanifold $\Sigma^k$ of $\mathbb{S}^n$ or $\mathbb{H}^n$

$$\Delta \cos r = -k \cos r \text{ on } \Sigma \subset \mathbb{S}^n, \quad \Delta \cosh r = k \cosh r \text{ on } \Sigma \subset \mathbb{H}^n.$$

Moreover $\vec{H} r = 0$, so

$$\Delta \cos r = -k \cos r \text{ on } p \divideontimes \partial\Sigma \subset \mathbb{S}^n, \quad \Delta \cosh r = k \cosh r \text{ on } p \divideontimes \partial\Sigma \subset \mathbb{H}^n.$$

Hence it follows that

$$\Delta r = \cot r (k - |\nabla r|^2) \text{ on } \Sigma \subset \mathbb{S}^n, \quad \Delta r = \coth r (k - |\nabla r|^2) \text{ on } \Sigma \subset \mathbb{H}^n \tag{10}$$

and

$$\Delta r = (k-1)\cot r \ \text{ on } p \divideontimes \partial\Sigma \subset \mathbb{S}^n, \quad \Delta r = (k-1)\coth r \ \text{ on } p \divideontimes \partial\Sigma \subset \mathbb{H}^n. \tag{11}$$

We are now ready to compute the Laplacian of functions of distance which will yield the area and angle comparisons in non-Euclidean space. Since $\sin r$ is the length of a Jacobi field in $\mathbb{S}^k$, $k\omega_k \sin^{k-1} r$ is the volume of a geodesic sphere of radius $r$. Let $f_0(r)$ be a function of distance on $\mathbb{S}^k$ such that

$$\nabla f_0 = \sin^{1-k} r \, \nabla r.$$

Then

$$\Delta f_0 = k\omega_k \, \delta_p.$$

If $f(r)$ is a function on $\mathbb{S}^n$ (not on $\mathbb{S}^k$) such that

$$\nabla f = \sin^{1-k} r \, \nabla r,$$

then by (10) and (11) we have for minimal $\Sigma^k \subset \mathbb{S}^n$

$$\Delta f \geq k\omega_k \, \delta_p \ \text{ on } \Sigma\,, \quad \Delta f = k\omega_k \, \Theta_{p \divideontimes \partial\Sigma} \, \delta_p \ \text{ on } p \divideontimes \partial\Sigma. \tag{12}$$

Similarly for $f(r)$ on $\mathbb{H}^n$ with $\nabla f = \sinh^{1-k} r \, \nabla r$, we get

$$\Delta f \geq k\omega_k \, \delta_p \ \text{ on minimal } \Sigma^k \subset \mathbb{H}^n, \quad \Delta f = k\omega_k \, \Theta_{p \divideontimes \partial\Sigma} \, \delta_p \ \text{ on } p \divideontimes \partial\Sigma. \tag{13}$$

Integrating (12), (13) on $\Sigma$ and $p \divideontimes \partial\Sigma$ and using (4), we conclude that for minimal $\Sigma^k \subset \mathbb{S}^n, \mathbb{H}^n$

$$k\omega_k \leq \int_{\partial\Sigma} \frac{\partial f}{\partial \nu} \leq \int_{\partial\Sigma} \frac{\partial f}{\partial \eta} = \int_{p \divideontimes \partial\Sigma} \Delta f = \text{Angle}(\partial\Sigma, p). \tag{14}$$

To get the area comparison let $\alpha(r)$ be the volume of a geodesic ball of radius $r$ in $\mathbb{H}^k$ and let $h(r)$ be a function on $\mathbb{H}^n$ such that

$$\nabla h = \frac{\alpha}{\alpha'} \nabla r.$$

Then it is easy to see that

$$\Delta h \geq 1 \ \text{ on minimal } \Sigma^k \subset \mathbb{H}^n, \quad \Delta h = 1 \ \text{ on } p \divideontimes \partial\Sigma.$$

Therefore

$$\text{Vol}(\Sigma) \leq \int_{\partial\Sigma} \frac{\partial h}{\partial \nu} \leq \int_{\partial\Sigma} \frac{\partial h}{\partial \eta} = \int_{p \divideontimes \partial\Sigma} \Delta h = \text{Vol}(p \divideontimes \partial\Sigma).$$

This area comparison does not hold in $\mathbb{S}^n$: $\text{Area}(\Sigma) > \text{Area}(p \divideontimes \partial\Sigma)$ when $\Sigma$ is the Clifford torus in the northern hemisphere and $p$ is the north pole.

Now that we have the angle and area comparisons we can apply the cutting and inserting argument as in Section 2 to prove

$$4\pi A \leq L^2 - A^2$$

for a minimal surface $\Sigma^2 \subset \mathbb{H}^n$ with $\partial\Sigma$ radially connected from a point $p \in \Sigma$. Instead of cutting and inserting, one could use the approximation argument that $p \divideontimes \partial\Sigma$ with $\text{Angle}(\partial\Sigma, p) \geq 2\pi$ can be approximated by a sequence of smooth surfaces $\{S_i\}$ with Gaussian curvature $K_{S_i} \leq -1$. One then uses Bol's isoperimetric inequality [Bo] $4\pi A \leq L^2 - A^2$ for a simply connected surface $S$ with $K_S \leq -1$.

## 5 Ray preserving metric

As mentioned above, Bol [Bo] has proved that a simply connected surface $S$ satisfies

$$4\pi A \leq L^2 + \left(\sup_S K_S\right) A^2.$$

Therefore one is tempted to conjecture that if $\Sigma$ is a minimal surface in a Riemannian manifold $M$ with nonconstant sectional curvature $K_M \leq K$, then

$$4\pi A \leq L^2 + KA^2.$$

For this conjecture, however, we cannot resort to the cone method because it is not possible to prove the area and angle comparisons

$$\text{Area}(\Sigma) \leq \text{Area}(p \divideontimes \partial\Sigma), \quad \text{Angle}(\partial\Sigma, p) \geq 2\pi.$$

Indeed, even if we have

$$\Delta \log(1 + \cosh r) \geq 1 \ \text{ on } \Sigma \subset M \text{ with } K_M \leq -1, \tag{15}$$

we cannot prove

$$\Delta \log(1 + \cosh r) = 1 \ \text{ on } p \divideontimes \partial\Sigma.$$

Hence we can only say that

$$\text{Area}(\Sigma) \leq \int_{\partial\Sigma} \frac{\sinh r}{1 + \cosh r} \frac{\partial r}{\partial \nu} \leq \int_{\partial\Sigma} \frac{\sinh r}{1 + \cosh r} \frac{\partial r}{\partial \eta} \neq \text{Area}(p \divideontimes \partial\Sigma). \tag{16}$$

So let's forget about $p \divideontimes \Sigma$ and consider the integral $\int_{\partial\Sigma} \frac{\sinh r}{1+\cosh r} \frac{\partial r}{\partial \eta}$. This integral is to be computed on the 1-dimensional set $\partial\Sigma$ but let's move $\partial\Sigma$ to a more agreeable ambient space(i.e., space form) and compute the integral there; or equivalently, let's give a constant curvature ($K = -1$) metric to $p \divideontimes \partial\Sigma$.

Among infinitely many constant curvature metrics on $p \divideontimes \partial\Sigma$ we want the one that preserves $\int_{\partial\Sigma} \frac{\sinh r}{1+\cosh r} \frac{\partial r}{\partial \eta}$. So we want the constant curvature metric $\hat{g}$ on $p \divideontimes \partial\Sigma$ that preserves all of the distance function $r$, $\frac{\partial r}{\partial \eta}$ and the arclength element $ds$ of $\partial\Sigma$. Let's call $\hat{g}$ the *ray preserving metric*. Hence (i) every geodesic from $p$ under the original metric $g$ on $p \divideontimes \partial\Sigma$ remains a geodesic of equal length under $\hat{g}$, (ii) the angles between the tangent vctor to $\partial\Sigma$ and the geodesic from $p$ remain unchanged, and (iii) the length of any arc of $\partial\Sigma$ remains the same.

As a matter of fact, condition (ii) follows from (i) and (iii), and conversely, condition (i) follows from (ii) and (iii). Therefore one can find the ray preserving metric $\hat{g}$ by developing $p \divideontimes \partial\Sigma$ on the hyperbolic plane $\mathbb{H}^2$ when $K_M \leq -1$ (or on $\mathbb{R}^2$ when $K_M \leq 0$) as follows. Fix a point $q_0$ in $\partial\Sigma$ and choose two points $\hat{p}$ and $\hat{q}_0$ in $\mathbb{H}^2$ (or in $\mathbb{R}^2$) such that $\text{dist}(p, q_0) = \text{dist}(\hat{p}, \hat{q}_0)$. Let $\Gamma$ be a curve in $\mathbb{H}^2$ (or in $\mathbb{R}^2$) starting from $\hat{q}_0$. Parametrize both $\partial\Sigma$ and $\Gamma$ by arc length $s$ such that $\partial\Sigma(0) = q_0$ and $\Gamma(0) = \hat{q}_0$. Define $\theta(s)(\hat{\theta}(s)$, respectively) to be the angle between $\partial\Sigma$ and the geodesic from $p$ to $\partial\Sigma(s)$ (between $\Gamma$ and the geodesic from $\hat{p}$ to $\Gamma(s)$, respectively). Assume further that $\Gamma$ satisfies

$$\theta(s) = \hat{\theta}(s).$$

This is a first order ODE and so such a $\Gamma$ exists uniquely. Thus the map $\phi$ mapping the geodesic from $p$ to $\partial\Sigma(s)$ to the geodesic from $\hat{p}$ to $\Gamma(s)$ can pull back the constant curvature metric of $\hat{p} \divideontimes \Gamma$ to $p \divideontimes \partial\Sigma$, which is the desired ray preserving metric $\hat{g}$.

Denote $(p \divideontimes \partial\Sigma, \hat{g})$ by $\hat{C}$. We have the following comparison results between $\Sigma$, $p \divideontimes \partial\Sigma$ and $\hat{C}$.

**Theorem 2** ([CG3]) *Let $\Sigma^2$ be a minimal surface in a simply connected Riemannian manifold $M^n$ of sectional curvature bounded above by a constant $K (= 0$ or $-1)$. Then*
(a) Area $(\Sigma) \leq$ Area$(\hat{C})$;
(b) $\Theta_\Sigma(p) \leq \Theta_{\hat{C}}(p)$, *with equality if and only if* $\Sigma = \hat{C}$;
(c) Area$(p \divideontimes \partial\Sigma) \leq$ Area$(\hat{C})$;
(d) $\Theta_{p \divideontimes \partial\Sigma}(p) \leq \Theta_{\hat{C}}(p)$.
(e) *Let $k(s)$ and $\hat{k}(s)$ be the inward geodesic curvatures of $\partial\Sigma$ in $p \divideontimes \partial\Sigma$ and in $\hat{C}$, respectively. Then $k(p) \geq \hat{k}(s)$.*

In order to prove Theorem 2 and inequality (15) we need to compute the Laplacian of some distance functions with conditions $K_M \leq K$ and $K_{\hat{C}} = K$. This will be done in the following section.

# 6 Varying curvature

The distance function on a Riemannian manifold $M$, being the simplest geometric function on $M$, implicitly gives us many pieces of information on the geometry of $M$. Actually all the results obtained so far in this article have arisen from the Laplacian of functions of distance. In this section, given the curvature comparison $K_M \leq K_{\overline{M}}$, let's see what one can say about the distance functions of $M$ and $\overline{M}$.

Let $r(x)$ be the distance from a fixed point $p$ to $x$ in $M$. Assume that $\gamma$ is a geodesic from $p$ to $q$, $v$ a vector at $q$ perpendicular to $\gamma$, and $X$ the Jacobi field along $\gamma$ satisfying $X(p) = 0$ and $X(q) = v$. Then

$$\begin{aligned} \operatorname{Hess} r(X, X) &= \left\langle X, \nabla_X \frac{\partial}{\partial r} \right\rangle = \int_0^r \frac{d}{dr} \left\langle X, \nabla_{\frac{\partial}{\partial r}} X \right\rangle \\ &= \int_0^r \left( \left| \nabla_{\frac{\partial}{\partial r}} X \right|^2 - \left\langle R\left( X, \frac{\partial}{\partial r} \right) \frac{\partial}{\partial r}, X \right\rangle \right), \end{aligned}$$

which is the second variation of the length of $\gamma$ associated with $X$. The Jacobi field minimizes the second variation among all vector fields along $\gamma$ with the same boundary conditions. Therefore if the sectional curvature of $M^n$ is bounded from above by that of a Riemannian manifold $\overline{M}^n$ which has a distance function $\bar{r}$ with $\bar{r}(x) = \operatorname{dist}(\bar{p}, x)$, then one gets the Hessian comparison

$$\operatorname{Hess} r(v, v) \;\geq\; \operatorname{Hess} \bar{r}(u, u), \tag{17}$$

where $u$ is the vector at $\bar{q} \in \overline{M}$ perpendicular to the geodesic $\bar{\gamma}$ from $\bar{p}$ to $\bar{q}$ with $\bar{r}(\bar{q}) = r(q)$ and $|u| = |v|$.

Let $\Sigma^m$ be a submanifold of $M$ with mean curvature vector $\vec{H}$. Given a smooth function $f$ on $M$, there are two types of Laplacian of $f|_\Sigma$ on $\Sigma$, $\bar{\Delta} f$ and $\Delta f$: for an orthonormal basis $\{e_1, ..., e_m\}$ of $\Sigma$ define

$$\bar{\Delta} f \;=\; \sum_{i=1}^{m} \operatorname{Hess} f(e_i, e_i), \quad \Delta f \;=\; \sum_{i=1}^{m} \operatorname{Hess} f|_\Sigma(e_i, e_i).$$

From (9) one sees that

$$\bar{\Delta} f \;=\; \Delta f - \vec{H} f. \tag{18}$$

Therefore if $\Sigma$ is minimal in $M$, the intrinsic Laplacian $\Delta f$ can be replaced with $\bar{\Delta} f$ which is more extrinsic and easier to compute. With (17) and (18) we are now ready to compute the Laplacian of functions of distance.

**Lemma 1** *Let $\Sigma^2$ be a minimal surface in a simply connected Riemannian manifold $M^n$ of sectional curvature $\leq K$ $(= 0$ or $-1)$.*

*If $K = 0$, we have on $\Sigma$*

(a) $\Delta r^2 \geq 4$;

(b) $\Delta \log r \geq 2\pi \Theta_\Sigma(p) \delta_p$ *for* $p \in \Sigma$.

*If $K = -1$,*

(c) $\Delta \log(1 + \cosh r) \geq 1$;

(d) $\Delta \log \frac{\sinh r}{1+\cosh r} \geq 2\pi \Theta_\Sigma(p) \delta_p$ *for* $p \in \Sigma$.

*Proof.* Assume that $\overline{M}^n$ is a complete simply connected Riemannian manifold of constant sectional curvature $K$. Let $g$, $\bar{g}$ denote the metrics of $M$ and $\overline{M}$, respectively, and let $\nabla$, $\overline{\nabla}$ denote the Riemannian connections of $M$ and $\overline{M}$, respectively. When $\overline{M} = \mathbb{R}^n$, we have

$$\operatorname{Hess} \bar{r}^2 = 2\bar{g}.$$

Since

$$\nabla^2 r^2 \;=\; 2r\nabla^2 r + 2\nabla r \otimes \nabla r \;\text{ and }\; \overline{\nabla}^2 \bar{r}^2 \;=\; 2\bar{r}\overline{\nabla}^2 \bar{r} + 2\overline{\nabla}\bar{r} \otimes \overline{\nabla}\bar{r},$$

(17) and (18) yield (a). Then

$$\Delta r \;=\; \operatorname{div} \frac{1}{2r} \nabla r^2 \;=\; \frac{1}{2r}\Delta r^2 - \frac{1}{2r^2}\langle \nabla r,\; 2r\nabla r\rangle \;\geq\; \frac{1}{r}\left(2 - |\nabla r|^2\right),$$

and hence

$$\Delta \log r \;=\; \operatorname{div} \frac{1}{r}\nabla r \;=\; \frac{1}{r}\Delta r - \frac{1}{r^2}|\nabla r|^2 \;\geq\; \frac{2}{r^2}(1 - |\nabla r|^2) \;\geq\; 0.$$

Near $p$, $\Sigma$ is approximately $T_p\Sigma$ or the tangent cone of $\Sigma$ at $p$, on which $\Delta \log r = 2\pi\Theta_\Sigma(p)\delta_p$. Therefore on $\Sigma$ we have (b). From (8) we get

$$\operatorname{Hess} \bar{r} = \coth \bar{r}(\bar{g} - \overline{\nabla}\bar{r} \otimes \overline{\nabla}\bar{r}),$$

and so

$$\Delta r \geq (2 - |\nabla r|^2) \coth r.$$

Then

$$\begin{aligned}\Delta\log(1+\cosh r) &= \operatorname{div}\frac{\sinh r}{1+\cosh r}\nabla r = \frac{1}{1+\cosh r}|\nabla r|^2 + \frac{\sinh r}{1+\cosh r}\Delta r \\ &\geq \frac{2\cosh r + |\nabla r|^2(1-\cosh r)}{1+\cosh r} \geq 1,\end{aligned}$$

which gives (c). Now we have

$$\begin{aligned}\Delta\log\frac{\sinh r}{1+\cosh r} &= \operatorname{div}\frac{1}{\sinh r}\nabla r = -\frac{\cosh r}{\sinh^2 r}|\nabla r|^2 + \frac{1}{\sinh r}\Delta r \\ &\geq \frac{2\cosh r(1-|\nabla r|^2)}{\sinh^2 r} \geq 0.\end{aligned}$$

Note that $f(r) := \frac{1}{2\pi}\log\frac{\sinh r}{1+\cosh r}$ is a fundamental solution of the Laplacian on the hyperbolic plane since $\frac{1}{2\pi f'(r)} = \sinh r$ is the length of a Jacobi field. So (d) follows. $\square$

**Lemma 2** *Let $\Sigma^2$ be a minimal surface in a simply connected Riemannian manifold $M^n$ of sectional curvature $\leq K$ $(= 0$ or $-1)$. Assume that $r$ is the distance from $p$ and $\hat{C}$ denotes the cone $p⨳\partial\Sigma$ with the ray preserving metric $\hat{g}$ of constant curvature $K_{\hat{C}} = 0$ or $-1$. Then on $\hat{C}$*

(a) $\Delta r^2 = 4$, *if* $K_{\hat{C}} = 0$; $\qquad \Delta\log(1+\cosh r) = 1$, *if* $K_{\hat{C}} = -1$;

(b) $\Delta\log r = 2\pi\Theta_{\hat{C}}(p)\delta_p$, *if* $K_{\hat{C}} = 0$; $\qquad \Delta\log\frac{\sinh r}{1+\cosh r} = 2\pi\Theta_{\hat{C}}(p)\delta_p$, *if* $K_{\hat{C}} = -1$.

*Proof.* On $\hat{C}$ $\nabla r$ is perpendicular to the mean curvature vector of $\hat{C}$. Hence (18) implies that for any function $f$ of distance $r$, $\bar{\Delta}f = \Delta f$ on $\hat{C}$. Moreover $|\nabla r| \equiv 1$ on $\hat{C}$. It follows that all the inequalities in the proof of Lemma 1 become equalities. This proves the lemma except for the constant $\alpha$. The constant $2\pi$ that appears in the Laplacian of the fundamental solution on $\mathbb{R}^2$ and $\mathbb{H}^2$ comes from the limit as $a \to 0$ of the circumference of the circle of radius $a$ with center at $\bar{p}$ divided by $a$. Similarly, if $S_a(p)$ denotes the geodesic sphere of radius $a$ with center at $p$, $\alpha$ equals $\lim_{a\to 0}\frac{1}{a}\text{Length}(\hat{C}\cap S_a(p))$, which is $2\pi\Theta_{\hat{C}}(p)$. $\square$

Now that we have proved Lemmas 1 and 2, we can modify (16) when $K_M \leq -1$ as follows:

$$\text{Area}(\Sigma) \leq \int_{\partial\Sigma}\frac{\sinh r}{1+\cosh r}\frac{\partial r}{\partial\nu} \leq \int_{\partial\Sigma}\frac{\sinh r}{1+\cosh r}\frac{\partial r}{\partial\eta} = \int_{\hat{C}}\Delta\log(1+\cosh r) = \text{Area}(\hat{C}).$$

Moreover

$$\begin{aligned}2\pi &\leq 2\pi\Theta_\Sigma(p) \leq \int_\Sigma \Delta\log\frac{\sinh r}{1+\cosh r} = \int_{\partial\Sigma}\frac{1}{\sinh r}\frac{\partial r}{\partial\nu} \\ &\leq \int_{\partial\Sigma}\frac{1}{\sinh r}\frac{\partial r}{\partial\eta} = \int_{\hat{C}}\Delta\log\frac{\sinh r}{1+\cosh r} = 2\pi\Theta_{\hat{C}}(p).\end{aligned}$$

Similarly, when $K_M \leq 0$,

$$4\,\text{Area}(\Sigma) \leq \int_{\partial\Sigma} 2r\frac{\partial r}{\partial\nu} \leq \int_{\partial\Sigma} 2r\frac{\partial r}{\partial\eta} = \int_{\hat{C}}\Delta r^2 = 4\,\text{Area}(\hat{C}),$$

and

$$\begin{aligned}2\pi &\leq 2\pi\Theta_\Sigma(p) \leq \int_\Sigma\Delta\log r = \int_{\partial\Sigma}\frac{1}{r}\frac{\partial r}{\partial\nu} \\ &\leq \int_{\partial\Sigma}\frac{1}{r}\frac{\partial r}{\partial\eta} = \int_{\hat{C}}\Delta\log r = 2\pi\Theta_{\hat{C}}(p).\end{aligned}$$

Therefore as in Sections 2 and 4 we can prove

$$4\pi \text{Area}(\hat{C}) \leq \text{Length}(\partial\Sigma)^2 + K\text{Area}(\hat{C})^2.$$

Then from the monotonicity of the function $f(A) = 4\pi A + A^2$ for $A > 0$ we get the following:

**Theorem 3** *Let $\Sigma$ be a minimal surface in a simply connected Riemannian manifold $M$ with sectional curvature $\leq K (= 0$ or $-1)$. If $\partial\Sigma$ is radially connected from a point of $\Sigma$, then*

$$4\pi A \leq L^2 + KA^2.$$

It remains to prove (c),(d),(e) of Theorem 2. Let $V$ be a Jacobi field along a geodesic $\gamma$ from $p$ to $q$ on $p ⨳ \partial\Sigma$ satisfying $V(p) = 0$, $V \perp \gamma'$ and let $\hat{V}$ be a Jacobi field along $\gamma$ on $\hat{C}$ satisfying $\hat{V}(p) = 0$ and $\hat{V}(q) = V(q)$. Since $K_{p ⨳ \partial\Sigma} \leq K_{\hat{C}}$, it follows from the Jacobi equation that

$$|V(\gamma(t))| \leq |\hat{V}(\gamma(t))|. \tag{19}$$

Therefore (c) and (d) follow easily. Remember that the angles that the rays from $p$ make with $\partial\Sigma$ remain the same when the metric of $p ⨳ \partial\Sigma$ is changed into the ray preserving metric of $\hat{C}$. However, (19) implies that the angle between the rays at $p$ increases. Hence it is not difficult to conclude that the geodesic curvature of $\partial\Sigma$ decreases when the metric of $p ⨳ \partial\Sigma$ is changed into the ray preserving metric of $\hat{C}$. See Proposition 4 of [CG3] for the details.

# 7 Embeddedness

In this article so far we have seen that the area comparison and angle comparison both played a key role in proving the isoperimetric inequality of minimal surfaces. We have also seen that the area comparison alone gave us unexpected information about the stability and index of a minimal surface. In this section we will see that the angle comparison alone can provide us with some valuable information about the embeddedness of a minimal surface and knottedness of a Jordan curve.

The Douglas solution of the Plateau problem gave rise to three interesting problems about a minimal surface $\Sigma$ spanning a given Jordan curve $\Gamma$ in $\mathbb{R}^n$: If the total curvature of $\Gamma$ is not bigger than $4\pi$, is $\Sigma$ unique? Is $\Sigma$ embedded? Does $\Sigma$ have no genus? The uniqueness of a disk-type $\Sigma$ was proved by Nitsche [N], the embeddedness of $\Sigma$ was proved in 2002 by Ekholm-White-Wienholtz [EWW], and the third problem is still open.

One can understand Ekholm-White-Wienholtz's proof of the embeddedness more easily by using the angle comparison as follows. Theorem 2 (b) gives

$$2\pi\Theta_\Sigma(p) \leq 2\pi\Theta_{p ⨳ \partial\Sigma}(p) = \int_{\partial\Sigma} \langle \vec{k}, \eta \rangle \leq \int_{\partial\Sigma} |\vec{k}| \leq 4\pi,$$

where the equality is due to the Gauss-Bonnet theorem with $\vec{k}$ the curvature vector of $\partial\Sigma$ and $\eta$ the outward unit conormal to $\partial\Sigma$ on $p ⨳ \partial\Sigma$. Hence $\Theta_\Sigma(p) < 2$ and $\Sigma$ is embedded.

R. Gulliver and I generalized Ekholm-White-Wienholtz's theorem to minimal surfaces in a Riemannian manifold of sectional curvature bounded above by a nonpositive constant $K$. The key difference comes from the Gauss-Bonnet theorem on $\hat{C}$:

$$2\pi\Theta_{\hat{C}}(p) = \int_{\partial\Sigma} \langle \vec{k}, \eta \rangle + K \text{Area}(\hat{C}),$$

which, compared with $\Sigma \subset \mathbb{R}^n$, has an extra term $K\text{Area}(\hat{C})$. Consequently Theorem 2 (b),(c),(d),(e) yield the following:

**Theorem 4** *Let $\Sigma^2$ be a minimal surface in a complete simply connected Riemannian manifold $M^n$ with $K_M \le K \le 0$. If*

$$\int_{\partial\Sigma} |\vec{k}| \le 4\pi - K \inf_{p \in \mathsf{CH}(\partial\Sigma)} \text{Area}(p \divideontimes \partial\Sigma),$$

*where $\mathsf{CH}(\partial\Sigma)$ is the convex hull of $\partial\Sigma$, then $\Sigma$ is embedded.*

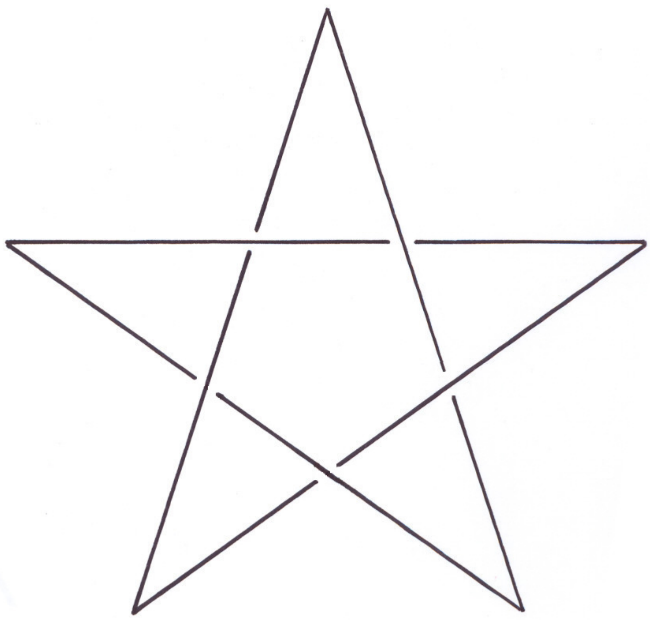

Figure 1: Nonexistent Star

Ekholm-White-Wienholtz remarked that the embeddedness of a minimal disk $\Sigma$ (when $\int_{\partial\Sigma} |\vec{k}| \le 4\pi$) implies the unknottedness of $\partial\Sigma$, giving a new proof of Fáry-Milnor's theorem. Recently a student of mine named Sung-Hong Min[M] has obtained a new criterion for the unknottedness of a Jordan curve:

**Theorem 5** *A piecewise-linear Jordan curve $\Gamma_5$ with five vertices in $\mathbb{R}^3$ is unknotted.*

By this theorem the well-known star knot of Figure 1 cannot exist! Min's proof is based on the observation that the tangent indicatrix of $\Gamma_5 \subset \mathbb{R}^n$ is a closed curve consisting of 5 geodesics of length $< \pi$ on $\mathbb{S}^{n-1}$ and hence

$$\int_{\Gamma_5} |\vec{k}| < 4\pi.$$

A remarkable fact is that this theorem holds in $\mathbb{H}^3$ and $\mathbb{S}^3$ as well, that is, *every piecewise-geodesic Jordan curve* $\Gamma_5$ *with five vertices in* $\mathbb{H}^3$ *or in a geodesic ball of radius* $\pi/4$ *in* $\mathbb{S}^3$ *is unknotted.*

This fact is surprising when we note that Choe-Gulliver's theorem needs an extra term for the non-Euclidean space whereas Min's theorem requires no extra assumption for $\Gamma_5$ in non-Euclidean space. A clue can be found in the angle comparison: $\Gamma_5$ should bound a minimal disk $\Sigma$ in $\mathbb{H}^3$ or $\mathbb{S}^3$, and $\Sigma$ must be embedded due to the angle comparison. Why? Because (i) inequality (14) and Theorem 2 (b) for $\Sigma$ can be rewritten as

$$\Theta_\Sigma(p) \le \Theta_{p \divideontimes \partial\Sigma}(p),$$

(ii) when $\partial\Sigma$ is piecewise-geodesic, $p \divideontimes \partial\Sigma$ is the union of 5 totally geodesic triangles, (iii) $\Theta_{p \divideontimes \partial\Sigma}(p)$ is the same whether $\partial\Sigma = \Gamma_5$ is in $\mathbb{H}^3, \mathbb{S}^3$ or $\mathbb{R}^3$, (iv) $2\pi\Theta_{p \divideontimes \partial\Sigma}(p) =$ total curvature of $\Gamma_5$ when $\Gamma_5$ is in $\mathbb{R}^3$, which has been proved to be less than $4\pi$.

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