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# The relative isoperimetric inequality in Cartan-Hadamard 3-manifolds

***To the memory of José F. Escobar***

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**Abstract.** We prove that a region $D$ outside a convex set $C$ with smooth boundary in a three-dimensional Cartan-Hadamard manifold $M$ satisfies the relative isoperimetric inequality $\operatorname{area}(\partial D \sim \partial C)^3 \geqq 18\pi \operatorname{vol}(D)^2$, with equality if and only if $D$ is isometric to a Euclidean half ball. We also prove similar sharp inequalities when the sectional curvature of $M$ is bounded above by a negative constant.

## 1. Introduction

Let $\mathbb{H} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geqq 0\}$ be the closed upper half space of $\mathbb{R}^n$. Given a set $D \subset \mathbb{H}$, let $D' \subset \mathbb{R}^n \sim \mathbb{H}$ be the mirror image of $D$ across $\partial\mathbb{H}$, where $\sim$ denotes the set exclusion operator. Then the classical isoperimetric inequality implies that

$$\operatorname{area}\bigl(\partial(D \cup D')\bigr)^n \geqq n^n \omega_n \operatorname{vol}(D \cup D')^{n-1},$$

with equality if and only if $D \cup D'$ is a ball. Here $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$, and area and vol, respectively, the $(n-1)$ and $n$ dimensional Hausdorff measures. As a result one gets

$$\operatorname{area}(\partial D \sim \partial\mathbb{H})^n \geqq \frac{1}{2} n^n \omega_n \operatorname{vol}(D)^{n-1}, \tag{1.1}$$

with equality if and only if $D$ is a half ball and $\partial D$ meets $\partial\mathbb{H}$ in an orthogonal way.

This proposes the following interesting problems: Given a set $D$ outside a convex set $C$ in $\mathbb{R}^n$, does $D$ satisfy inequality (1.1)? Does equality hold if and only if $D$ is a half ball? Is inequality (1.1) still true outside a convex set in a Cartan-Hadamard manifold?

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Inequality (1.1) will be called the *relative isoperimetric inequality* for $D$. The proof of (1.1) in $\mathbb{R}^2$ is easy once one reflects the convex hull of $D$ about its linear boundary. For $\mathbb{R}^n$, $n \geqq 3$, some partial results have been obtained in [K], [C1]. I. Kim [K] proved (1.1) for the epigraph of a $C^2$ convex function and the first-named author [C1] proved the relative isoperimetric inequality when $\partial D \cap \partial C$ is a graph which is symmetric about $(n-1)$ hyperplanes of $\mathbb{R}^n$.

In this paper we prove that the relative isoperimetric inequality (1.1) holds outside a closed convex set $C$ with smooth boundary in a three-dimensional Cartan-Hadamard manifold $M$. More precisely, let $C$ be a closed convex set with interior points in a Cartan-Hadamard manifold $M^3$ with sectional curvatures bounded above by a nonpositive constant $\kappa$. Let us denote by $M(\kappa)$ the three-dimensional simply connected space form with sectional curvature $\kappa$ (Euclidean space for $\kappa = 0$ or hyperbolic space of sectional curvatures equal to $\kappa$ otherwise), then we prove in Theorem 3.2 that, for any bounded set $D \subset M^3 \sim C$ with piecewise smooth boundary, we have

$$\text{area}(\partial D \sim \partial C) \geqq \frac{1}{2}\,\text{area}(\partial B), \tag{1.2}$$

where $B$ is a geodesic ball in $M(\kappa)$ with $\text{vol}(B) = 2\,\text{vol}(D)$, and area is the 2 dimensional Hausdorff measure. Moreover, we show that equality holds in (1.2) if and only if $D$ is isometric to a half ball in $M(\kappa)$.

The two-dimensional relative isoperimetric problem (1.2) was solved affirmatively in [C1]. The four-dimensional relative isoperimetric inequality (1.1) was obtained in [C2] following C. Croke [Cr].

Our proof for the three-dimensional case follows the scheme of B. Kleiner [Kl]: the isoperimetric inequality we intend to prove is equivalent to the comparison $I_C \geqq I_{\mathbb{H}}$ of the isoperimetric profiles $I_C$ of $M \sim C$ and $I_{\mathbb{H}}$ of the complement of a closed half space $\mathbb{H} \subset M(\kappa)$. When an isoperimetric region $\Omega_0$ of volume $v_0$ exists in $M \sim C$ and the profile $I_C$ is smooth at $v_0$ then the derivative $I_C'(v_0)$ is equal to the constant mean curvature of the boundary $\partial\Omega_0 \sim \partial C$. Hence, in order to compare $I_C$ and $I_{\mathbb{H}}$ we only need to compare their derivatives, that is, the mean curvatures of isoperimetric regions in both spaces for given volumes. Since both $I_C$ and $I_{\mathbb{H}}$ are increasing functions, we only need to make this comparison for given areas. A main problem for this approach is that isoperimetric regions need not exist in the noncompact manifold $M \sim C$ but, as in [Kl], we avoid it by considering an exhaustion of $M \sim C$ by relatively compact sets where isoperimetric regions can be found. The mean curvature comparison is obtained in Proposition 3.1 by techniques similar to the ones used in [R], which are inspired by the lower bound for the Willmore functional obtained by Li and Yau [LY].

## 2. Preliminaries

Throughout this paper, $M = M^3$ will be a three-dimensional *Cartan-Hadamard manifold*, i.e., a complete, simply connected Riemannian manifold with nonpositive sectional curvatures. If $\kappa$ is a nonpositive constant, then $M^3(\kappa)$ will denote the three-dimensional

Cartan-Hadamard manifold with sectional curvatures equal to $\kappa$. $M(0)$ is the Euclidean space $\mathbb{R}^3$ and $M^3(\kappa)$, for $\kappa < 0$, is the hyperbolic space with sectional curvatures equal to $\kappa$.

A *proper convex domain* will be the closure of an open convex set or, equivalently, a closed convex set with nonempty interior. For notational simplicity in the setting of relative geometry, let us adopt the following: if $C \subset M$ then we define $M_C := M \sim C$.

For $C \subset M$, define the *relative isoperimetric profile* of $M_C, I_C : \mathbb{R}^+ \to \mathbb{R}^+$, by

$$I_C(v) = \inf_D \{\text{area}(\partial D)_C : D \subset M_C, \partial D \text{ piecewise smooth, } \text{vol}(D) = v\},$$

where area is the 2 dimensional Hausdorff measure and vol denotes the Riemannian volume in $M$. An *isoperimetric region* $D \subset M_C$ is one for which the equality

$$\text{area}(\partial D)_C = I_C\big(\text{vol}(D)\big)$$

holds. We remark that isoperimetric regions need not be unique for a given volume. One of the problems in our approach is that isoperimetric regions need not exist in $M_C$. We solve this problem by considering isoperimetric regions in bounded sets and letting these sets exhaust $M_C$.

Let $\mathbb{H} \subset M(\kappa)$ be a closed half space. From the reflection argument of the introduction and the isoperimetric inequality in $M(\kappa)$ we easily obtain that the isoperimetric regions in $M(\kappa) \sim \mathbb{H}$ are half balls touching orthogonally the boundary of $M(\kappa) \sim \mathbb{H}$.

If $M$ is a three-dimensional Cartan-Hadamard manifold with sectional curvatures $K_M \leqq \kappa$, then the relative isoperimetric inequality (1.2) in $M_C$ is equivalent to

$$\text{area}(\partial D)_C \geqq I_{\mathbb{H}}\big(\text{vol}(D)\big), \tag{2.1}$$

which in turn is equivalent to

$$I_C(v) \geqq I_{\mathbb{H}}(v), \tag{2.2}$$

for all $v > 0$. We also want to prove that equality holds in (2.1) if and only if $D$ is a half ball in $M(\kappa)$.

Let $C \subset M$ be a closed set. Denote by $C_0^1(TM_C)$ the set of $C^1$ vector fields with compact support in $M_C$. For any $D \subset M_C$ define the *perimeter* of $D$ relative to $C$ as

$$\mathscr{P}_C(D) = \sup\left\{ \int_D \text{div}\, X : X \in C_0^1(TM_C), |X| \leqq 1 \right\},$$

where $|X|$ is the sup norm. If $\partial D$ is piecewise smooth then $\mathscr{P}_C(D)$ coincides with $\text{area}(\partial D)_C$ since the supremum is taken over vector fields with compact support in $M_C$, and so the area of $\partial D \cap \partial C$ does not contribute to the perimeter.

Let $E$ be the closure of a bounded open set in $M_C$. We wish to minimize the perimeter $\mathscr{P}_C$ amongst sets included in $E$ subject to a volume constraint, i.e., given $v \in \big(0, \text{vol}(E)\big)$, we want to find $\Omega_0 \subset E$ such that

$$\mathscr{P}_C(\Omega_0) \leqq \mathscr{P}_C(\Omega)$$

for any $\Omega \subset E$ with $\mathrm{vol}(\Omega) = \mathrm{vol}(\Omega_0) = v$. The existence of $\Omega_0$ is guaranteed by the boundedness of $E$; see [G]. In fact we get

**Lemma 2.1** (Existence and regularity). *Let $C \subset M^3$ be a proper convex domain with smooth boundary and let $E$ be the closure of a bounded domain in $M_C$ with $\overline{(\partial E)_C}$ smooth. Then, for any $v_0 \in \big(0, \mathrm{vol}(E)\big)$ there is a set $\Omega_0 \subset E$ of volume $v_0$ minimizing the perimeter relative to $C$. Moreover*:

(i) ([GMT]) *$\partial\Omega_0$ has constant mean curvature and is smooth in the interior of $E$.*

(ii) ([Gr2], p. 263) *$\overline{(\partial\Omega_0)_C}$ meets $\partial C \sim \overline{(\partial E)_C}$ orthogonally and it is smooth.*

(iii) ([Wh], [StZ], Thm. 3.6) *If $(\partial E)_C$ is strictly convex then $(\partial\Omega_0)_C$ meets $(\partial E)_C$ tangentially and it is $C^{1,1}$ in a neighbourhood of $(\partial E)_C$.*

**Remark 2.2.** Observe that the above regularity theorem does not apply to points of $\overline{(\partial\Omega_0)_C} \cap \partial C \cap \overline{(\partial E)_C}$.

## 3. The relative isoperimetric inequality in $M^3$

The key method in proving the relative isoperimetric inequality is to obtain a mean curvature comparison in $M$. A minor problem in this method is that the boundary of a perimeter-minimizing domain $\Omega_0$ inside a set $E$ is only $C^{1,1}$ (see [Wh], [StZ]). As we shall see later, however, this is not a true problem, since a $C^{1,1}$ surface is $C^2$ almost everywhere by Rademacher's Theorem and a weak divergence theorem still holds for this kind of surfaces. We prove the mean curvature comparison in the following

**Proposition 3.1.** *Let $M$ be a three-dimensional Cartan-Hadamard manifold with sectional curvatures bounded above by a nonpositive constant $\kappa$. Let $C \subset M$ be a proper convex domain with smooth boundary, $\Omega \subset M_C$ a domain such that $\Sigma := \overline{(\partial\Omega)_C}$ is a $C^{1,1}$ surface perpendicular to $\partial C$ with mean curvature $H$, and let $H_\Sigma := \sup_\Sigma H$. Then*

$$(\kappa + H_\Sigma^2)\,\mathrm{area}(\Sigma) \geqq 2\pi, \tag{3.1}$$

*with equality if and only if $\Omega$ is isometric to a geodesic half ball in $M(\kappa)$ of the same volume as $\Omega$.*

*Proof.* For the proof we shall follow the arguments in [R]. We assume first that $\Sigma$ is a smooth surface. Consider a family $\{g_b\}$ of metrics, conformal to the Riemannian metric $g$ of $M$, given by

$$g_b = e^{2u_b} g.$$

The relation between the sectional curvatures of $g$ and $g_b$ yields, at every point $p \in \Sigma$,

$$e^{2u_b}(K_s)_b = K_s - \left(|\nabla u_b|^2 - \sum_{i=1}^{2}\big(e_i(u_b)\big)^2\right) - \sum_{i=1}^{2} \nabla^2 u_b(e_i, e_i), \tag{3.2}$$

where $(K_s)_b$ and $K_s$ are the sectional curvatures of the tangent plane to $\Sigma$ for the metrics $g_b$ and $g$, respectively, and $\{e_i\}$ is a $g$-orthonormal basis of the tangent plane to $\Sigma$. The gradient and the Hessian of $u_b$ in $(M,g)$ have been denoted by $\nabla u_b$ and $\nabla^2 u_b$, respectively.

The surface integral $\int_\Sigma (\lambda_1 - \lambda_2)^2 \, dA$, where $\lambda_i$ are the principal curvatures of $\Sigma$, is invariant under conformal changes of the ambient metric. Since

$$(\lambda_1 - \lambda_2)^2 = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = 4(H^2 + K_s - K),$$

where $K$ is the intrinsic Gauss curvature of the surface, we have

$$\int_\Sigma H^2 \, dA = \int_\Sigma H_b^2 \, dA_b + \int_\Sigma \big(e^{2u_b}(K_s)_b - K_s\big) \, dA + \int_\Sigma K \, dA - \int_\Sigma K_b \, dA_b,$$

where $H_b$, $K_b$ are the mean and Gauss curvature of $\Sigma$ with respect to the metric $g_b$, and $dA$, $dA_b$ the area forms on $\Sigma$ induced by $g$, $g_b$, respectively.

Let $\partial\Sigma$ be the boundary of the surface $\Sigma$. Integrating the relation $\Delta_\Sigma u_b = K - K_b e^{2u_b}$ on $\Sigma$, where $\Delta_\Sigma$ is the Laplacian for the metric induced on $\Sigma$ by $g$, and applying the divergence theorem we get

$$\int_\Sigma K \, dA - \int_\Sigma K_b \, dA_b = \int_\Sigma \Delta_\Sigma u_b \, dA = -\int_{\partial\Sigma} \nu(u_b) \, ds,$$

where $\nu$ is the inner unit normal to $\partial\Sigma$ on $\Sigma$. So we finally obtain

$$\int_\Sigma H^2 \, dA = \int_\Sigma H_b^2 \, dA_b + \int_\Sigma \big(e^{2u_b}(K_s)_b - K_s\big) \, dA - \int_{\partial\Sigma} \nu(u_b) \, ds. \tag{3.3}$$

Let us consider first the case $\kappa = 0$. Let $p_0 \in \partial\Sigma \cap \partial C$ and let $d$ be the distance function to $p_0$. For any $b > 0$ we consider the family of conformal metrics $g_b = e^{2u_b} g$ given by the functions

$$u_b = \log\left(\frac{2b}{1 + b^2 d^2}\right). \tag{3.4}$$

In case $M^3 = \mathbb{R}^3$ these metrics are obtained by homothetically blowing up the spherical metric $\big(2/(1+d^2)\big)^2 g$ by a factor of $b$.

Equation (3.2) gives in this case, for any $p \in \Sigma$,

$$e^{2u_b}(K_s)_b = K_s - \left(\frac{b^2}{1 + b^2 d^2}\right)^2 4d^2 + \left(\frac{b^2}{1 + b^2 d^2}\right) \sum_{i=1}^2 \nabla^2 d^2(e_i, e_i). \tag{3.5}$$

By the Hessian Comparison Theorem we have $\nabla^2 d^2 \geqq 2$, so that from (3.5) we get

$$e^{2u_b}(K_s)_b \geqq K_s + e^{2u_b}. \tag{3.6}$$

Equality in (3.6) holds if and only if $\nabla^2 d^2(e_i, e_i) = 2$ along $\Sigma$, for $i = 1, 2$. By equation (3.3) and inequality (3.6) we obtain

$$H_\Sigma^2 \operatorname{area}(\Sigma) \geqq \int_\Sigma H^2\, dA \geqq \int_\Sigma dA_b - \int_{\partial\Sigma} v(u_b)\, ds. \tag{3.7}$$

Let us see that

$$\lim_{b\to\infty} \int_\Sigma dA_b = \lim_{b\to\infty} \int_\Sigma e^{2u_b}\, dA = 2\pi. \tag{3.8}$$

Let $r(p) := d(p, p_0)$ be the extrinsic distance to $p_0$. Fix $r_0 > 0$ small. Observe that $e^{2u_b}$ converges uniformly to 0 in $\Sigma$ out of the ball $B(p_0, r_0)$ when $b \to \infty$. For $r_0$ small enough, the modulus of the gradient of $r|_\Sigma$ over $\Sigma \cap B(p_0, r_0)$ is approximately 1, and the length of $\Sigma \cap \partial B(p_0, r)$, for $r \in (0, r_0)$, is approximately $\pi r$. By applying the coarea formula to $r|_\Sigma$ we get

$$\int_{\Sigma\cap B(p_0,r_0)} dA_b = \pi \int_0^{r_0} r\left(\frac{2b}{1+b^2r^2}\right)^2 dr + o(r_0) = 2\pi + o(r_0),$$

where $o(r_0)$ converges to 0 when $r_0 \to 0$. Letting $r_0 \to 0$ we get (3.8).

On the other hand, $\partial\Sigma$ is a curve (possibly disconnected) that bounds a region $R \subset \partial C$. Recall that $v$, the inner normal to $\partial\Sigma$ on $\Sigma$, is also the outer normal to $\partial C$. As $p_0 \in \partial C$ we see that a geodesic $\gamma_p : [0, t_p] \to M$ from $p_0$ to any other point $p \in \partial\Sigma \subset \partial C$ must satisfy $g(\gamma_p'(t_p), v(p)) \geqq 0$ by the convexity of $C$. Hence $g(\nabla d, v) \geqq 0$ for all $p \in \partial\Sigma$ and we deduce that

$$-v(u_b) = \frac{2b^2 d g(\nabla d, v)}{1 + b^2 d^2}$$

is a nonnegative function. Discarding the last summand in (3.7) and taking limits we obtain (3.1).

Let us consider now the equality case

$$H_\Sigma^2 \operatorname{area}(\Sigma) = 2\pi. \tag{3.9}$$

If (3.9) holds then there is also equality in (3.6), which implies that

$$\nabla^2 d^2(v, v) = 2g(v, v), \quad \text{for any } p \in \Sigma,\ v \in T_p\Sigma. \tag{3.10a}$$

We also have that $\lim_{b\to\infty} \int_{\partial\Sigma} (-v(u_b))\, ds = 0$ since, after taking limits, there is equality in (3.7). The sequence $\{-v(u_b)\}_b$ is pointwise increasing and converges to $2v(d)/d$. By the Monotone Convergence Theorem

$$0 \leqq \int_{\partial\Sigma} \frac{2v(d)}{d}\, ds = \lim_{b\to\infty} \int_{\partial\Sigma} (-v(u_b))\, ds = 0,$$

and so

$$g(\nabla d, v) \equiv 0 \quad \text{along } \partial\Sigma. \tag{3.10b}$$

Finally, if (3.9) holds then

$$(3.10c) \qquad H_\Sigma = H,$$

so that $\Sigma$ is a constant mean curvature surface.

Let us see first that (3.10b) implies that $R$ is totally geodesic. For any $p \in M$, define $\gamma_p : [0, t_p] \to M$ as the arc-length minimizing geodesic joining $p_0$ and $p$. If $p \in \partial\Sigma$, (3.10b) implies that $\gamma_p$ is tangent to $\partial C$ at $p$. By the convexity of $C$, we have that $\gamma_p([0, t_p]) \subset \partial C$. So we have $C_{p_0} := \bigcup_{p \in \partial\Sigma} \gamma_p([0, t_p]) \subset \partial C$. Hence $R$, the closed set in $\partial C$ bounded by $\partial\Sigma$, satisfies

$$R \subset C_{p_0},$$

since the intersection of $R$ and $C_{p_0}$ is open and closed in $R$ and every component of $R$ intersects $C_{p_0}$ as $\partial\Sigma \subset C_{p_0}$. Observe that when (3.9) holds, (3.10a), (3.10b), (3.10c) are satisfied for any choice of $p_0 \in \partial\Sigma$ and so $R \subset C_{p_0}$ *for any* $p_0 \in \partial\Sigma$. Let $q$ be a point in the interior of $R$ in $\partial C$. Joining $q$ with three different points $p_i$, $i = 1, 2, 3$, of $\partial\Sigma$ not lying in the same geodesic, we have $R \subset C_{p_i}$ for all $i$. We get three vectors $v_i \in T_q\partial C$, $i = 1, 2, 3$, pairwise independent, so that $\sigma(v_i, v_i) = 0$, $i = 1, 2, 3$, where $\sigma$ is the second fundamental form of $\partial C$. We conclude that $\partial C$ is totally geodesic at $q$ and, hence, over all of $R$.

Let us see now that (3.10a) implies that $\Omega$ is a flat region. With the same notation as in the previous paragraph, let $p \in \Sigma \sim \partial\Sigma$. If $\gamma_p$ is transverse to $\Sigma$ at $p$ then $\nabla^2 d^2 = 2g$ at $p$ and, by the standard comparison theorems in Riemannian Geometry, $\nabla^2 d^2 \equiv 2g$ along $\gamma_p([0, t_p])$. Let $U$ be an open neighbourhood of $p$ in $\Sigma \sim \partial\Sigma$ so that $\gamma_q$ is tranverse to $\Sigma$ at $q$ for every $q \in U$. Then on the cone $\bigcup_{q \in U} \gamma_q([0, t_q])$ we have $\nabla^2 d^2 \equiv 2g$, so this cone is flat.

On the other hand, the set of unit vectors $v \in T_{p_0}M$ for which there is $p \in \Sigma \sim \partial\Sigma$ with $\gamma_p'(0) = v$ and $\gamma_p$ does not meet $\Sigma$ transversally at $p$, has measure zero on the unit sphere $S_{p_0} \subset T_{p_0}M$. This can be easily seen by considering the smooth map $\Sigma \sim \partial\Sigma \to S_{p_0}$ given by $p \mapsto \gamma_p'(0)$, whose critical points are precisely the points $p \in \Sigma \sim \partial\Sigma$ for which $\gamma_p$ is not transverse to $\Sigma$ at $p$, and by applying Sard's Theorem. We conclude that the set of points in $\bigcup_{p \in \Sigma \sim \partial\Sigma} \gamma_p([0, t_p])$ where the metric is flat is dense, and so $\bigcup_{p \in \Sigma} \gamma_p([0, t_p])$ is flat. Since $\Omega \subset \bigcup_{p \in \Sigma} \gamma_p([0, t_p])$, we conclude that $\Omega$ is flat.

Finally, let us consider (3.10c). Since $\Omega$ is flat, $R$ is totally geodesic, and $H = H_\Sigma > 0$, we can apply the theorem by S. Montiel and A. Ros [MoR] which implies, for regions of this type, that

$$3\,\mathrm{vol}(\Omega) \leqq \frac{1}{H}\,\mathrm{area}(\Sigma),$$

with equality if and only if $\Omega$ is a half-ball in Euclidean space. But on $\Sigma$ we have

$$(3.11) \qquad \Delta_\Sigma d^2 = 4 + 4Hd\langle \nabla d, N \rangle,$$

where $N$ is the unit normal to $\Sigma$ in the direction of the mean curvature vector of $\Sigma$. Integrating (3.11) on $\Sigma$, we obtain the classical Minkowski formula

$$3\,\mathrm{vol}(\Omega) = \frac{1}{H}\,\mathrm{area}(\Sigma).$$

This concludes the proof in the Euclidean case.

Let us consider now the hyperbolic case. By scaling the metric $g$ of $M$ we may assume that $\kappa = -1$, so that the inequality we intend to prove is

$$(-1 + H_\Sigma^2)\,\mathrm{area}(\Sigma) \geqq 2\pi, \tag{3.12}$$

with equality if and only if $\Omega$ is a half-ball in the hyperbolic space $M^3(-1)$ of constant sectional curvature equal to $-1$.

Consider again a point $p_0 \in \partial\Sigma$ and let $d$ be the distance to $p_0$. We define the family of conformal metrics $g_b = e^{2u_b}g$, given by the functions

$$u_b = \log\left(\frac{2b}{(1-b^2) + (1+b^2)\cosh(d)}\right), \quad b > 1. \tag{3.13}$$

When $M$ is the three-dimensional hyperbolic space, this family of metrics is obtained by homothetically expanding the spherical metric

$$\left(\frac{1-\tanh^2(d/2)}{1+\tanh^2(d/2)}\right)^2 g.$$

From (3.2) we obtain the following relation for the sectional curvatures of the tangent plane to $\Sigma$.

$$\begin{aligned} e^{2u_b}(K_s)_b = K_s + 1 + e^{2u_b} + \left(\frac{1+b^2}{(1-b^2)+(1+b^2)\cosh(d)}\right) \\ \times \left(\sum_{i=1}^{2} \nabla^2\cosh(d)(e_i, e_i) - 2\cosh(d)\right), \end{aligned} \tag{3.14}$$

where $\{e_1, e_2\}$ is a $g$-orthonormal basis of the tangent plane to $\Sigma$. By the Hessian Comparison Theorem we know that $\nabla^2\cosh(d) \geqq \cosh(d)g$. So we obtain

$$e^{2u_b}(K_s)_b \geqq K_s + e^{2u_b} + 1. \tag{3.15}$$

Equality holds in (3.15) if and only if $\nabla^2\cosh(d)(e_i, e_i) = \cosh(d)$ for $i = 1, 2$. From equation (3.3) and inequality (3.15) we obtain

$$\int_\Sigma (-1 + H^2)\,dA \geqq \int_\Sigma dA_b - \int_{\partial\Sigma} \nu(u_b)\,ds.$$

As in the previous case one shows that $\lim_{b\to\infty} \int_\Sigma dA_b \to 2\pi$ and, by the convexity of $C$, that $-\nu(u_b) \geqq 0$, which yields the desired estimate (3.12).

If equality holds in (3.12) then we conclude, as in the Euclidean case, that $\nabla^2 \cosh(d)(v,v) = \cosh(d)$ for any unit tangent vector $v$ to $\Sigma$ at any point of $\Sigma$, that $H$ is constant, and that $g(\nabla d, v) = 0$ at any point of $\partial\Sigma$.

Condition $\nabla^2 \cosh(d)(v,v) = \cosh(d)$ for any unit tangent vector $v$ to $\Sigma$ at any point of $\Sigma$ implies, by the standard comparison theorems, that the metric $g$ of $\Omega$ (i.e., on the cone over $\Sigma$ with vertex $p_0$), has constant sectional curvatures equal to $-1$. Moreover, condition $g(\nabla d, v) = 0$ at any point of $\partial\Sigma$ implies that $R$ is a totally geodesic surface. Finally, as $\Sigma$ has constant mean curvature $H > 1$, we see as in [Mo], Theorem 9, by taking inner parallels, that

$$\int_\Sigma \big(\cosh(d) + H \sinh(d)\langle \nabla d, N\rangle\big)\, dA \geqq 0, \quad N \perp \Sigma$$

with equality if and only if $\Omega$ is a half ball in hyperbolic space. But since the metric of $\Omega$ is hyperbolic, we have $\nabla^2 \cosh(d) = \cosh(d) g$ in $\Omega$ so that integrating

$$\Delta_\Sigma \cosh(d) = 2\cosh(d) + 2H \sinh(d)\langle \nabla d, N\rangle$$

on $\Sigma$ we get

$$\int_\Sigma \big(\cosh(d) + H \sinh(d)\langle \nabla d, N\rangle\big)\, dA = 0.$$

This completes the proof in the hyperbolic case $\kappa = -1$.

Finally, if $\Sigma$ is merely $C^{1,1}$, the arguments in the proof apply without changes since the principal curvatures of $\Sigma$, and hence the mean and the Gauss curvature, are defined almost everywhere. The divergence theorem still holds under our weak hypothesis. $\square$

By using the mean comparison result in Proposition 3.1 we can now prove the isoperimetric comparison theorem.

**Theorem 3.2.** *Let $M$ be a three-dimensional Cartan-Hadamard manifold with sectional curvatures bounded above by a nonpositive constant $\kappa$, and let $M(\kappa)$ be the three-dimensional Cartan-Hadamard manifold with constant sectional curvatures equal to $\kappa$. Assume that $C \subset M$ is a proper convex domain with smooth boundary and that $\mathbb{H} \subset M(\kappa)$ is a half space. Then we have*

$$I_C \geqq I_{\mathbb{H}}, \tag{3.16}$$

*which in turn implies that for any bounded finite perimeter set $D \subset M_C$ we have*

$$\operatorname{area}(\partial D)_C \geqq I_{\mathbb{H}}\big(\operatorname{vol}(D)\big). \tag{3.17}$$

*Moreover, equality holds in* (3.17) *if and only if $D$ is isometric to a half ball in $M(\kappa)$.*

*Proof.* Since the existence of isoperimetric regions is a crucial point of our arguments, but it is not guaranteed in the noncompact $M_C$, we first construct an exhaustion of $M_C$ by relatively compact sets $\{E_k\}_{k\in\mathbb{N}}$. We take $p_0 \in \partial C$, and define $E_k = \bar{B}(p_0, r_k)_C$, where $r_k$ is an increasing diverging sequence of positive numbers, and $\bar{B}(p_0, r_k)$ is the closed ball centered at $p_0$ of radius $r_k$.

Since $E_k$ is bounded, isoperimetric regions exist on $E_k$ for any given volume $v \in (0, \operatorname{vol} E_k)$. The boundary $\Sigma$ of any isoperimetric region $\Omega$ in $E_k$ satisfies the regularity properties of Lemma 2.1 provided there is no point in $\Sigma \cap \partial C \cap \overline{(\partial E_k)_C}$. Assume there is $q_0 \in \Sigma \cap \partial C \cap \overline{(\partial E_k)_C}$. The set $\partial E_k$ is contained in the geodesic sphere $\partial B(p_0, r_k)$, which meets $\partial C$ at $q_0$ at an angle less than or equal to $\pi/2$. Reflecting locally $\Omega$ with respect to $\partial C$ and blowing up $\Omega$ and the metric from $q_0$, [Gr1], we obtain a cone in $\mathbb{R}^3$ which is area minimizing in a wedge of $\mathbb{R}^3$. This implies regularity if the angle is $\pi/2$ and it is not possible if the angle is less than $\pi/2$; see [G], Thm. 15.5.

Fix $k \in \mathbb{N}$ and assume that $\Omega_k$ is an isoperimetric region in $E_k$. The set $\Omega_k$ may have several components. Let $\Omega_k = \Omega_k^1 \cup \Omega_k^2$, where $\Omega_k^1$ consists of the components of $\Omega_k$ touching $\partial C$ (in an orthogonal way), and $\Omega_k^2$ consists of the components of $\Omega_k$ disjoint from $\partial C$. Then by (3.1)

$$(\kappa + H^2_{(\partial\Omega_k^1)_C}) \operatorname{area}(\partial\Omega_k^1)_C \geqq 2\pi[\#\{\text{components of } \Omega_k^1\}] \geqq 2\pi.$$

On the other hand, by [Kl] or [R],

$$(\kappa + H^2_{(\partial\Omega_k^2)_C}) \operatorname{area}(\partial\Omega_k^2)_C \geqq 4\pi[\#\{\text{components of } \Omega_k^2\}] \geqq 0.$$

Adding both inequalities we get

$$(\kappa + H^2_{(\partial\Omega_k)_C}) \operatorname{area}(\partial\Omega_k)_C \geqq 2\pi,$$

and hence

$$H_{(\partial\Omega_k)_C} \geqq H_\kappa\big(\operatorname{area}(\partial\Omega_k)_C\big), \tag{3.18}$$

where $H_\kappa(a)$ denotes the mean curvature of a geodesic sphere of area $2a$ in $M(\kappa)$. By Proposition 3.1, equality holds in (3.18) if and only if $\Omega_k$ is isometric to a geodesic half ball in $M(\kappa)$.

Denote by $I_k$ the isoperimetric profile of $E_k$. By standard arguments, [Hs], pp. 170–172, we have

- $I_k$ is continuous and increasing,

- when $I_k$ is smooth at $v_0$ then $I_k'(v_0) = 2H$, where $H$ is the constant mean curvature in the interior of $E_k$ of *any* isoperimetric region of volume $v_0$, and

- left and right derivatives of $I_k$ exist everywhere.

Since $I_k$ is a continuous monotone function with left and right derivatives at every point, it is absolutely continuous.

The continuity of $I_k$ follows from the convergence of isoperimetric regions. To prove the monotonicity of $I_k$ we just need to show that the constant mean curvature $H$ of the boundary of an isoperimetric region $\Omega$ in the interior of $E_k$ is positive. Let $\Sigma = \overline{\partial\Omega_C}$. If $\Sigma$ does not touch $(\partial E_k)_C$ then there is an outer parallel $\partial C_t$ to $\partial C$, which is tangent to $\Sigma$ at some point and leaves $\Sigma$ on one side. Standard comparison theorems show that the principal curvatures of $\partial C_t$ are nonnegative. By the maximum principle, $H \geqq 0$. But in case $H = 0$, we obtain from the maximum principle that $\Sigma$ and $\partial C_t$ locally coincide and, by a connectedness argument, that a connected component of $\Sigma$ is contained in $\partial C_t$, which gives us a contradiction. If $\Sigma \cap (\partial E_k)_C$ is not empty then the maximum principle shows that $H$ is larger than or equal to the mean curvature of $(\partial E_k)_C$ at some point, which is strictly positive since geodesic spheres in a Cartan-Hadamard manifold are strictly convex.

The computation of the derivative $I_k'(v_0)$ is standard by making a variation supported around a point of $\Sigma \cap \operatorname{int}(E_k)$. The fact that left and right derivatives always exist follows from the convergence of isoperimetric regions of volumes $v_k \to v_0$ to an isoperimetric region of volume $v_0$, see [Hs], p. 171.

Let $J_k$ be the restriction of the isoperimetric profile of $M(\kappa)$ to the interval $(0, \operatorname{vol} E_k)$. Let $f(a)$, $g(a)$ be the inverse functions of $I_k$, $J_k$, respectively. We know that $g'(a) = J_k'(a)^{-1} = \big(2H_\kappa(a)\big)^{-1}$, and that, when $f'$ exists, $f'(a) = I_k'(a)^{-1} = (2H)^{-1}$, where $H$ is the mean curvature in the interior of $E_k$ of any isoperimetric region of volume $f(a)$. By Proposition 3.1, we obtain $g'(a) \geqq f'(a)$ a.e. Since $f$ is absolutely continuous (and $g$ is smooth), we have $g(a) \geqq f(a)$. It follows that $I_k \geqq J_k$.

If equality holds for some $v_0$, then for $a_0 = J_k(v_0) = I_k(v_0)$ we have $g(a_0) = f(a_0)$. Since $g' \geqq f'$ we obtain that $f \equiv g$ in the interval $(0, a_0)$ and so $H_\kappa(a_0)^{-1} = H(a_0)^{-1}$. If $\Omega_0$ is any isoperimetric region of volume $v_0$ then Proposition 3.1 implies that $\Omega_0$ is isometric to a half ball in $M(\kappa)$ of volume $v_0$.

Finally let $\Omega \subset M_C$ be relatively compact with smooth boundary. Then $\Omega \subset E_k$ for some $k$, and

$$\mathscr{P}(\Omega) \geqq I_k\big(\operatorname{vol}(\Omega)\big) \geqq I_{\mathbb{H}}\big(\operatorname{vol}(\Omega)\big).$$

If equality holds then $\Omega$ is an isoperimetric region in $E_k$ and $I_k\big(\operatorname{vol}(\Omega)\big) = I_{\mathbb{H}}\big(\operatorname{vol}(\Omega)\big)$. By the discussion in the above paragraph we have that $\Omega$ is isometric to a half ball in $M(\kappa)$ of volume $\operatorname{vol}(\Omega)$. $\square$

**Corollary 3.3.** *Suppose that $M$ is a three-dimensional Cartan-Hadamard manifold, $C \subset M$ a proper convex domain with smooth boundary, and $D$ bounded and of finite perimeter in $M \sim C$. Then*

$$\operatorname{area}(\partial D \sim \partial C)^3 \geqq 18\pi \operatorname{vol}(D)^2,$$

*and equality holds if and only if $D$ is a flat half ball.*

## 4. The relative isoperimetric inequality for a general convex set

In this section, $C$ will denote a bounded strictly convex body in $\mathbb{R}^3$. No assumption on the regularity of its boundary is made. We say that a convex body is *strictly convex* if its boundary does not contain a nontrivial segment [Sch], p. 77. Recall that $p$ is an *extreme point* of $C$ if it cannot be written in the form $p = \lambda x + (1 - \lambda) y$, with $x, y \in C$, $x \neq y$, and

$\lambda \in (0,1)$. Any point in the boundary of a strictly convex set is an extreme point. A *cap of $C$ around $p$* is a set of the form $C \cap H^+$, where $H^+$ is a closed half space with $p \in \operatorname{int} H^+$, [Sch], pp. 18–19.

For strictly convex sets we have the following result:

**Theorem 4.1.** *Let $C \subset \mathbb{R}^3$ be a proper convex domain which is strictly convex, and $\mathbb{H} \subset \mathbb{R}^3$ a half space. Then, for any $v > 0$,*

$$I_C(v) > I_{\mathbb{H}}(v).$$

*That is, equality never holds in the above inequality for these convex bodies.*

*Proof.* Using standard results on the Hausdorff metric, we can find a sequence of convex bodies with smooth boundary $C_k \subset \mathbb{R}^3$, with $C \subset C_k$ for all $n \in \mathbb{N}$, converging in the Hausdorff distance to $C$. Let $\Omega \subset (\mathbb{R}^3)_C$ be a relatively compact set and define $\Omega_k = \Omega \cap (\mathbb{R}^3)_{C_k}$. Then $\mathrm{vol}(\Omega_k) \to \mathrm{vol}(\Omega)$ and $\mathscr{P}_C(\Omega) \geqq \mathscr{P}_{C_k}(\Omega_k)$. Since the relative isoperimetric inequality (2.1) is satisfied in $(\mathbb{R}^3)_{C_k}$, we have $\mathscr{P}_C(\Omega) \geqq \mathscr{P}_{C_k}(\Omega_k) \geqq I_{\mathbb{H}}\big(\mathrm{vol}(\Omega_k)\big)$. Taking limits, we get $\mathscr{P}_C(\Omega) \geqq I_{\mathbb{H}}\big(\mathrm{vol}(\Omega)\big)$, and the isoperimetric inequality $I_C(v) \geqq I_{\mathbb{H}}(v)$ holds in $(\mathbb{R}^3)_C$.

To see that this inequality is always strict, consider a region $\Omega \subset (\mathbb{R}^3)_C$ such that equality $\mathscr{P}_C(\Omega) = I_{\mathbb{H}}\big(\mathrm{vol}(\Omega)\big)$ holds. Let $p$ be a point in the interior, relative to $\partial C$, of $\partial\Omega \cap \partial C$. The strict convexity implies that $p$ is an extreme point of $C$. By [Sch], Lemma 1.4.6, there is a cap $P^+$ of $C$ around $p$ contained in the interior of $\partial\Omega \cap \partial C$. Let $P^-$ be the half space obtained as the closure of the complement of the half space determining $P^+$. Consider the convex set $C' = C \cap P^-$, and $\Omega' = \Omega \cup (C \cap P^+)$. We have $\mathscr{P}_{C'}(\Omega') = \mathscr{P}_C(\Omega)$, and $\mathrm{vol}(\Omega) < \mathrm{vol}(\Omega')$. Hence

$$\mathscr{P}_{C'}(\Omega') = \mathscr{P}_C(\Omega) = I_{\mathbb{H}}\big(\mathrm{vol}(\Omega)\big) < I_{\mathbb{H}}\big(\mathrm{vol}(\Omega')\big),$$

and so $I_C\big(\mathrm{vol}(\Omega')\big) < I_{\mathbb{H}}\big(\mathrm{vol}(\Omega')\big)$. This is a contradiction, since we have already proved that in $(\mathbb{R}^3)_{C'}$ the isoperimetric inequality $I_C(v) \geqq I_{\mathbb{H}}(v)$ holds. $\square$

**Remark 4.2.** The first paragraph in the proof of Theorem 4.1 shows that the isoperimetric inequality $\mathrm{area}(\partial D \sim \partial C)^3 \geqq 18\pi\,\mathrm{vol}(D)^2$ holds for any region $D$ outside a convex set $C$ with nonsmooth boundary in $\mathbb{R}^3$. The authors have not characterized what happens in the equality case, and they believe that techniques different from the ones used in this paper should be employed.

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