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The relative isoperimetric inequality in Cartan-Hadamard 3-manifolds

To the memory of José F. Escobar

By *Jaigyoung Choe* at Seoul and *Manuel Ritoré* at Granada

Abstract. We prove that a region D outside a convex set C with smooth boundary in a three-dimensional Cartan-Hadamard manifold M satisfies the relative isoperimetric inequality $\operatorname{area}(\partial D \sim \partial C)^3 \geqq 18\pi \operatorname{vol}(D)^2$, with equality if and only if D is isometric to a Euclidean half ball. We also prove similar sharp inequalities when the sectional curvature of M is bounded above by a negative constant.

1. Introduction

Let $\mathbb{H} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geqq 0\}$ be the closed upper half space of $\mathbb{R}^n$. Given a set $D \subset \mathbb{H}$, let $D' \subset \mathbb{R}^n \sim \mathbb{H}$ be the mirror image of D across $\partial\mathbb{H}$, where $\sim$ denotes the set exclusion operator. Then the classical isoperimetric inequality implies that

$$\operatorname{area}\big(\partial(D \cup D')\big)^n \geqq n^n \omega_n \operatorname{vol}(D \cup D')^{n-1},$$

with equality if and only if $D \cup D'$ is a ball. Here ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, and area and vol, respectively, the $(n-1)$ and n dimensional Hausdorff measures. As a result one gets

$$\operatorname{area}(\partial D \sim \partial\mathbb{H})^n \geqq \frac{1}{2} n^n \omega_n \operatorname{vol}(D)^{n-1}, \tag{1.1}$$

with equality if and only if D is a half ball and ∂D meets $\partial\mathbb{H}$ in an orthogonal way.

This proposes the following interesting problems: Given a set D outside a convex set C in $\mathbb{R}^n$, does D satisfy inequality (1.1)? Does equality hold if and only if D is a half ball? Is inequality (1.1) still true outside a convex set in a Cartan-Hadamard manifold?

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Inequality (1.1) will be called the *relative isoperimetric inequality* for D. The proof of (1.1) in $\mathbb{R}^2$ is easy once one reflects the convex hull of D about its linear boundary. For $\mathbb{R}^n$, $n \geqq 3$, some partial results have been obtained in [K], [C1]. I. Kim [K] proved (1.1) for the epigraph of a C^2 convex function and the first-named author [C1] proved the relative isoperimetric inequality when $\partial D \cap \partial C$ is a graph which is symmetric about $(n-1)$ hyperplanes of $\mathbb{R}^n$.

In this paper we prove that the relative isoperimetric inequality (1.1) holds outside a closed convex set C with smooth boundary in a three-dimensional Cartan-Hadamard manifold M. More precisely, let C be a closed convex set with interior points in a Cartan-Hadamard manifold M^3 with sectional curvatures bounded above by a nonpositive constant κ. Let us denote by $M(\kappa)$ the three-dimensional simply connected space form with sectional curvature κ (Euclidean space for $\kappa = 0$ or hyperbolic space of sectional curvatures equal to κ otherwise), then we prove in Theorem 3.2 that, for any bounded set $D \subset M^3 \sim C$ with piecewise smooth boundary, we have

$$\text{area}(\partial D \sim \partial C) \geqq \frac{1}{2}\,\text{area}(\partial B), \tag{1.2}$$

where B is a geodesic ball in $M(\kappa)$ with $\text{vol}(B) = 2\,\text{vol}(D)$, and area is the 2 dimensional Hausdorff measure. Moreover, we show that equality holds in (1.2) if and only if D is isometric to a half ball in $M(\kappa)$.

The two-dimensional relative isoperimetric problem (1.2) was solved affirmatively in [C1]. The four-dimensional relative isoperimetric inequality (1.1) was obtained in [C2] following C. Croke [Cr].

Our proof for the three-dimensional case follows the scheme of B. Kleiner [Kl]: the isoperimetric inequality we intend to prove is equivalent to the comparison $I_C \geqq I_{\mathbb{H}}$ of the isoperimetric profiles I_C of $M \sim C$ and $I_{\mathbb{H}}$ of the complement of a closed half space $\mathbb{H} \subset M(\kappa)$. When an isoperimetric region Ω_0 of volume v_0 exists in $M \sim C$ and the profile I_C is smooth at v_0 then the derivative $I_C'(v_0)$ is equal to the constant mean curvature of the boundary $\partial\Omega_0 \sim \partial C$. Hence, in order to compare I_C and $I_{\mathbb{H}}$ we only need to compare their derivatives, that is, the mean curvatures of isoperimetric regions in both spaces for given volumes. Since both I_C and $I_{\mathbb{H}}$ are increasing functions, we only need to make this comparison for given areas. A main problem for this approach is that isoperimetric regions need not exist in the noncompact manifold $M \sim C$ but, as in [Kl], we avoid it by considering an exhaustion of $M \sim C$ by relatively compact sets where isoperimetric regions can be found. The mean curvature comparison is obtained in Proposition 3.1 by techniques similar to the ones used in [R], which are inspired by the lower bound for the Willmore functional obtained by Li and Yau [LY].

2. Preliminaries

Throughout this paper, $M = M^3$ will be a three-dimensional *Cartan-Hadamard manifold*, i.e., a complete, simply connected Riemannian manifold with nonpositive sectional curvatures. If κ is a nonpositive constant, then $M^3(\kappa)$ will denote the three-dimensional

Cartan-Hadamard manifold with sectional curvatures equal to κ. $M(0)$ is the Euclidean space $\mathbb{R}^3$ and $M^3(\kappa)$, for $\kappa < 0$, is the hyperbolic space with sectional curvatures equal to κ.

A *proper convex domain* will be the closure of an open convex set or, equivalently, a closed convex set with nonempty interior. For notational simplicity in the setting of relative geometry, let us adopt the following: if $C \subset M$ then we define $M_C := M \sim C$.

For $C \subset M$, define the *relative isoperimetric profile* of $M_C, I_C : \mathbb{R}^+ \to \mathbb{R}^+$, by

$$I_C(v) = \inf_D \{\text{area}(\partial D)_C : D \subset M_C, \partial D \text{ piecewise smooth, } \text{vol}(D) = v\},$$

where area is the 2 dimensional Hausdorff measure and vol denotes the Riemannian volume in M. An *isoperimetric region* $D \subset M_C$ is one for which the equality

$$\text{area}(\partial D)_C = I_C(\text{vol}(D))$$

holds. We remark that isoperimetric regions need not be unique for a given volume. One of the problems in our approach is that isoperimetric regions need not exist in M_C. We solve this problem by considering isoperimetric regions in bounded sets and letting these sets exhaust M_C.

Let $\mathbb{H} \subset M(\kappa)$ be a closed half space. From the reflection argument of the introduction and the isoperimetric inequality in $M(\kappa)$ we easily obtain that the isoperimetric regions in $M(\kappa) \sim \mathbb{H}$ are half balls touching orthogonally the boundary of $M(\kappa) \sim \mathbb{H}$.

If M is a three-dimensional Cartan-Hadamard manifold with sectional curvatures $K_M \leqq \kappa$, then the relative isoperimetric inequality (1.2) in M_C is equivalent to

$$\text{area}(\partial D)_C \geqq I_{\mathbb{H}}(\text{vol}(D)), \tag{2.1}$$

which in turn is equivalent to

$$I_C(v) \geqq I_{\mathbb{H}}(v), \tag{2.2}$$

for all $v > 0$. We also want to prove that equality holds in (2.1) if and only if D is a half ball in $M(\kappa)$.

Let $C \subset M$ be a closed set. Denote by $C_0^1(TM_C)$ the set of C^1 vector fields with compact support in M_C. For any $D \subset M_C$ define the *perimeter* of D relative to C as

$$\mathscr{P}_C(D) = \sup\left\{ \int_D \text{div}\, X : X \in C_0^1(TM_C), |X| \leqq 1 \right\},$$

where $|X|$ is the sup norm. If ∂D is piecewise smooth then $\mathscr{P}_C(D)$ coincides with $\text{area}(\partial D)_C$ since the supremum is taken over vector fields with compact support in M_C, and so the area of $\partial D \cap \partial C$ does not contribute to the perimeter.

Let E be the closure of a bounded open set in M_C. We wish to minimize the perimeter $\mathscr{P}_C$ amongst sets included in E subject to a volume constraint, i.e., given $v \in (0, \text{vol}(E))$, we want to find $\Omega_0 \subset E$ such that

$$\mathscr{P}_C(\Omega_0) \leqq \mathscr{P}_C(\Omega)$$

for any $\Omega \subset E$ with $\mathrm{vol}(\Omega) = \mathrm{vol}(\Omega_0) = v$. The existence of Ω_0 is guaranteed by the boundedness of E; see [G]. In fact we get

Lemma 2.1 (Existence and regularity). *Let $C \subset M^3$ be a proper convex domain with smooth boundary and let E be the closure of a bounded domain in M_C with $\overline{(\partial E)_C}$ smooth. Then, for any $v_0 \in (0, \mathrm{vol}(E))$ there is a set $\Omega_0 \subset E$ of volume v_0 minimizing the perimeter relative to C. Moreover*:

(i) ([GMT]) *$\partial\Omega_0$ has constant mean curvature and is smooth in the interior of E.*

(ii) ([Gr2], p. 263) *$\overline{(\partial\Omega_0)_C}$ meets $\partial C \sim \overline{(\partial E)_C}$ orthogonally and it is smooth.*

(iii) ([Wh], [StZ], Thm. 3.6) *If $(\partial E)_C$ is strictly convex then $(\partial\Omega_0)_C$ meets $(\partial E)_C$ tangentially and it is $C^{1,1}$ in a neighbourhood of $(\partial E)_C$.*

Remark 2.2. Observe that the above regularity theorem does not apply to points of $\overline{(\partial\Omega_0)_C} \cap \partial C \cap \overline{(\partial E)_C}$.

3. The relative isoperimetric inequality in M^3

The key method in proving the relative isoperimetric inequality is to obtain a mean curvature comparison in M. A minor problem in this method is that the boundary of a perimeter-minimizing domain Ω_0 inside a set E is only $C^{1,1}$ (see [Wh], [StZ]). As we shall see later, however, this is not a true problem, since a $C^{1,1}$ surface is C^2 almost everywhere by Rademacher's Theorem and a weak divergence theorem still holds for this kind of surfaces. We prove the mean curvature comparison in the following

Proposition 3.1. *Let M be a three-dimensional Cartan-Hadamard manifold with sectional curvatures bounded above by a nonpositive constant κ. Let $C \subset M$ be a proper convex domain with smooth boundary, $\Omega \subset M_C$ a domain such that $\Sigma := \overline{(\partial\Omega)_C}$ is a $C^{1,1}$ surface perpendicular to ∂C with mean curvature H, and let $H_\Sigma := \sup_\Sigma H$. Then*

$$(\kappa + H_\Sigma^2)\,\mathrm{area}(\Sigma) \geqq 2\pi, \tag{3.1}$$

with equality if and only if Ω is isometric to a geodesic half ball in $M(\kappa)$ of the same volume as Ω.

Proof. For the proof we shall follow the arguments in [R]. We assume first that Σ is a smooth surface. Consider a family $\{g_b\}$ of metrics, conformal to the Riemannian metric g of M, given by

$$g_b = e^{2u_b} g.$$

The relation between the sectional curvatures of g and g_b yields, at every point $p \in \Sigma$,

$$e^{2u_b}(K_s)_b = K_s - \left(|\nabla u_b|^2 - \sum_{i=1}^{2} \big(e_i(u_b)\big)^2\right) - \sum_{i=1}^{2} \nabla^2 u_b(e_i, e_i), \tag{3.2}$$

where $(K_s)_b$ and K_s are the sectional curvatures of the tangent plane to Σ for the metrics g_b and g, respectively, and $\{e_i\}$ is a g-orthonormal basis of the tangent plane to Σ. The gradient and the Hessian of u_b in (M, g) have been denoted by ∇u_b and $\nabla^2 u_b$, respectively.

The surface integral $\int_\Sigma (\lambda_1 - \lambda_2)^2 \, dA$, where λ_i are the principal curvatures of Σ, is invariant under conformal changes of the ambient metric. Since

$$(\lambda_1 - \lambda_2)^2 = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = 4(H^2 + K_s - K),$$

where K is the intrinsic Gauss curvature of the surface, we have

$$\int_\Sigma H^2 \, dA = \int_\Sigma H_b^2 \, dA_b + \int_\Sigma \left(e^{2u_b}(K_s)_b - K_s\right) dA + \int_\Sigma K \, dA - \int_\Sigma K_b \, dA_b,$$

where H_b, K_b are the mean and Gauss curvature of Σ with respect to the metric g_b, and dA, dA_b the area forms on Σ induced by g, g_b, respectively.

Let $\partial\Sigma$ be the boundary of the surface Σ. Integrating the relation $\Delta_\Sigma u_b = K - K_b e^{2u_b}$ on Σ, where Δ_Σ is the Laplacian for the metric induced on Σ by g, and applying the divergence theorem we get

$$\int_\Sigma K \, dA - \int_\Sigma K_b \, dA_b = \int_\Sigma \Delta_\Sigma u_b \, dA = -\int_{\partial\Sigma} v(u_b) \, ds,$$

where v is the inner unit normal to $\partial\Sigma$ on Σ. So we finally obtain

$$\int_\Sigma H^2 \, dA = \int_\Sigma H_b^2 \, dA_b + \int_\Sigma \left(e^{2u_b}(K_s)_b - K_s\right) dA - \int_{\partial\Sigma} v(u_b) \, ds. \tag{3.3}$$

Let us consider first the case $\kappa = 0$. Let $p_0 \in \partial\Sigma \cap \partial C$ and let d be the distance function to p_0. For any $b > 0$ we consider the family of conformal metrics $g_b = e^{2u_b} g$ given by the functions

$$u_b = \log\left(\frac{2b}{1 + b^2 d^2}\right). \tag{3.4}$$

In case $M^3 = \mathbb{R}^3$ these metrics are obtained by homothetically blowing up the spherical metric $\left(2/(1 + d^2)\right)^2 g$ by a factor of b.

Equation (3.2) gives in this case, for any $p \in \Sigma$,

$$e^{2u_b}(K_s)_b = K_s - \left(\frac{b^2}{1 + b^2 d^2}\right)^2 4d^2 + \left(\frac{b^2}{1 + b^2 d^2}\right) \sum_{i=1}^{2} \nabla^2 d^2(e_i, e_i). \tag{3.5}$$

By the Hessian Comparison Theorem we have $\nabla^2 d^2 \geqq 2$, so that from (3.5) we get

$$e^{2u_b}(K_s)_b \geqq K_s + e^{2u_b}. \tag{3.6}$$

Equality in (3.6) holds if and only if $\nabla^2 d^2(e_i, e_i) = 2$ along Σ, for $i = 1, 2$. By equation (3.3) and inequality (3.6) we obtain

$$H_\Sigma^2 \operatorname{area}(\Sigma) \geqq \int_\Sigma H^2\, dA \geqq \int_\Sigma dA_b - \int_{\partial\Sigma} v(u_b)\, ds. \tag{3.7}$$

Let us see that

$$\lim_{b\to\infty} \int_\Sigma dA_b = \lim_{b\to\infty} \int_\Sigma e^{2u_b}\, dA = 2\pi. \tag{3.8}$$

Let $r(p) := d(p, p_0)$ be the extrinsic distance to p_0. Fix $r_0 > 0$ small. Observe that e^{2u_b} converges uniformly to 0 in Σ out of the ball $B(p_0, r_0)$ when $b \to \infty$. For r_0 small enough, the modulus of the gradient of $r|_\Sigma$ over $\Sigma \cap B(p_0, r_0)$ is approximately 1, and the length of $\Sigma \cap \partial B(p_0, r)$, for $r \in (0, r_0)$, is approximately πr. By applying the coarea formula to $r|_\Sigma$ we get

$$\int_{\Sigma\cap B(p_0,r_0)} dA_b = \pi \int_0^{r_0} r\left(\frac{2b}{1+b^2r^2}\right)^2 dr + o(r_0) = 2\pi + o(r_0),$$

where $o(r_0)$ converges to 0 when $r_0 \to 0$. Letting $r_0 \to 0$ we get (3.8).

On the other hand, $\partial\Sigma$ is a curve (possibly disconnected) that bounds a region $R \subset \partial C$. Recall that v, the inner normal to $\partial\Sigma$ on Σ, is also the outer normal to ∂C. As $p_0 \in \partial C$ we see that a geodesic $\gamma_p : [0, t_p] \to M$ from p_0 to any other point $p \in \partial\Sigma \subset \partial C$ must satisfy $g(\gamma_p'(t_p), v(p)) \geqq 0$ by the convexity of C. Hence $g(\nabla d, v) \geqq 0$ for all $p \in \partial\Sigma$ and we deduce that

$$-v(u_b) = \frac{2b^2 d g(\nabla d, v)}{1 + b^2 d^2}$$

is a nonnegative function. Discarding the last summand in (3.7) and taking limits we obtain (3.1).

Let us consider now the equality case

$$H_\Sigma^2 \operatorname{area}(\Sigma) = 2\pi. \tag{3.9}$$

If (3.9) holds then there is also equality in (3.6), which implies that

$$\nabla^2 d^2(v, v) = 2g(v, v), \quad \text{for any } p \in \Sigma,\ v \in T_p\Sigma. \tag{3.10a}$$

We also have that $\lim_{b\to\infty} \int_{\partial\Sigma} \big(-v(u_b)\big)\, ds = 0$ since, after taking limits, there is equality in (3.7). The sequence $\{-v(u_b)\}_b$ is pointwise increasing and converges to $2v(d)/d$. By the Monotone Convergence Theorem

$$0 \leqq \int_{\partial\Sigma} \frac{2v(d)}{d}\, ds = \lim_{b\to\infty} \int_{\partial\Sigma} \big(-v(u_b)\big)\, ds = 0,$$

and so

$$g(\nabla d, v) \equiv 0 \quad \text{along } \partial\Sigma. \tag{3.10b}$$

Finally, if (3.9) holds then

(3.10c) $$H_\Sigma = H,$$

so that Σ is a constant mean curvature surface.

Let us see first that (3.10b) implies that R is totally geodesic. For any $p \in M$, define $\gamma_p : [0, t_p] \to M$ as the arc-length minimizing geodesic joining p_0 and p. If $p \in \partial\Sigma$, (3.10b) implies that γ_p is tangent to ∂C at p. By the convexity of C, we have that $\gamma_p([0, t_p]) \subset \partial C$. So we have $C_{p_0} := \bigcup_{p \in \partial\Sigma} \gamma_p([0, t_p]) \subset \partial C$. Hence R, the closed set in ∂C bounded by $\partial\Sigma$, satisfies

$$R \subset C_{p_0},$$

since the intersection of R and C_{p_0} is open and closed in R and every component of R intersects C_{p_0} as $\partial\Sigma \subset C_{p_0}$. Observe that when (3.9) holds, (3.10a), (3.10b), (3.10c) are satisfied for any choice of $p_0 \in \partial\Sigma$ and so $R \subset C_{p_0}$ *for any* $p_0 \in \partial\Sigma$. Let q be a point in the interior of R in ∂C. Joining q with three different points p_i, $i = 1, 2, 3$, of $\partial\Sigma$ not lying in the same geodesic, we have $R \subset C_{p_i}$ for all i. We get three vectors $v_i \in T_q\partial C$, $i = 1, 2, 3$, pairwise independent, so that $\sigma(v_i, v_i) = 0$, $i = 1, 2, 3$, where σ is the second fundamental form of ∂C. We conclude that ∂C is totally geodesic at q and, hence, over all of R.

Let us see now that (3.10a) implies that Ω is a flat region. With the same notation as in the previous paragraph, let $p \in \Sigma \sim \partial\Sigma$. If γ_p is transverse to Σ at p then $\nabla^2 d^2 = 2g$ at p and, by the standard comparison theorems in Riemannian Geometry, $\nabla^2 d^2 \equiv 2g$ along $\gamma_p([0, t_p])$. Let U be an open neighbourhood of p in $\Sigma \sim \partial\Sigma$ so that γ_q is tranverse to Σ at q for every $q \in U$. Then on the cone $\bigcup_{q \in U} \gamma_q([0, t_q])$ we have $\nabla^2 d^2 \equiv 2g$, so this cone is flat.

On the other hand, the set of unit vectors $v \in T_{p_0}M$ for which there is $p \in \Sigma \sim \partial\Sigma$ with $\gamma_p'(0) = v$ and γ_p does not meet Σ transversally at p, has measure zero on the unit sphere $S_{p_0} \subset T_{p_0}M$. This can be easily seen by considering the smooth map $\Sigma \sim \partial\Sigma \to S_{p_0}$ given by $p \mapsto \gamma_p'(0)$, whose critical points are precisely the points $p \in \Sigma \sim \partial\Sigma$ for which γ_p is not transverse to Σ at p, and by applying Sard's Theorem. We conclude that the set of points in $\bigcup_{p \in \Sigma \sim \partial\Sigma} \gamma_p([0, t_p])$ where the metric is flat is dense, and so $\bigcup_{p \in \Sigma} \gamma_p([0, t_p])$ is flat. Since $\Omega \subset \bigcup_{p \in \Sigma} \gamma_p([0, t_p])$, we conclude that Ω is flat.

Finally, let us consider (3.10c). Since Ω is flat, R is totally geodesic, and $H = H_\Sigma > 0$, we can apply the theorem by S. Montiel and A. Ros [MoR] which implies, for regions of this type, that

$$3\,\mathrm{vol}(\Omega) \leqq \frac{1}{H}\,\mathrm{area}(\Sigma),$$

with equality if and only if Ω is a half-ball in Euclidean space. But on Σ we have

(3.11) $$\Delta_\Sigma d^2 = 4 + 4Hd\langle \nabla d, N \rangle,$$

where N is the unit normal to Σ in the direction of the mean curvature vector of Σ. Integrating (3.11) on Σ, we obtain the classical Minkowski formula

$$3\,\mathrm{vol}(\Omega) = \frac{1}{H}\,\mathrm{area}(\Sigma).$$

This concludes the proof in the Euclidean case.

Let us consider now the hyperbolic case. By scaling the metric g of M we may assume that $\kappa = -1$, so that the inequality we intend to prove is

$$(-1 + H_\Sigma^2)\,\mathrm{area}(\Sigma) \geqq 2\pi, \tag{3.12}$$

with equality if and only if Ω is a half-ball in the hyperbolic space $M^3(-1)$ of constant sectional curvature equal to -1.

Consider again a point $p_0 \in \partial\Sigma$ and let d be the distance to p_0. We define the family of conformal metrics $g_b = e^{2u_b} g$, given by the functions

$$u_b = \log\left(\frac{2b}{(1-b^2) + (1+b^2)\cosh(d)}\right), \quad b > 1. \tag{3.13}$$

When M is the three-dimensional hyperbolic space, this family of metrics is obtained by homothetically expanding the spherical metric

$$\left(\frac{1-\tanh^2(d/2)}{1+\tanh^2(d/2)}\right)^2 g.$$

From (3.2) we obtain the following relation for the sectional curvatures of the tangent plane to Σ.

$$\begin{aligned} e^{2u_b}(K_s)_b = K_s + 1 + e^{2u_b} + \left(\frac{1+b^2}{(1-b^2) + (1+b^2)\cosh(d)}\right) \\ \times \left(\sum_{i=1}^{2} \nabla^2 \cosh(d)(e_i, e_i) - 2\cosh(d)\right), \end{aligned} \tag{3.14}$$

where $\{e_1, e_2\}$ is a g-orthonormal basis of the tangent plane to Σ. By the Hessian Comparison Theorem we know that $\nabla^2 \cosh(d) \geqq \cosh(d) g$. So we obtain

$$e^{2u_b}(K_s)_b \geqq K_s + e^{2u_b} + 1. \tag{3.15}$$

Equality holds in (3.15) if and only if $\nabla^2 \cosh(d)(e_i, e_i) = \cosh(d)$ for $i = 1, 2$. From equation (3.3) and inequality (3.15) we obtain

$$\int_\Sigma (-1 + H^2)\,dA \geqq \int_\Sigma dA_b - \int_{\partial\Sigma} \nu(u_b)\,ds.$$

As in the previous case one shows that $\lim_{b\to\infty} \int_\Sigma dA_b \to 2\pi$ and, by the convexity of C, that $-\nu(u_b) \geqq 0$, which yields the desired estimate (3.12).

If equality holds in (3.12) then we conclude, as in the Euclidean case, that $\nabla^2 \cosh(d)(v,v) = \cosh(d)$ for any unit tangent vector v to Σ at any point of Σ, that H is constant, and that $g(\nabla d, v) = 0$ at any point of $\partial\Sigma$.

Condition $\nabla^2 \cosh(d)(v,v) = \cosh(d)$ for any unit tangent vector v to Σ at any point of Σ implies, by the standard comparison theorems, that the metric g of Ω (i.e., on the cone over Σ with vertex p_0), has constant sectional curvatures equal to -1. Moreover, condition $g(\nabla d, v) = 0$ at any point of $\partial\Sigma$ implies that R is a totally geodesic surface. Finally, as Σ has constant mean curvature $H > 1$, we see as in [Mo], Theorem 9, by taking inner parallels, that

$$\int_{\Sigma} \big(\cosh(d) + H \sinh(d)\langle \nabla d, N\rangle\big)\, dA \geqq 0, \quad N \perp \Sigma$$

with equality if and only if Ω is a half ball in hyperbolic space. But since the metric of Ω is hyperbolic, we have $\nabla^2 \cosh(d) = \cosh(d) g$ in Ω so that integrating

$$\Delta_{\Sigma} \cosh(d) = 2\cosh(d) + 2H \sinh(d)\langle \nabla d, N\rangle$$

on Σ we get

$$\int_{\Sigma} \big(\cosh(d) + H \sinh(d)\langle \nabla d, N\rangle\big)\, dA = 0.$$

This completes the proof in the hyperbolic case $\kappa = -1$.

Finally, if Σ is merely $C^{1,1}$, the arguments in the proof apply without changes since the principal curvatures of Σ, and hence the mean and the Gauss curvature, are defined almost everywhere. The divergence theorem still holds under our weak hypothesis. □

By using the mean comparison result in Proposition 3.1 we can now prove the isoperimetric comparison theorem.

Theorem 3.2. *Let M be a three-dimensional Cartan-Hadamard manifold with sectional curvatures bounded above by a nonpositive constant κ, and let $M(\kappa)$ be the three-dimensional Cartan-Hadamard manifold with constant sectional curvatures equal to κ. Assume that $C \subset M$ is a proper convex domain with smooth boundary and that $\mathbb{H} \subset M(\kappa)$ is a half space. Then we have*

$$I_C \geqq I_{\mathbb{H}}, \tag{3.16}$$

which in turn implies that for any bounded finite perimeter set $D \subset M_C$ we have

$$\operatorname{area}(\partial D)_C \geqq I_{\mathbb{H}}\big(\operatorname{vol}(D)\big). \tag{3.17}$$

Moreover, equality holds in (3.17) *if and only if D is isometric to a half ball in $M(\kappa)$.*

Proof. Since the existence of isoperimetric regions is a crucial point of our arguments, but it is not guaranteed in the noncompact M_C, we first construct an exhaustion of M_C by relatively compact sets $\{E_k\}_{k \in \mathbb{N}}$. We take $p_0 \in \partial C$, and define $E_k = \bar{B}(p_0, r_k)_C$, where r_k is an increasing diverging sequence of positive numbers, and $\bar{B}(p_0, r_k)$ is the closed ball centered at p_0 of radius r_k.

Since E_k is bounded, isoperimetric regions exist on E_k for any given volume $v \in (0, \operatorname{vol} E_k)$. The boundary Σ of any isoperimetric region Ω in E_k satisfies the regularity properties of Lemma 2.1 provided there is no point in $\Sigma \cap \partial C \cap \overline{(\partial E_k)_C}$. Assume there is $q_0 \in \Sigma \cap \partial C \cap \overline{(\partial E_k)_C}$. The set ∂E_k is contained in the geodesic sphere $\partial B(p_0, r_k)$, which meets ∂C at q_0 at an angle less than or equal to $\pi/2$. Reflecting locally Ω with respect to ∂C and blowing up Ω and the metric from q_0, [Gr1], we obtain a cone in $\mathbb{R}^3$ which is area minimizing in a wedge of $\mathbb{R}^3$. This implies regularity if the angle is $\pi/2$ and it is not possible if the angle is less than $\pi/2$; see [G], Thm. 15.5.

Fix $k \in \mathbb{N}$ and assume that Ω_k is an isoperimetric region in E_k. The set Ω_k may have several components. Let $\Omega_k = \Omega_k^1 \cup \Omega_k^2$, where Ω_k^1 consists of the components of Ω_k touching ∂C (in an orthogonal way), and Ω_k^2 consists of the components of Ω_k disjoint from ∂C. Then by (3.1)

$$(\kappa + H^2_{(\partial\Omega_k^1)_C}) \operatorname{area}(\partial\Omega_k^1)_C \geqq 2\pi[\#\{\text{components of } \Omega_k^1\}] \geqq 2\pi.$$

On the other hand, by [Kl] or [R],

$$(\kappa + H^2_{(\partial\Omega_k^2)_C}) \operatorname{area}(\partial\Omega_k^2)_C \geqq 4\pi[\#\{\text{components of } \Omega_k^2\}] \geqq 0.$$

Adding both inequalities we get

$$(\kappa + H^2_{(\partial\Omega_k)_C}) \operatorname{area}(\partial\Omega_k)_C \geqq 2\pi,$$

and hence

$$H_{(\partial\Omega_k)_C} \geqq H_\kappa\big(\operatorname{area}(\partial\Omega_k)_C\big), \tag{3.18}$$

where $H_\kappa(a)$ denotes the mean curvature of a geodesic sphere of area $2a$ in $M(\kappa)$. By Proposition 3.1, equality holds in (3.18) if and only if Ω_k is isometric to a geodesic half ball in $M(\kappa)$.

Denote by I_k the isoperimetric profile of E_k. By standard arguments, [Hs], pp. 170–172, we have

- I_k is continuous and increasing,
- when I_k is smooth at v_0 then $I_k'(v_0) = 2H$, where H is the constant mean curvature in the interior of E_k of *any* isoperimetric region of volume v_0, and
- left and right derivatives of I_k exist everywhere.

Since I_k is a continuous monotone function with left and right derivatives at every point, it is absolutely continuous.

The continuity of I_k follows from the convergence of isoperimetric regions. To prove the monotonicity of I_k we just need to show that the constant mean curvature H of the boundary of an isoperimetric region Ω in the interior of E_k is positive. Let $\Sigma = \overline{\partial\Omega_C}$. If Σ does not touch $(\partial E_k)_C$ then there is an outer parallel ∂C_t to ∂C, which is tangent to Σ at some point and leaves Σ on one side. Standard comparison theorems show that the principal curvatures of ∂C_t are nonnegative. By the maximum principle, $H \geqq 0$. But in case $H = 0$, we obtain from the maximum principle that Σ and ∂C_t locally coincide and, by a connectedness argument, that a connected component of Σ is contained in ∂C_t, which gives us a contradiction. If $\Sigma \cap (\partial E_k)_C$ is not empty then the maximum principle shows that H is larger than or equal to the mean curvature of $(\partial E_k)_C$ at some point, which is strictly positive since geodesic spheres in a Cartan-Hadamard manifold are strictly convex.

The computation of the derivative $I_k'(v_0)$ is standard by making a variation supported around a point of $\Sigma \cap \operatorname{int}(E_k)$. The fact that left and right derivatives always exist follows from the convergence of isoperimetric regions of volumes $v_k \to v_0$ to an isoperimetric region of volume v_0, see [Hs], p. 171.

Let J_k be the restriction of the isoperimetric profile of $M(\kappa)$ to the interval $(0, \operatorname{vol} E_k)$. Let $f(a)$, $g(a)$ be the inverse functions of I_k, J_k, respectively. We know that $g'(a) = J_k'(a)^{-1} = \big(2H_\kappa(a)\big)^{-1}$, and that, when f' exists, $f'(a) = I_k'(a)^{-1} = (2H)^{-1}$, where H is the mean curvature in the interior of E_k of any isoperimetric region of volume $f(a)$. By Proposition 3.1, we obtain $g'(a) \geqq f'(a)$ a.e. Since f is absolutely continuous (and g is smooth), we have $g(a) \geqq f(a)$. It follows that $I_k \geqq J_k$.

If equality holds for some v_0, then for $a_0 = J_k(v_0) = I_k(v_0)$ we have $g(a_0) = f(a_0)$. Since $g' \geqq f'$ we obtain that $f \equiv g$ in the interval $(0, a_0)$ and so $H_\kappa(a_0)^{-1} = H(a_0)^{-1}$. If Ω_0 is any isoperimetric region of volume v_0 then Proposition 3.1 implies that Ω_0 is isometric to a half ball in $M(\kappa)$ of volume v_0.

Finally let $\Omega \subset M_C$ be relatively compact with smooth boundary. Then $\Omega \subset E_k$ for some k, and

$$\mathscr{P}(\Omega) \geqq I_k\big(\operatorname{vol}(\Omega)\big) \geqq I_{\mathbb{H}}\big(\operatorname{vol}(\Omega)\big).$$

If equality holds then Ω is an isoperimetric region in E_k and $I_k\big(\operatorname{vol}(\Omega)\big) = I_{\mathbb{H}}\big(\operatorname{vol}(\Omega)\big)$. By the discussion in the above paragraph we have that Ω is isometric to a half ball in $M(\kappa)$ of volume $\operatorname{vol}(\Omega)$. $\square$

Corollary 3.3. *Suppose that M is a three-dimensional Cartan-Hadamard manifold, $C \subset M$ a proper convex domain with smooth boundary, and D bounded and of finite perimeter in $M \sim C$. Then*

$$\operatorname{area}(\partial D \sim \partial C)^3 \geqq 18\pi \operatorname{vol}(D)^2,$$

and equality holds if and only if D is a flat half ball.

4. The relative isoperimetric inequality for a general convex set

In this section, C will denote a bounded strictly convex body in $\mathbb{R}^3$. No assumption on the regularity of its boundary is made. We say that a convex body is *strictly convex* if its boundary does not contain a nontrivial segment [Sch], p. 77. Recall that p is an *extreme point* of C if it cannot be written in the form $p = \lambda x + (1 - \lambda) y$, with $x, y \in C$, $x \neq y$, and

$\lambda \in (0,1)$. Any point in the boundary of a strictly convex set is an extreme point. A *cap of C around p* is a set of the form $C \cap H^+$, where H^+ is a closed half space with $p \in \operatorname{int} H^+$, [Sch], pp. 18–19.

For strictly convex sets we have the following result:

Theorem 4.1. *Let $C \subset \mathbb{R}^3$ be a proper convex domain which is strictly convex, and $\mathbb{H} \subset \mathbb{R}^3$ a half space. Then, for any $v > 0$,*

$$I_C(v) > I_{\mathbb{H}}(v).$$

That is, equality never holds in the above inequality for these convex bodies.

Proof. Using standard results on the Hausdorff metric, we can find a sequence of convex bodies with smooth boundary $C_k \subset \mathbb{R}^3$, with $C \subset C_k$ for all $n \in \mathbb{N}$, converging in the Hausdorff distance to C. Let $\Omega \subset (\mathbb{R}^3)_C$ be a relatively compact set and define $\Omega_k = \Omega \cap (\mathbb{R}^3)_{C_k}$. Then $\operatorname{vol}(\Omega_k) \to \operatorname{vol}(\Omega)$ and $\mathscr{P}_C(\Omega) \geqq \mathscr{P}_{C_k}(\Omega_k)$. Since the relative isoperimetric inequality (2.1) is satisfied in $(\mathbb{R}^3)_{C_k}$, we have $\mathscr{P}_C(\Omega) \geqq \mathscr{P}_{C_k}(\Omega_k) \geqq I_{\mathbb{H}}\big(\operatorname{vol}(\Omega_k)\big)$. Taking limits, we get $\mathscr{P}_C(\Omega) \geqq I_{\mathbb{H}}\big(\operatorname{vol}(\Omega)\big)$, and the isoperimetric inequality $I_C(v) \geqq I_{\mathbb{H}}(v)$ holds in $(\mathbb{R}^3)_C$.

To see that this inequality is always strict, consider a region $\Omega \subset (\mathbb{R}^3)_C$ such that equality $\mathscr{P}_C(\Omega) = I_{\mathbb{H}}\big(\operatorname{vol}(\Omega)\big)$ holds. Let p be a point in the interior, relative to ∂C, of $\partial\Omega \cap \partial C$. The strict convexity implies that p is an extreme point of C. By [Sch], Lemma 1.4.6, there is a cap P^+ of C around p contained in the interior of $\partial\Omega \cap \partial C$. Let P^- be the half space obtained as the closure of the complement of the half space determining P^+. Consider the convex set $C' = C \cap P^-$, and $\Omega' = \Omega \cup (C \cap P^+)$. We have $\mathscr{P}_{C'}(\Omega') = \mathscr{P}_C(\Omega)$, and $\operatorname{vol}(\Omega) < \operatorname{vol}(\Omega')$. Hence

$$\mathscr{P}_{C'}(\Omega') = \mathscr{P}_C(\Omega) = I_{\mathbb{H}}\big(\operatorname{vol}(\Omega)\big) < I_{\mathbb{H}}\big(\operatorname{vol}(\Omega')\big),$$

and so $I_C\big(\operatorname{vol}(\Omega')\big) < I_{\mathbb{H}}\big(\operatorname{vol}(\Omega')\big)$. This is a contradiction, since we have already proved that in $(\mathbb{R}^3)_{C'}$ the isoperimetric inequality $I_C(v) \geqq I_{\mathbb{H}}(v)$ holds. □

Remark 4.2. The first paragraph in the proof of Theorem 4.1 shows that the isoperimetric inequality $\operatorname{area}(\partial D \sim \partial C)^3 \geqq 18\pi \operatorname{vol}(D)^2$ holds for any region D outside a convex set C with nonsmooth boundary in $\mathbb{R}^3$. The authors have not characterized what happens in the equality case, and they believe that techniques different from the ones used in this paper should be employed.

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Korea Institute for Advanced Study, 207-43 Cheongryangri 2-dong, Dongdaemun-gu,
Seoul 130-722, South-Korea
e-mail: choe@kias.re.kr

Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada,
18071 Granada, España
e-mail: ritore@ugr.es

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