RELATIVE ISOPERIMETRIC INEQUALITY FOR DOMAINS OUTSIDE A CONVEX SET

JAIGYOUNG CHOE

ABSTRACT. Given a convex set $C \subset \mathbf{R}^n$ and a set $D \subset \mathbf{R}^n \sim C$, the inequality $\frac{1}{2}n^n\omega_n\mathrm{Volume}(D)^{n-1} \leq \mathrm{Volume}(\partial D \sim \partial C)^n$ is called the relative isoperimetric inequality. We prove this inequality in three cases: i) when C and D are symmetric about $n-1$ mutually orthogonal vertical hyperplanes and $\partial D \cap \partial C$ is a graph over a horizontal hyperplane; ii) when $\partial D \sim \partial C$ and $\partial D \cap \partial C$ are graphs over a subset A of a horizontal hyperplane such that A is symmetric about $n-1$ mutually orthogonal vertical hyperplanes; iii) when C is an n-dimensional ball. Also, if S is a disk type surface of nonpositive Gaussian curvature and $\Gamma \subset \partial S$ is connected and concave, it is proved that $2\pi\mathrm{Area}(S) \leq \mathrm{Length}(\partial S \sim \Gamma)^2$. These relative isoperimetric inequalities are sharp.

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1. Introduction

The classical isoperimetric inequality states that if D is a set in $\mathbf{R}^n$ and ω_n is the volume of a unit ball in $\mathbf{R}^n$, then

$$n^n\omega_n\mathrm{Volume}(D)^{n-1} \leq \mathrm{Volume}(\partial D)^n \tag{1}$$

and equality holds if and only if D is a ball. An immediate consequence of this inequality is that if H is a closed half space of $\mathbf{R}^n$ and D is a subset of H then

$$\frac{1}{2}n^n\omega_n\mathrm{Volume}(D)^{n-1} \leq \mathrm{Volume}(\partial D \sim \partial H)^n$$

and equality holds if and only if D is a half ball with the flat part of its boundary contained in ∂H. This follows if one applies (1) to the union of D and its mirror image across ∂H. Then a natural question to ask is the following. If $C \subset \mathbf{R}^n$ is a convex set and D is a subset of $\mathbf{R}^n \sim C$, does D satisfy the isoperimetric inequality

$$\frac{1}{2}n^n\omega_n\mathrm{Volume}(D)^{n-1} \leq \mathrm{Volume}(\partial D \sim \partial C)^n? \tag{2}$$

Does equality hold if and only if $C = H$ and D is a half ball with the flat part of its boundary lying in ∂H? (2) is called the *relative* isoperimetric inequality, C is called the supporting set of D, and $\mathrm{Volume}(\partial D \sim \partial C)$ is called the relative volume of ∂D. For $n = 2$ one can easily prove (2) by reflecting the convex hull of D about its linear boundary.

A partial answer for $n \geq 3$ was recently obtained by I. Kim [7]; he showed that if $U = \{(x, y) \in \mathbf{R}^2 : y \geq f(x), f'' \geq 0\}$, then (2) holds for $C = U \times \mathbf{R}^{n-2}$. In this paper

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we prove that the relative isoperimetric inequality holds if C is a graph which is symmetric about $n-1$ hyperplanes of $\mathbf{R}^n$. In particular, we prove (2) when C is a ball. The tools we use are Gromov's method of using the divergence theorem and Steiner's method of symmetrization.

Moreover we prove the relative isoperimetric inequality on a disk type surface S of nonpositive Gaussian curvature. In 1926 Weil [10] showed that such a surface S satisfies

$$4\pi\text{Area}(S) \le \text{Length}(\partial S)^2.$$

By contrast we prove that if Γ is a connected subset of ∂S on which the geodesic curvature is not positive with respect to the inward unit normal, then

$$2\pi\text{Area}(S) \le \text{Length}(\partial S \sim \Gamma)^2$$

and equality holds if and only if S is a flat half disk.

Added in proof: The author and M. Ritoré [11] have recently proved the relative isoperimetric inequality in the general setting using a different method.

2. Gromov's method

In [5] Gromov gave a new proof of the classical isoperimetric inequality. His proof is based on a volume-preserving map whose divergence is bigger than or equal to the dimension of space. In this section we shall see how Gromov's method can be adapted for our purpose and why the convexity of the supporting set is necessary.

Theorem 1. *Let C be a convex set in $\mathbf{R}^n$ and D a subset of $\mathbf{R}^n \sim C$ with piecewise C^1 boundary. Suppose that every normal vector η to $\partial D \cap \partial C$ toward the exterior of D does not point upward, that is, $\langle \eta, \frac{\partial}{\partial x^n}\rangle \le 0$ for the unit vertical vector $\frac{\partial}{\partial x^n}$. Suppose also that there exist vertical hyperplanes $\Pi_1, ..., \Pi_{n-1}$ which are mutually perpendicular such that C and D are symmetric about each of them. Then*

$$\frac{1}{2}n^n\omega_n\text{Volume}(D)^{n-1} \le \text{Volume}(\partial D \sim \partial C)^n,$$

where equality holds if and only if D is a half ball.

Proof. First let us define a C^1 map $\phi_D : D \to [0,1]^n$ by

$$\phi_D(x^1, ..., x^n) = (\phi^1, ..., \phi^n),\ \phi^i = \frac{\overline{v}_i}{v_i},$$

$$v_i = L^{n-i+1}\{(a^1, ..., a^n) \in D : a^j = x^j, 1 \le j \le i-1, -\infty \le a^k \le \infty, i \le k \le n\},$$

$$\overline{v}_i = L^{n-i+1}\{(a^1, ..., a^n) \in D : a^j = x^j, 1 \le j \le i-1, -\infty \le a^i \le x^i, -\infty \le a^k \le \infty, i+1 \le k\},$$

where L^k is the k-dimensional Lebesgue measure. Then $\phi^i = \phi^i(x^1, ..., x^i)$ and the Jacobian matrix of ϕ_D, $\left(\frac{\partial\phi^i}{\partial x^j}\right)$, is lower triangular with diagonal entries $\frac{\partial\phi^i}{\partial x^i} = \frac{v_{i+1}}{v_i}$ and $\frac{\partial\phi^n}{\partial x^n} = \frac{1}{v_n}$. Therefore

$$\det\left(\frac{\partial\phi^i}{\partial x^j}\right) = \frac{1}{v_1}.$$

Similarly, define $\phi_B : B \to [0,1]^n$ where B is the half ball

$$\{(x^1, ..., x^n) \in \mathbf{R}^n : x^n \geq 0, \sum (x^i)^2 \leq (2\omega_n^{-1}\mathrm{Volume}(D))^{2/n}\}. \tag{3}$$

Note that $\mathrm{Volume}(B) = \mathrm{Volume}(D) = v_1$. Like ϕ_D the Jacobian determinant of ϕ_B equals $1/v_1$. Let $\psi : D \to B$ be defined by $\psi = \phi_B^{-1} \circ \phi_D$. Then the Jacobian determinant of ψ equals 1. In other words, ψ is a volume-preserving map.

Now let us consider a vector field V on D defined by $V(x)$=*the position vector of* $\psi(x)$, $x \in D$. Since the Jacobian matrix of ψ is also lower triangular, it follows from the arithmetic-geometric mean inequality that

$$n = n(\det D\psi)^{1/n} \leq \mathrm{div}V. \tag{4}$$

Let Π_n be the horizontal hyperplane $\{x^n = 0\}$ and let $U_1, ..., U_{2^{n-1}}$ be the congruent subsets of Π_n separated by the vertical hyperplanes $\Pi_1, ..., \Pi_{n-1}$. Translating C and D in a suitable way we may assume that each Π_i contains $(0, ..., 0)$. Define the projection $p : \mathbf{R}^n \to \Pi_n$ by $p(x^1, ..., x^n) = (x^1, ..., x^{n-1}, 0)$. By the divergence theorem applied to (4), we have

$$n\mathrm{Volume}(D) \leq \int_{\partial D \sim \partial C} \langle V, \eta \rangle + \int_{\partial D \cap \partial C} \langle V, \eta \rangle, \tag{5}$$

where η is the outward unit normal to ∂D. By (3) we have

$$|V| \leq (2\omega_n^{-1}\mathrm{Volume}(D))^{1/n} \text{ on } \partial D \sim \partial C. \tag{6}$$

By the symmetry of C and D about $\Pi_1, ..., \Pi_{n-1}$ and by the convexity of C, we get

$$\langle V, \eta \rangle \leq 0 \text{ on } \partial D \cap \partial C. \tag{7}$$

This is because if $x \in \partial D \cap \partial C$ and $p(x) \in U_k, 1 \leq k \leq 2^{n-1}$, then both $\psi(x)$ and $-p(q_\eta)$ lie in U_k, where $q_\eta \in \mathbf{R}^n$ is the point whose position vector is η. Therefore it follows from (5),(6), and (7) that

$$n\mathrm{Volume}(D) \leq (2\omega_n^{-1}\mathrm{Volume}(D))^{1/n}\mathrm{Volume}(\partial D \sim \partial C),$$

which imples (2).

Now let us assume that equality holds in (2). Then we have equality in (4),(5),(6), and (7). Hence

$$\frac{\partial \psi^i}{\partial x^i} = 1 \text{ on } D, \tag{8}$$

$$V = (2\omega_n^{-1}\mathrm{Volume}(D))^{1/n}\eta \text{ on } \partial D \sim \partial C, \tag{9}$$

and

$$V \perp \eta \text{ on } \partial D \cap \partial C. \tag{10}$$

Therefore (8) and the fact that $D\psi$ is lower triangular imply that

$$\psi^i = x^i + f^i(x^1, ..., x^{i-1}), \ i = 1, ..., n. \tag{11}$$

Let A_1 and A_2 be disjoint subsets of $\partial D \sim \partial C$ such that $x^n(\psi(A_1)) > 0$, $x^n(\psi(A_2)) = 0$, and $A_1 \cup A_2 = \partial D \sim \partial C$. By (9) $\sum_{i=1}^n (\psi^i)^2 = (2\omega_n^{-1}\text{Volume}(D))^{2/n}$ on A_1 and hence

$$\begin{aligned} 0 &= \sum_{i=1}^n \psi^i d\psi^i = \sum_{i=1}^n \psi^i \left(dx^i + \sum_{j=1}^{i-1} \frac{\partial f^i}{\partial x^j} dx^j \right) \\ &= \sum_{i=1}^n \left(\psi^i + \sum_{j=i+1}^n \psi^j \frac{\partial f^j}{\partial x^i} \right) dx^i. \end{aligned}$$

Therefore the vector

$$\left(\psi^1 + \sum_{j=2}^n \psi^j \frac{\partial f^j}{\partial x^1},\ \psi^2 + \sum_{j=3}^n \psi^j \frac{\partial f^j}{\partial x^2}, \dots,\ \psi^{n-1} + \psi^n \frac{\partial f^n}{\partial x^{n-1}},\ \psi^n \right)$$

is normal to A_1. (9) then implies that there is a constant a depending on the point of A_1 such that

$$\left(\psi^1 + \sum_{j=2}^n \psi^j \frac{\partial f^j}{\partial x^1},\ \psi^2 + \sum_{j=3}^n \psi^j \frac{\partial f^j}{\partial x^2}, \dots,\ \psi^{n-1} + \psi^n \frac{\partial f^n}{\partial x^{n-1}},\ \psi^n \right) = (a\psi^1, \dots, a\psi^n).$$

Since $\psi^n > 0$ on A_1, comparing the last components of these vectors gives $a = 1$. Hence the second last components give us

$$\frac{\partial f^n}{\partial x^{n-1}} = 0 \text{ on } A_1. \tag{12}$$

Let us now fix $x^1, \dots, x^{n-2}$. Then (11) and (12) imply that

$$\psi^1 = b^1, \dots,\ \psi^{n-2} = b^{n-2},\ \psi^{n-1} = x^{n-1} + b^{n-1},\ \psi^n = x^n + b^n, \text{ on } A_1, \tag{13}$$

where $b^1, \dots, b^n$ are constants. It follows that $\{x^i = \text{constant} : i = 1, \dots, n-2\} \cap A_1$ is a translate of a semicircle on $\partial B \sim \Pi_n$. Since we can arbitrarily move and rotate the rectangular coordinates $x^1, \dots, x^{n-1}$ while fixing x^n, we can say that the intersection of A_1 with any 2-plane perpendicular to the horizontal hyperplane Π_n is a semicircle.

Choose a point $q \in A_1$ such that $\psi(q)$ is the north pole of B. Then from (13) we conclude that the intersection of A_1 with any 2-plane P which passes through q and is perpendicular to Π_n is part of a semicircle of radius $(2\omega_n^{-1}\text{Volume}(D))^{1/n}$. Thus A_1 is congruent to a subset of the northern hemisphere $\partial B \sim \Pi_n$.

On the other hand (10) implies that $\partial D \cap \partial C$ is flat; for otherwise there should exist a point at which $\langle V, \eta \rangle < 0$. Note that

$$\psi\{A_2 \cup (\partial D \cap \partial C)\} = B \cap \Pi_n.$$

But by (9) we have

$$\psi(A_2) \subset \partial(B \cap \Pi_n).$$

Hence

$$\psi(\partial D \cap \partial C) = B \cap \Pi_n \text{ and } \psi(\partial D \sim \partial C) = \partial B \sim \Pi_n.$$

Therefore D is a half ball.

3. Symmetrization

One of the oldest and most powerful methods in isoperimetric inequalities is Steiner's symmetrization [9]. The key idea of this method is that given k functions $x^n = f_1(x^1, ..., x^{n-1})$, ..., $x^n = f_k(x^1, ..., x^{n-1})$, the volume of the graph of the average function of $f_1, ..., f_k$ is not bigger than the average of the volumes of the graphs of $f_1, ..., f_k$. This volume estimate is based on the simple inequality for k vectors in $\mathbf{R}^n$: $|v_1 + ... + v_k| \leq |v_1| + ... + |v_k|$. In this section, using the symmetrization method, we shall improve Theorem 1.

Theorem 2. *Let C be a convex set in $\mathbf{R}^n$, D a subset of $\mathbf{R}^n \sim C$ with piecewise C^1 boundary, and Π_n a horizontal hyperplane $\{x^n = 0\}$. Suppose that both $\partial D \sim \partial C$ and $\partial D \cap \partial C$ are graphs over a closed set $A \subset \Pi_n$. If A is symmetric about $n-1$ vertical hyperplanes $\Pi_1, ..., \Pi_{n-1}$ which are mutually perpendicular, then*

$$\frac{1}{2} n^n \omega_n \mathrm{Volume}(D)^{n-1} \leq \mathrm{Volume}(\partial D \sim \partial C)^n,$$

where equality holds if and only if D is a half ball.

Proof. Let $f_0, g_0 : A \to \mathbf{R}$ be the functions defined by $x^n = f_0(x^1, ..., x^{n-1}), x^n = g_0(x^1, ..., x^{n-1})$ such that $\partial D \sim \partial C, \partial D \cap \partial C$ are the graphs of f_0, g_0, respectively. Let G be the group of isometries of $\mathbf{R}^n$ generated by $n-1$ horizontal reflections which leave $\Pi_1, ..., \Pi_{n-1}$ fixed, respectively. G consists of 2^{n-1} elements, say, $r_1, ..., r_{2^{n-1}}$. Define $f_i = f_0 \circ r_i$ and $g_i = g_0 \circ r_i, i = 1, ..., 2^{n-1}$. Also define $f = 2^{1-n} \sum_{i=1}^{2^{n-1}} f_i, g = 2^{1-n} \sum_{i=1}^{2^{n-1}} g_i$. Since $f_0 \geq g_0$ on A and $f_0 = g_0$ on ∂A, we have $f \geq g$ on A and $f = g$ on ∂A. Hence graph(f) and graph(g) enclose a domain $\widehat{D}$, and it is easy to see that

$$\mathrm{Volume}(D) = \mathrm{Volume}(\widehat{D}). \tag{14}$$

Note also that $\widehat{D}$ is symmetric about $\Pi_1, ..., \Pi_{n-1}$ and graph$(g) \subset \partial \widehat{D}$ is a subset of $\partial \widehat{C}$ for some convex doamain $\widehat{C}$. Moreover

$$\begin{aligned}
\mathrm{Volume}(\partial \widehat{D} \sim \partial \widehat{C}) &= \mathrm{Volume}(\mathrm{graph}(f)) \\
&= \int_A \left| \left(2^{1-n} \sum_i \frac{\partial f_i}{\partial x^1}, ..., 2^{1-n} \sum_i \frac{\partial f_i}{\partial x^{n-1}}, 1 \right) \right| \\
&= 2^{1-n} \int_A \left| \left(\sum_i \frac{\partial f_i}{\partial x^1}, ..., \sum_i \frac{\partial f_i}{\partial x^{n-1}}, 2^{n-1} \right) \right| \\
&\leq 2^{1-n} \int_A \sum_{i=1}^{2^{n-1}} \left| \left(\frac{\partial f_i}{\partial x^1}, ..., \frac{\partial f_i}{\partial x^{n-1}}, 1 \right) \right| \\
&= 2^{1-n} \sum_{i=1}^{2^{n-1}} \mathrm{Volume}(\mathrm{graph}(f_i)) = \mathrm{Volume}(\mathrm{graph}(f_0))
\end{aligned}$$

$$= \mathrm{Volume}(\partial D \sim \partial C). \tag{15}$$

Therefore by Theorem 1 applied to $\widehat{C}, \widehat{D}$ and by (14) and (15), we get the desired inequality.

Suppose equality holds for D. Then by Theorem 1, (14), and (15) equality should also hold for $\widehat{D}$. Hence (15) becomes equality and $g \equiv \text{constant}$. So

$$\frac{\partial f_i}{\partial x^k} = \frac{\partial f_j}{\partial x^k} \text{ for } 1 \leq i, j \leq 2^{n-1},\ 1 \leq k \leq n-1,$$

and $\text{graph}(g_0)$ is a hyperplane. Therefore f_0 is symmetric about $\Pi_1, ..., \Pi_{n-1}$ and hence $g_0 \equiv \text{constant}$. Thus from Theorem 1 it follows that D is a half ball.

Although the symmetry assumption is required in Theorems 1 and 2, it is not necessary in case the convex set C is a ball:

Theorem 3. *If C is a ball in $\mathbf{R}^n$ and D is a subset of $\mathbf{R}^n \sim C$ with rectifiable boundary, then*

$$\frac{1}{2} n^n \omega_n \text{Volume}(D)^{n-1} \leq \text{Volume}(\partial D \sim \partial C)^n$$

with equality if and only if D is a half ball.

It is easy to prove this theorem once we know that the isoperimetric region of the complement of a ball is rotationally symmetric about a line through the center of the ball.

Lemma. *Outside a ball $C \subset \mathbf{R}^n$ there exists a set $\widetilde{D}$ whose boundary has the least relative volume $\text{Volume}(\partial\widetilde{D} \sim \partial C)$ among all sets outside C with the same volume as $\widetilde{D}$. In fact, $\partial\widetilde{D} \sim \partial C$ is a spherical cap perpendicular to ∂C and $\partial\widetilde{D} \cap \partial C$ lies in an open hemisphere of ∂C.*

Proof of Lemma. The existence of $\widetilde{D}$ can be obtained by following the compactness argument in [8], pp. 441-444. Obviously $\partial\widetilde{D} \sim \partial C$ has constant mean curvature and makes $90°$ with ∂C. We claim that $\widetilde{D}$ is rotationally symmetric about a line. Suppose not. Then there exists an $(n-3)$-dimensional great sphere S in ∂C such that $\widetilde{D}$ is not symmetric about any hyperplane containing S. Choose a hyperplane Π containing S that devides $\widetilde{D}$ into $\widetilde{D}_1$ and $\widetilde{D}_2$ of equal volume. Suppose without loss of generality that $\text{Volume}(\partial\widetilde{D}_1 \sim (\partial C \cup \Pi)) \leq \text{Volume}(\partial\widetilde{D}_2 \sim (\partial C \cup \Pi))$. Let $\widetilde{D}_3$ be the mirror image of $\widetilde{D}_1$ across Π and define $\widetilde{D}_{13}$ to be the union of the closures of $\widetilde{D}_1$ and $\widetilde{D}_3$. If $\partial\widetilde{D} \sim \partial C$ intersects Π at $90°$, then the unique continuation property of the constant mean curvature hypersurfaces implies that $\widetilde{D}$ is symmetric about Π, contradicting our hypothesis. Therefore some part of $\partial\widetilde{D}_{13} \sim \partial C$ should be not C^1 along Π. Then we can slightly perturb $\widetilde{D}_{13}$ along this singular part to get a set $D' \subset \mathbf{R}^n \sim C$ such that

$$\text{Volume}(D') = \text{Volume}(\widetilde{D}_{13}) = \text{Volume}(\widetilde{D}),$$

and

$$\text{Volume}(\partial D' \sim \partial C) < \text{Volume}(\partial\widetilde{D}_{13} \sim \partial C) \leq \text{Volume}(\partial\widetilde{D} \sim \partial C).$$

But this contradicts the least relative volume property of $\partial\widetilde{D}$. Hence $\widetilde{D}$ must be rotationally symmetric about a line l. Now let $\{q\} = (\partial\widetilde{D} \sim \partial C) \cap l$ and take a spherical cap A through q which is rotationally symmetric about l and has the same mean curvature as $\partial\widetilde{D} \sim \partial C$. Since $\partial\widetilde{D} \sim \partial C$ is tangent to A at q, we can apply the maximum principle and conclude that $\partial\widetilde{D} \sim \partial C$ itself is a spherical cap. Then $\partial\widetilde{D} \cap \partial C$ is a subset of an open hemisphere of ∂C.

Proof of Theorem 3. Let $\widetilde{D}$ be as in Lemma with $\mathrm{Volume}(\widetilde{D}) = \mathrm{Volume}(D)$. Let D^* be the convex hull of $\widetilde{D}$ and F the flat part of ∂D^*. Then

$$\begin{aligned}\frac{1}{2}n^n\omega_n\mathrm{Volume}(D)^{n-1} &\leq \frac{1}{2}n^n\omega_n\mathrm{Volume}(D^*)^{n-1} \leq \mathrm{Volume}(\partial D^* \sim F)^n \\ &= \mathrm{Volume}(\partial\widetilde{D} \sim \partial C)^n \leq \mathrm{Volume}(\partial D \sim \partial C)^n.\end{aligned}$$

If equality holds, then ∂D has the least relative volume and hence by Lemma $D = \widetilde{D}$. Also the inequalities above should become equality and so $\widetilde{D} = D^*$. Therefore D is a half ball and ∂C is a hyperplane.

4. Negatively curved surfaces

It was Carleman [4] who first showed that the classical isoperimetric inequality

$$4\pi\mathrm{Area}(S) \leq \mathrm{Length}(\partial S)^2 \tag{16}$$

remains valid for a disk type minimal surface S in space. Then in 1926 Weil [10] obtained the same result for a disk type surface of negative Gaussian curvature. Thereafter a variety of different methods were employed by a dozen mathematicians to prove the same or more general inequality; Bol [2] used parallel curves and Alexandrov [1] used the method of polyhedral approximation. Huber's method [6] was to improve the inequality of Carleman and its generalization to subharmonic functions by Beckenbach and Radó [3]. In this section we give a new simple proof of (16) using the maximum principle: Given a disk type negatively curved surface S, we construct a flat surface $\overline{D}$ with area larger than that of S and perimeter equal to that of S. Then (16) follows immediately from the classical isoperimetric inequality for $\overline{D}$. In fact, a more general theorem is proved: If ∂S is concave on $\Gamma_1 \subset \partial S$, then

$$2\pi\mathrm{Area}(S) \leq \mathrm{Length}(\partial S \sim \Gamma_1)^2$$

with equality if and only if S is a flat half disk.

Theorem 4. *Let S be a disk type surface of nonpositive Gaussian curvature. Suppose that ∂S is the disjoint union of Γ_1 and Γ_2 such that Γ_1 is connected and concave, i.e., if $c(s)$ is an arclength parametrization of Γ_1, then $c''(s)$ vanishes or points outward from S. Then*

$$2\pi\mathrm{Area}(S) \leq \mathrm{Length}(\Gamma_2)^2$$

and equality holds if and only if S is a flat half disk.

Proof. Let $D \subset \mathbf{R}^2$ be a half disk with the diameter C_1 and the semicircle C_2 such that $\partial D = C_1 \cup C_2$. Take the coordinates x and y of $\mathbf{R}^2$ such that $x = 0$ on C_1 and $x \geq 0$ on C_2. Assuming that x and y are also the isothermal coordinates of S via a conformal map $\varphi : D \to S$, we can write the metric of S as $g = e^{2f}(dx^2 + dy^2)$ for some function f on D. It is well known that the Gaussian curvature K of S satisfies

$$K = -e^{-2f} \triangle f.$$

So by the curvature hypothesis

$$\triangle f \geq 0 \text{ on } D. \tag{17}$$

Let h be the harmonic function on D satisfying the mixed boundary condition

$$h = f \text{ on } C_2 \tag{18}$$

and

$$\frac{\partial h}{\partial \nu} = 0 \text{ on } C_1 \tag{19}$$

where ν is the outward unit normal to C_1. The key point here is that the concavity of Γ_1 implies

$$\frac{\partial f}{\partial \nu} \leq 0. \tag{20}$$

This is because

$$\begin{aligned} 0 &\geq \left\langle \nabla_{e^{-f}\frac{\partial}{\partial y}} e^{-f}\frac{\partial}{\partial y}, e^{-f}\frac{\partial}{\partial x} \right\rangle = -\left\langle e^{-f}\frac{\partial}{\partial y}, \nabla_{e^{-f}\frac{\partial}{\partial y}} e^{-f}\frac{\partial}{\partial x} \right\rangle \\ &= -e^{-3f}\left\langle \frac{\partial}{\partial y}, \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} \right\rangle = -e^{-3f}\left\langle \frac{\partial}{\partial y}, \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} \right\rangle \\ &= -\frac{1}{2}e^{-3f}\frac{\partial}{\partial x}\left|\frac{\partial}{\partial y}\right|^2 = -e^{-f}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \nu}. \end{aligned}$$

Using the maximum principle, we can conclude from (17),(18),(19), and (20) that

$$h \geq f \text{ on } D. \tag{21}$$

Now let us introduce a surface $\overline{D}$ which is D equipped with the new flat metric $\overline{g} = e^{2h}(dx^2 + dy^2)$. Actually $\overline{D}$ is the image of D in the complex plane under the complex analytic function $\phi(z)$ such that $\log|\phi'(z)| = h(x, y), z = x + iy$. Denote by $\overline{C}_1, \overline{C}_2$ the parts of $\partial\overline{D}$ which correspond to C_1, C_2 of ∂D, respectively. From (19) it follows remarkably that $\overline{C}_1$ is also a line segment in $\partial\overline{D}$. Hence $\overline{D}$ satisfies the relative isoperimetric inequality

$$2\pi \text{Area}(\overline{D}) \leq \text{Length}(\overline{C}_2)^2.$$

However, (21) and (18) imply respectively that

$$\text{Area}(S) \leq \text{Area}(\overline{D}) \text{ and } \text{Length}(\Gamma_2) = \text{Length}(\overline{C}_2).$$

Therefore

$$2\pi \mathrm{Area}(S) \leq \mathrm{Length}(\Gamma_2)^2.$$

If equality holds here, then $\overline{D}$ is a half disk and $f = h$ on D. Thus S is also a half disk.

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Department of Mathematics
Seoul National University
Seoul, 151-742, Korea
email: choe@math.snu.ac.kr fax: 82-2-887-4694